Arbres, cartes et nombres de Hurwitz

GILLES SCHAEFFER CNRS & École Polytechnique

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Plan de l'exposé

Revêtements ramifiés et cartes

Cartes et arbres

Énumération d'arbres et formule d'Hurwitz

Revêtements et cartes aléatoires

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Ramified coverings of the sphere by itself

Let $B = \{z \mid |z| < 1\} \subset \mathbb{C}$ and let $\sim$ denote equivalence up to homeomorphisms

A mapping $\phi : \mathcal{D} \to \mathcal{I}$ is a covering if, for all x in $\mathcal{I}$ there exists $n \geq 1$ and a neighborhood V of x such that $\phi^{-1}(V) \sim B \times \{1, \ldots, n\}$,

and the restriction of ϕ to each sheet B_i (connected component of the preimage)
is an homeomorphism $\phi_{|B_i} : B_i \xrightarrow{\sim} B$.

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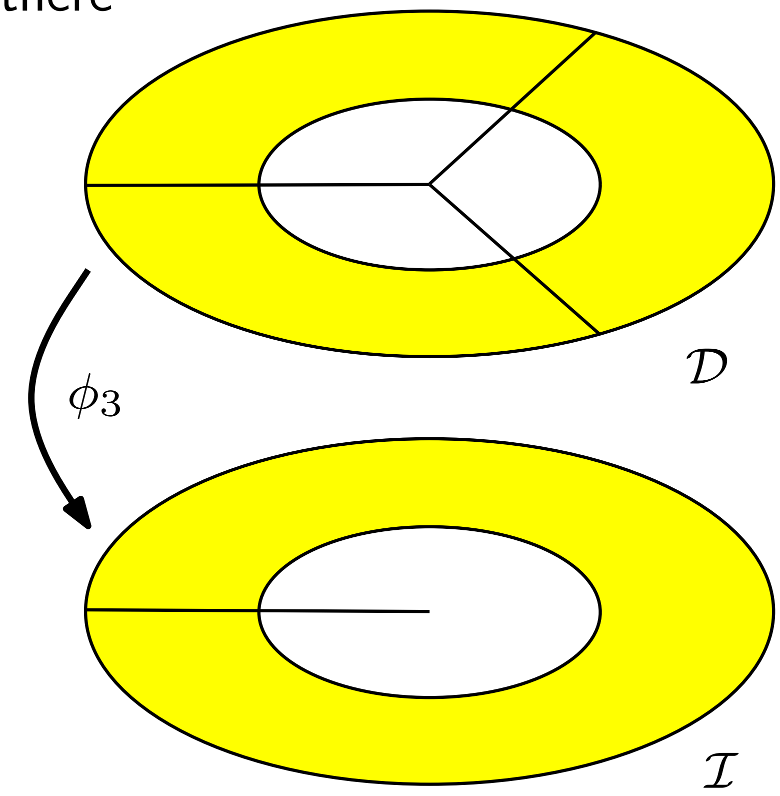

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Consider $\phi_k : A_r \to A_{r^k}$ with $\phi_k(z) = z^k$.

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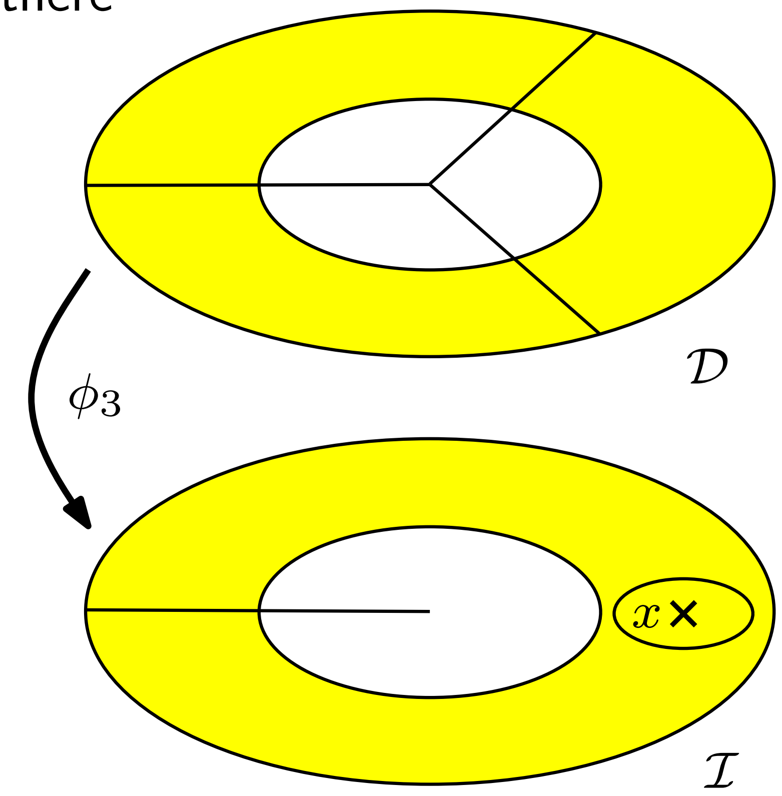

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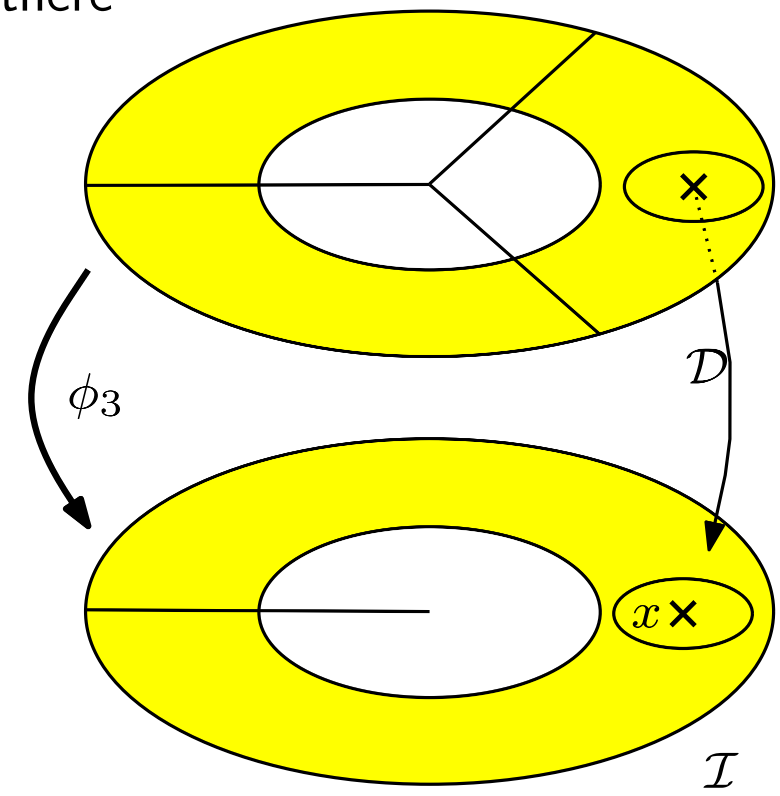

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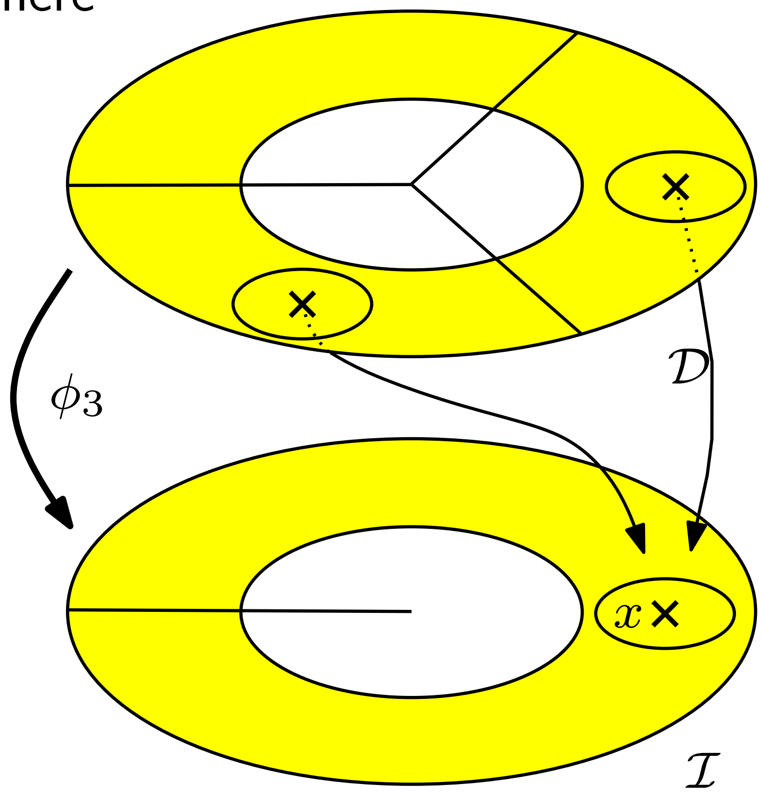

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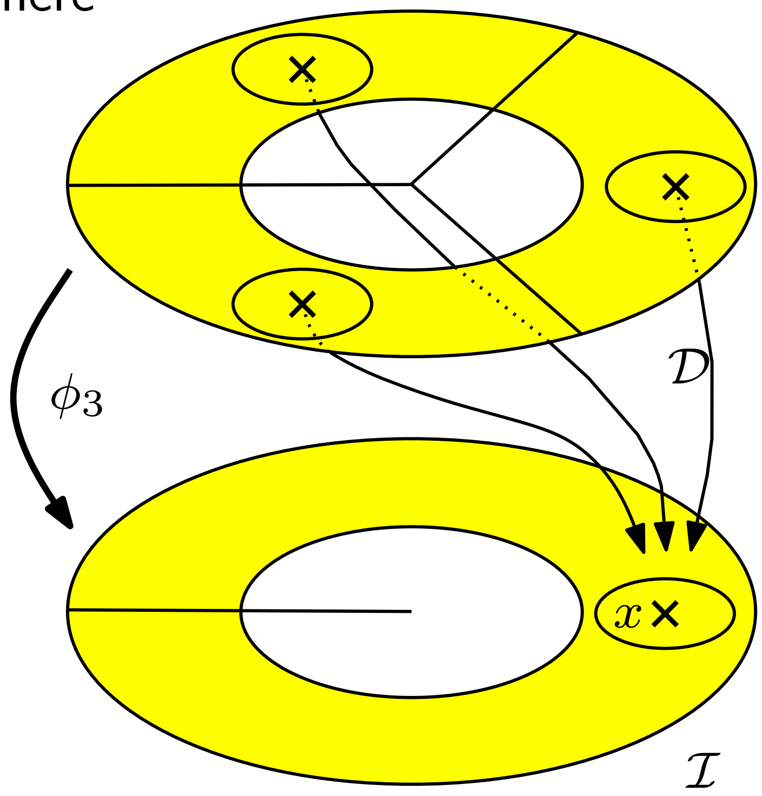

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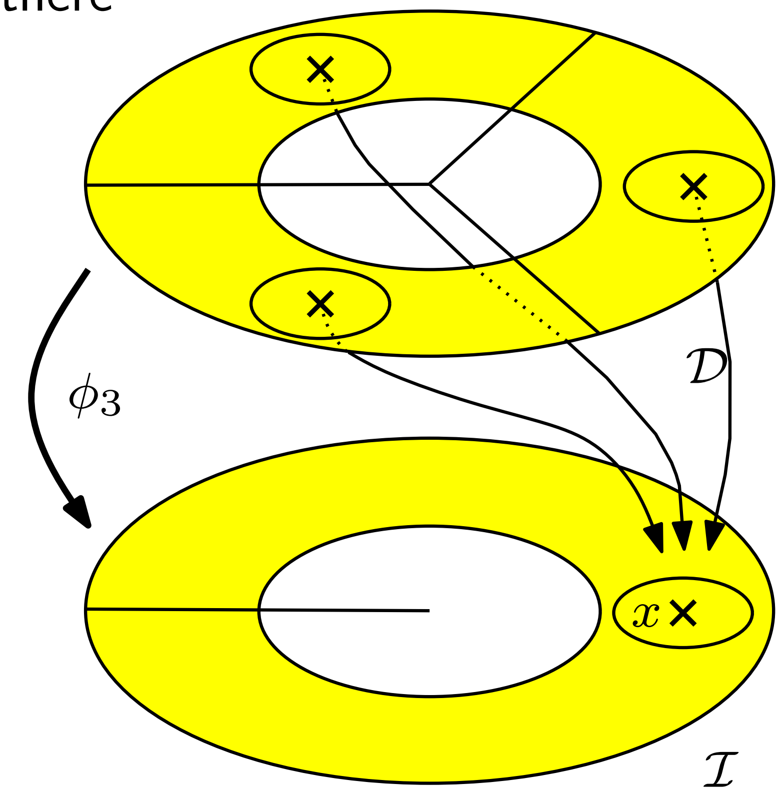

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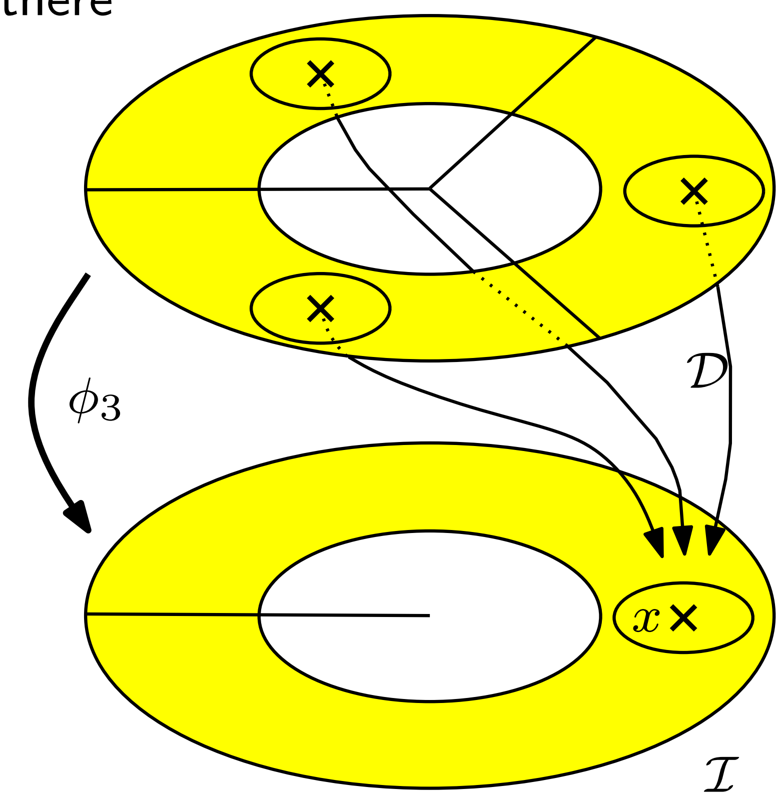

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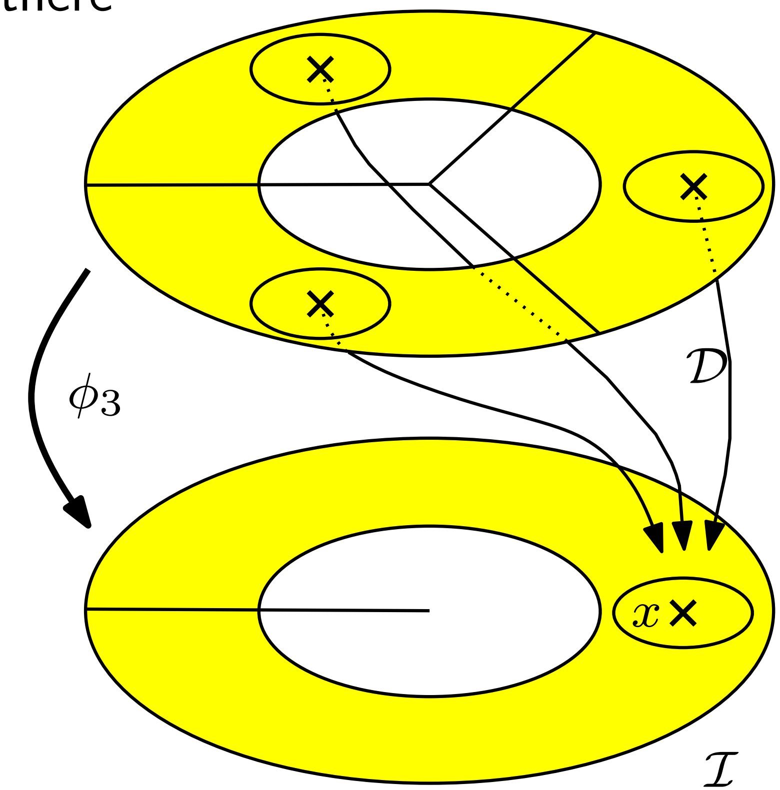

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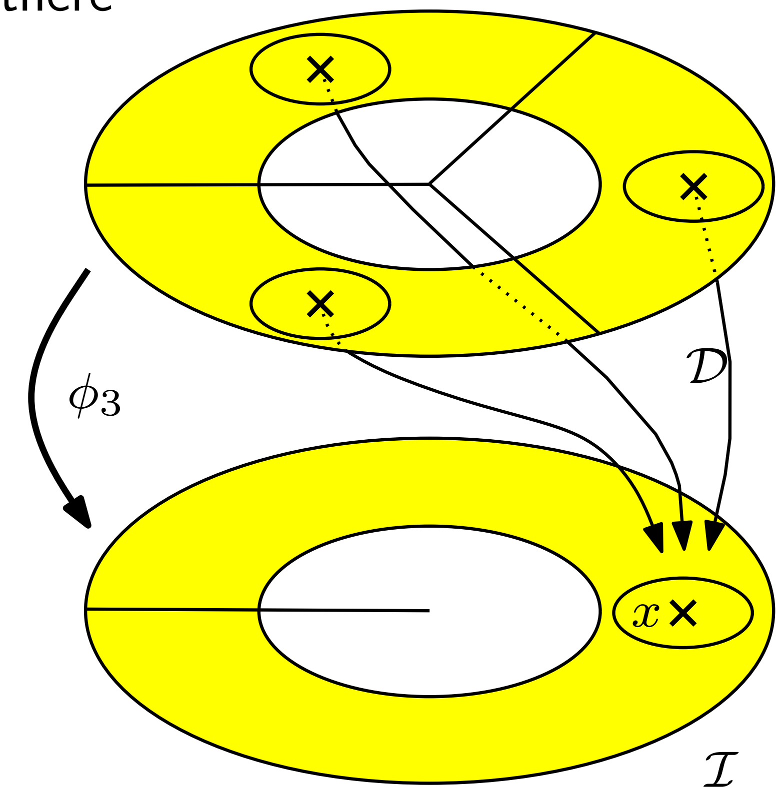

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What is we try to extend from A_r to B?

Ramified coverings of the sphere by itself

Recall $\phi_k : A_r \to A_{r^k}$ with $\phi_k(z) = z^k$.

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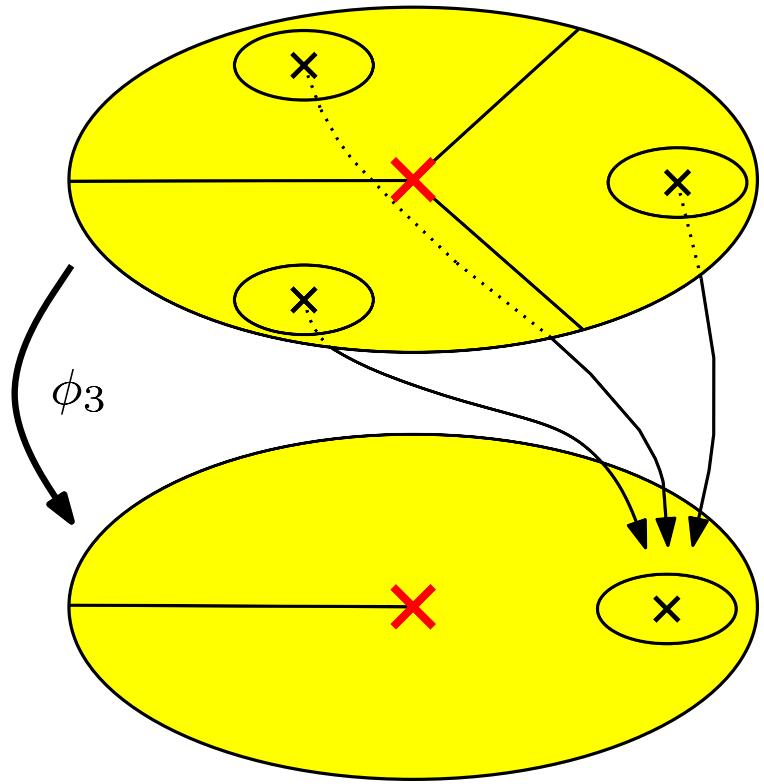
The mapping $\phi_k : B^* \to B^*$ is a covering,

but not $\phi_k : B \to B$.



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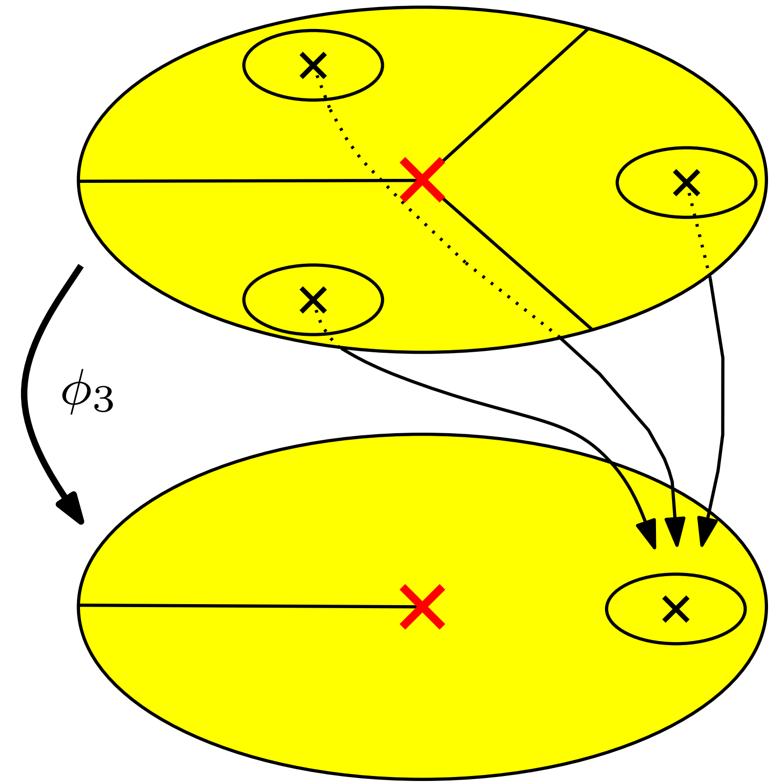

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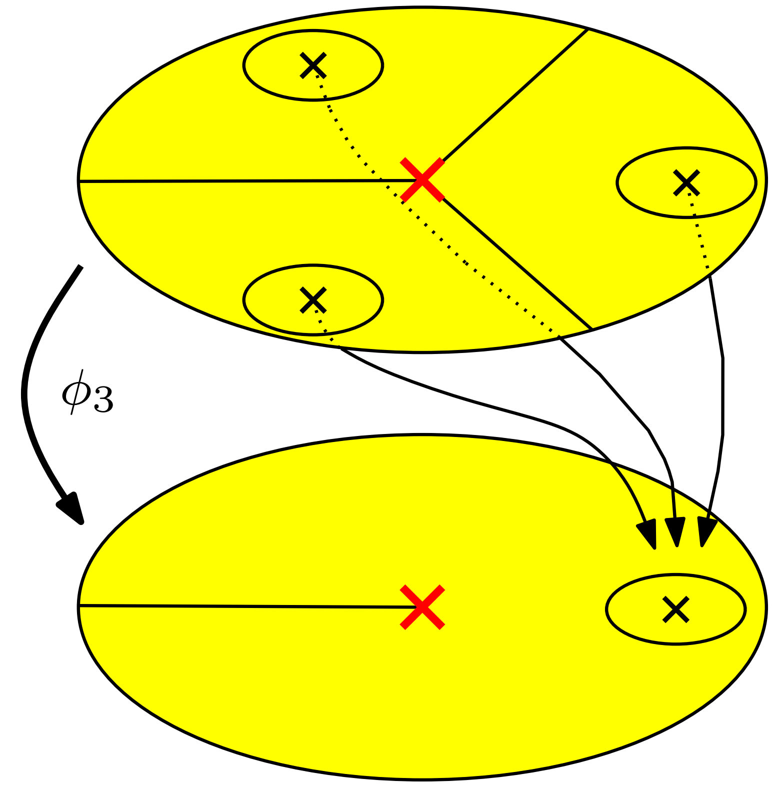

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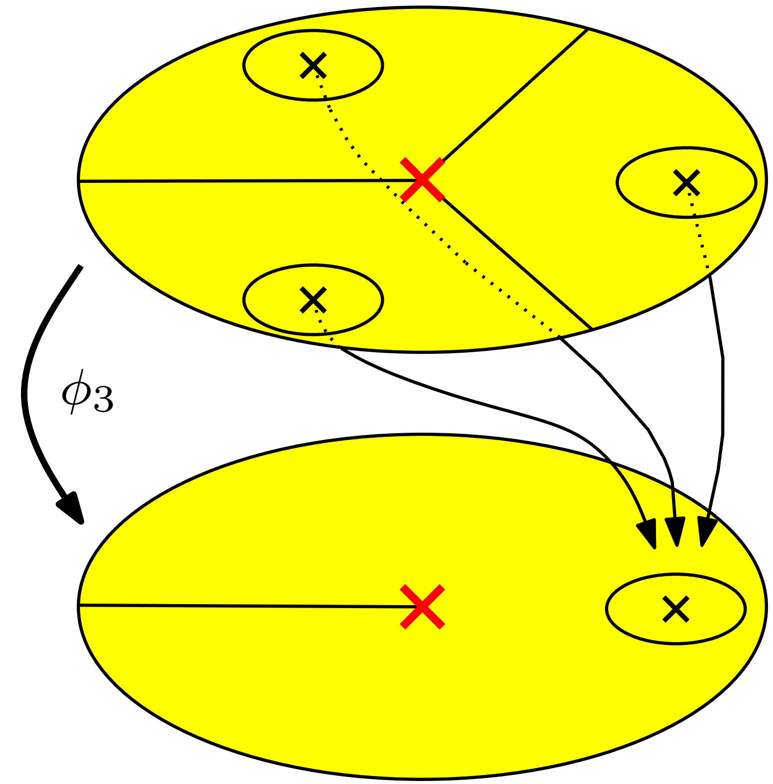

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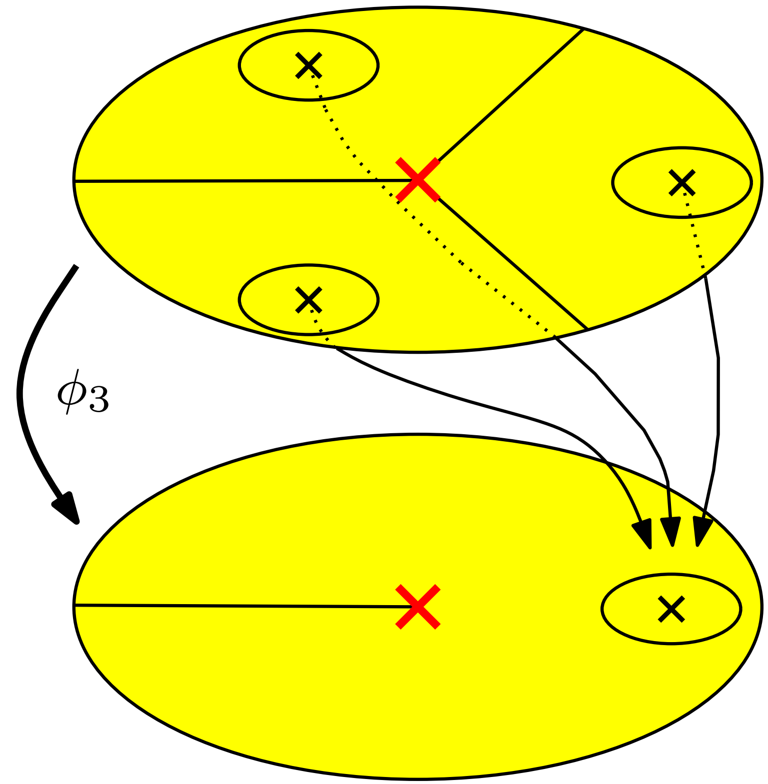

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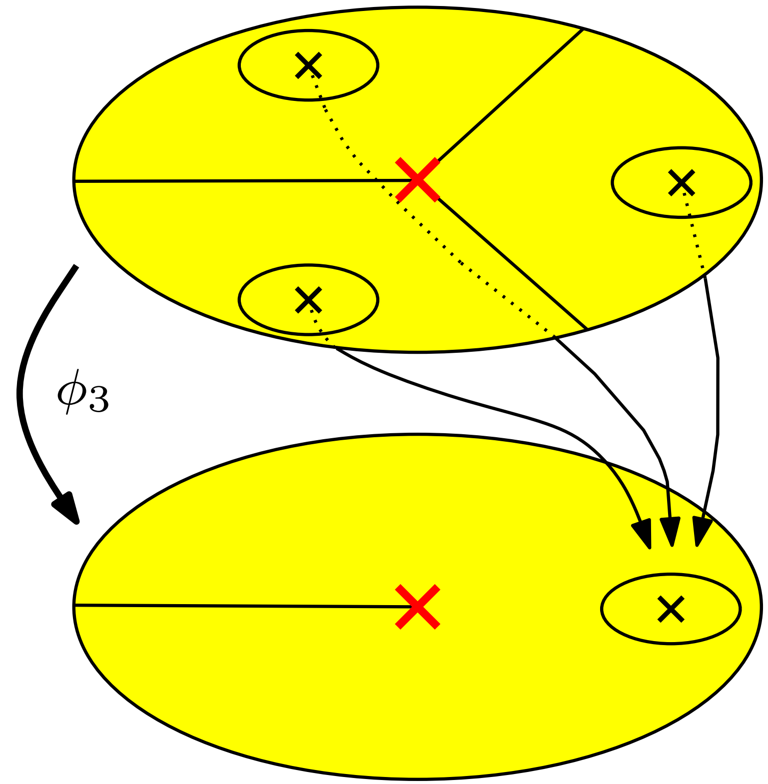

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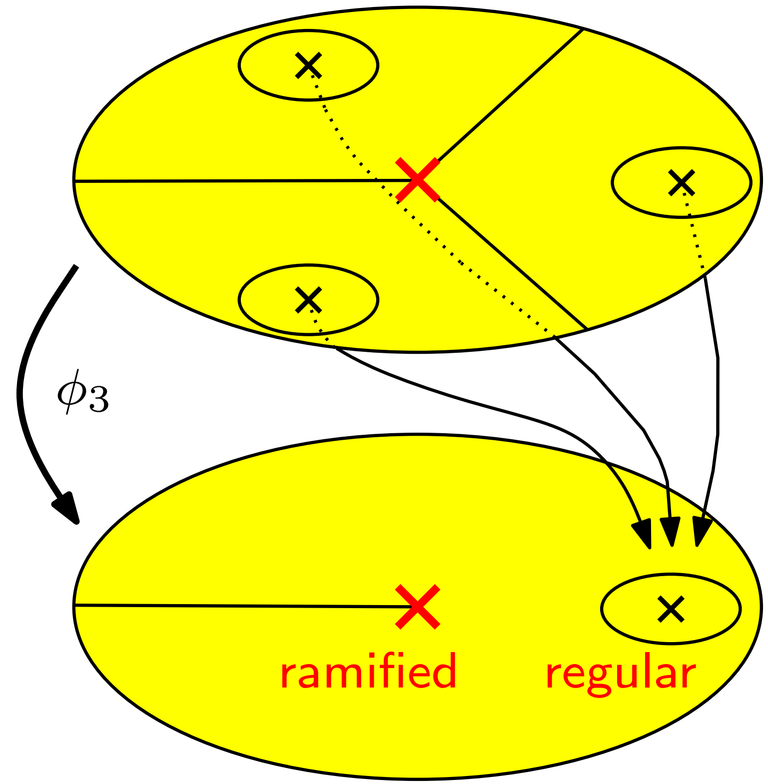

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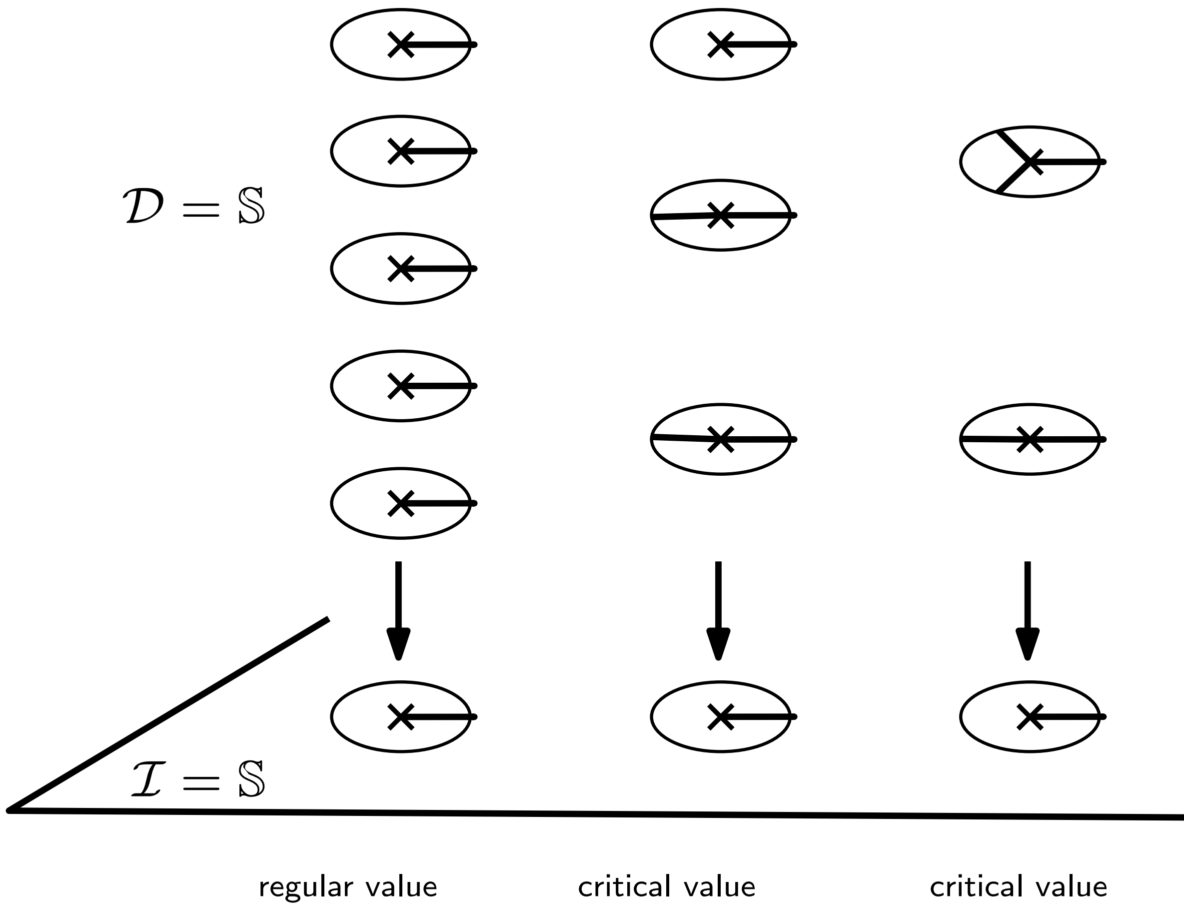
Ramified coverings of the sphere by itself (Cont'd)

A mapping ϕ is a ramified covering of $\mathbb{S}$ by $\mathbb{S}$ if there exists a finite subset $X = \{x_1, \ldots, x_p\}$ such that:

- $\phi_{\mathbb{S}\setminus\phi^{-1}(X)}$ is a covering, and
- ϕ is ramified over each x_i



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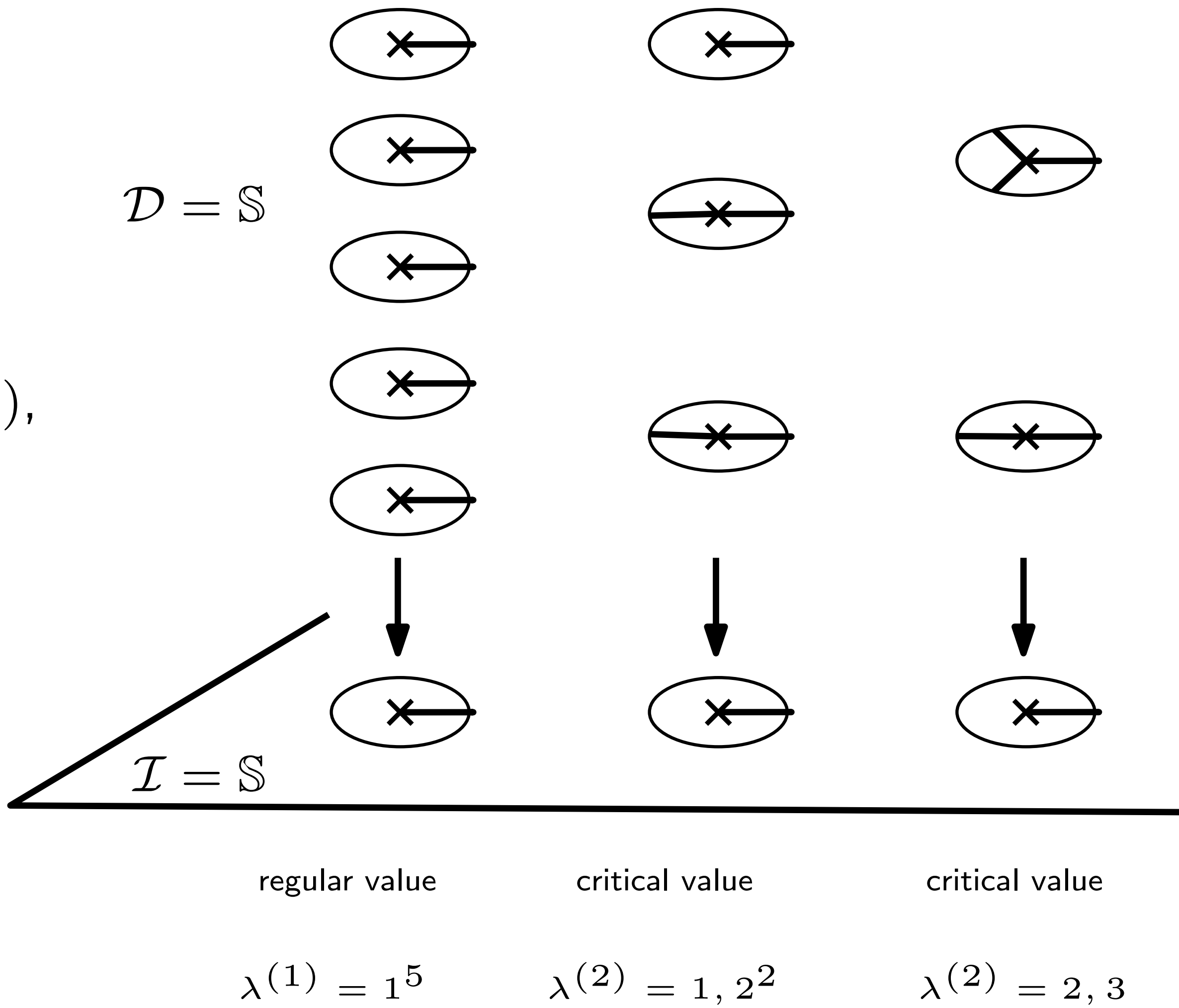

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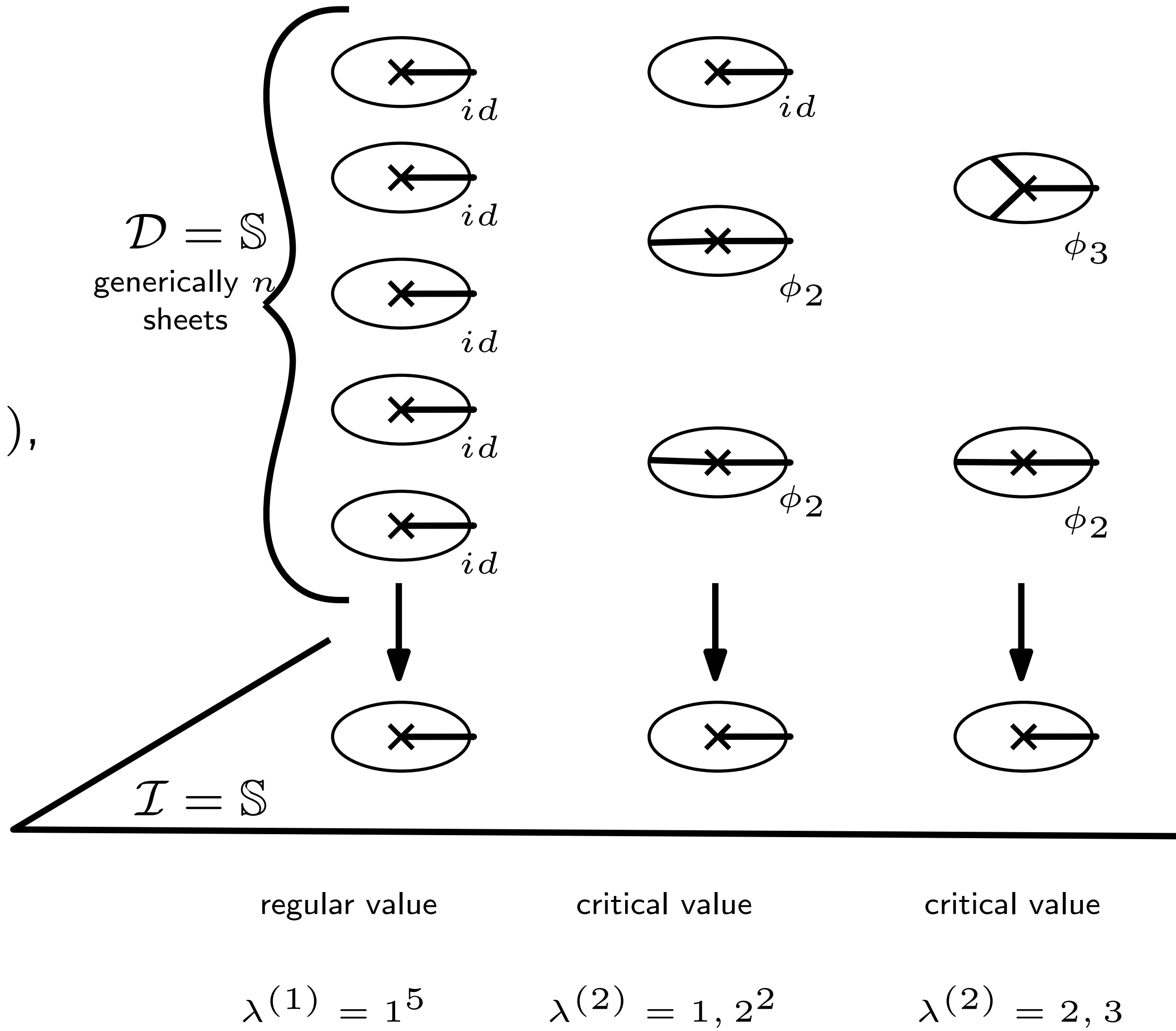

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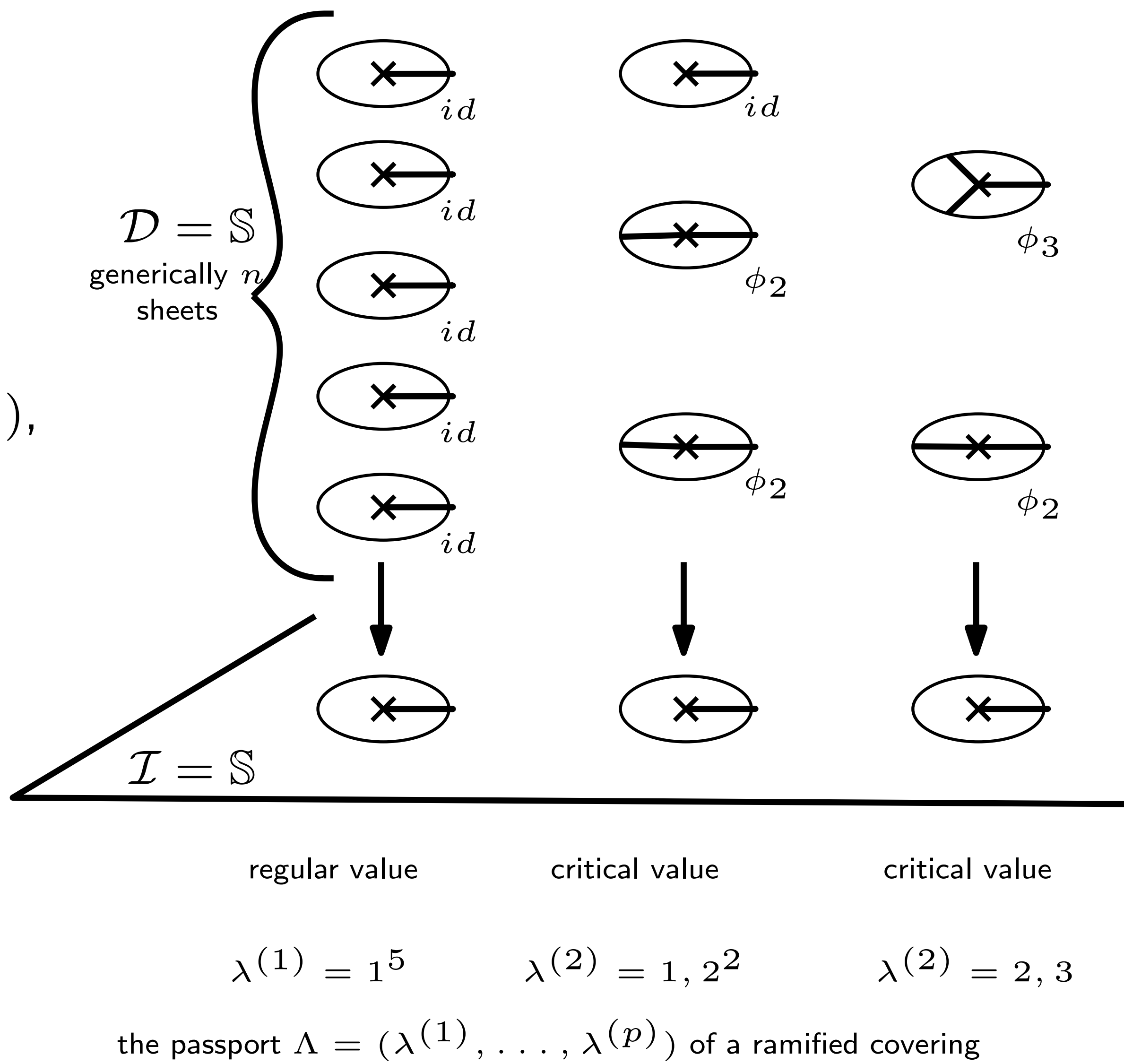
The ramification type over a critical value x_i is the partition $\lambda^{(i)}$

The passport of a ramified covering is the list $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$



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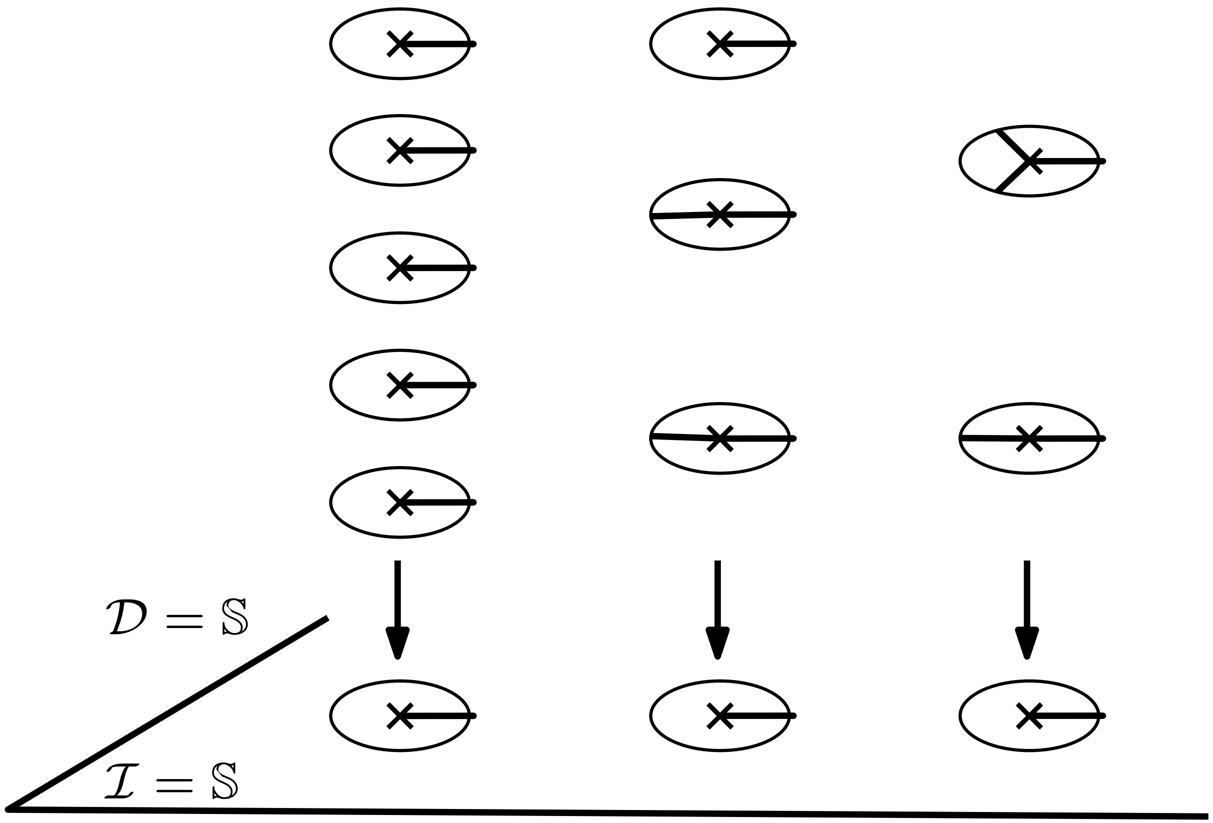

regular value | critical value | critical value

$\lambda^{(1)} = 1^5$ | $\lambda^{(2)} = 1, 2^2$ | $\lambda^{(2)} = 2, 3$

the passport $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ of a ramified covering

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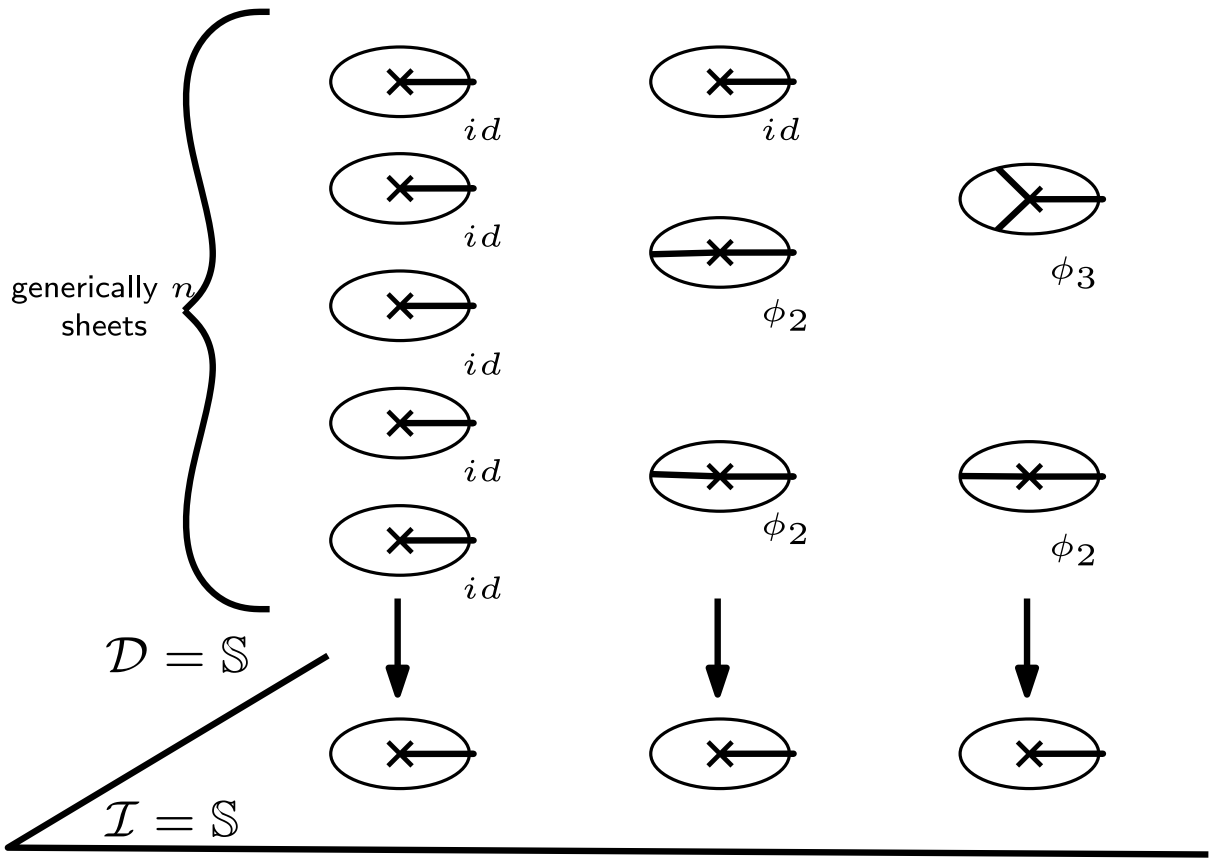

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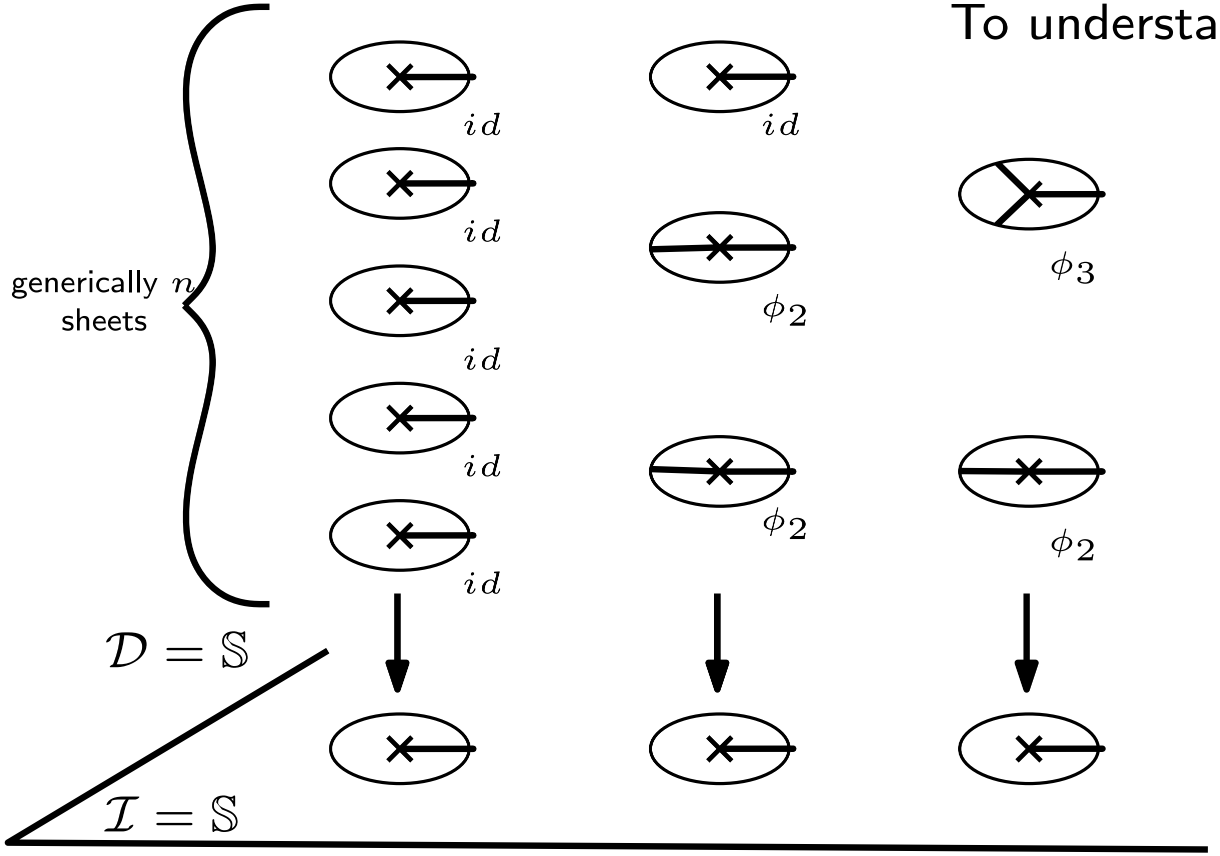

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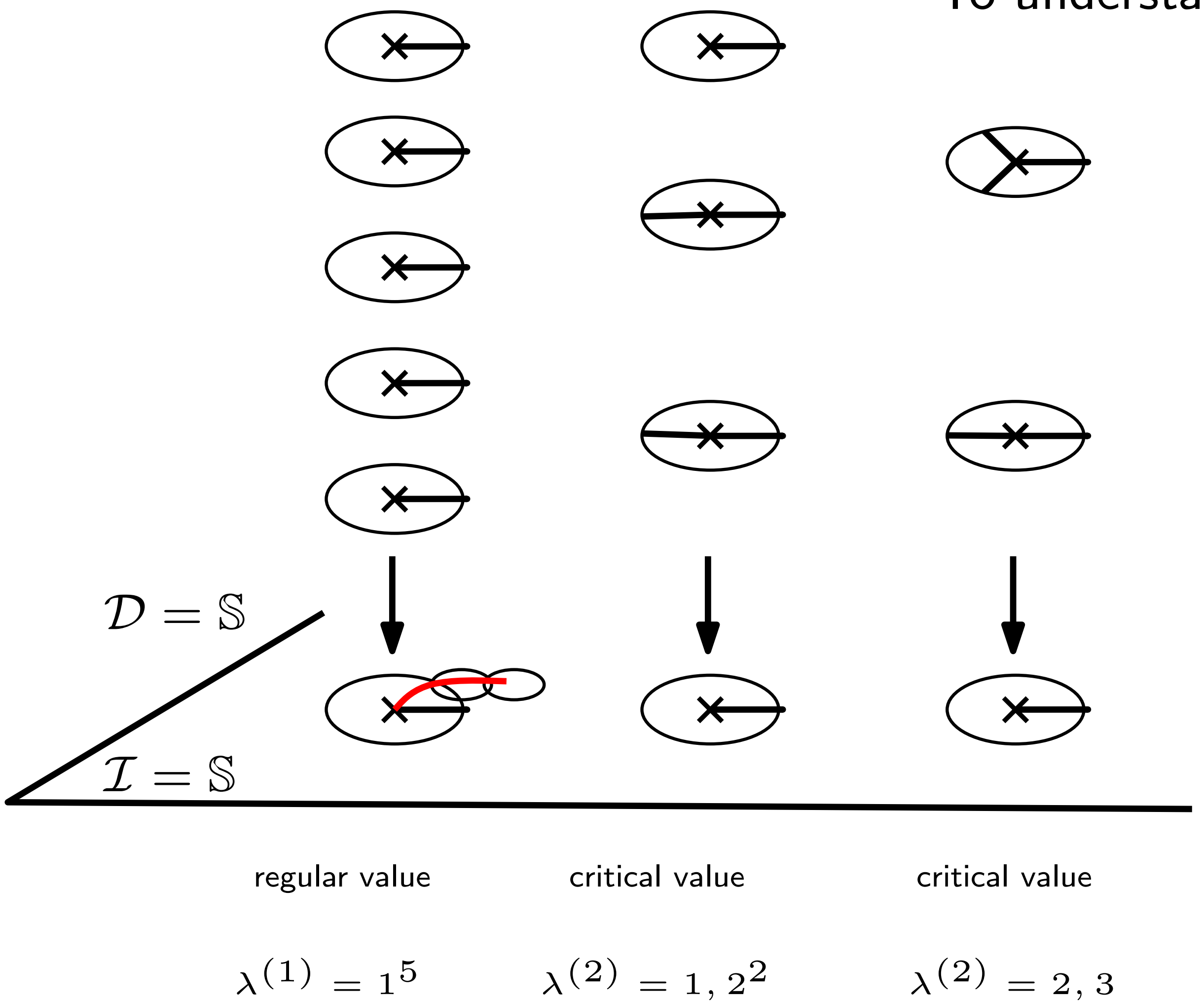

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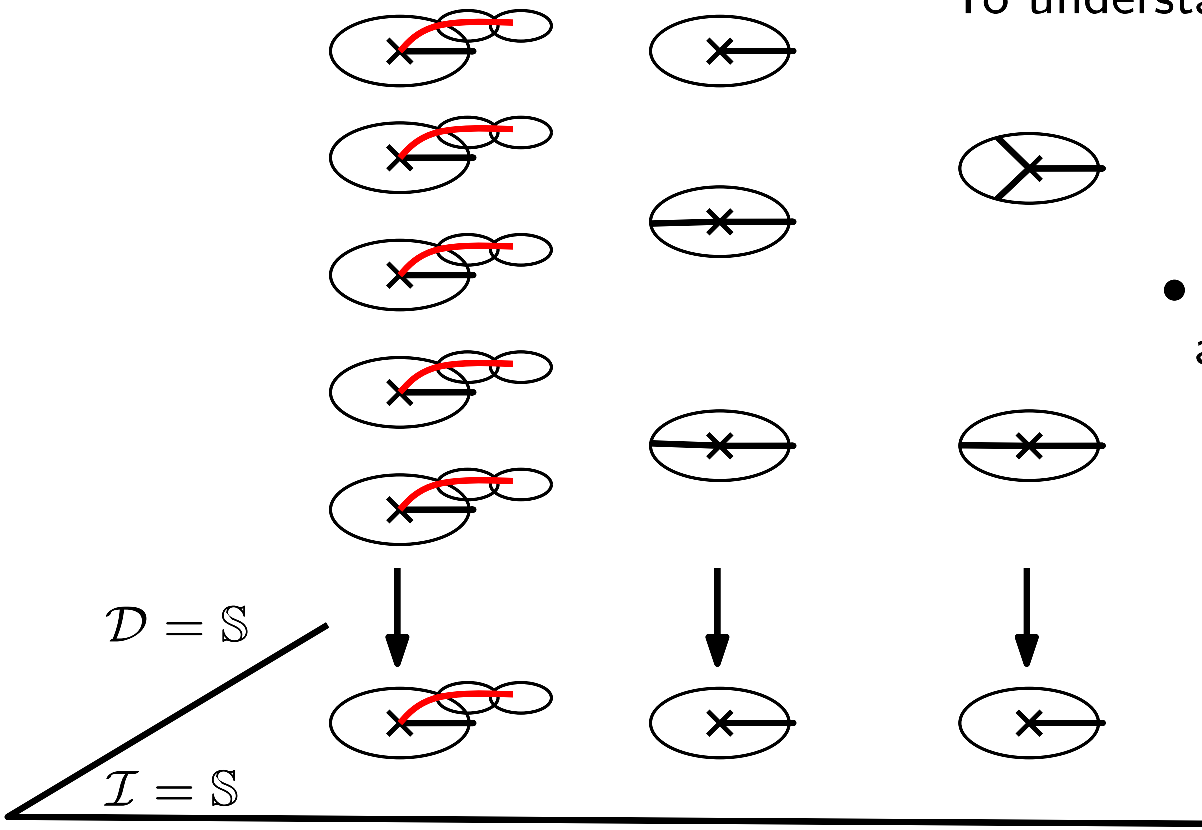

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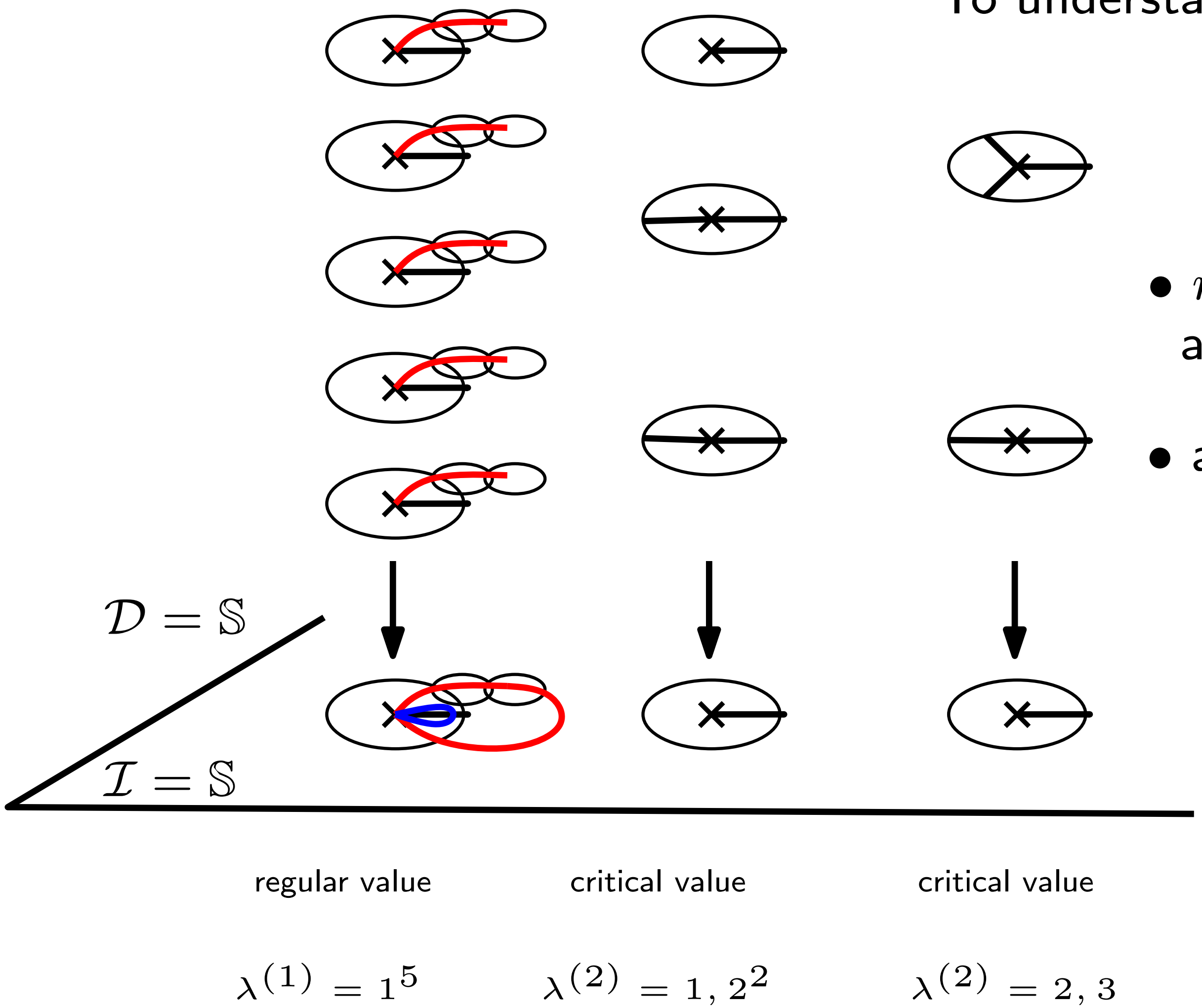

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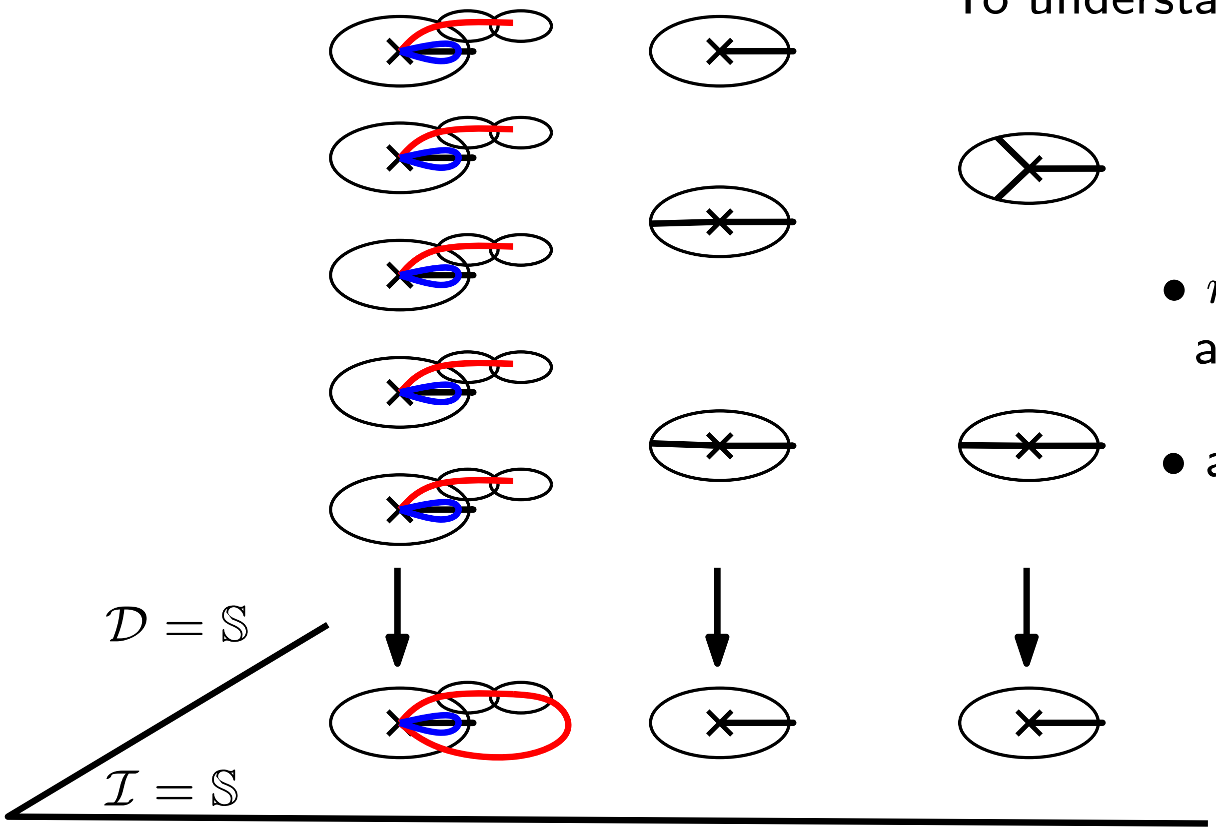

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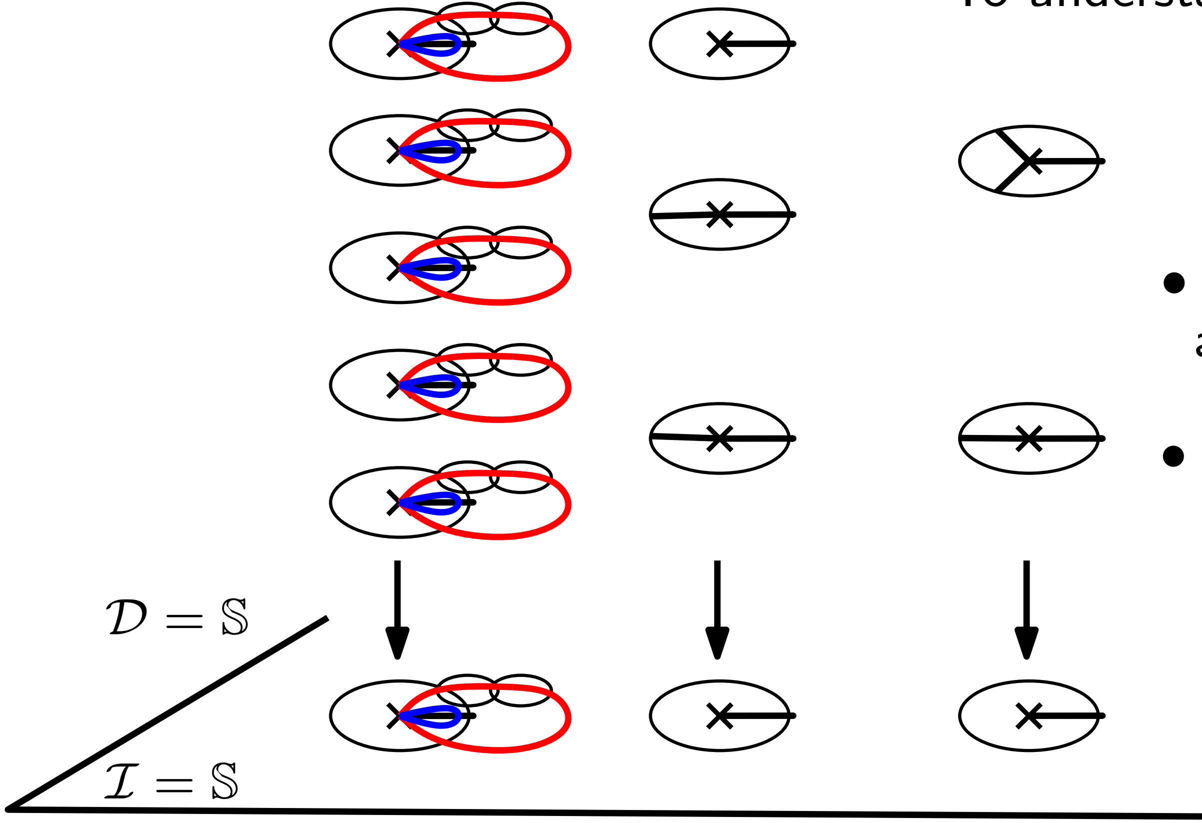

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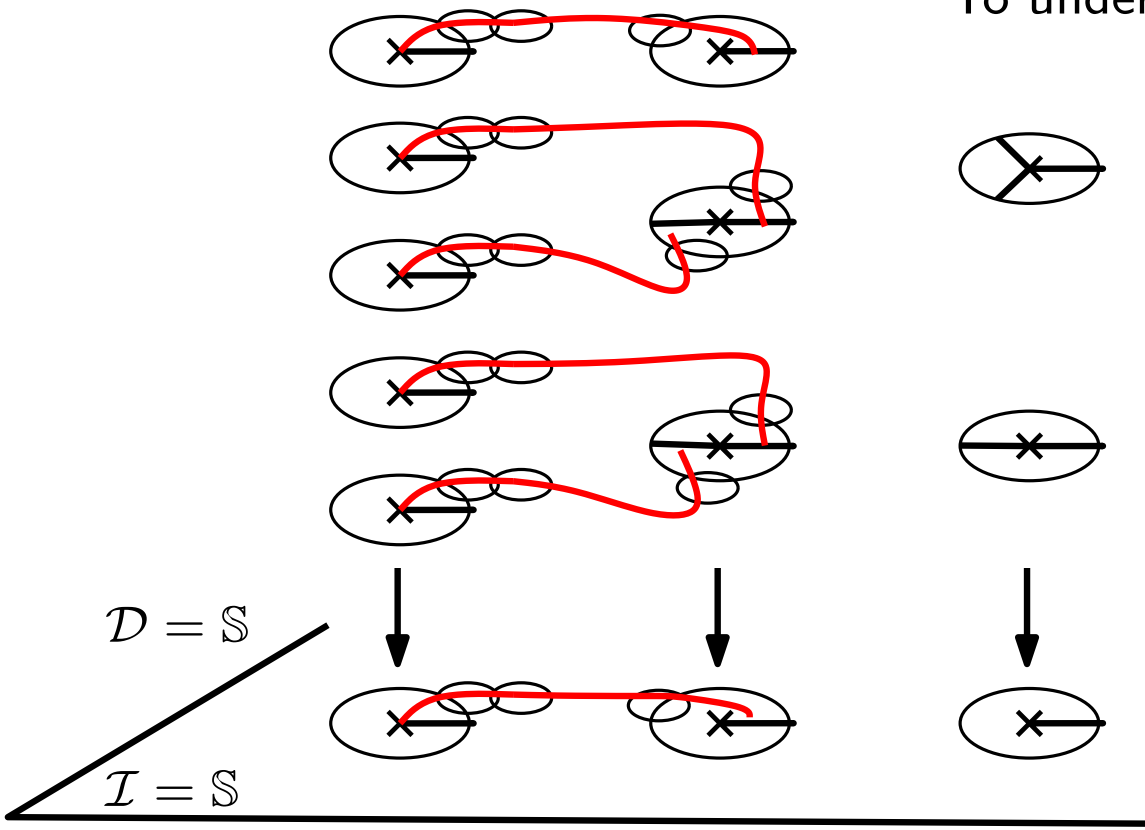
- a contractible loop on $\mathcal{I}$ yields n contractible loops on $\mathcal{D}$ but if we wind around critical points



regular value	critical value	critical value
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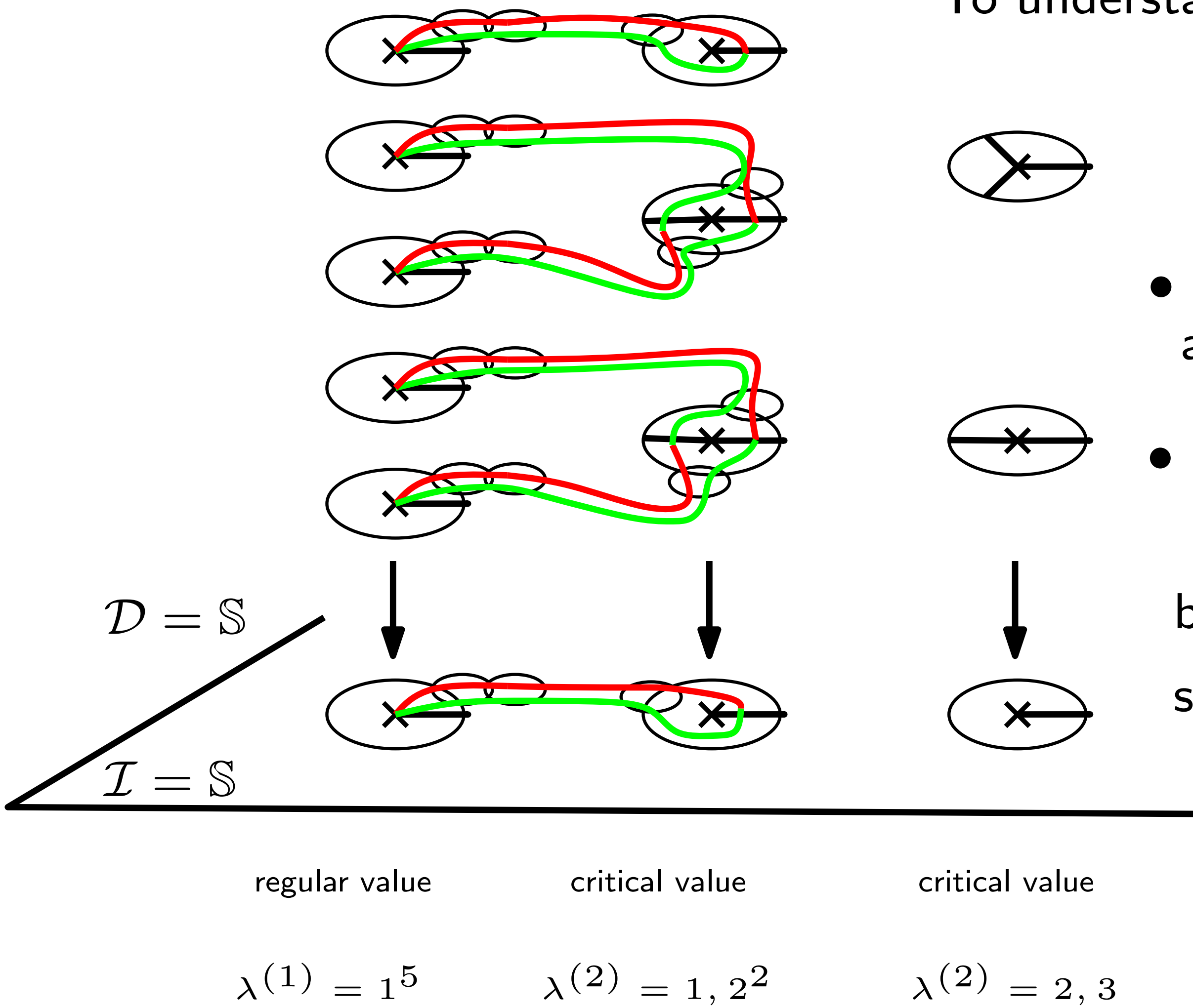
Ramified coverings of the sphere by itself (Cont'd)



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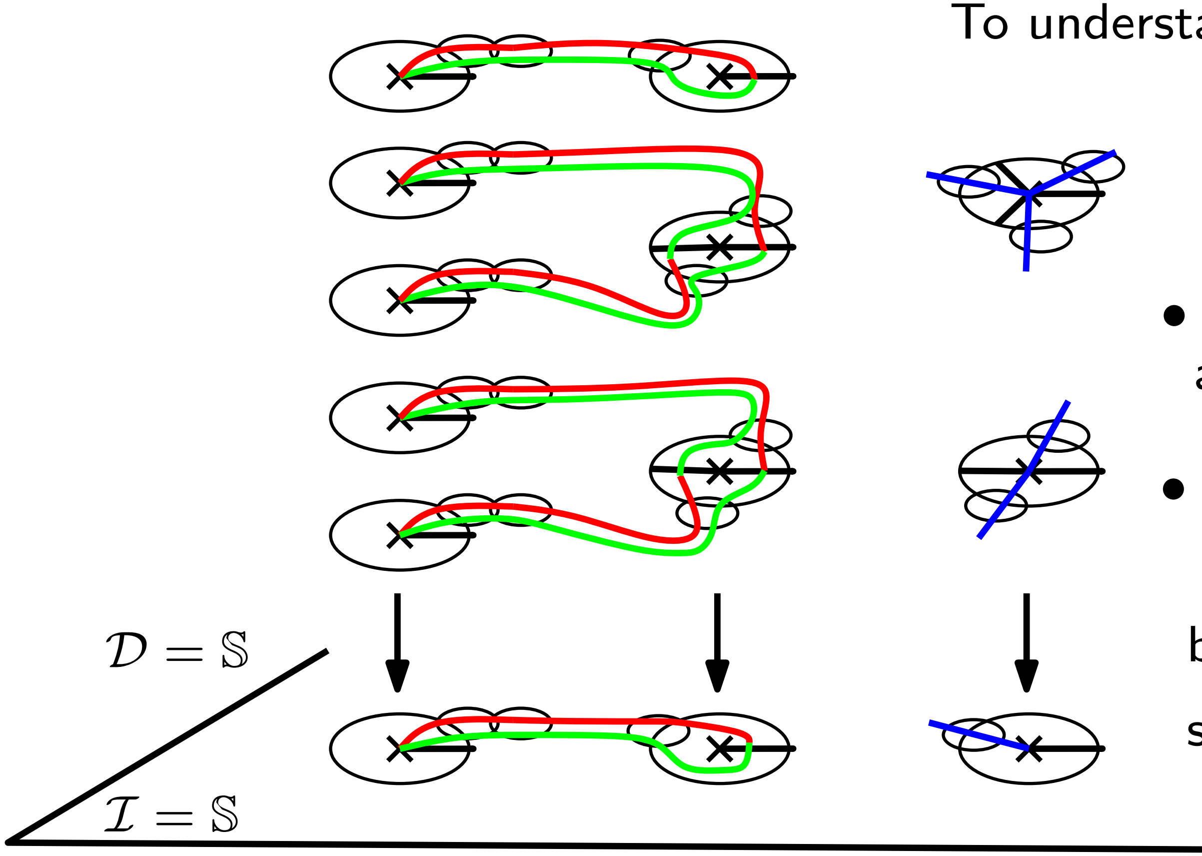

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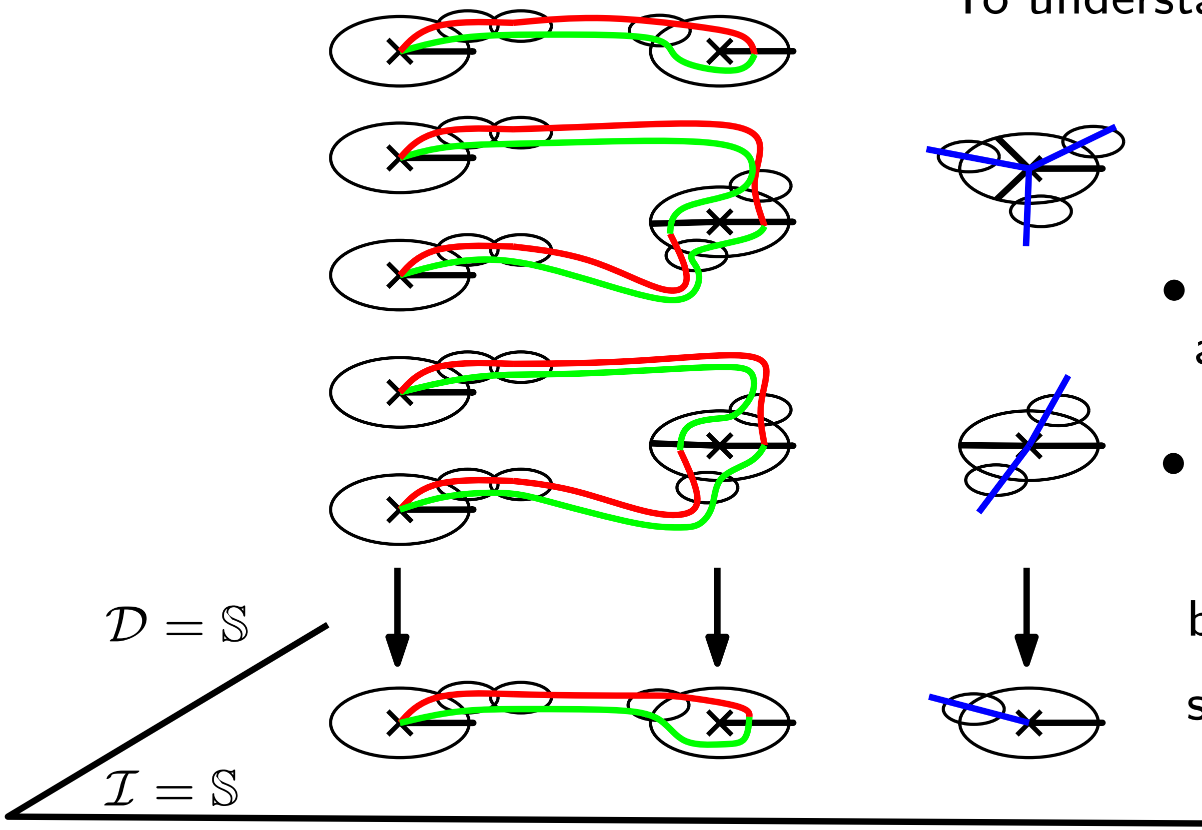

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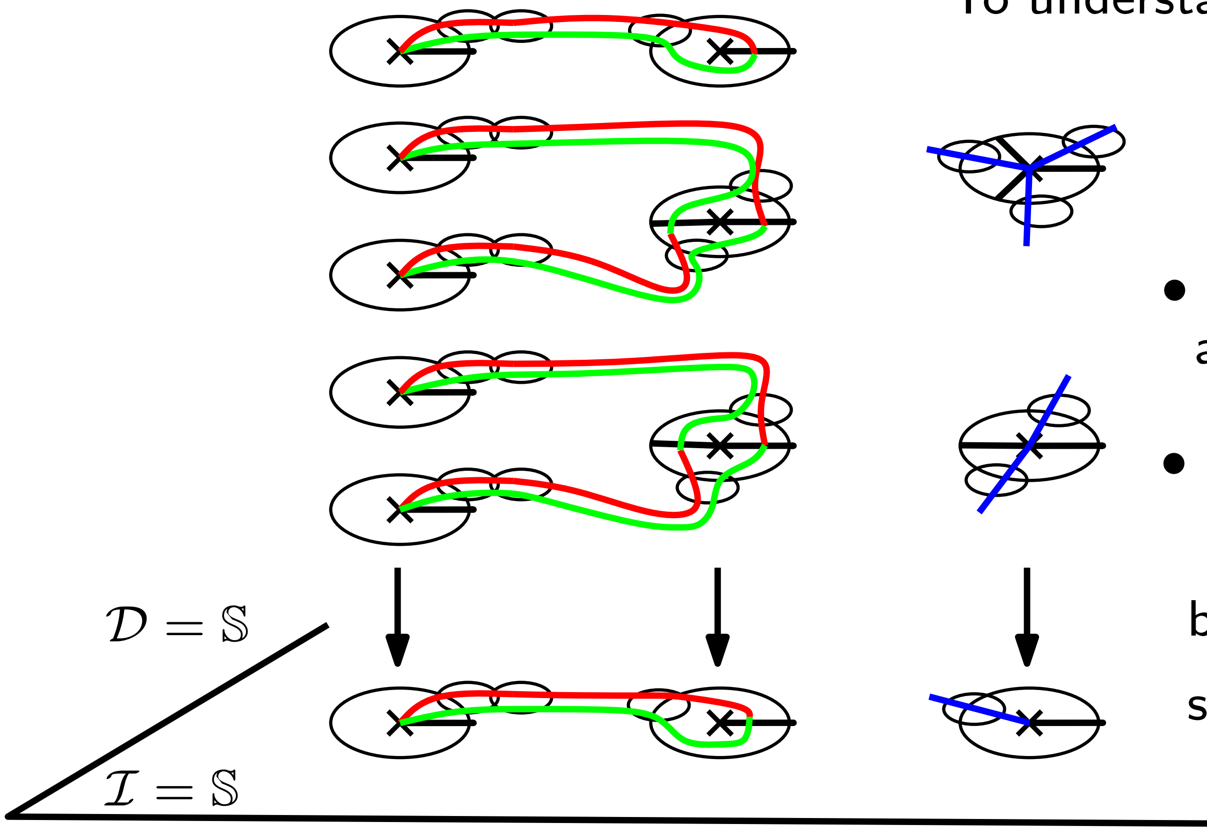

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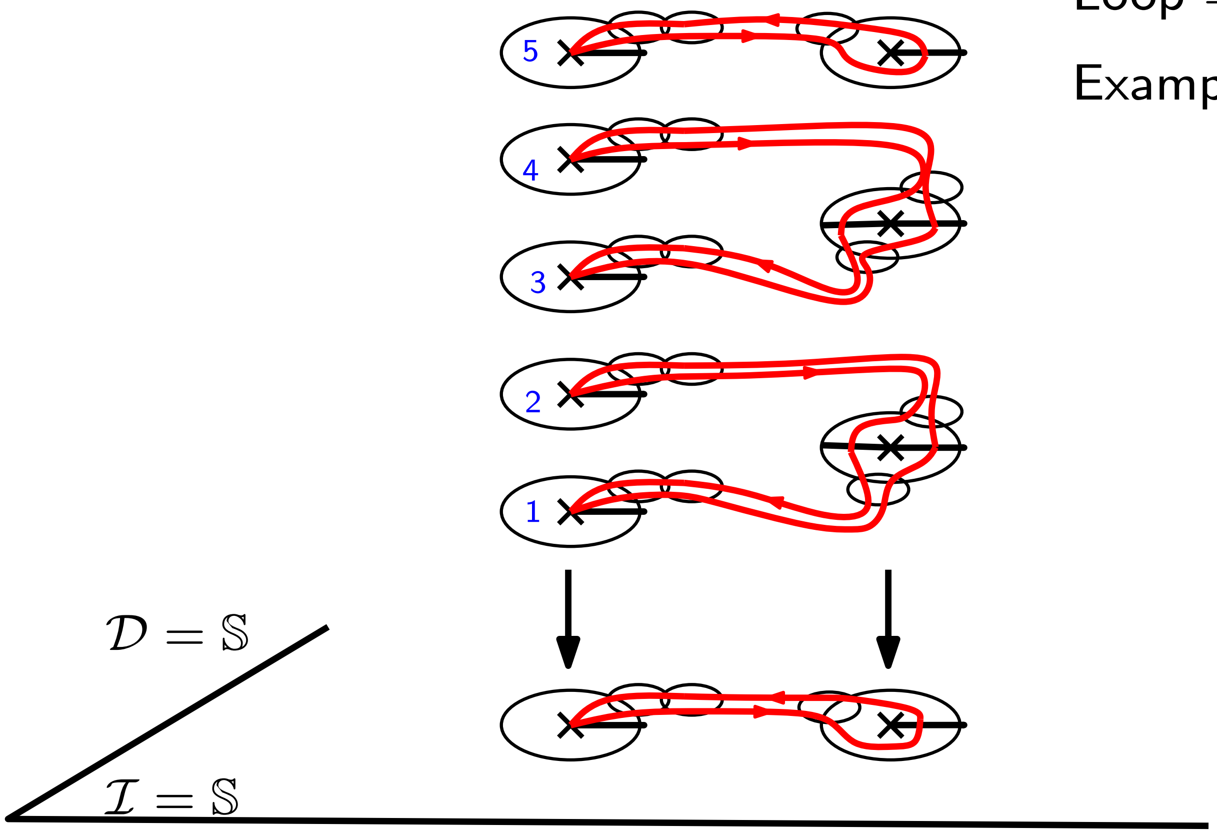
$\Rightarrow$ The partitions $\lambda^{(i)}$ are partitions of n, degree of the covering.

Monodromy, and permutations

Let us label $\{1, \ldots, n\}$ the preimages of a regular point.



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Example: $(1, 2)(3, 4)(5)$ in cyclic notation

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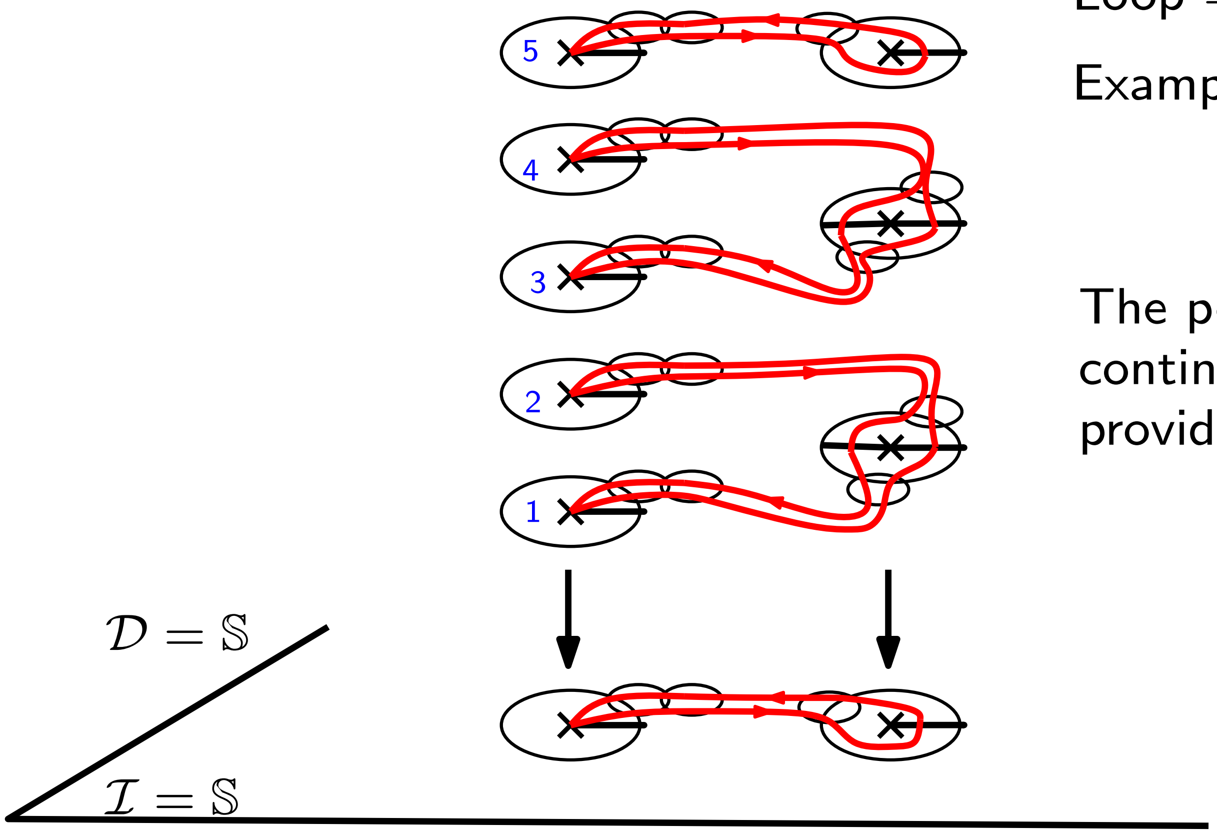

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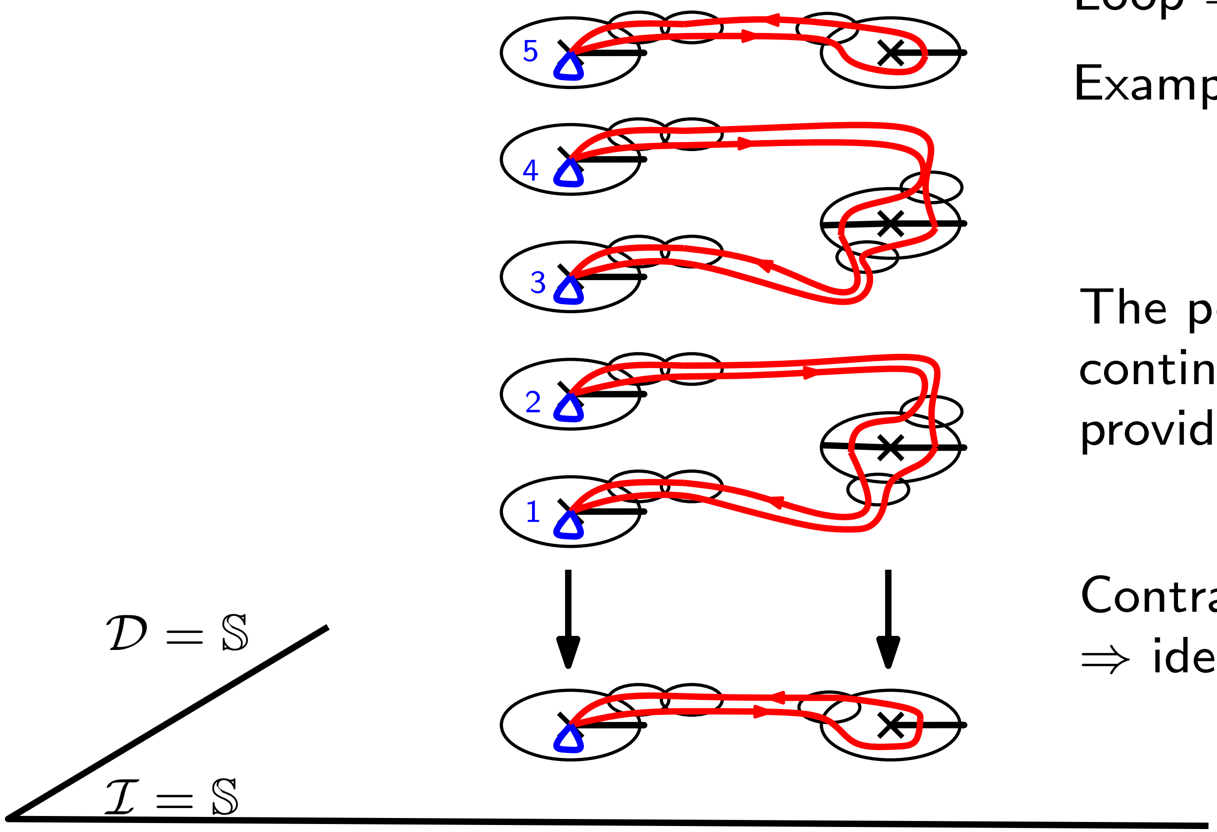

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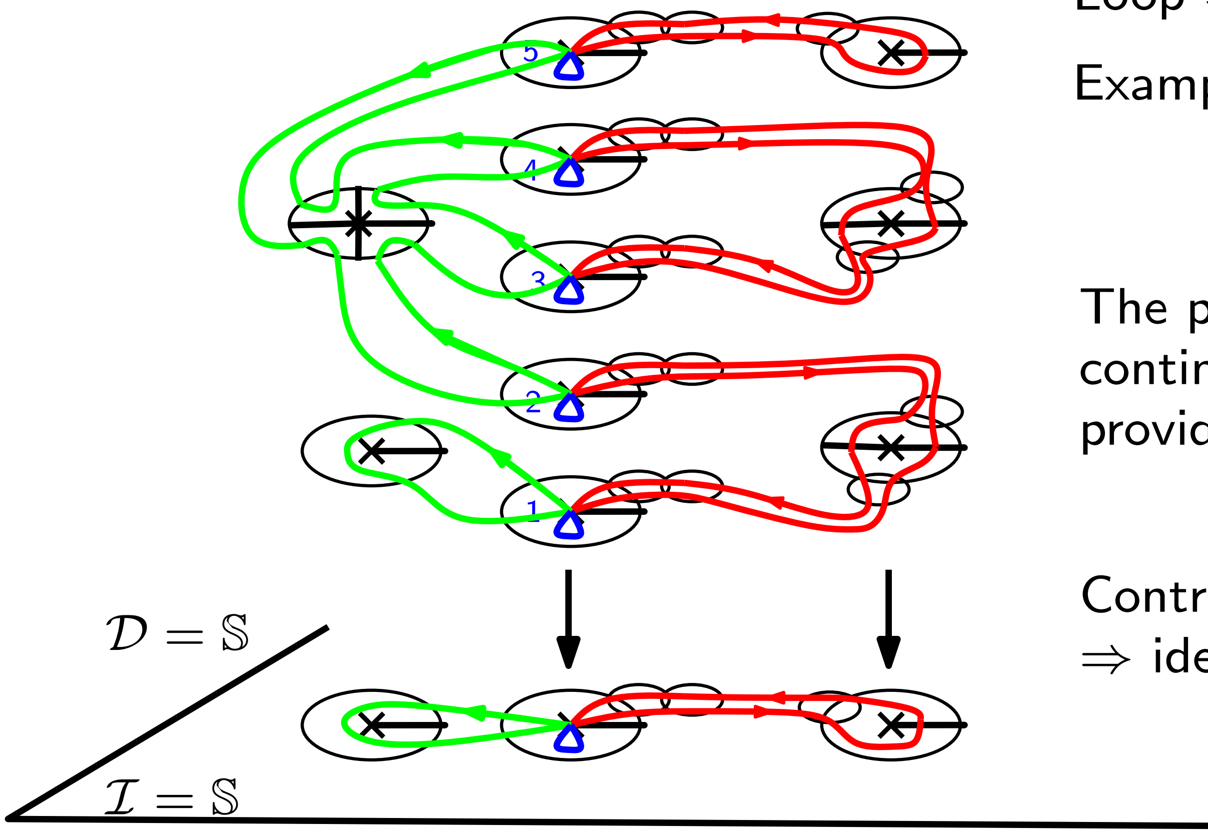

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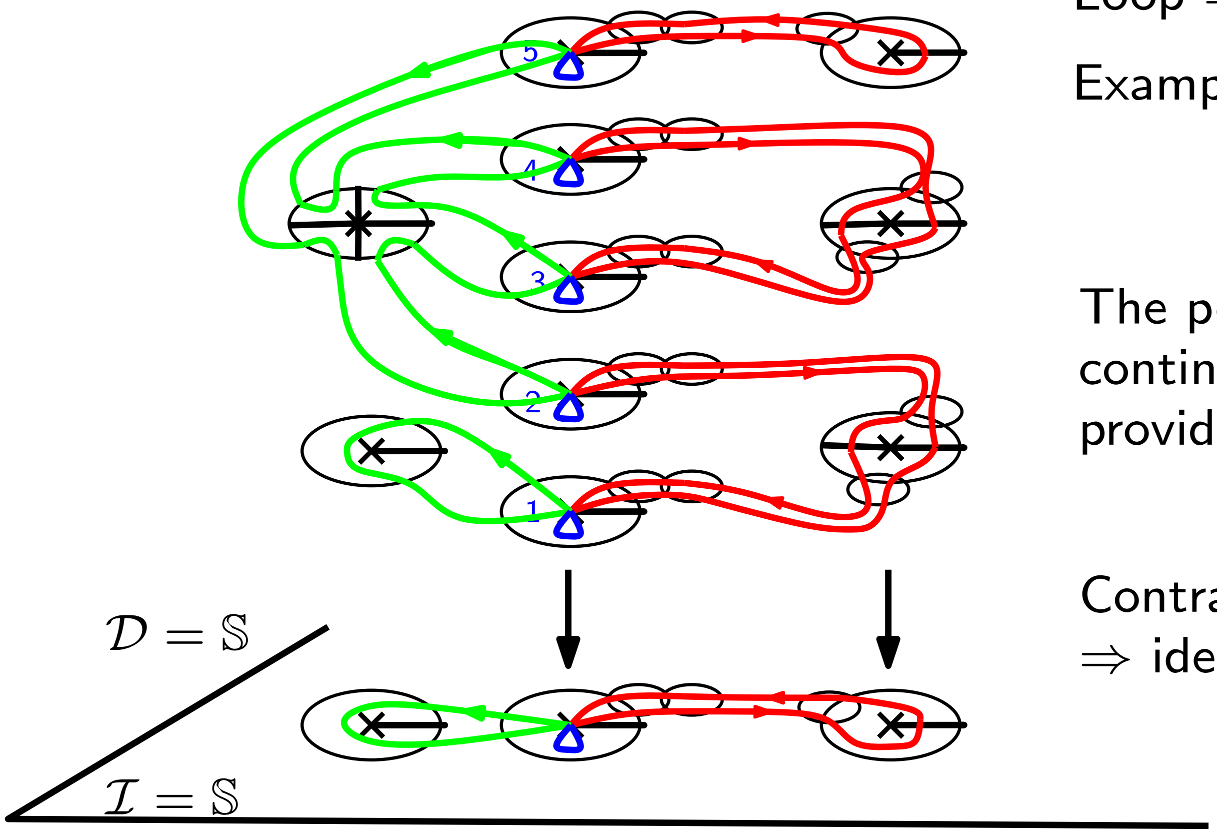

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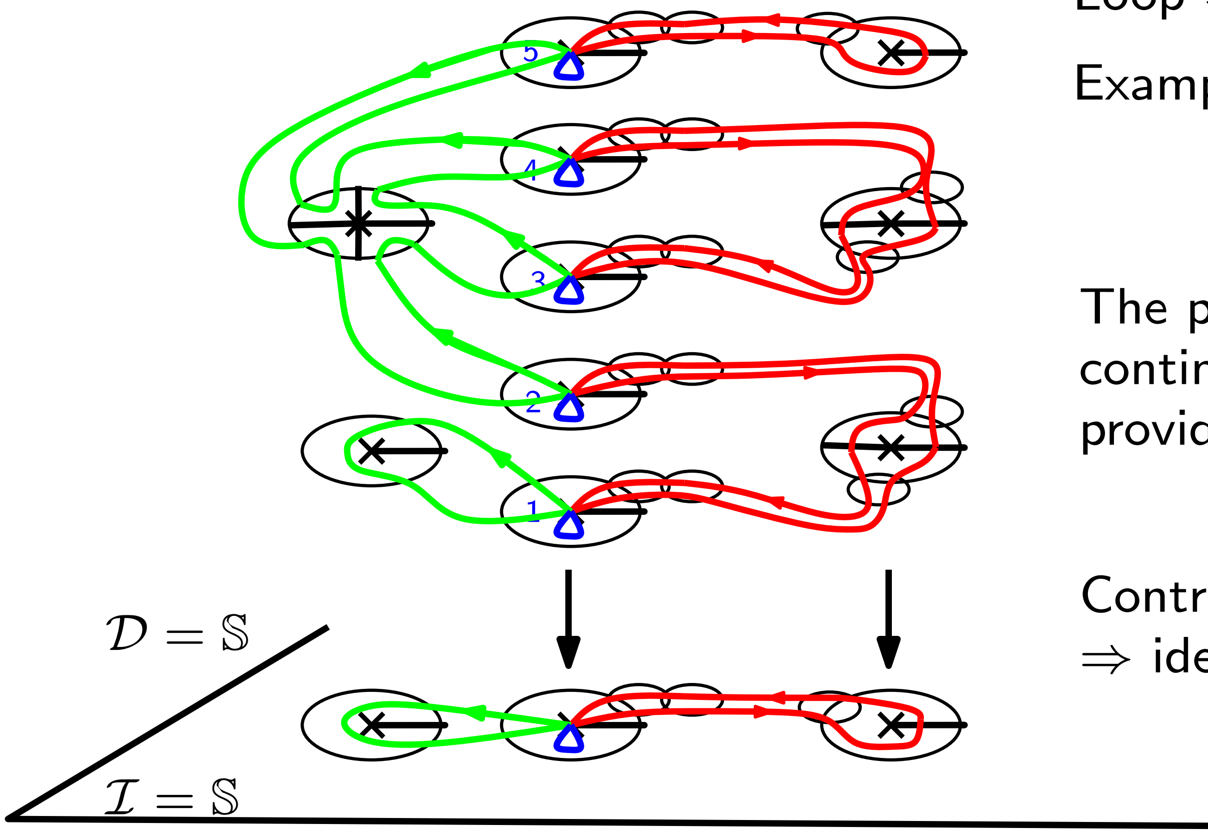

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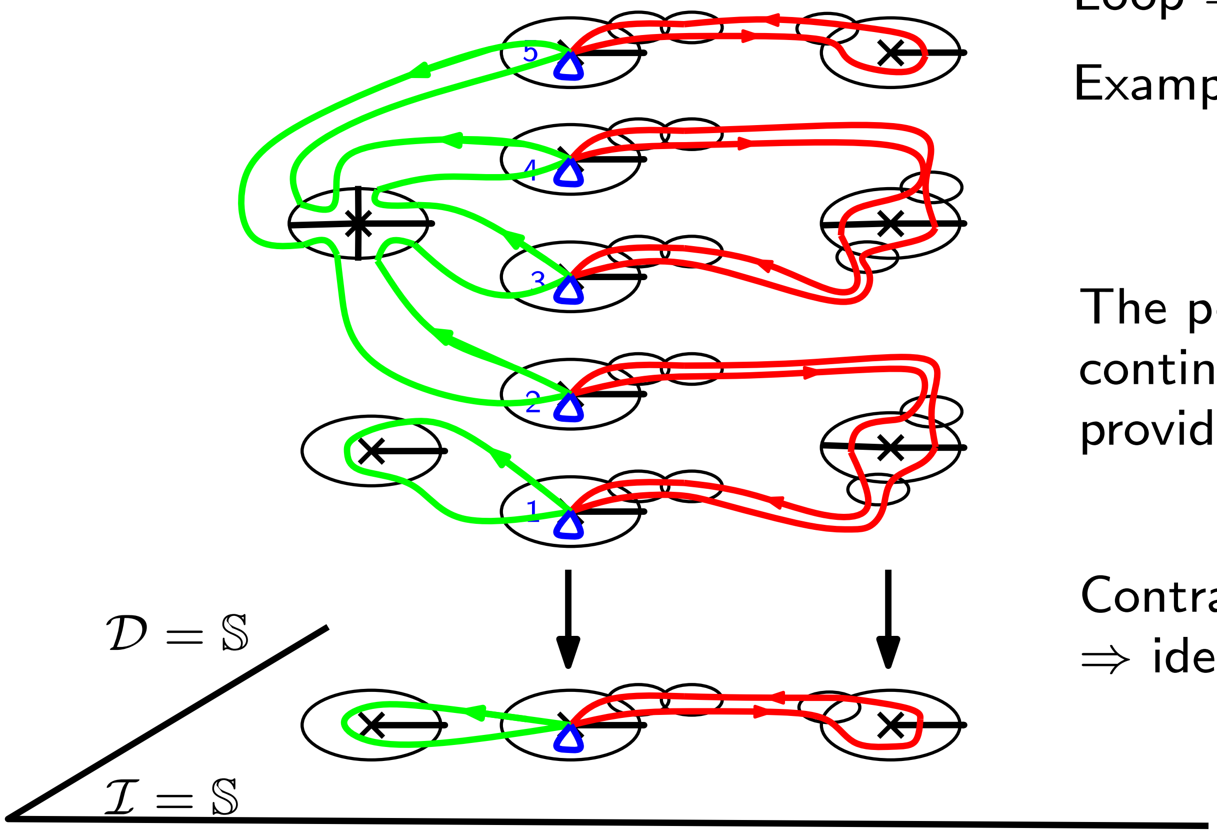

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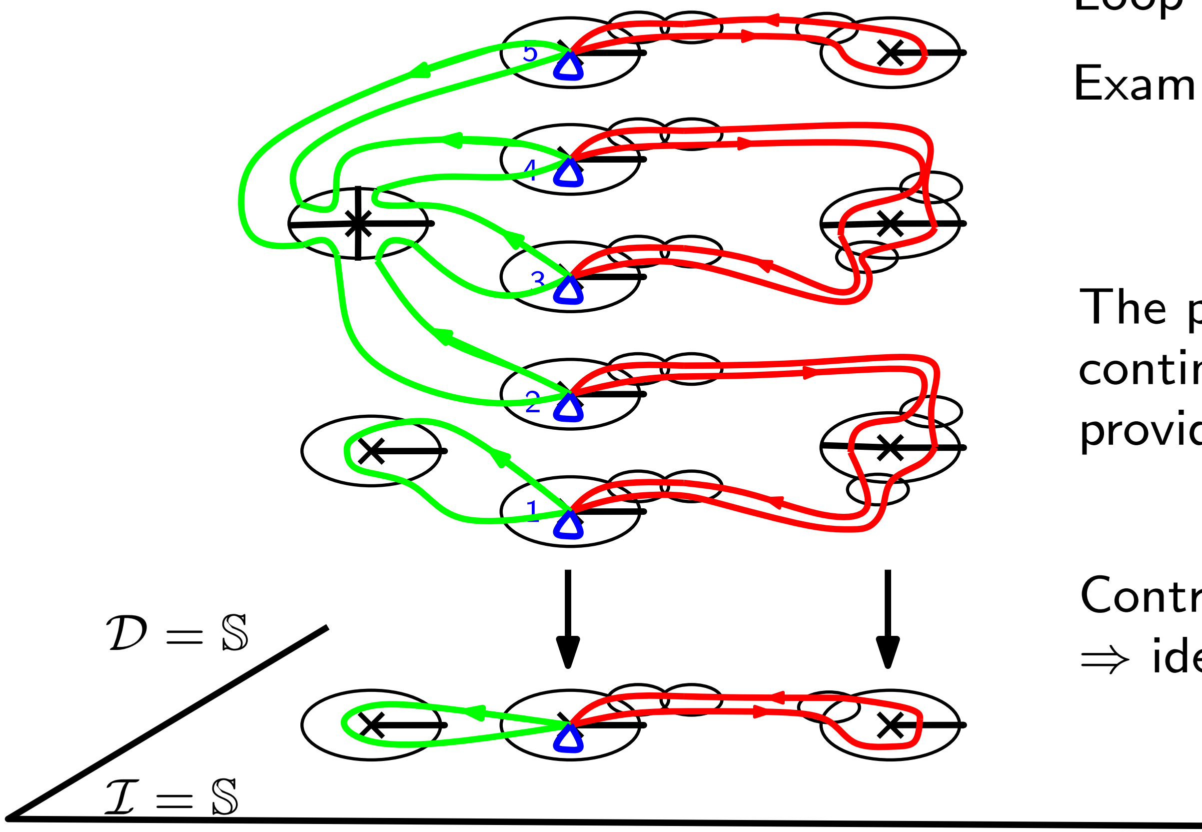

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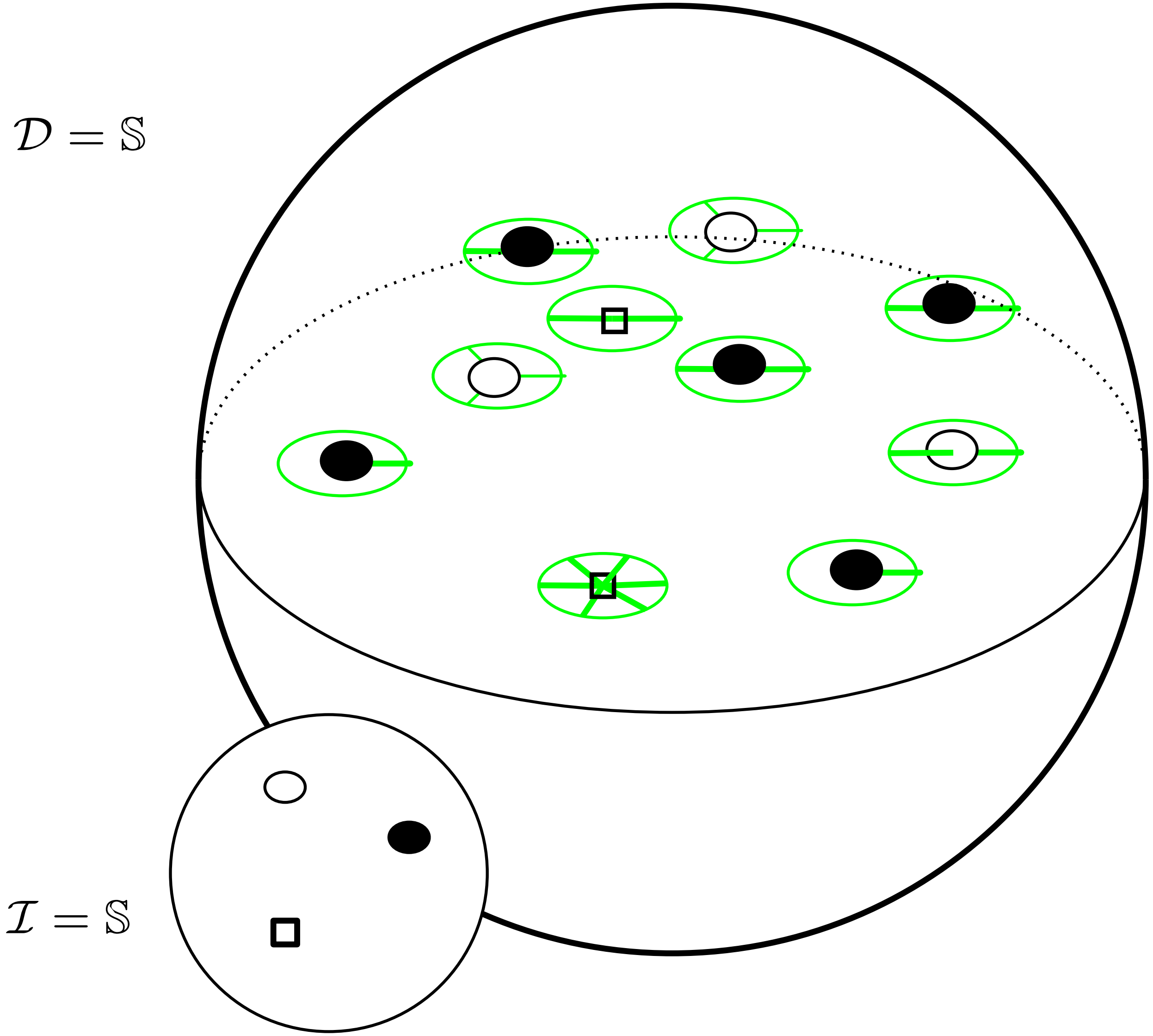
but geometric intuition is lost

coverings with 3 critical values and bipartite maps



3 critical values $\lambda^{\bullet} = 2^3 1^2$ $\lambda^{\circ} = 3^2 2$ $\lambda^{\square} = 62$

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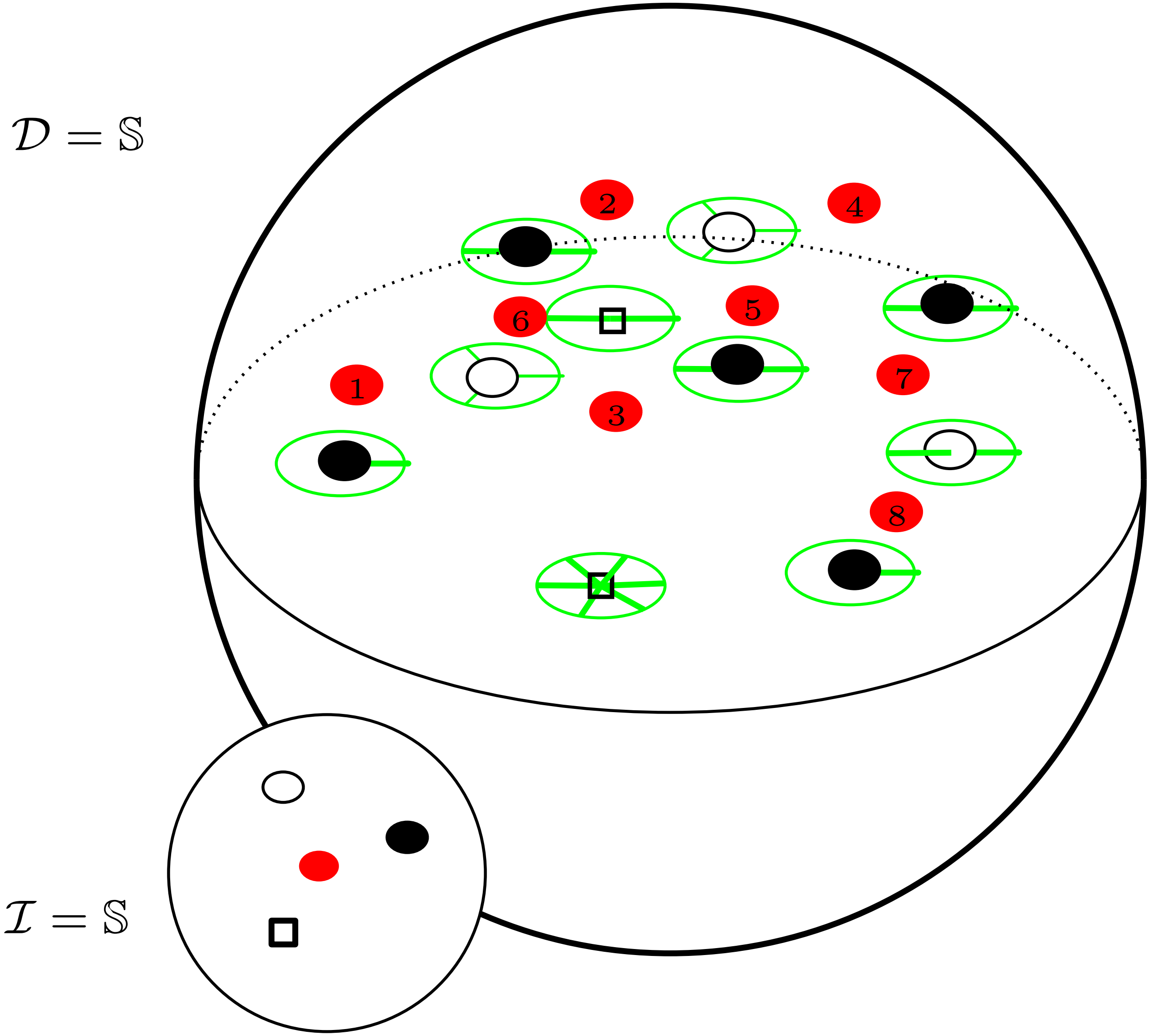

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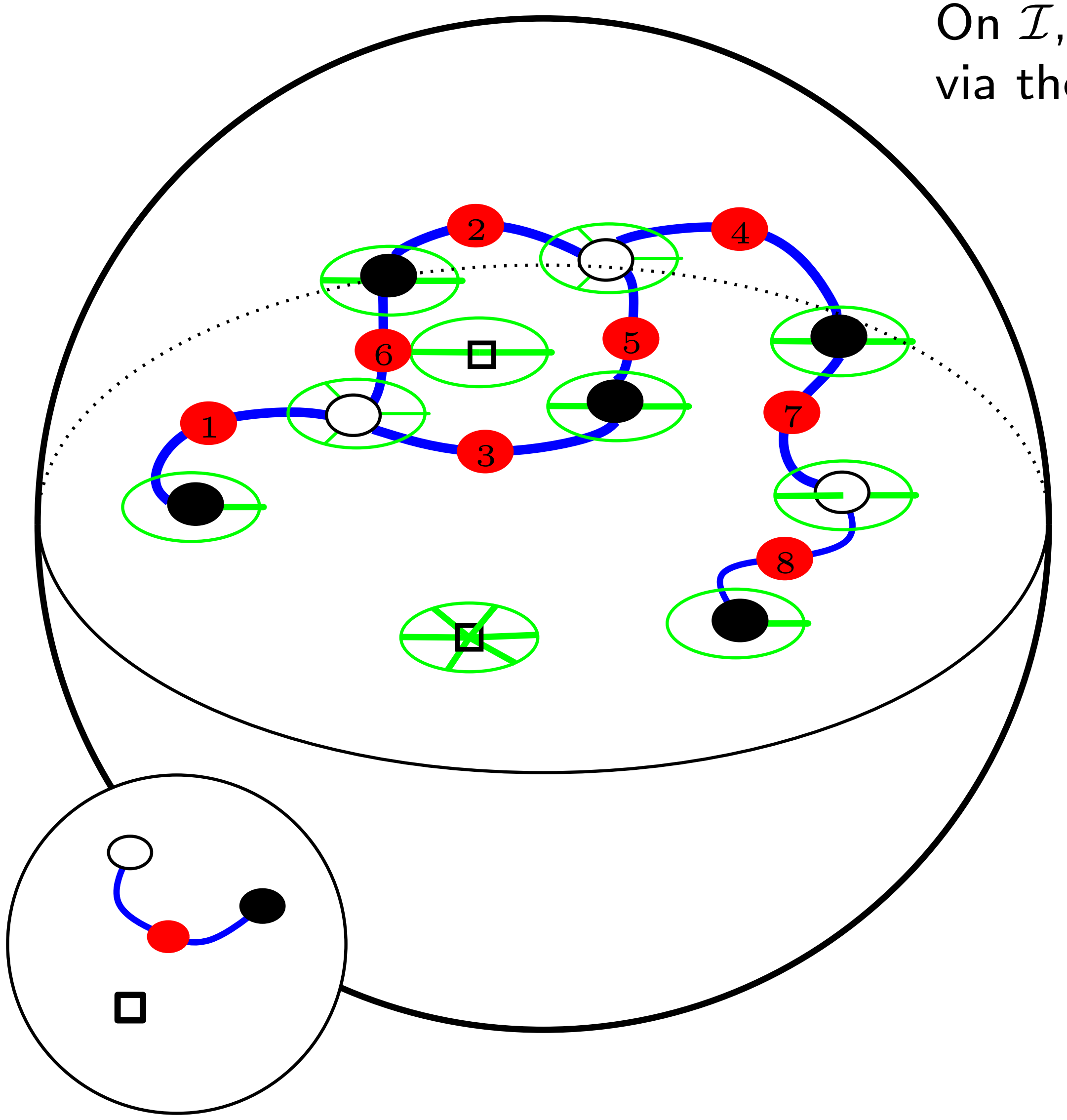
coverings with 3 critical values and bipartite maps

On $\mathcal{I}$, draw an edge between • and ◦ via the basepoint

$\mathcal{D} = \mathbb{S}$



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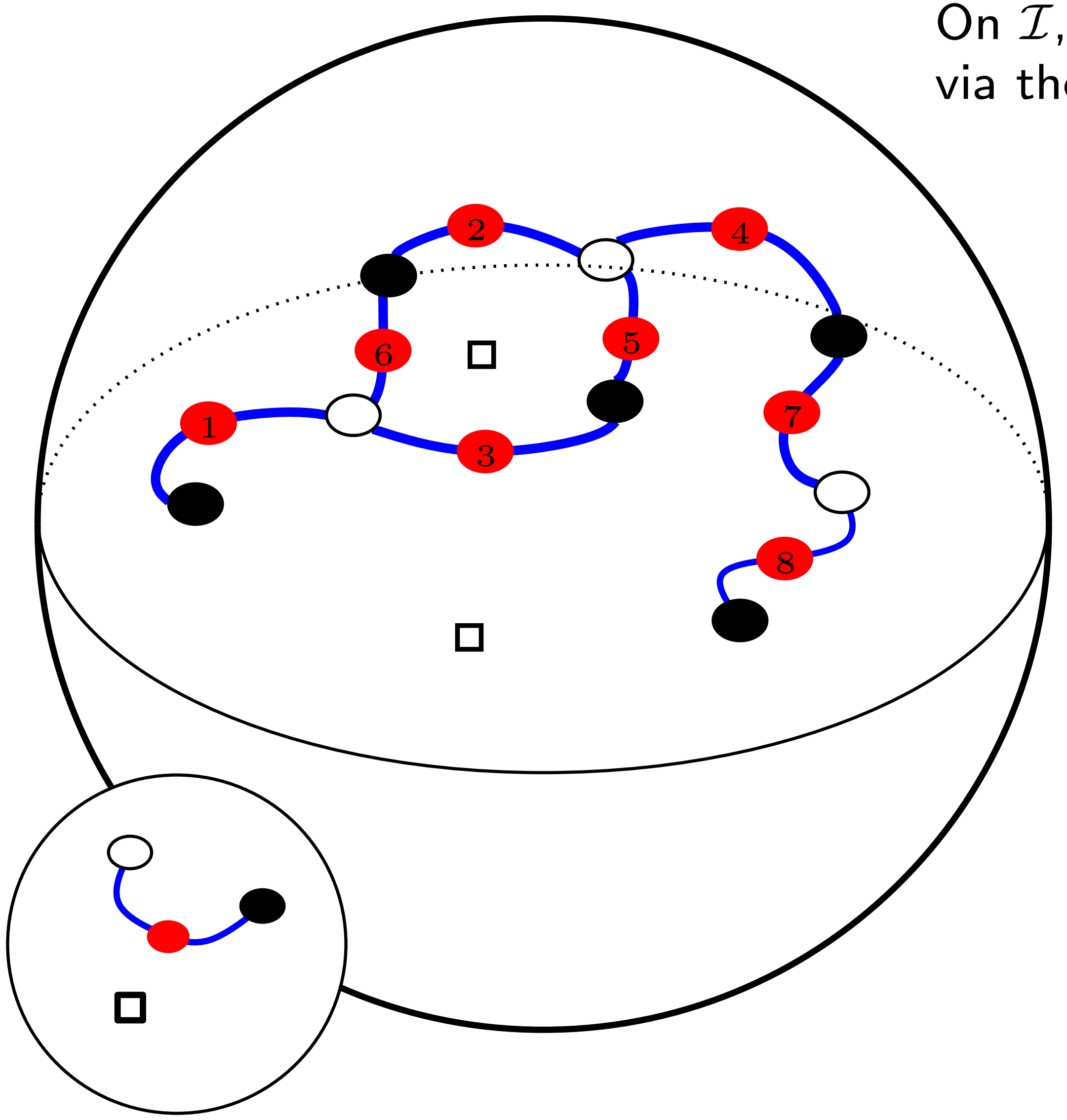

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On $\mathcal{I}$, draw an edge between $\bullet$ and $\circ$ via the basepoint

We get a planar map:

that is, a graph embedded on the sphere with simply connected faces

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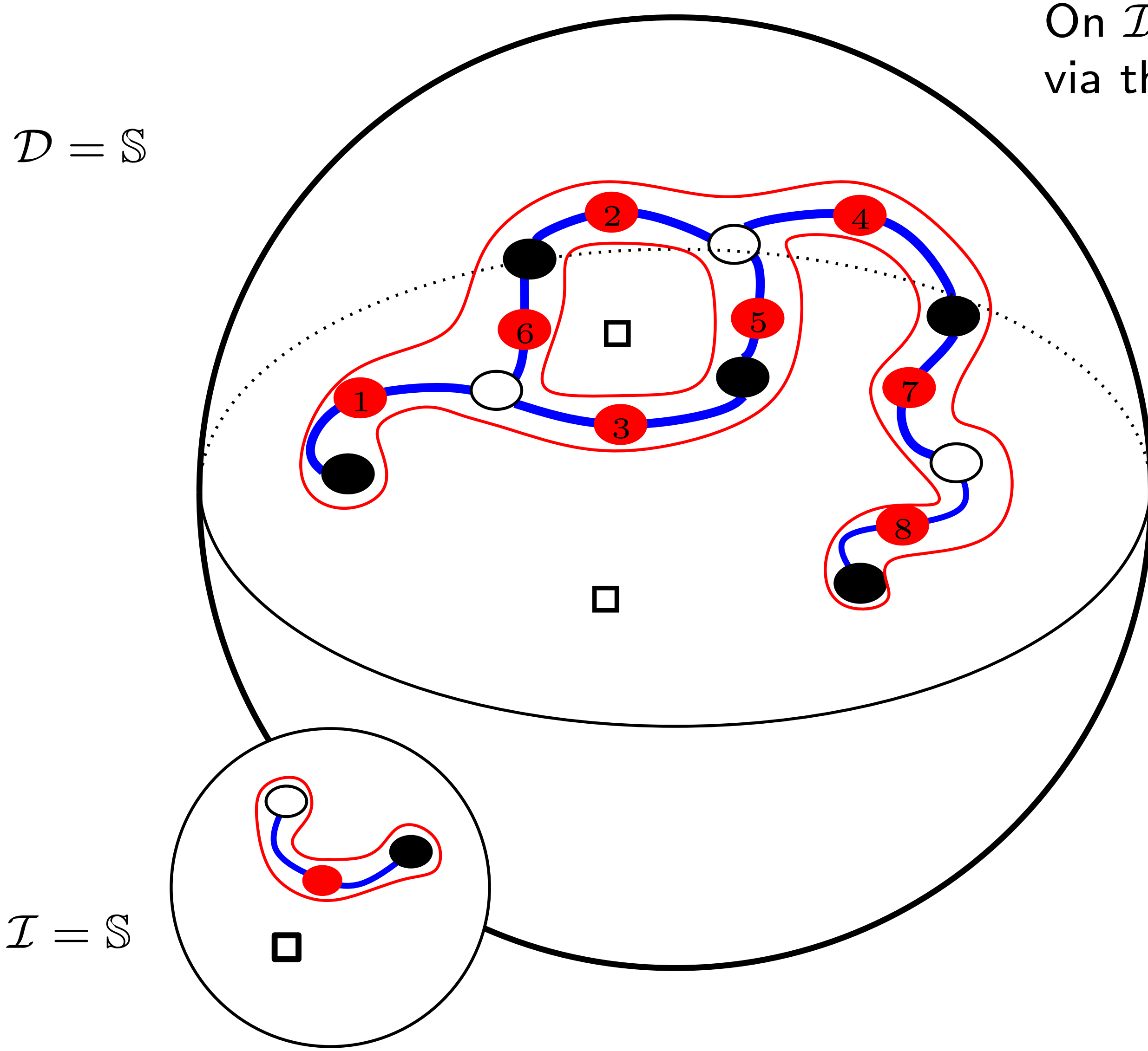

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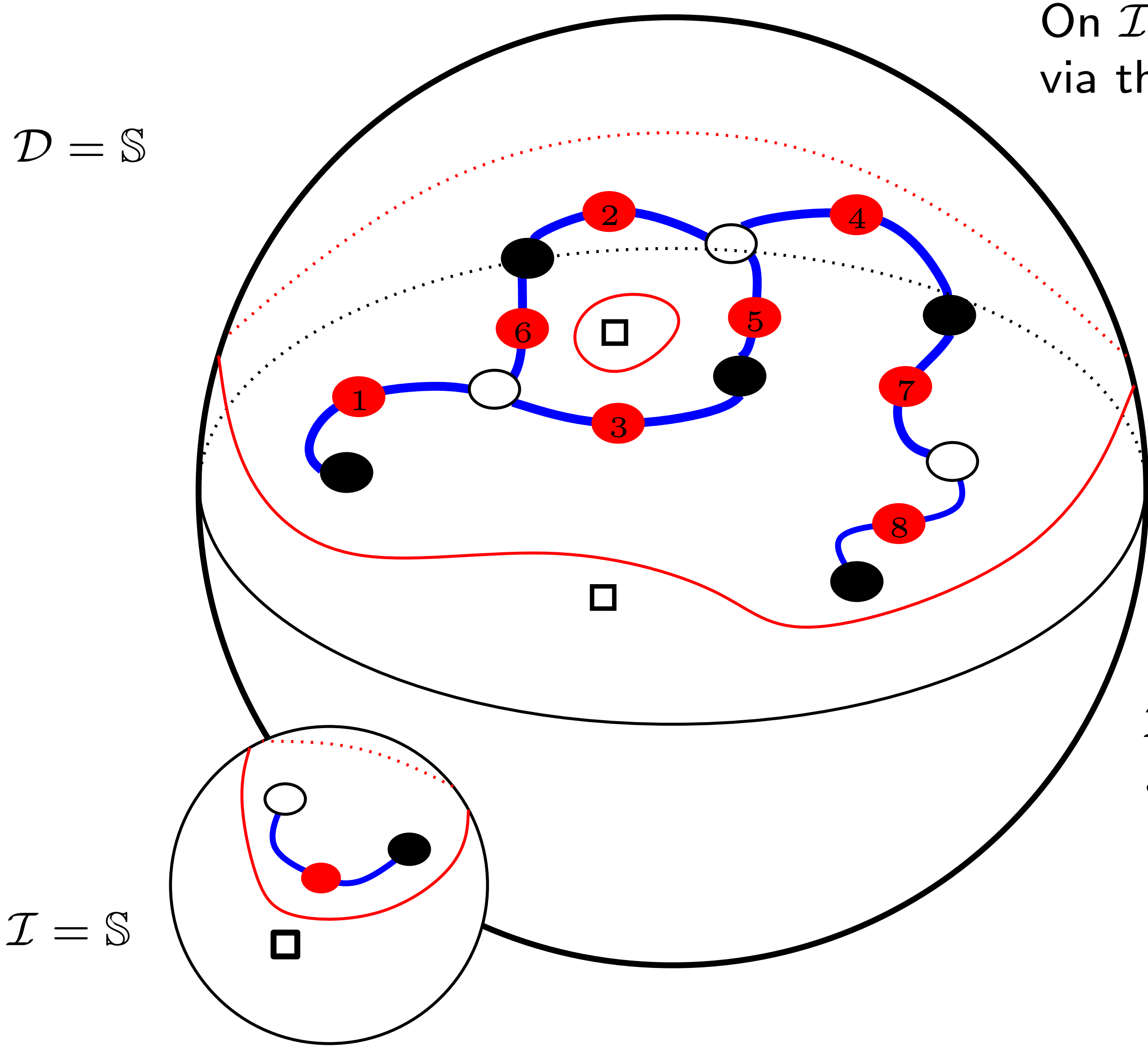

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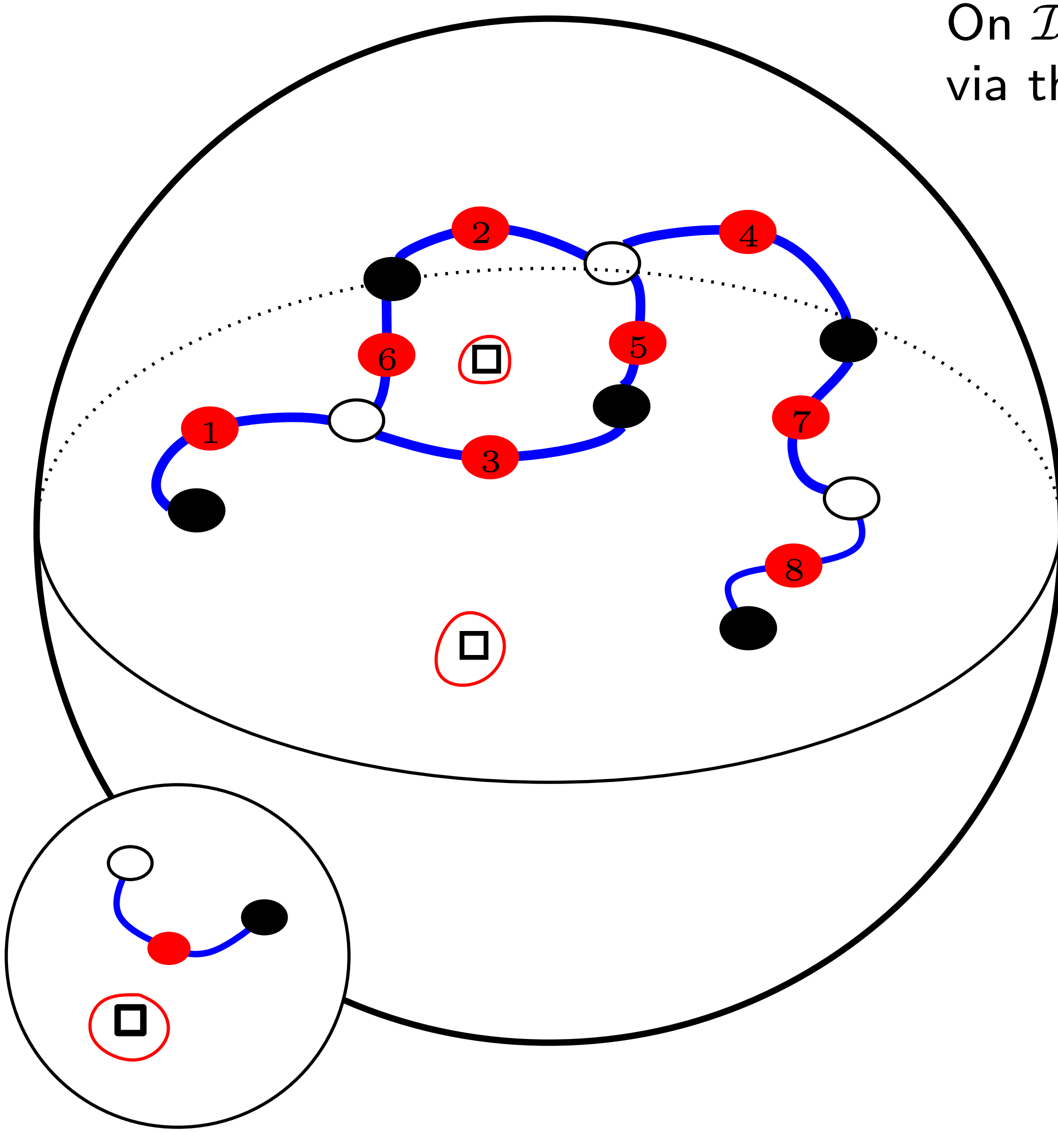

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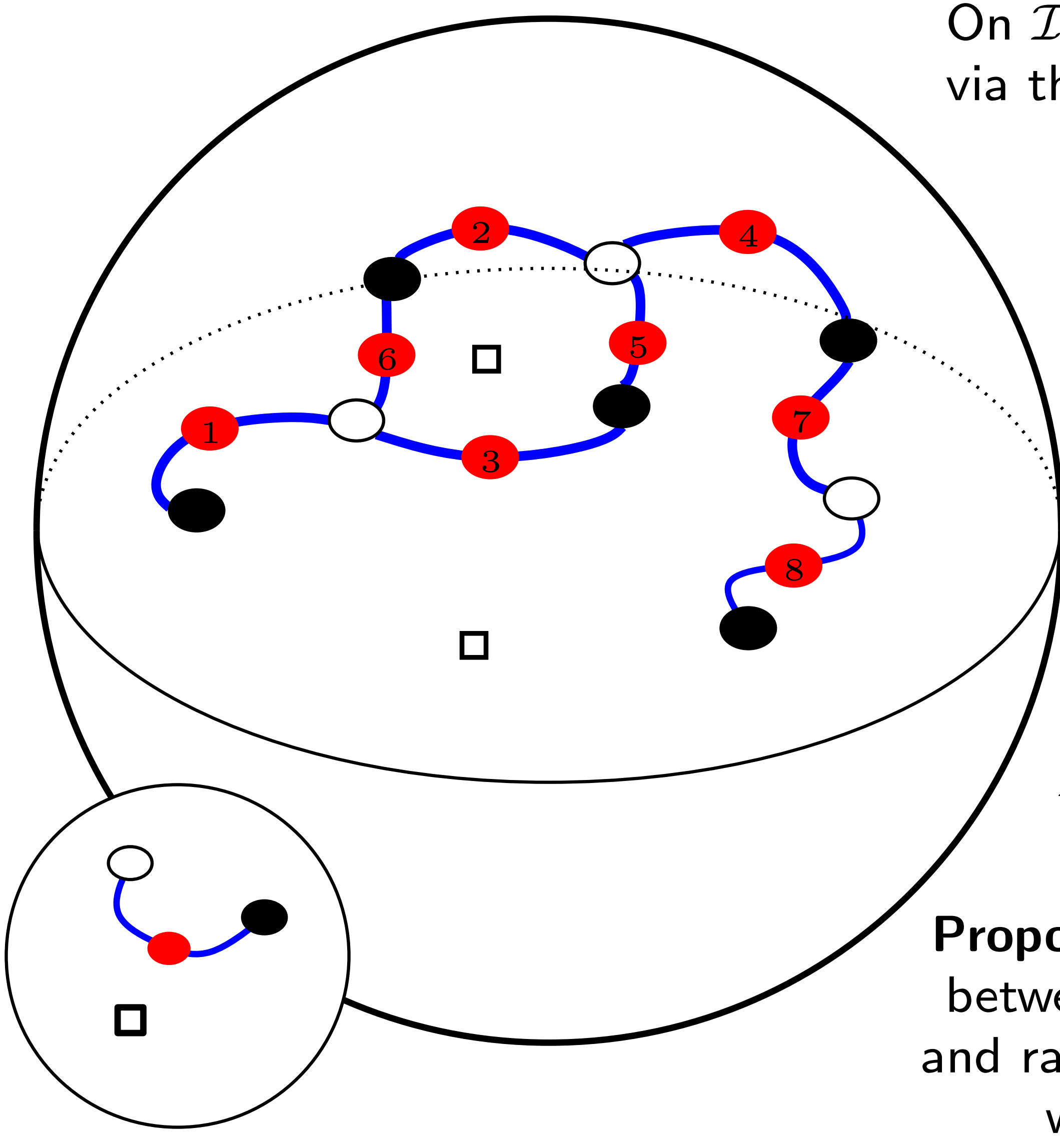

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Proposition. This is a bijection between bipartite planar maps and ramified coverings of $\mathbb{S}$ by $\mathbb{S}$ with 3 critical values.

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3 critical values, bipartite maps and permutations

A loop around a critical value yields a permutation

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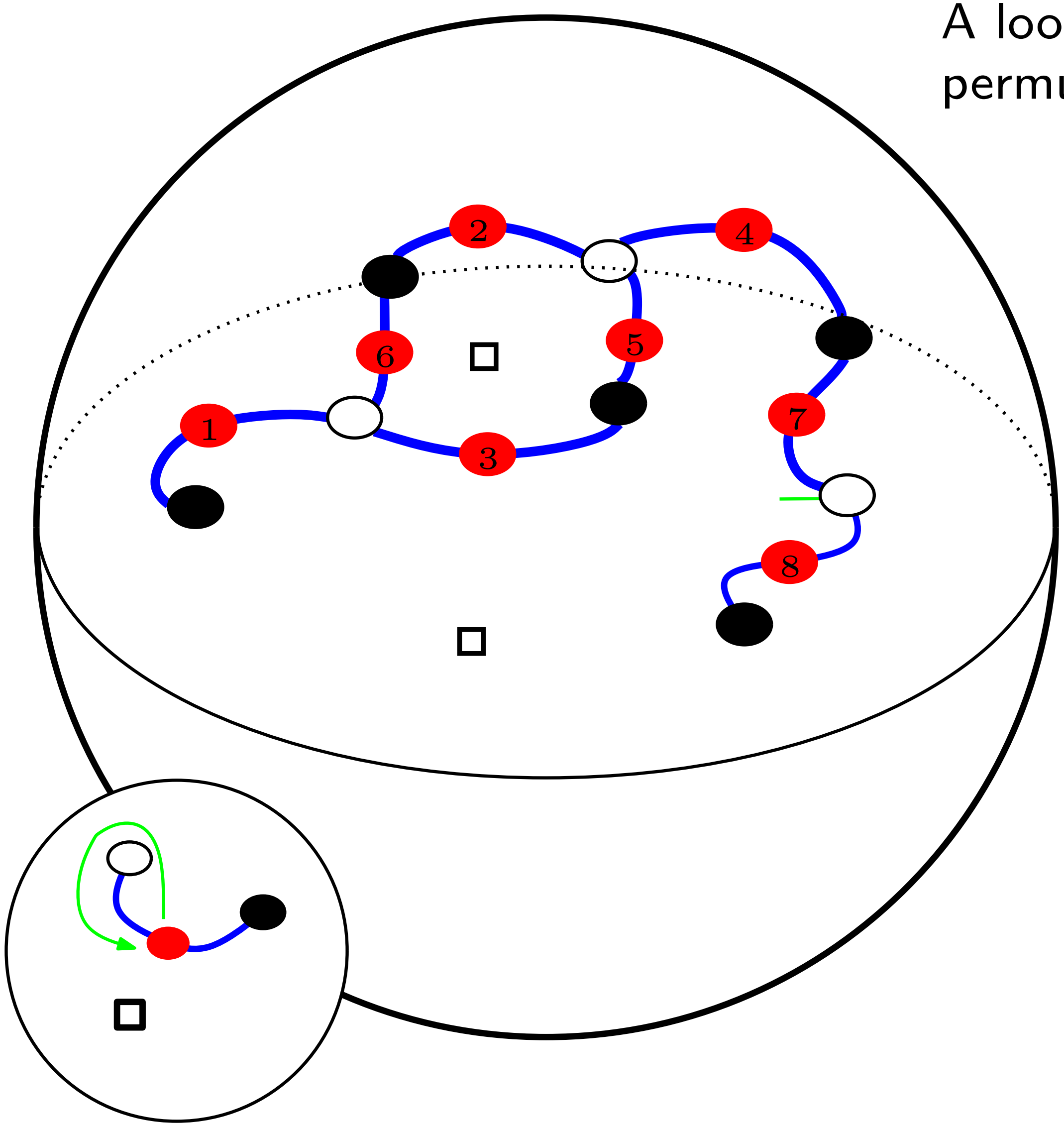

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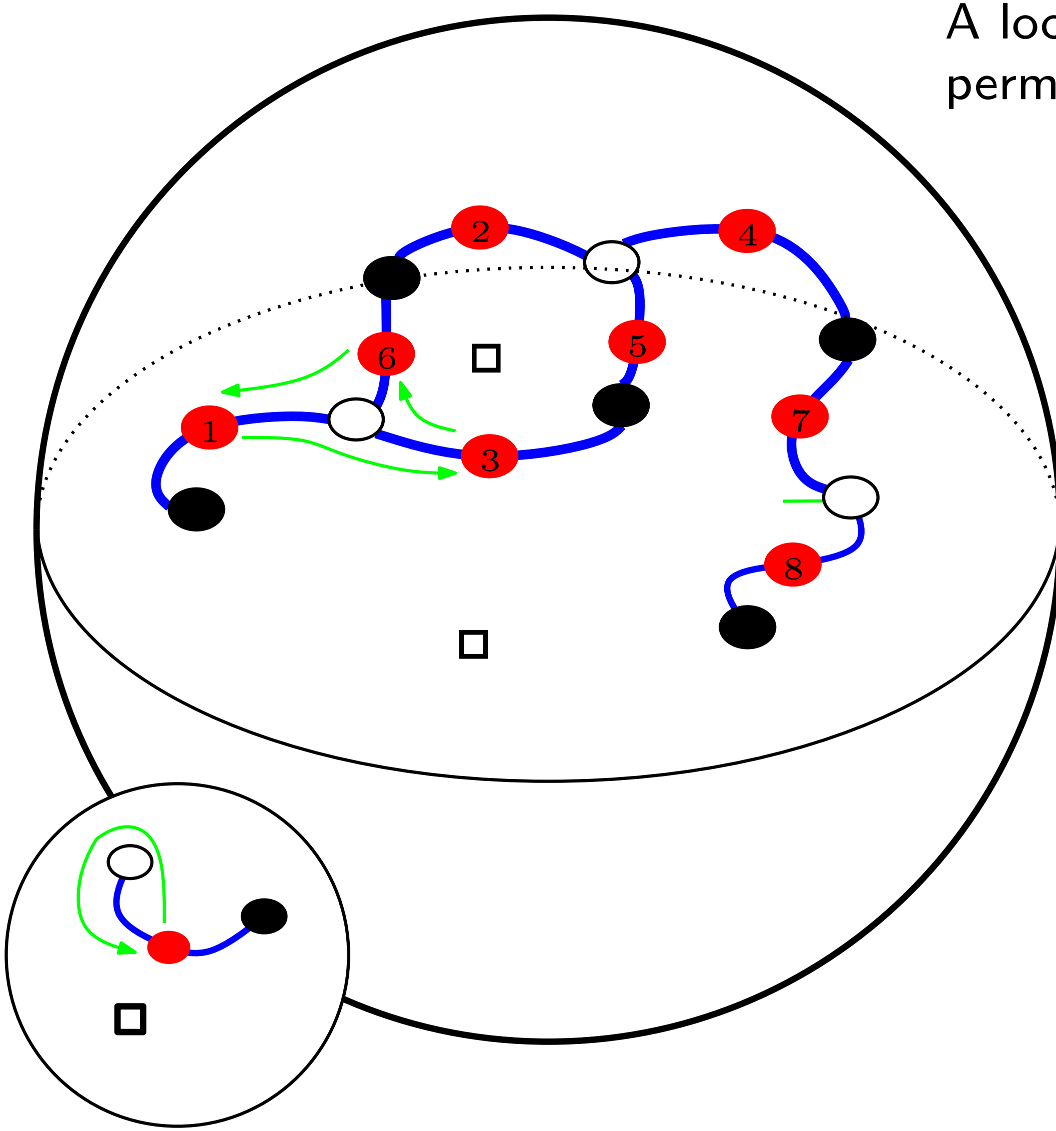

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A loop around a critical value yields a permutation

$$\sigma_\circ = (1,3,6)(2,5,4)(7,8)$$

with cyclic type λ°

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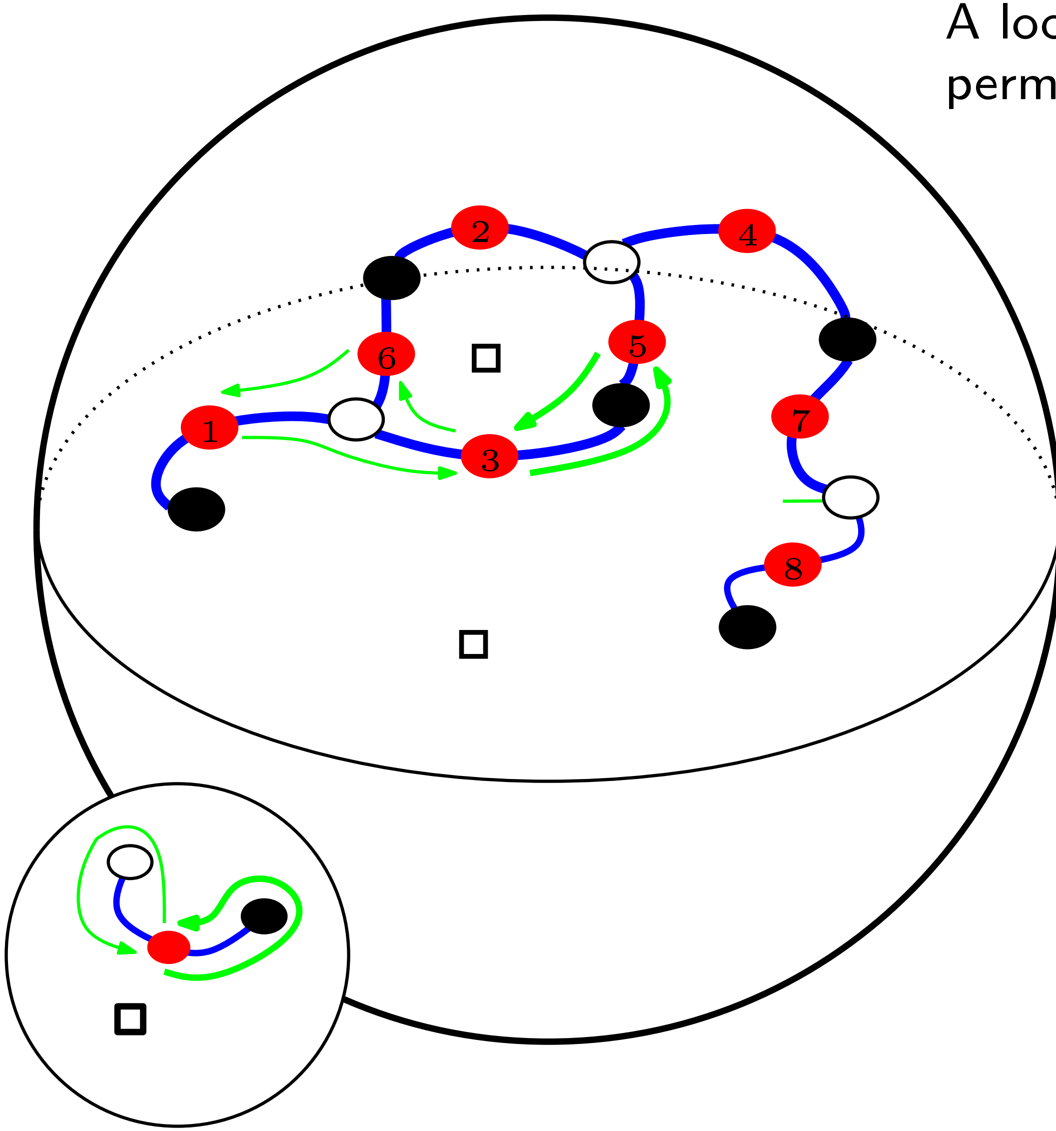

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Cycle types $\Leftrightarrow$ degree distributions

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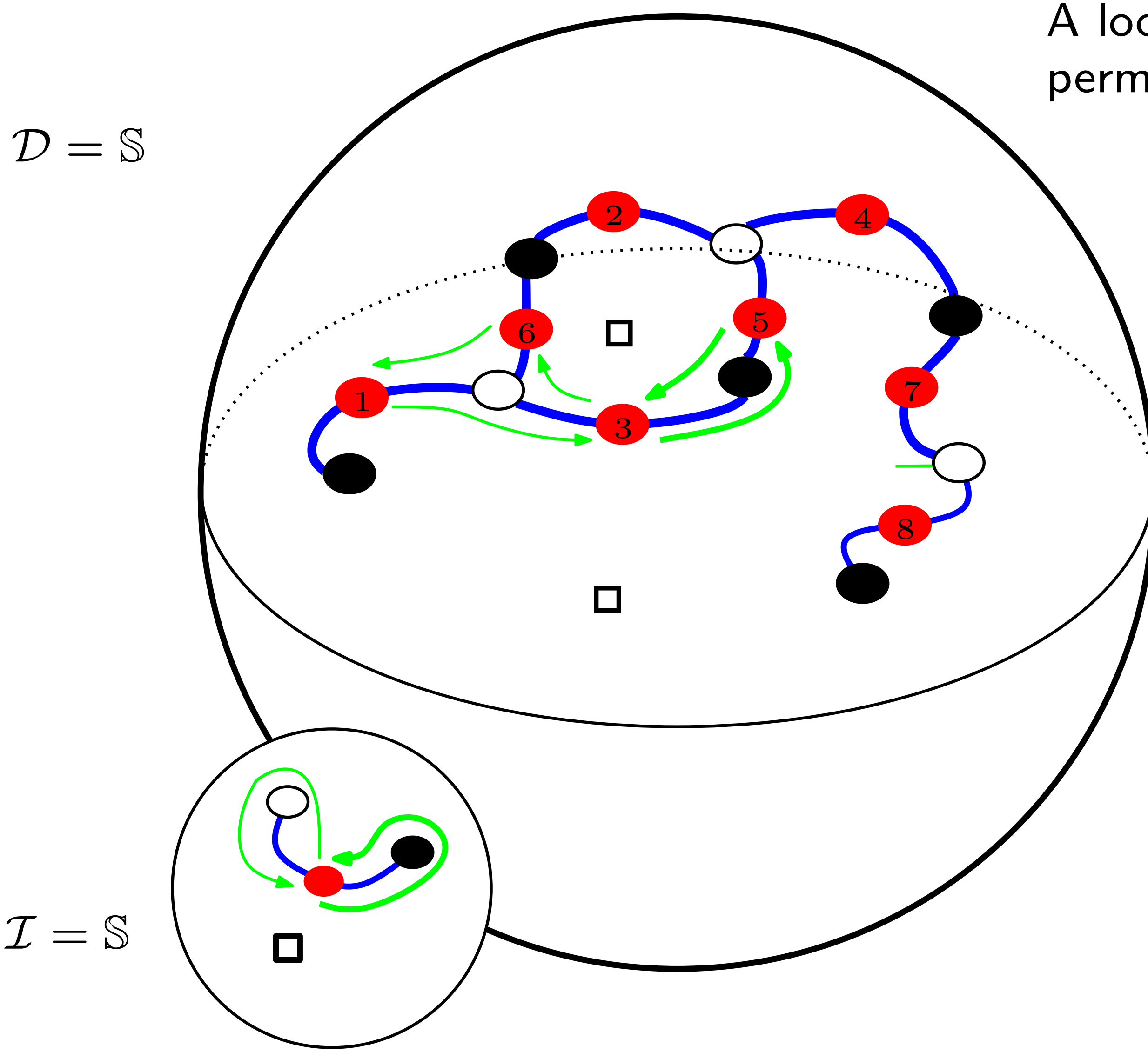
3 critical values, bipartite maps and permutations



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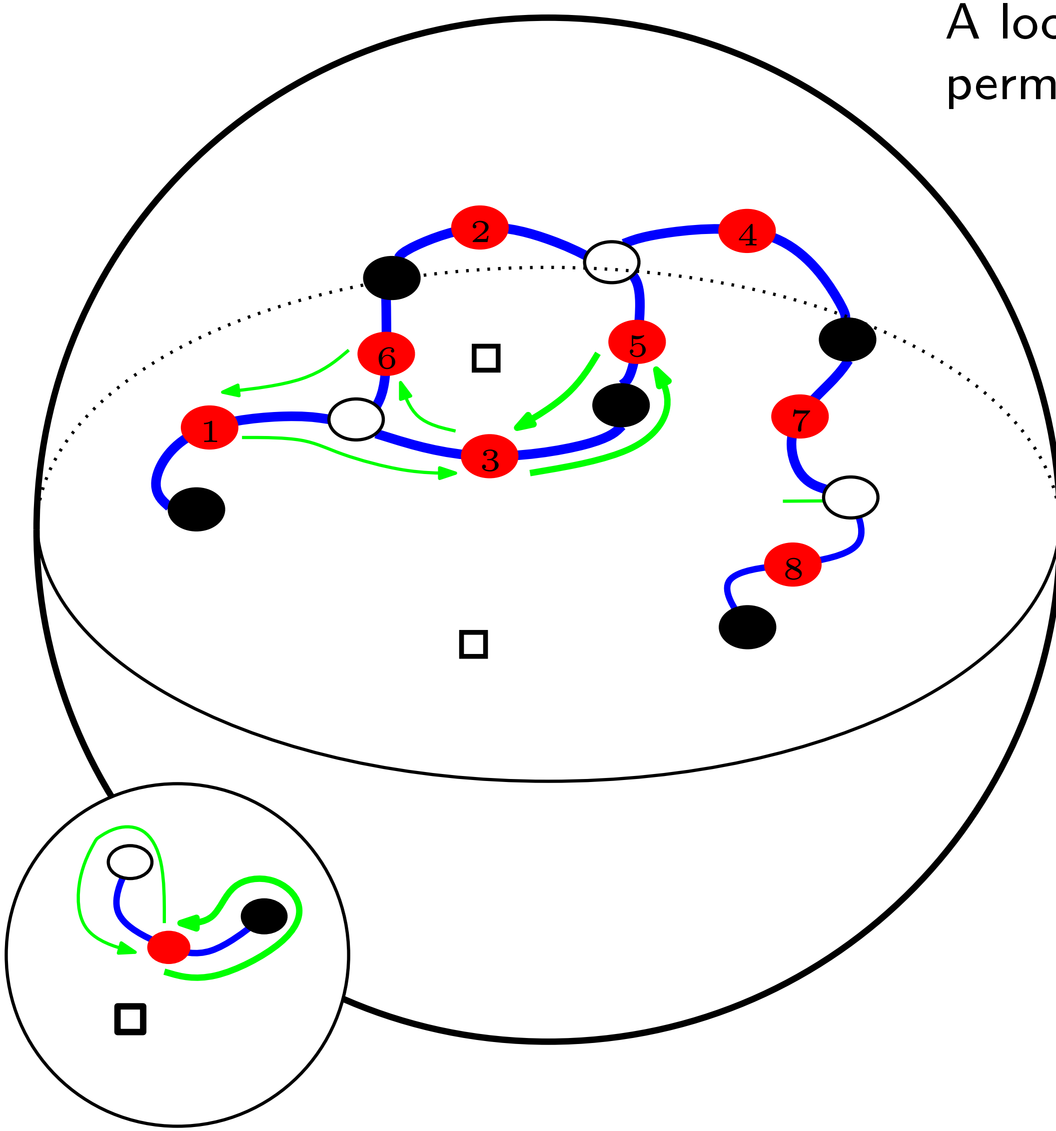

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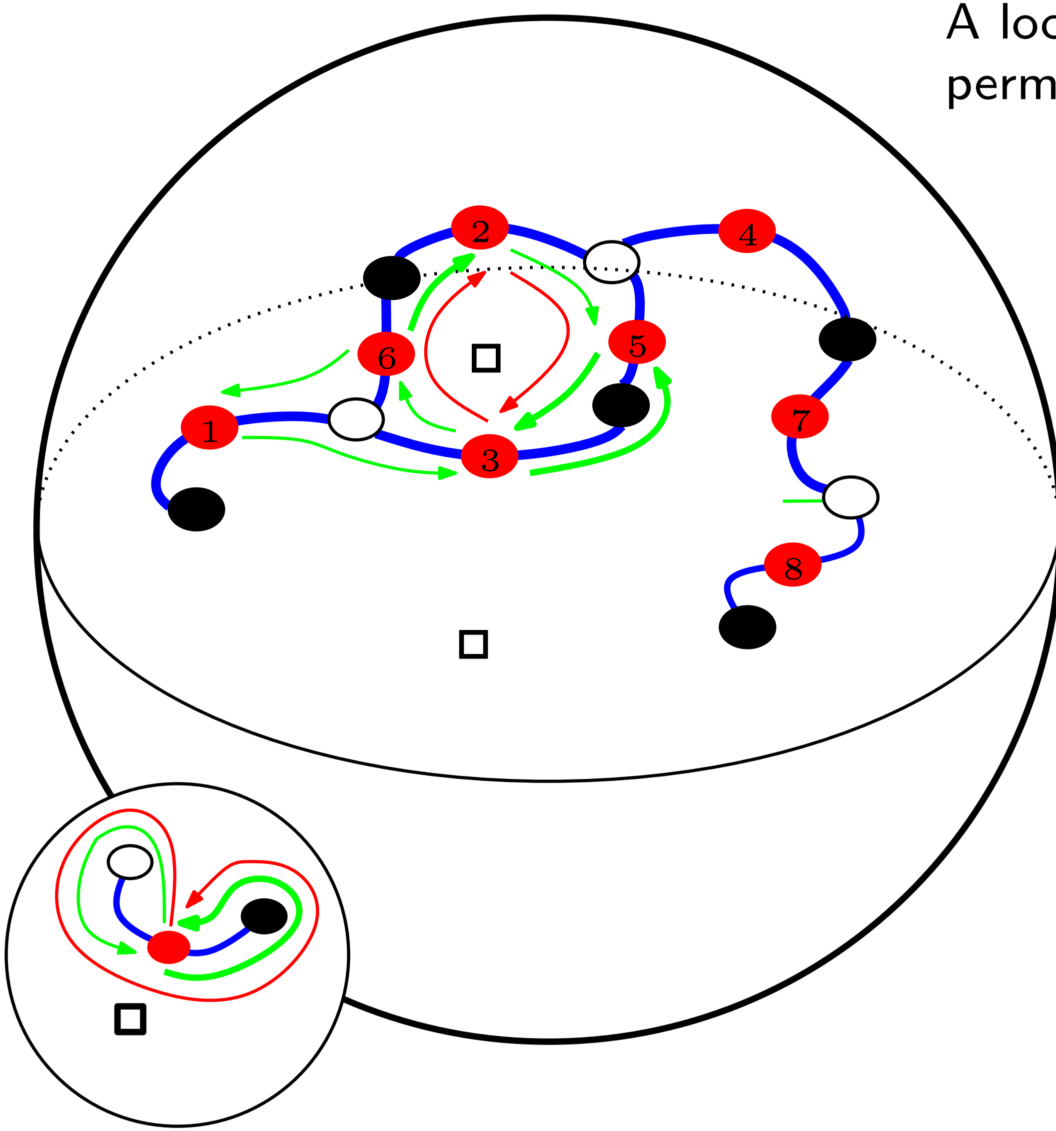

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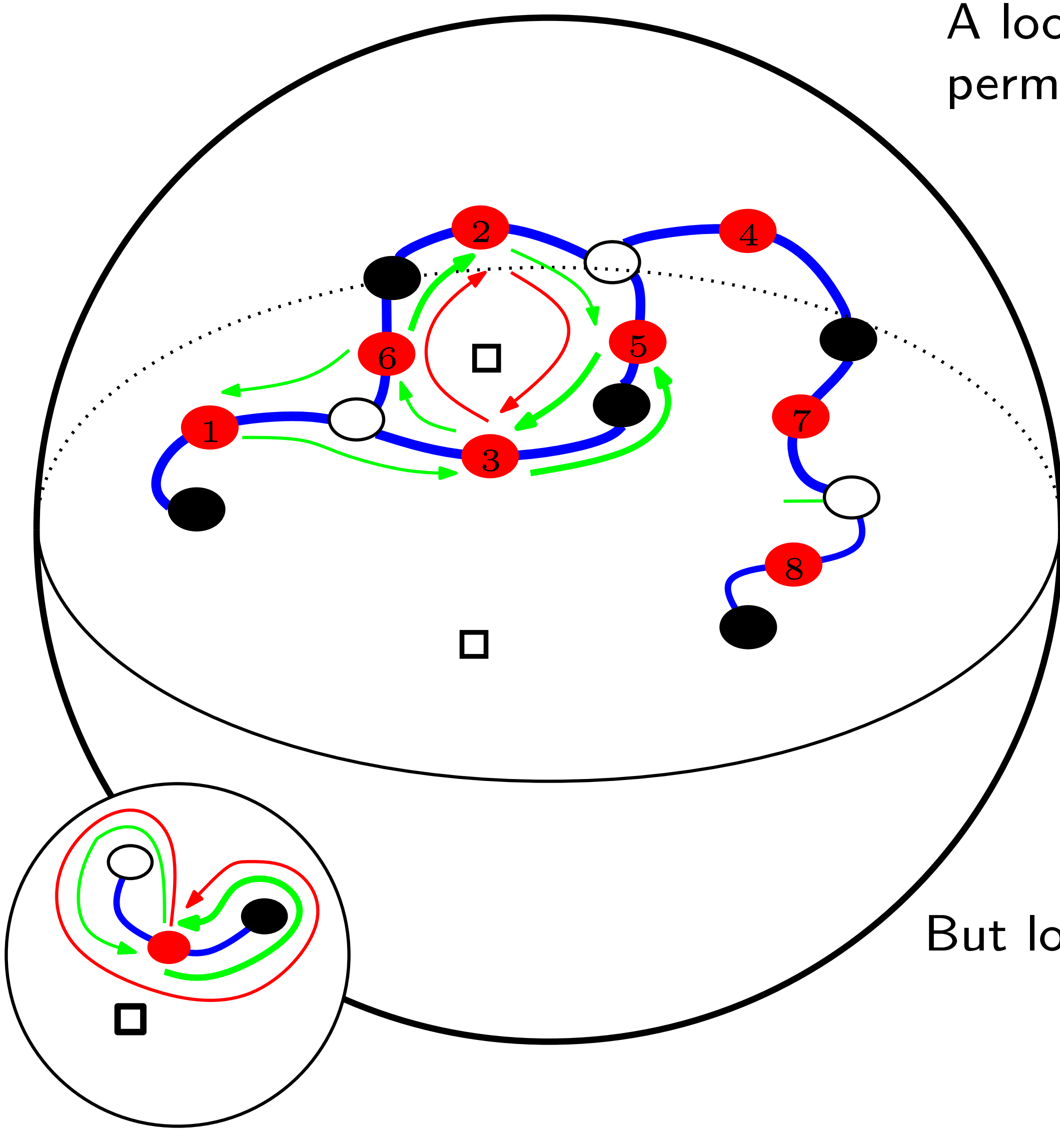

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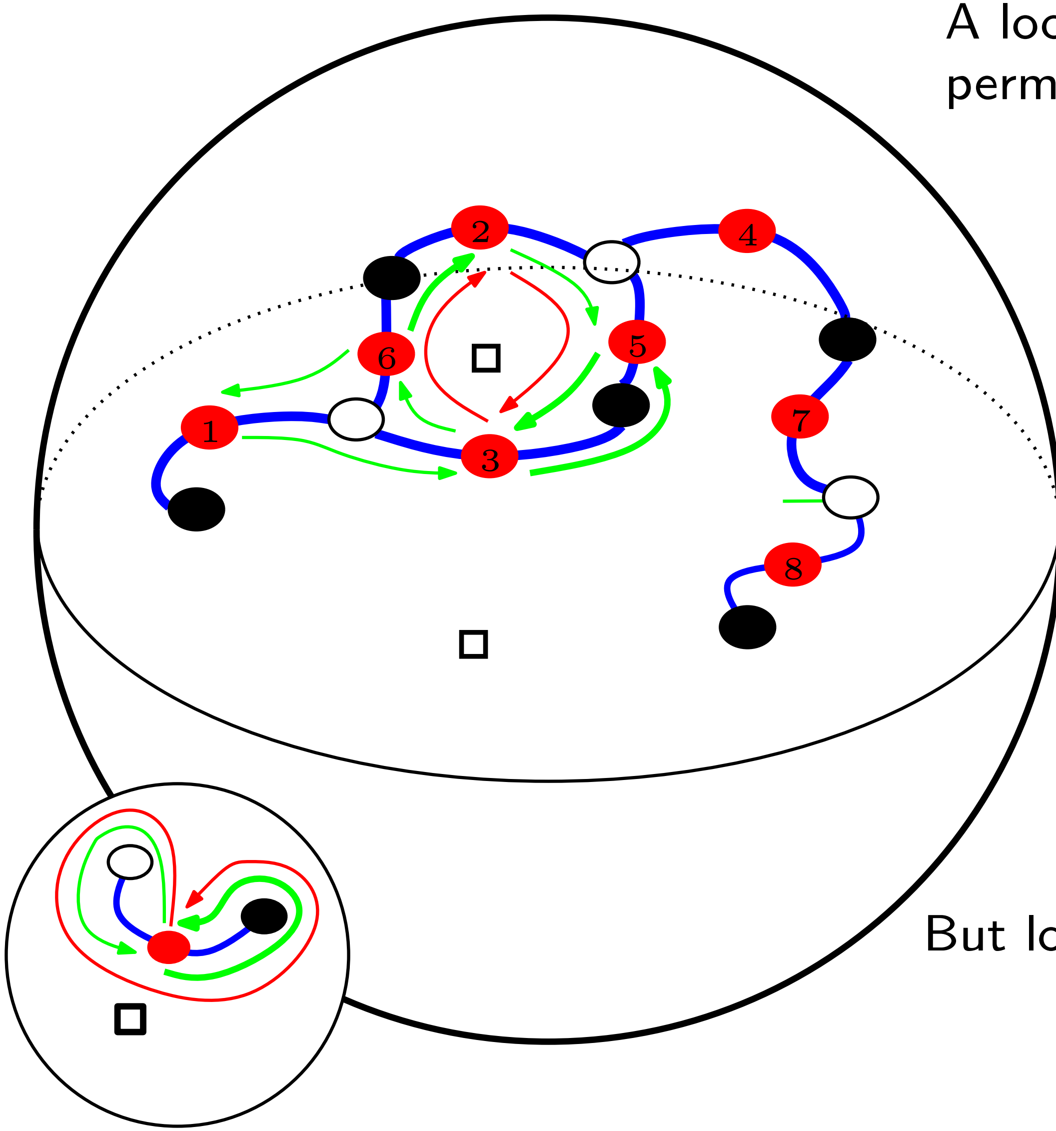

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Proposition: $\sigma_\circ \sigma_\bullet = \sigma_\square$.

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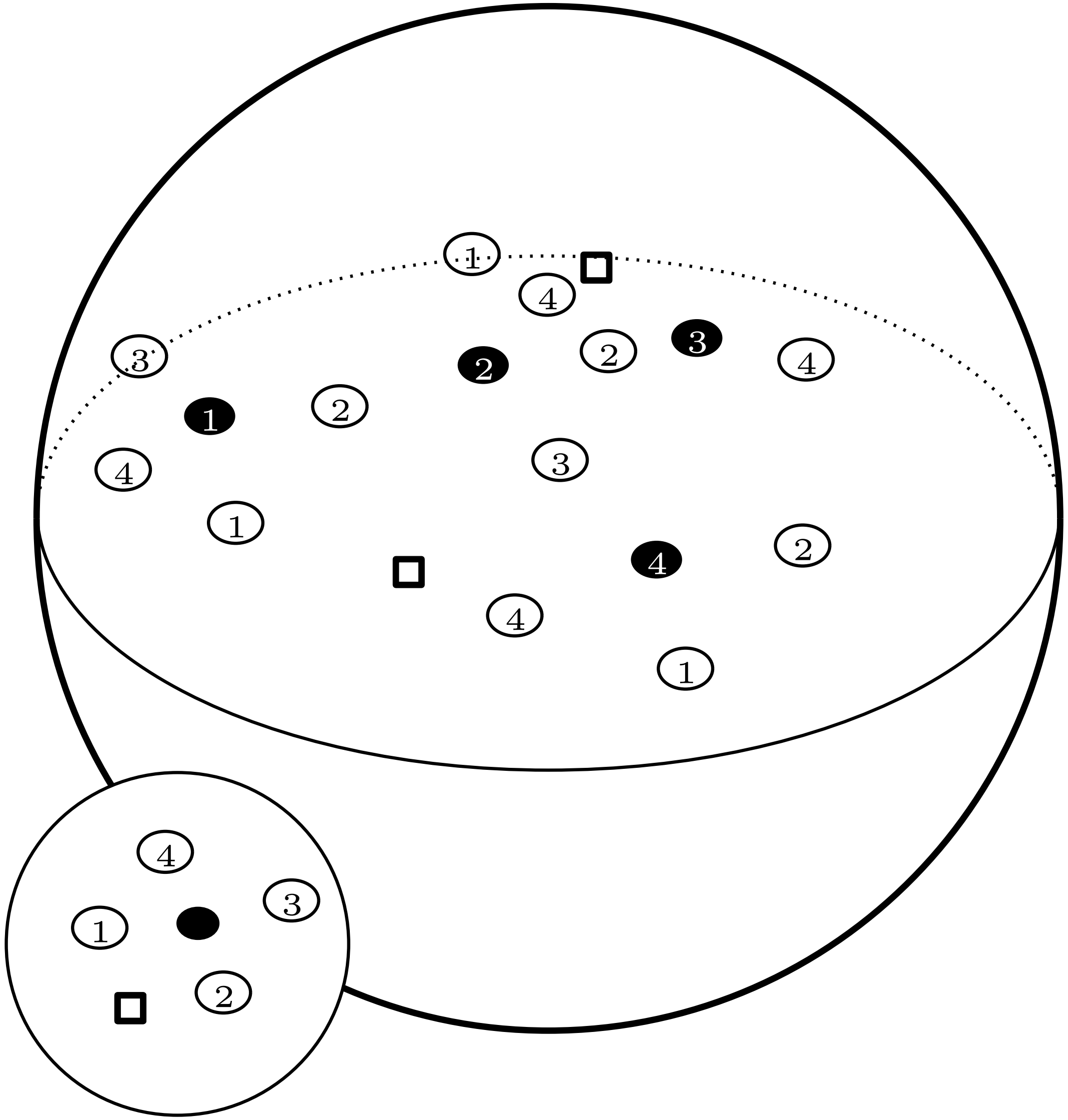
$m+1$ critical values, m-constellations, permutations



$m+1$ critical values ◻ ① ② ③ ④

1 regular value with labeled preimages ●

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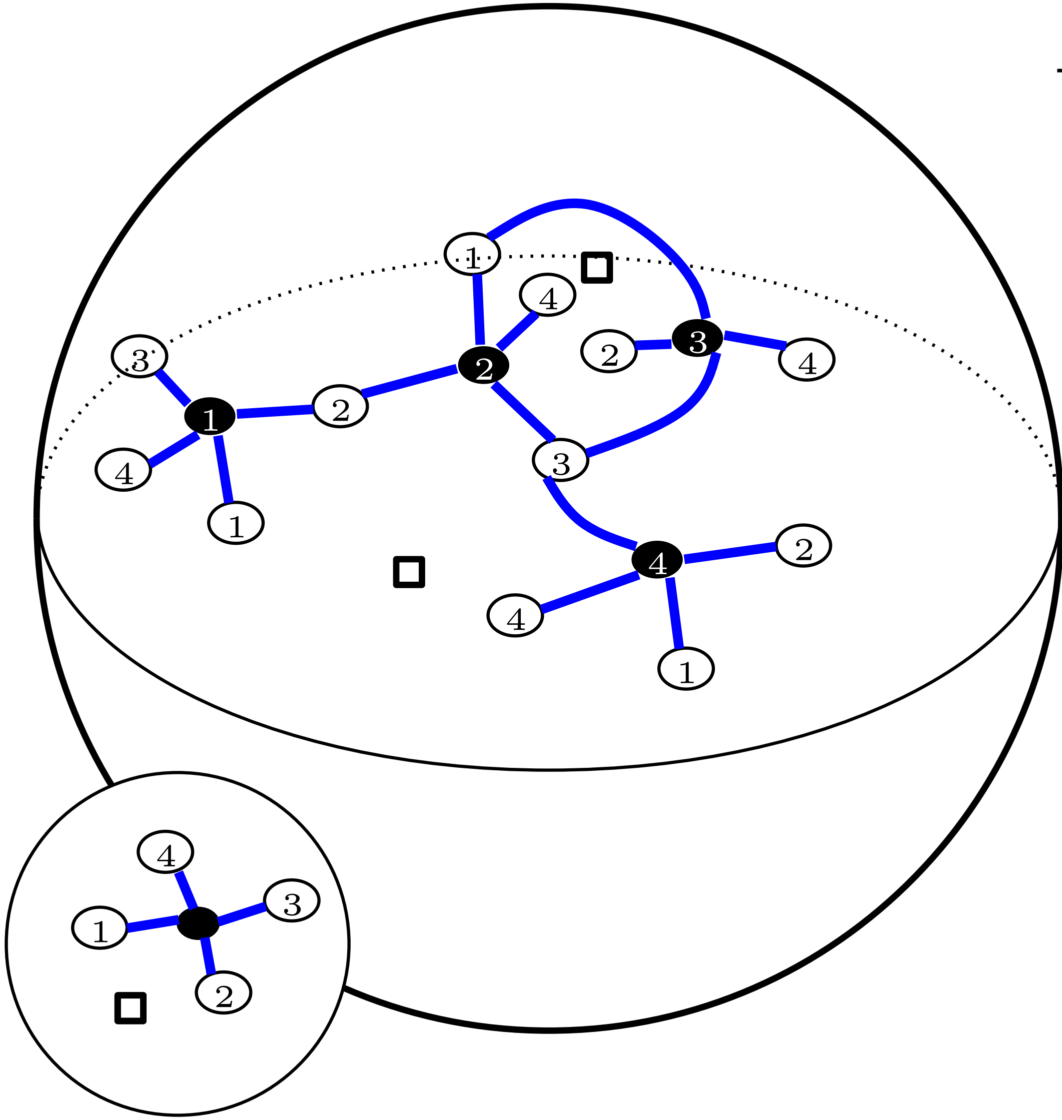

The preimage of the m-star is called a star-constellation.

Proposition. Planar star-constellations with:

– n labelled m-stars,

– $\lambda_j^{\square}$ faces of degree j,

– $\lambda_j^{(i)}$ color i vertices of degree j

are in bijection with minimal transitive factorizations $\sigma_1 \cdots \sigma_m = \sigma_{\square}$ with σ_i of cyclic type $\lambda^{(i)}$.

$m+1$ critical values ◻ ① ② ③ ④

1 regular value with labeled preimages ●

Monodromy, permutations, constellations: a summary

Theorem. There is a bijection between

- Labelled ramified covering of $\mathbb{S}$ of type $\Lambda = (\lambda_0, \ldots, \lambda_m)$
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Specializations.

— $m = 2$: bipartite maps with n edges

— $m = 2$, $\lambda_0 = 4^n$, all faces have degree 4: quadrangulations

$\Rightarrow$ Jean-François Le Gall's last year talk at this seminar

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— $m = 2$, $\lambda_0 = 4^n$, all faces have degree 4: quadrangulations

$\Rightarrow$ Jean-François Le Gall's last year talk at this seminar

— for all $i \geq 1$, $\lambda^{(i)} = 21^{n-2}$: factorizations in transpositions.

coverings with only simple branch points

Monodromy, permutations, constellations: a summary

Theorem. There is a bijection between

- Labelled ramified covering of $\mathbb{S}$ of type $\Lambda = (\lambda_0, \ldots, \lambda_m)$
- Factorizations $(\sigma_1 \cdots \sigma_m = \sigma_0)$ of type Λ
- labelled m-star-constellations of type Λ.

$\mathcal{D} = \mathbb{S} \Leftrightarrow$ minimality $\Leftrightarrow$ planarity.

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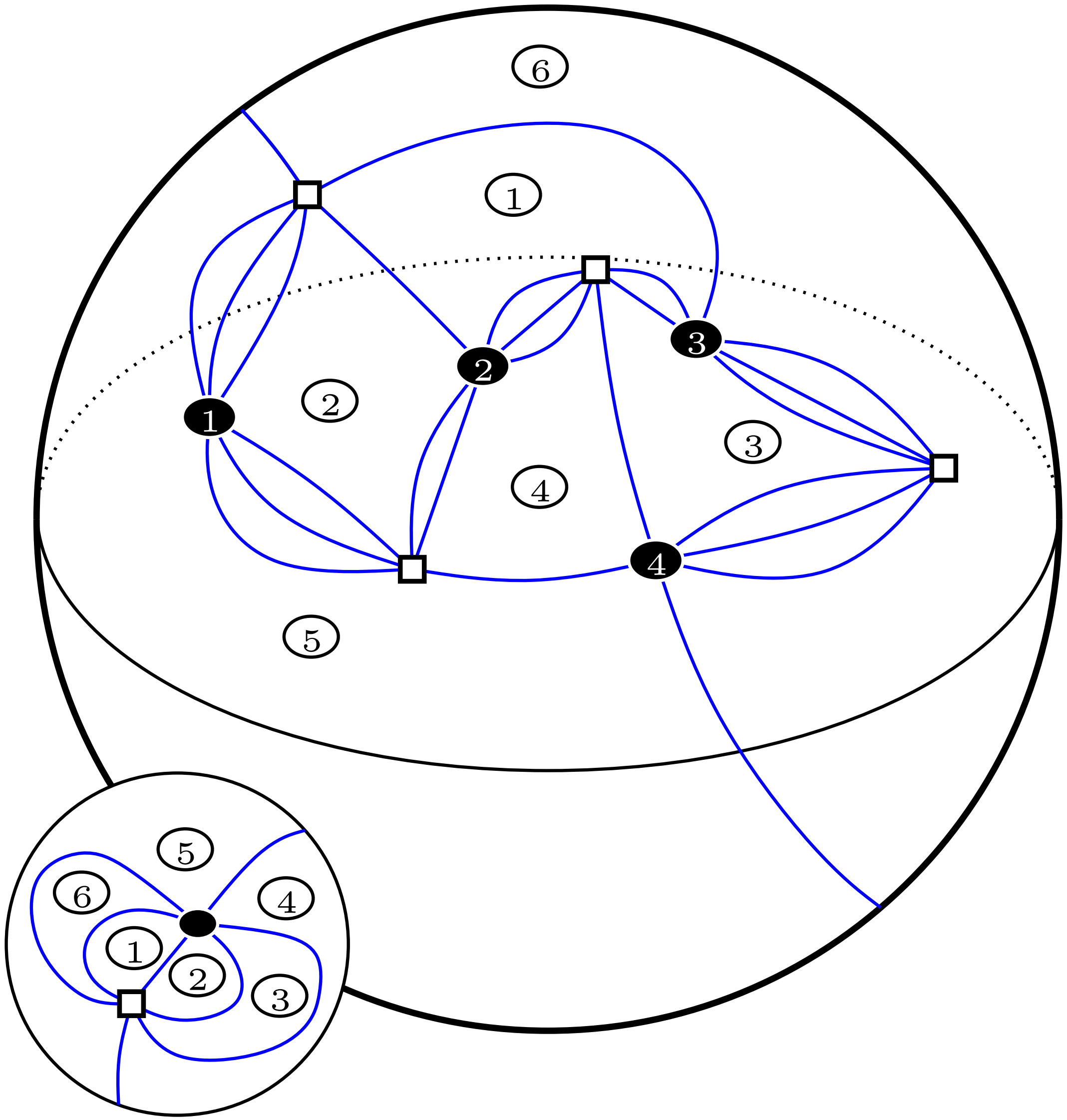
coverings with only simple branch points

Today's topic

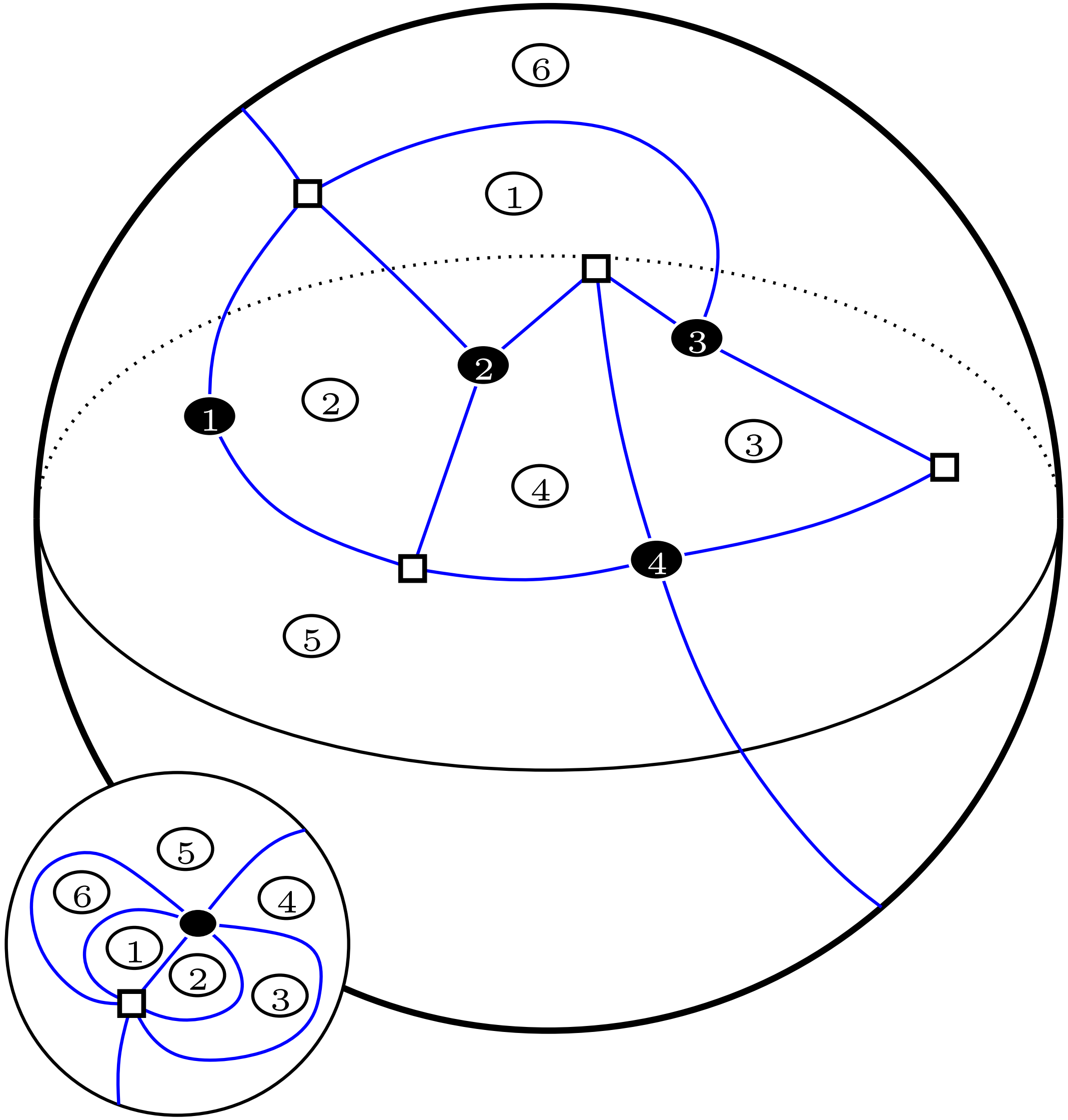
Simple ramified covers, increasing quadrangulations



A ramified cover is simple if its m ramifications have type 21^{n-2}.

Then each face of degree 2 on the image has $n-2$ preimages that are faces of degree 2, and 1 that is a quadrangle.

Simple ramified covers, increasing quadrangulations



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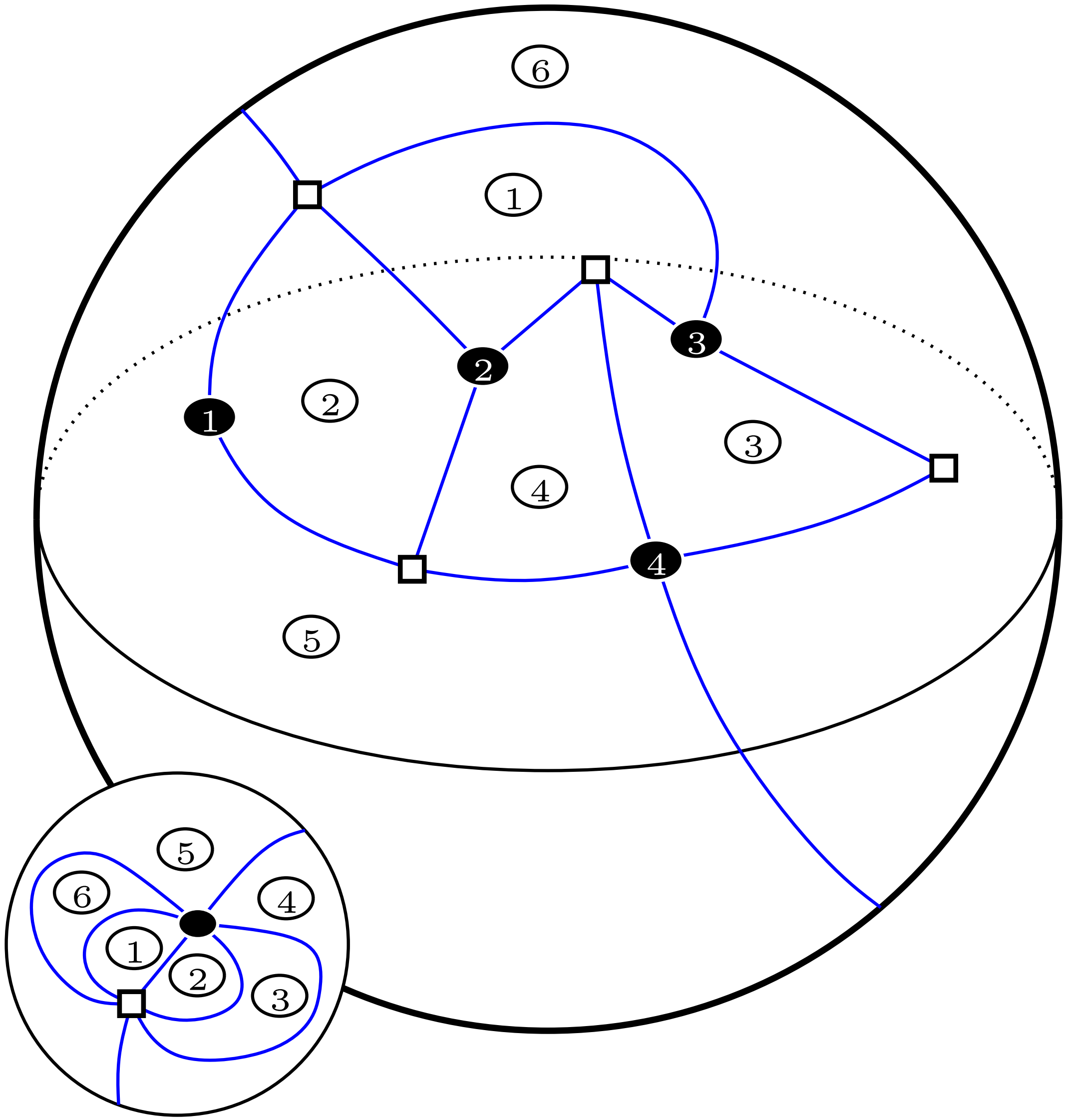

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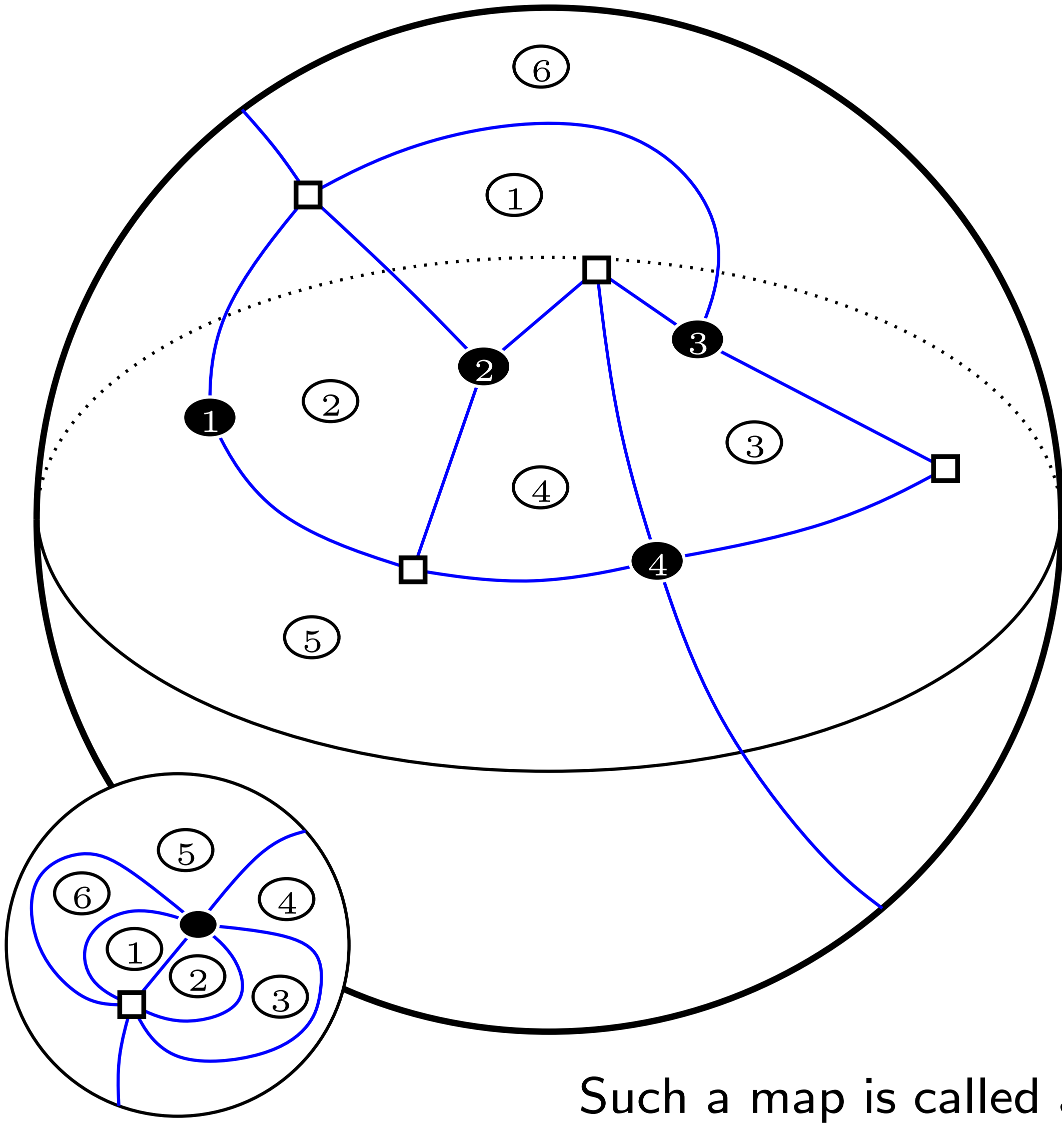

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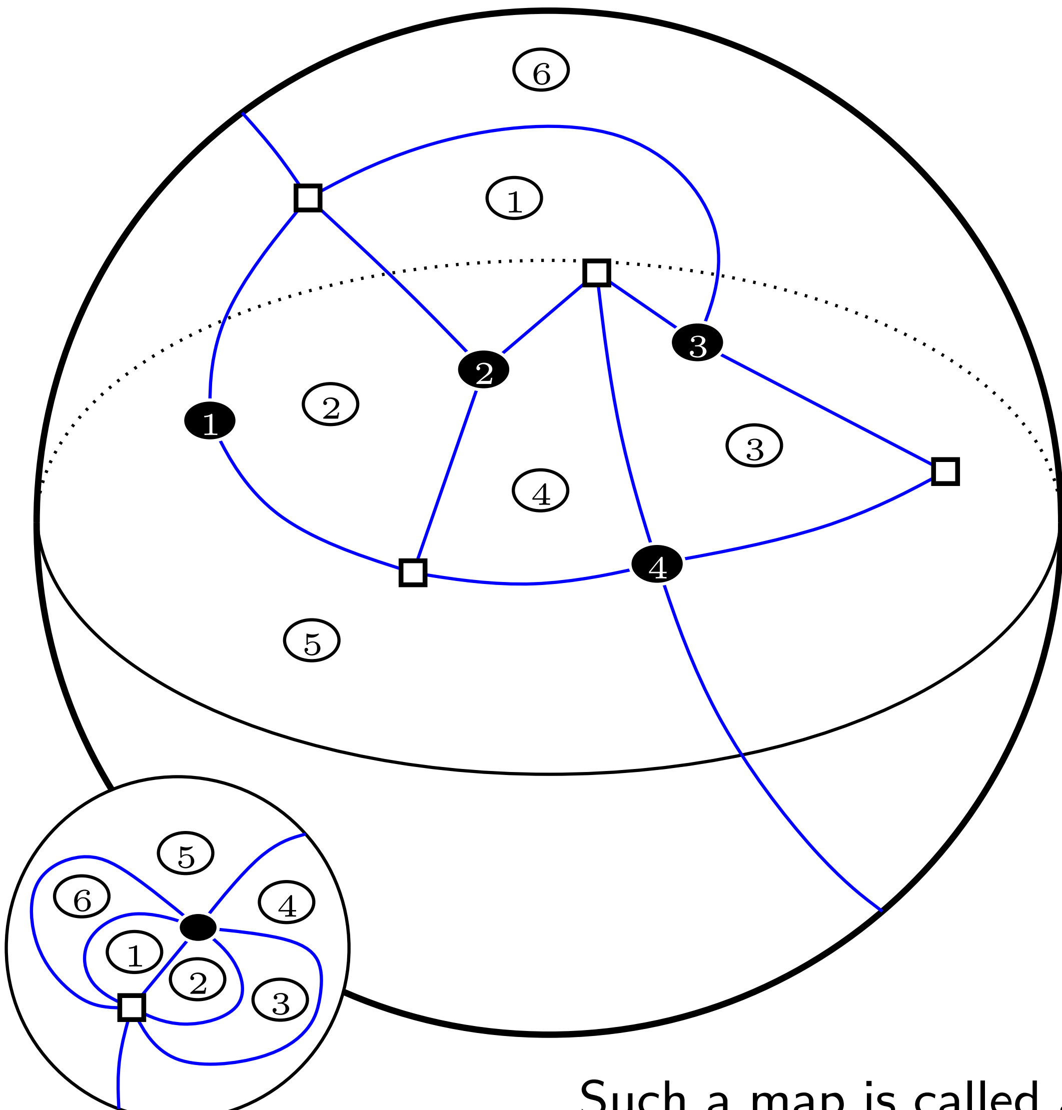

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Theorem. Simple ramified covers of $\mathbb{S}$ by itself with m ramifications points are in bijection with increasing labelled quadrangulations with m faces.

Résumé du 1er épisode

Compter des classes d'équivalence de revêtements ramifiés

$\Updownarrow$

compter certaines plongements de graphes

Plan de l'exposé

Revêtements ramifiés et cartes

Cartes et arbres

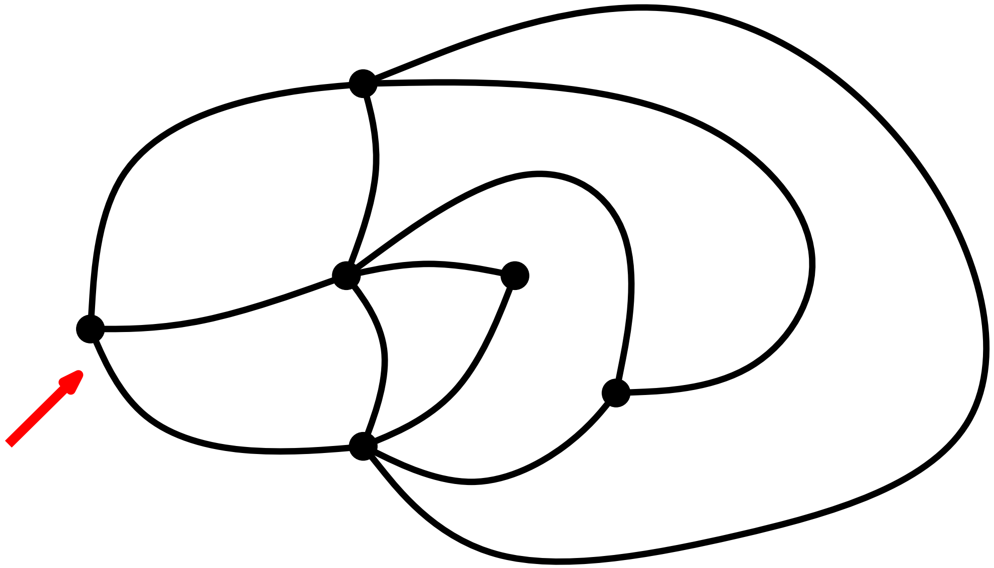
Énumération d'arbres et formule d'Hurwitz

Revêtements et cartes aléatoires

Planar maps, spanning trees and duality

A planar map is a proper embedding of a connected graph on the sphere (considered up to homeomorphisms).

From now on, map means rooted planar map.



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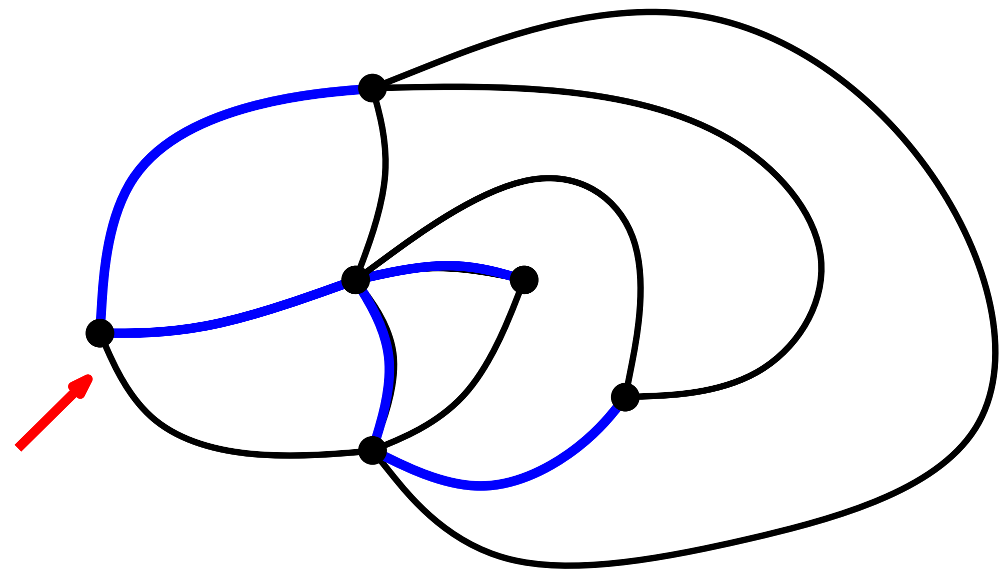
A tree-rooted map is a map with a spanning tree.



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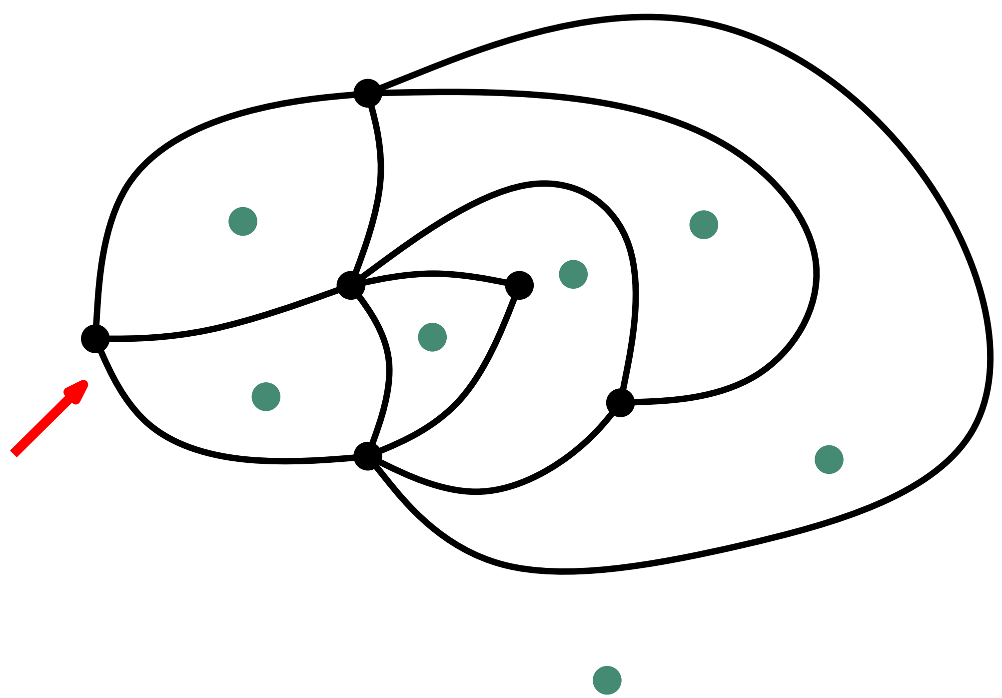

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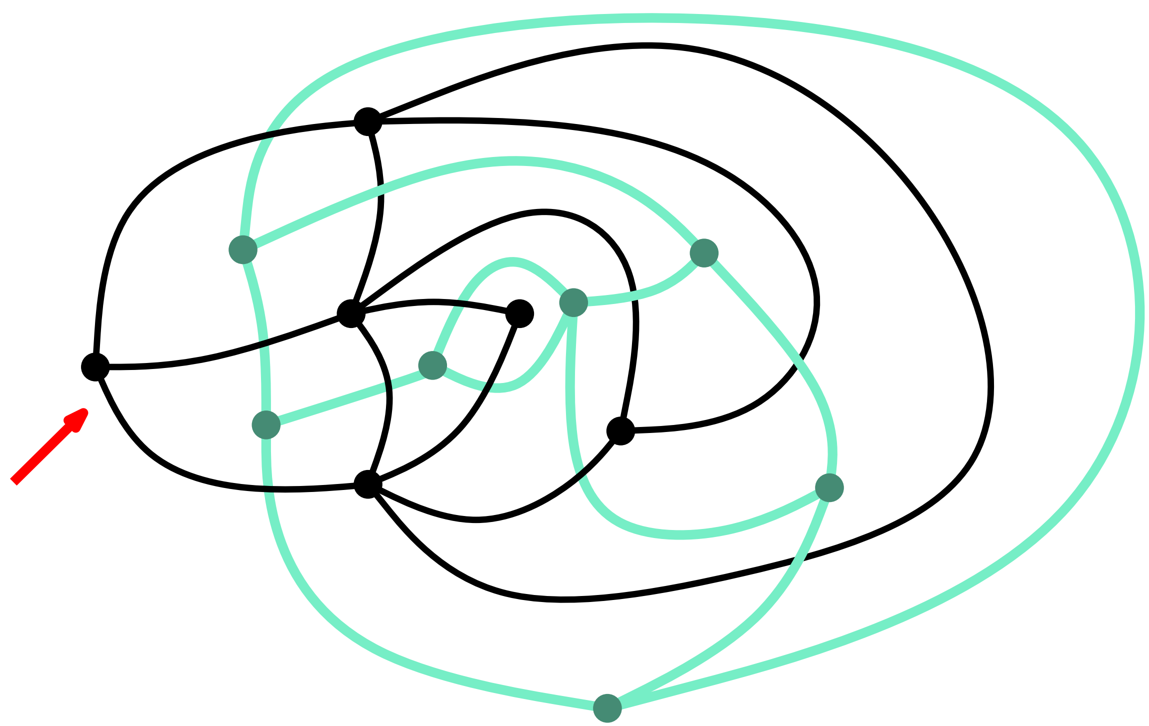

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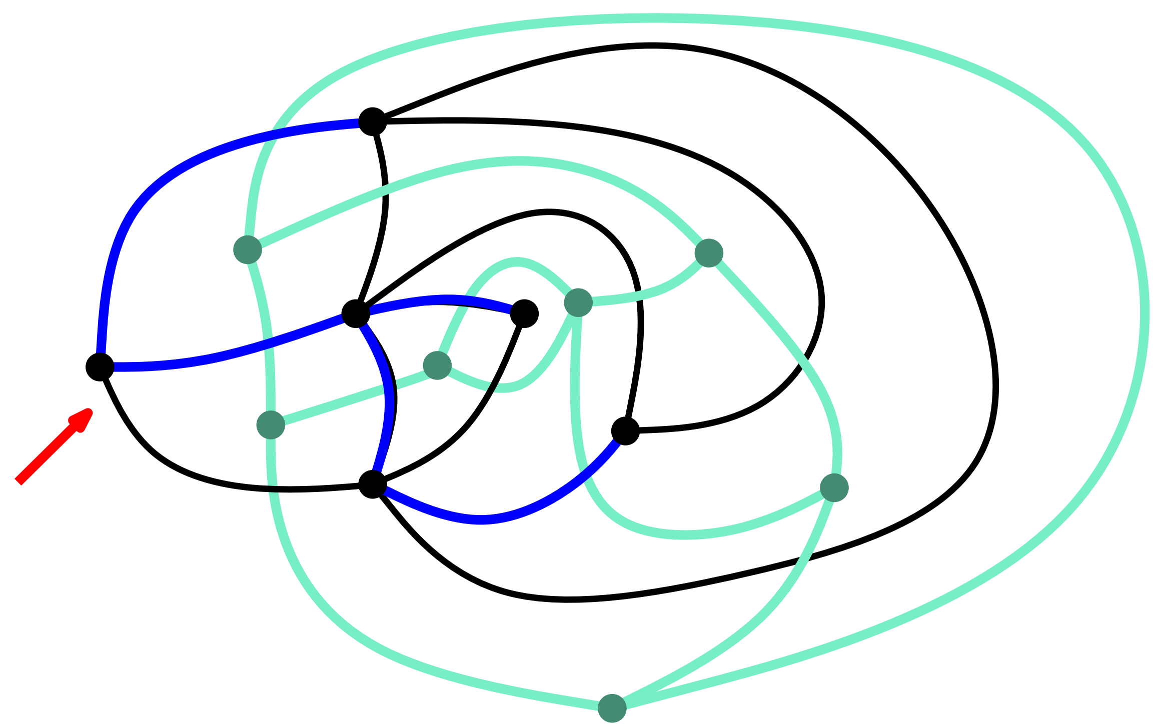

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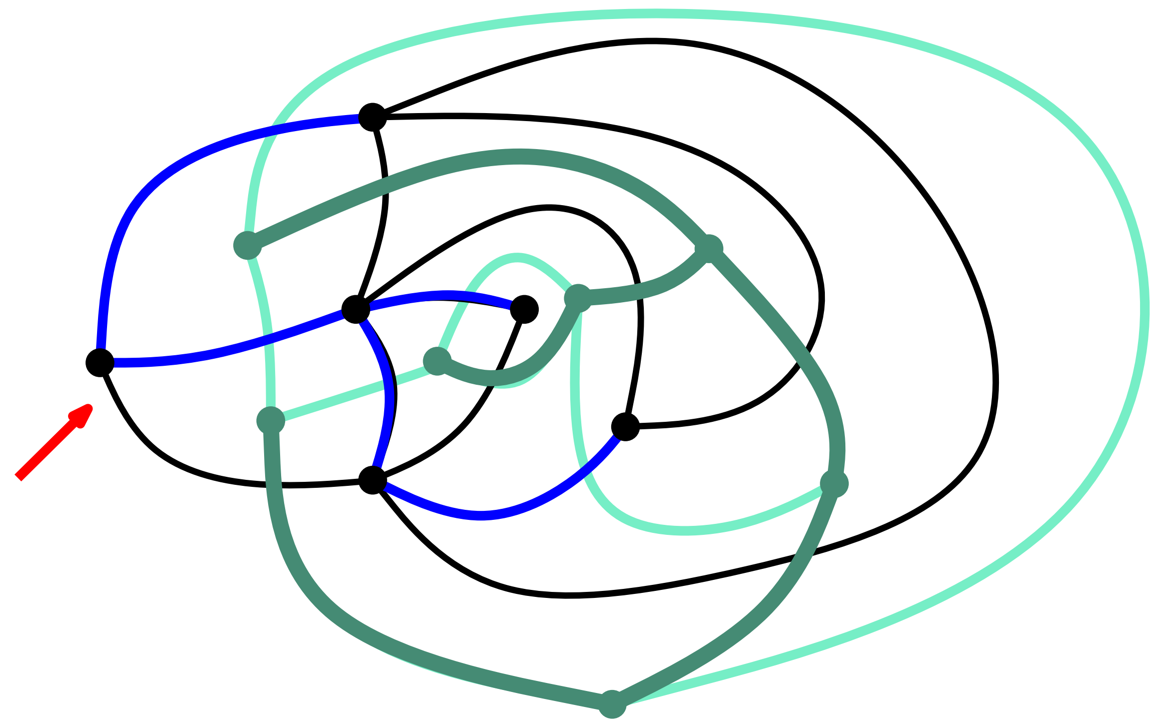

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Euler's relation:
(#vertices-1)+(#faces-1)
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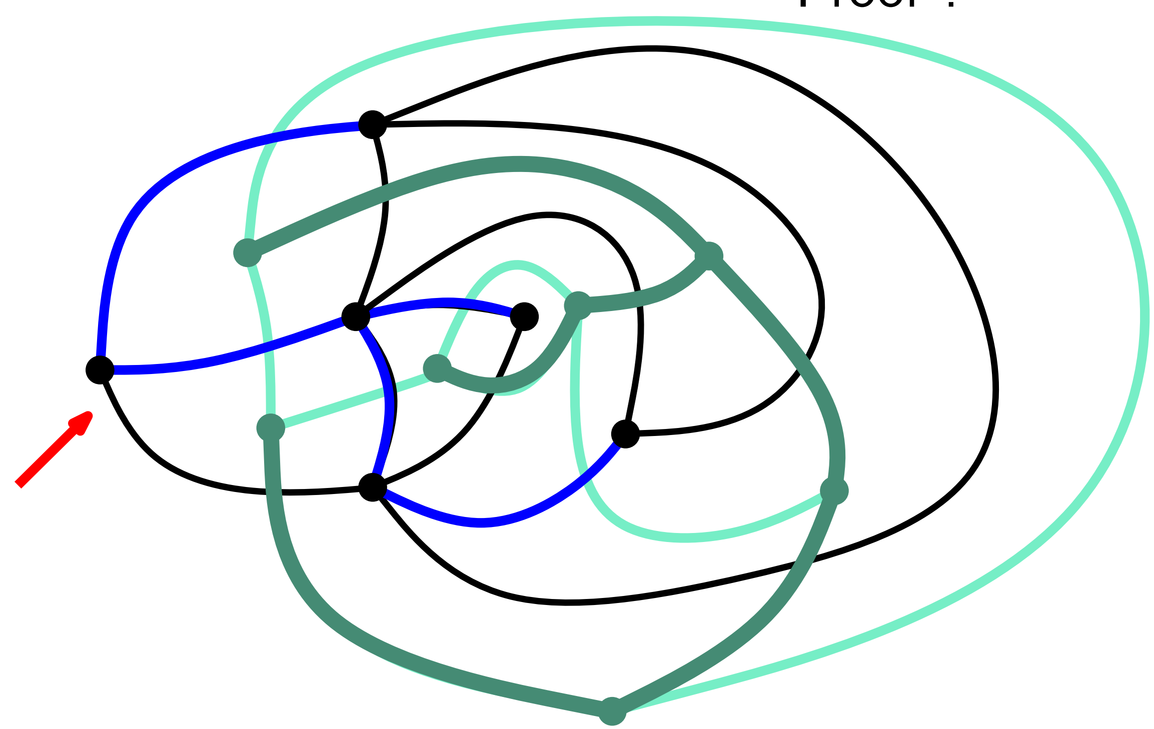

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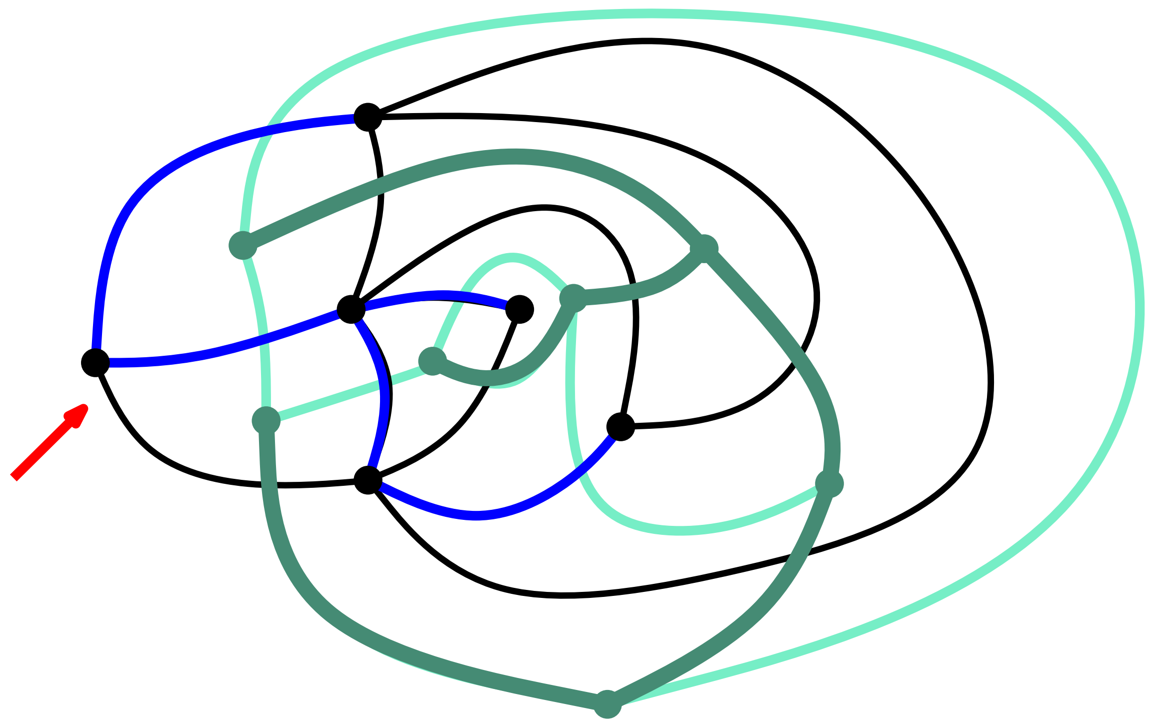
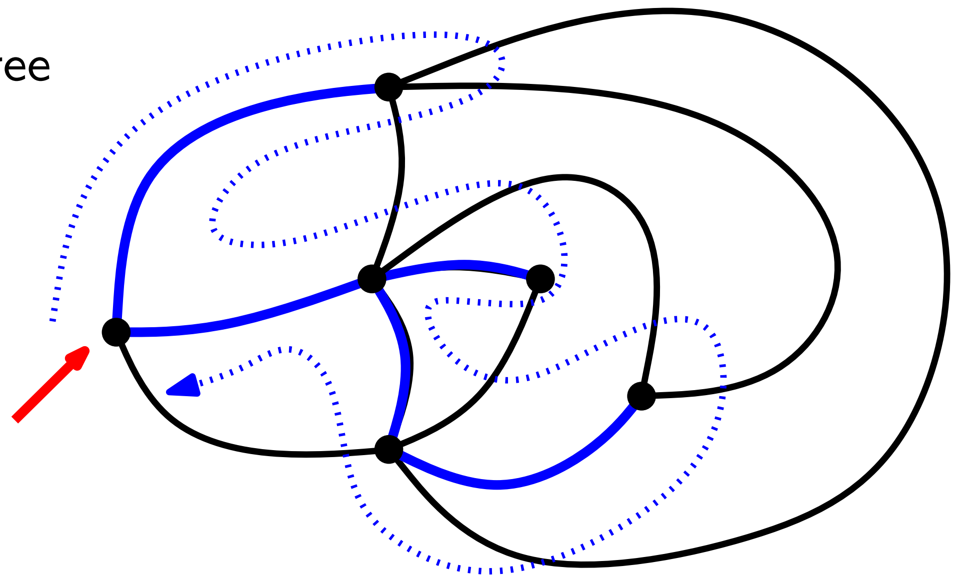


Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree



Encoding and counting tree-rooted maps

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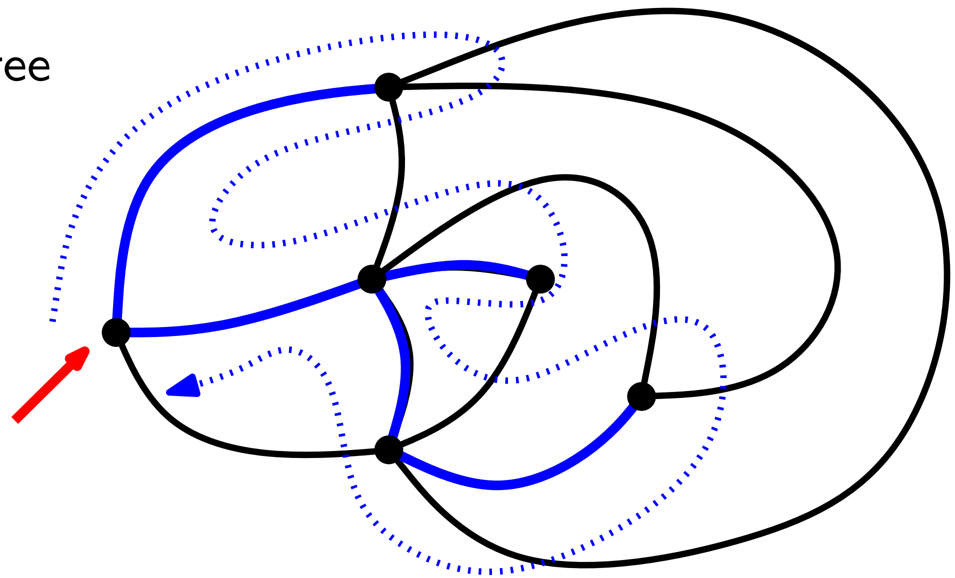
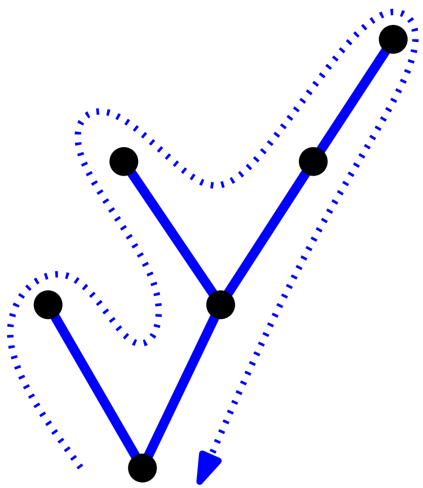
Rooted tree $\equiv$ balanced parenthesis word

$uduuduuddd$





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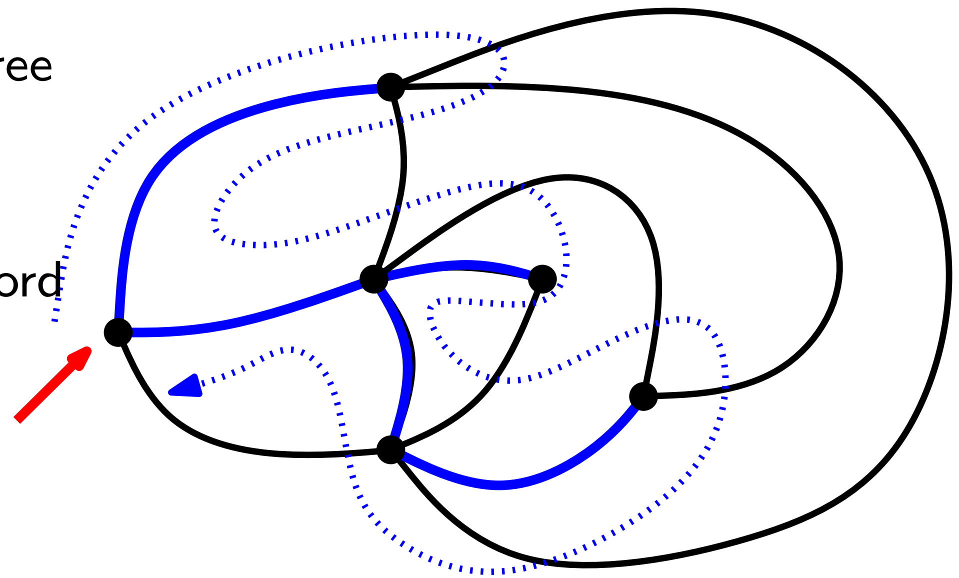

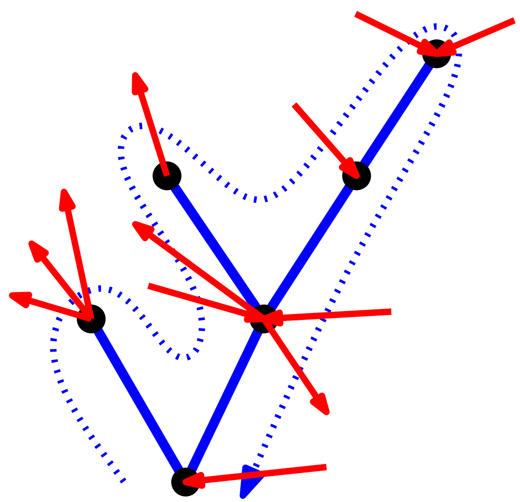

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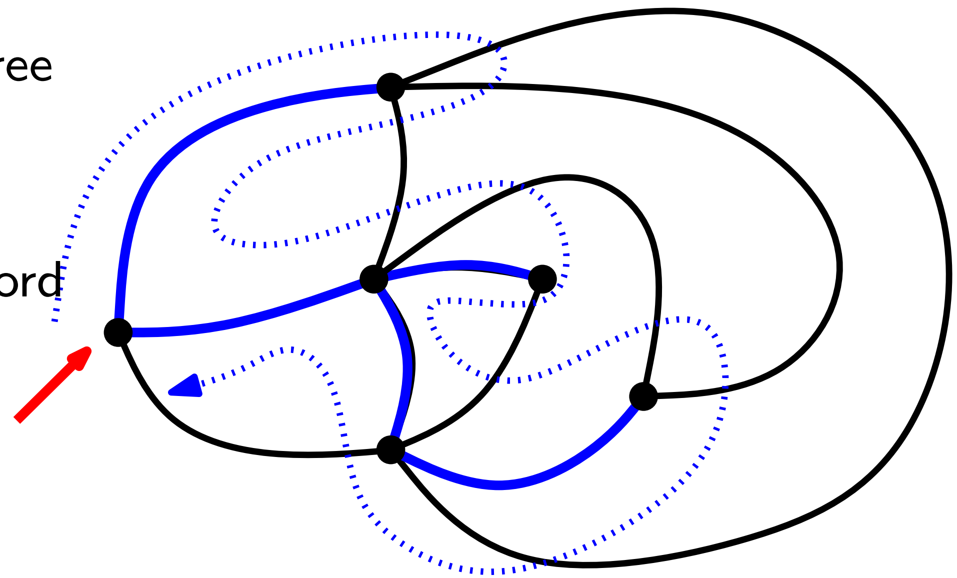

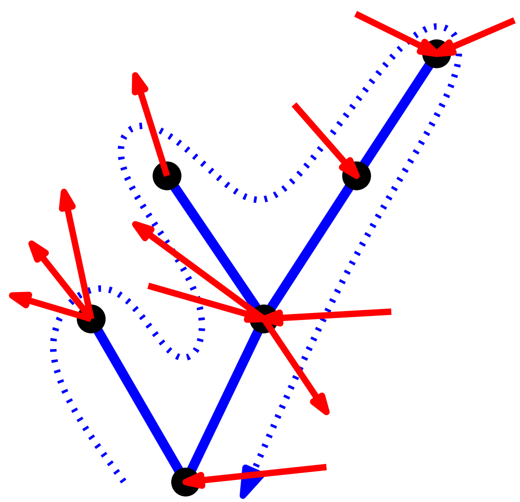

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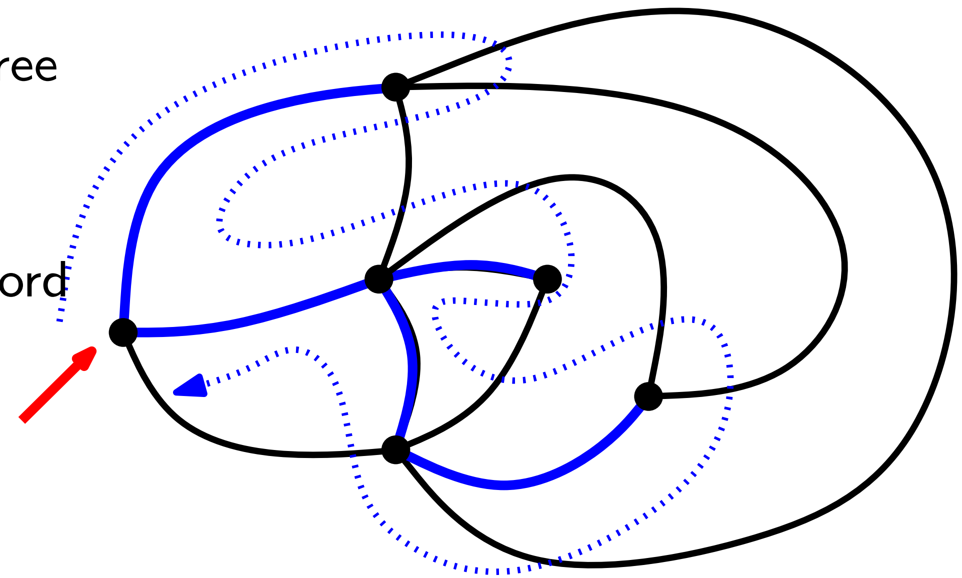

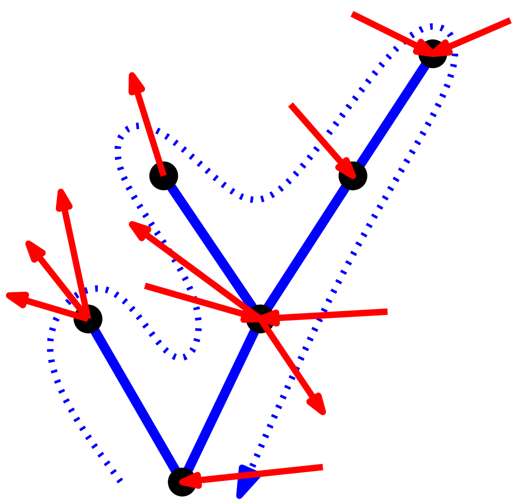

Code of the tree-rooted map $=$ tree decorated by a balanced parenthesis word

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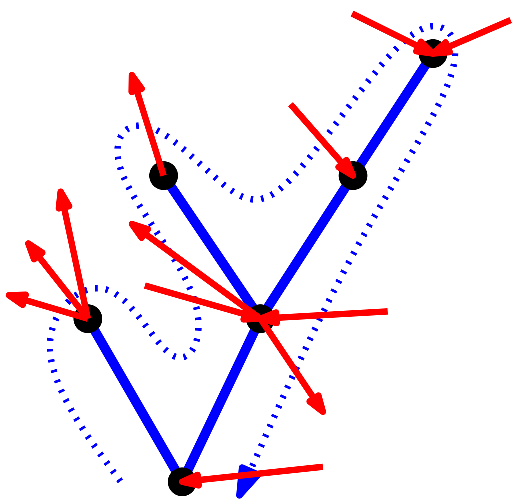

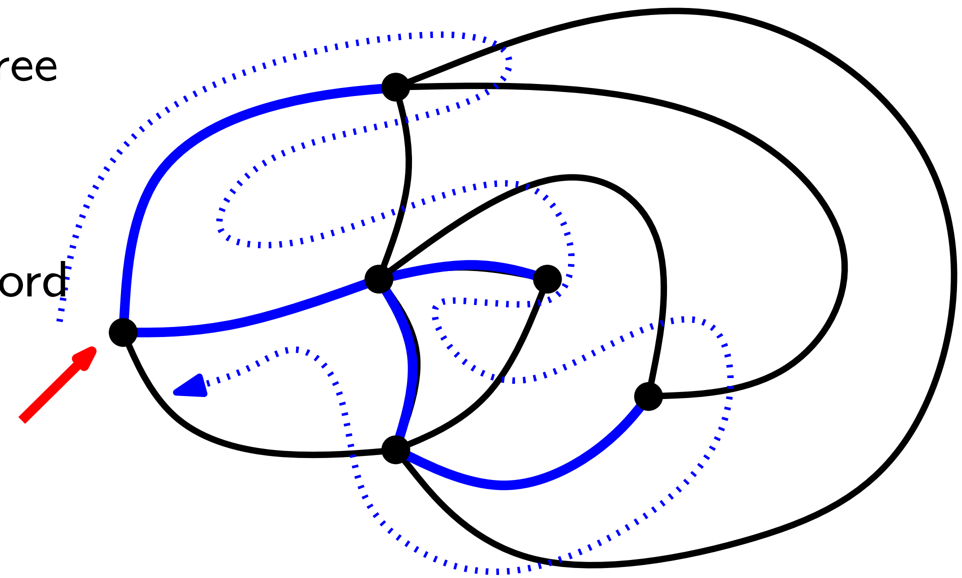

Writing the two codes during the walk:

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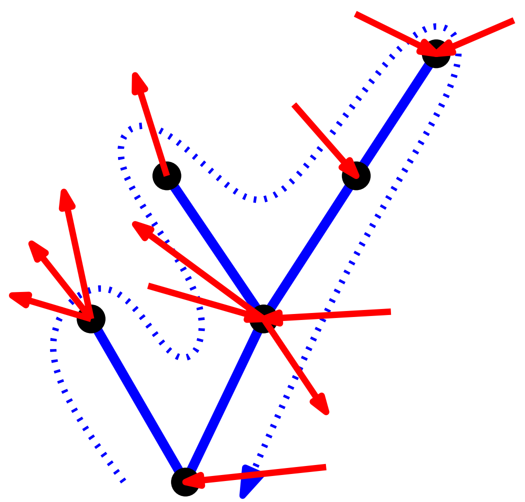

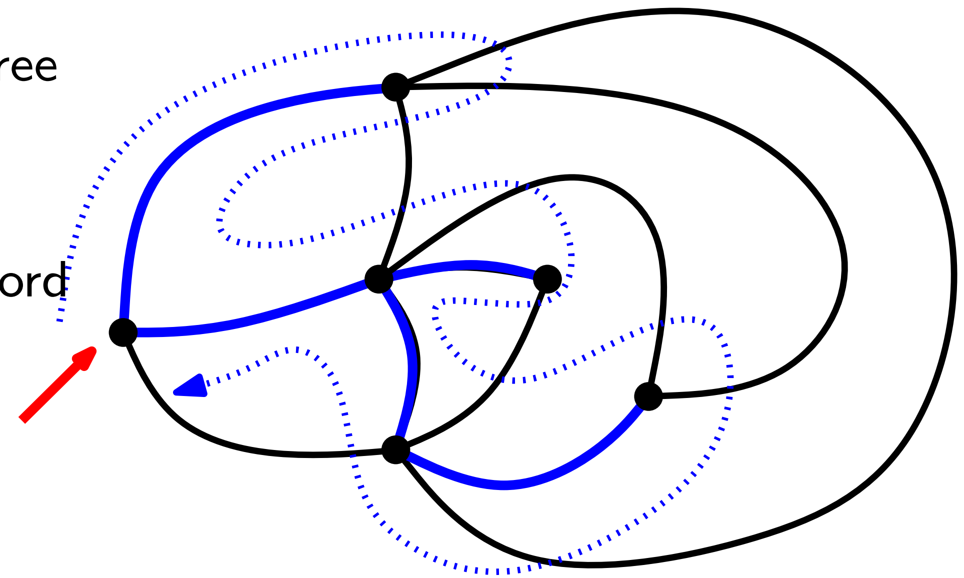

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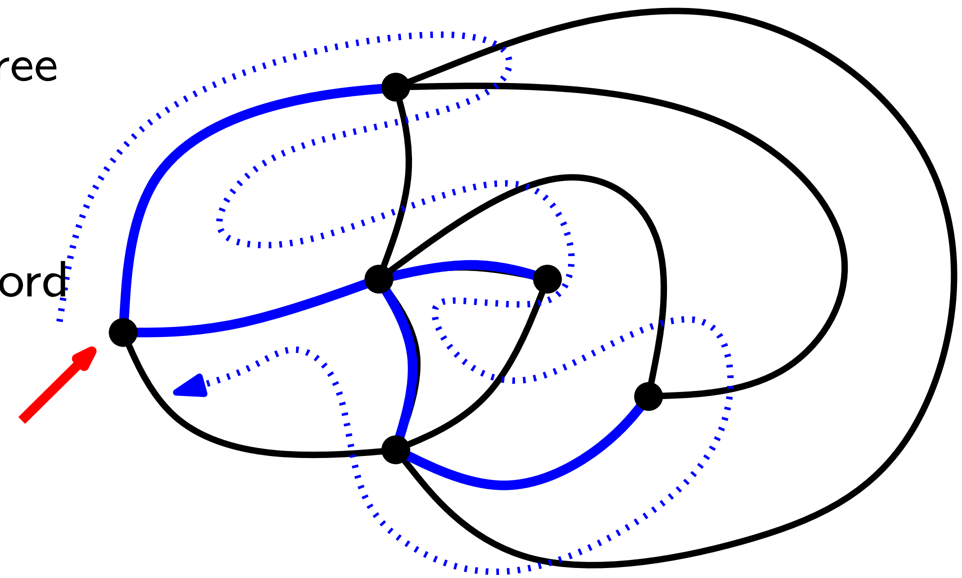

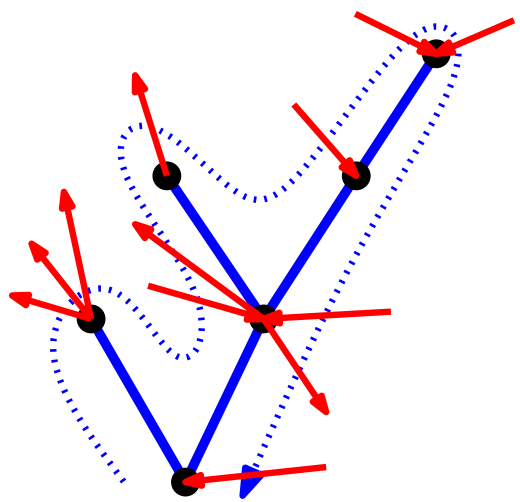

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The number of tree rooted planar maps with n edges is $\sum_{i=0}^{n} \binom{2n}{i} C_i C_{n-i}$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes Catalan numbers, counting balanced parenthesis words.

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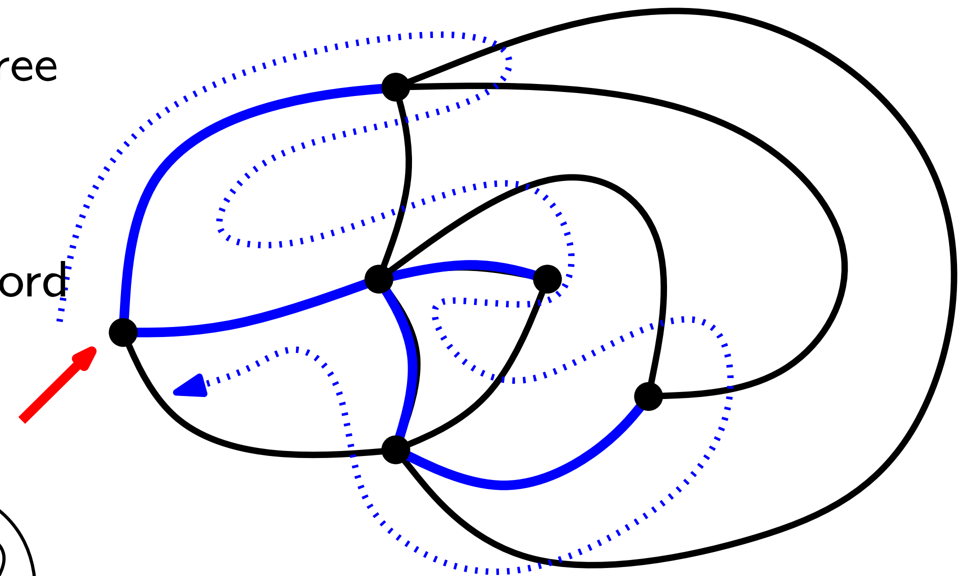

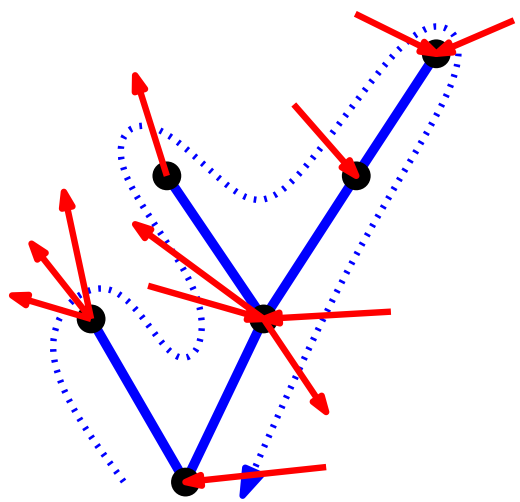

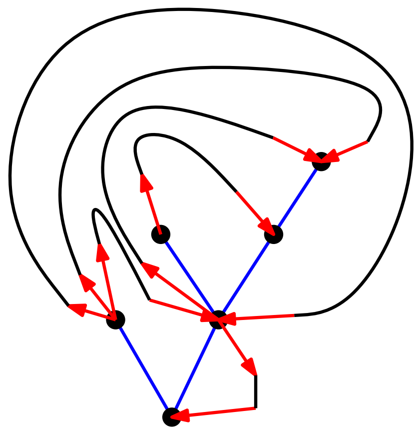

Observe that closure edges turn clockwise around the tree.

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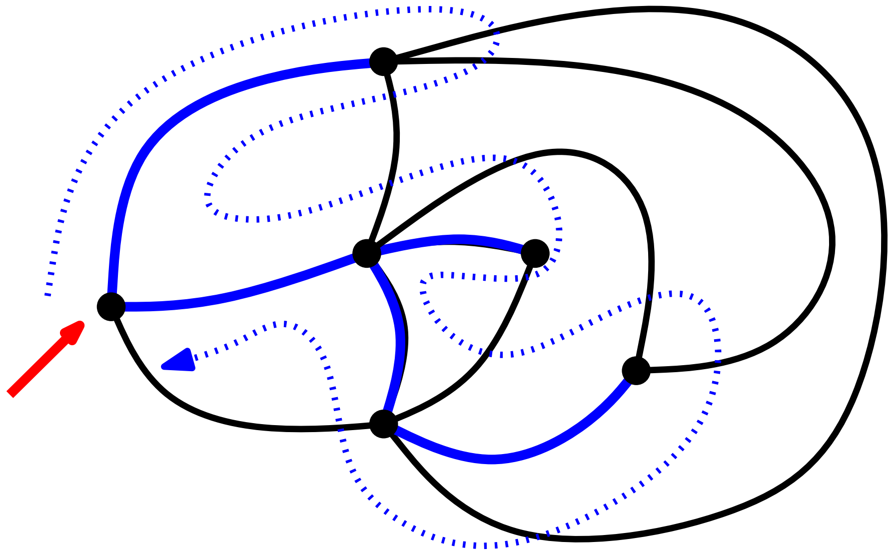
but we want rooted (not tree-rooted) maps

Let us recycle the idea used for tree-rooted maps, using a canonical spanning tree



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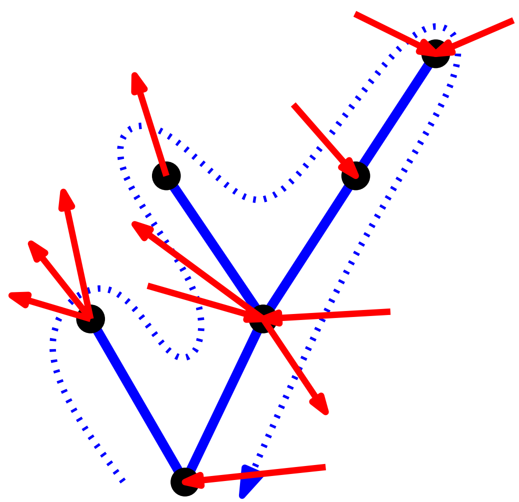

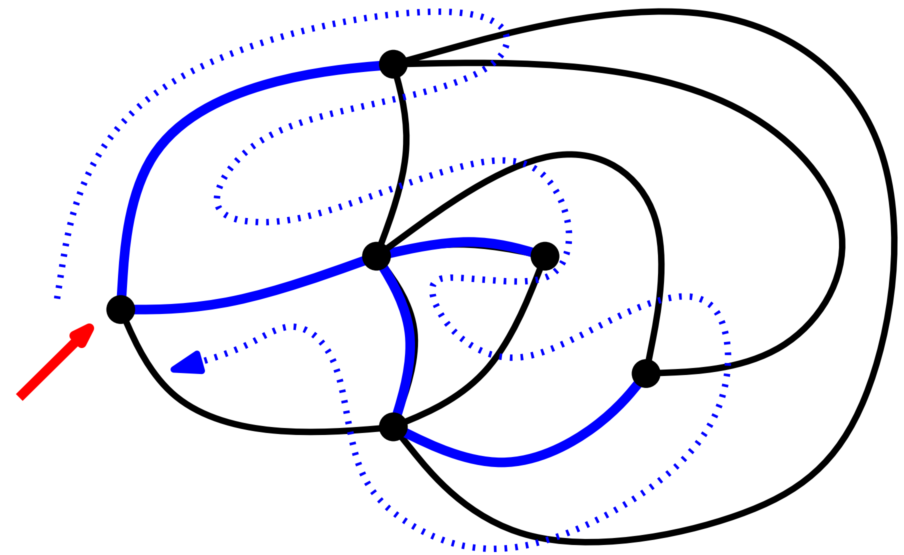

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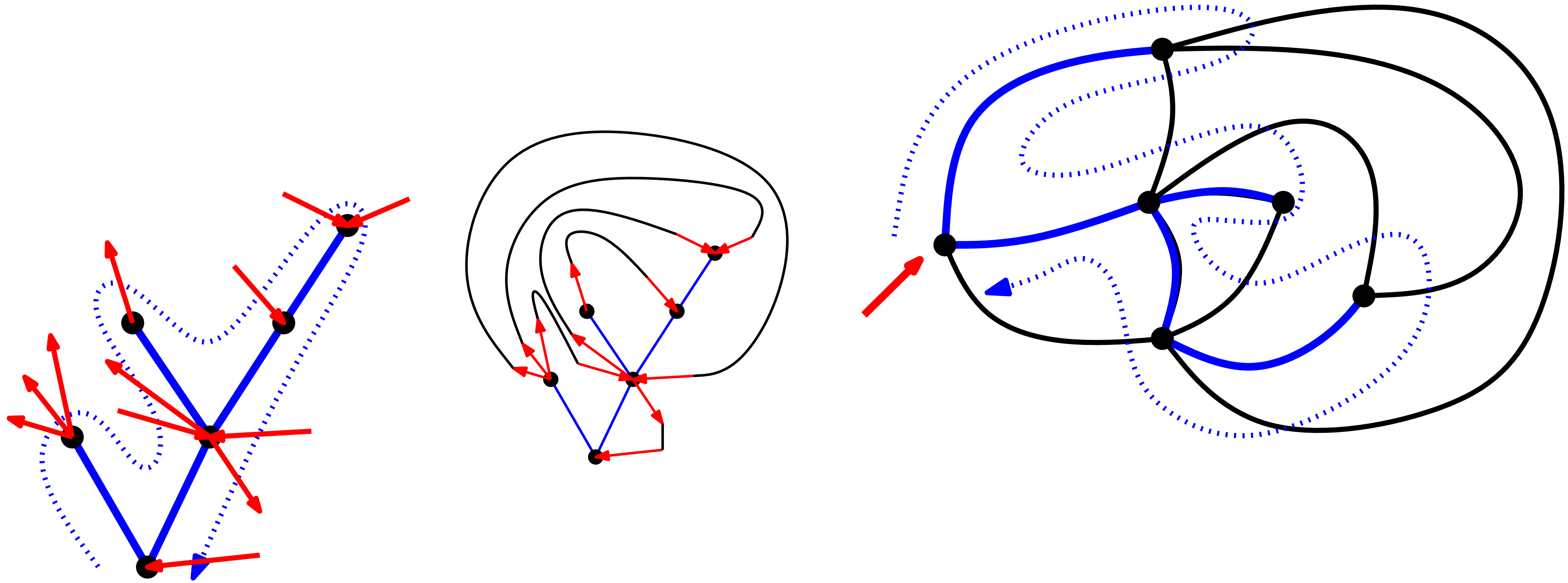

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The map is recovered from the code by *closure*.

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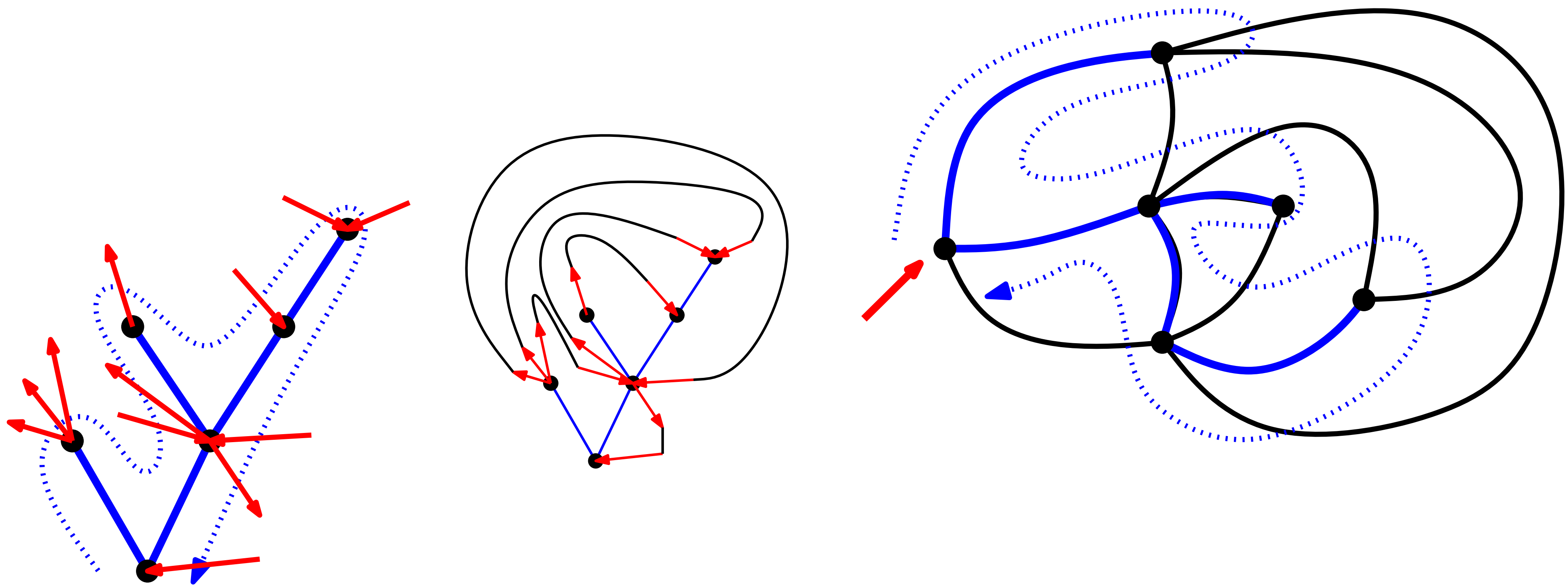

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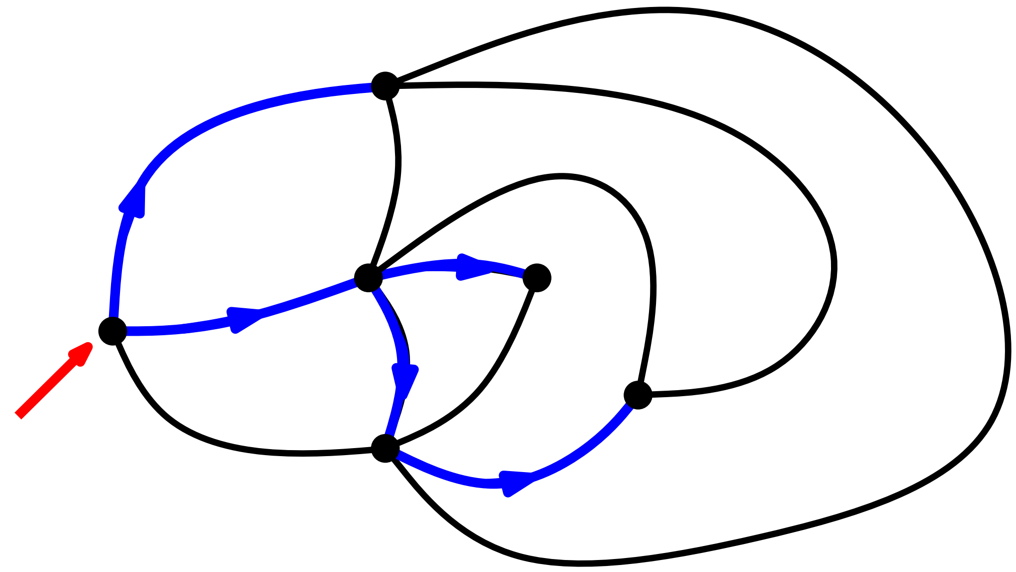
Our code of the map will be a canonical decorated tree

Question is How do we choose the canonical spanning tree
so that the resulting decorated trees can be described and counted ?

From tree-rooted maps to minimal accessible maps

Orient the tree edges away from the root



From tree-rooted maps to minimal accessible maps

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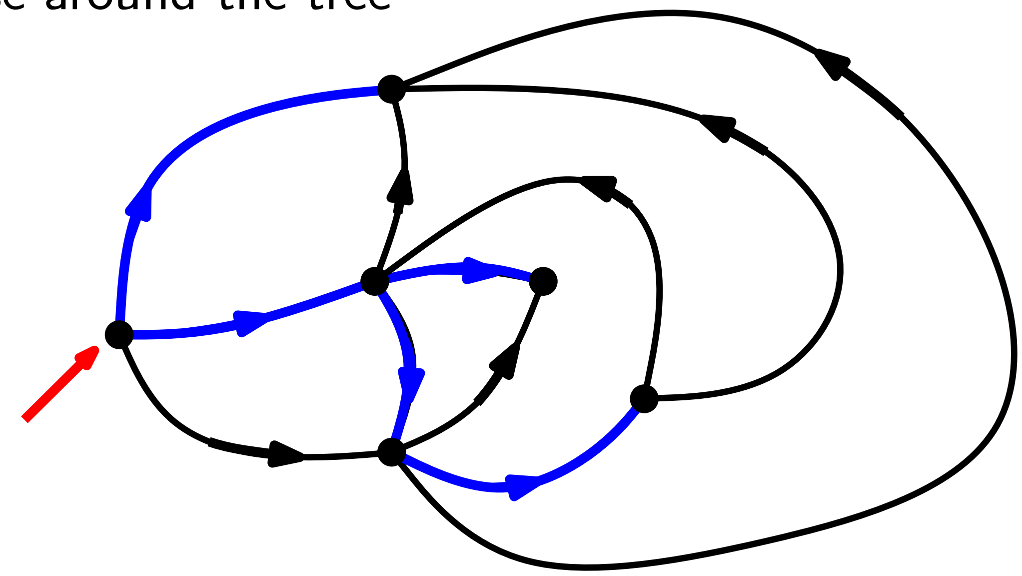
Orient the other edges couterclockwise around the tree



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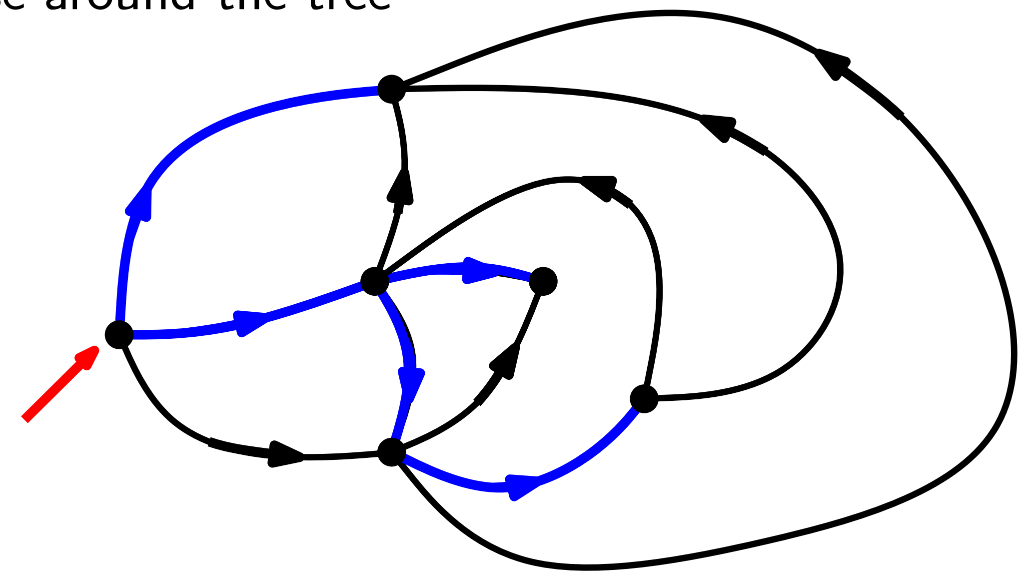

The resulting orientation has no clockwise circuit.

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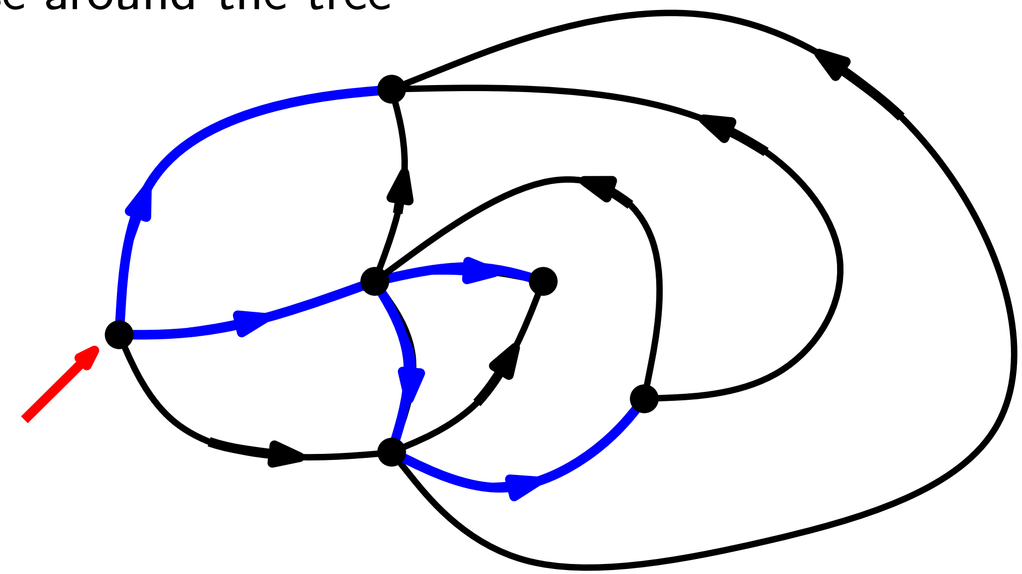

It is called a minimal orientation (for the order induced by circuit reversal).

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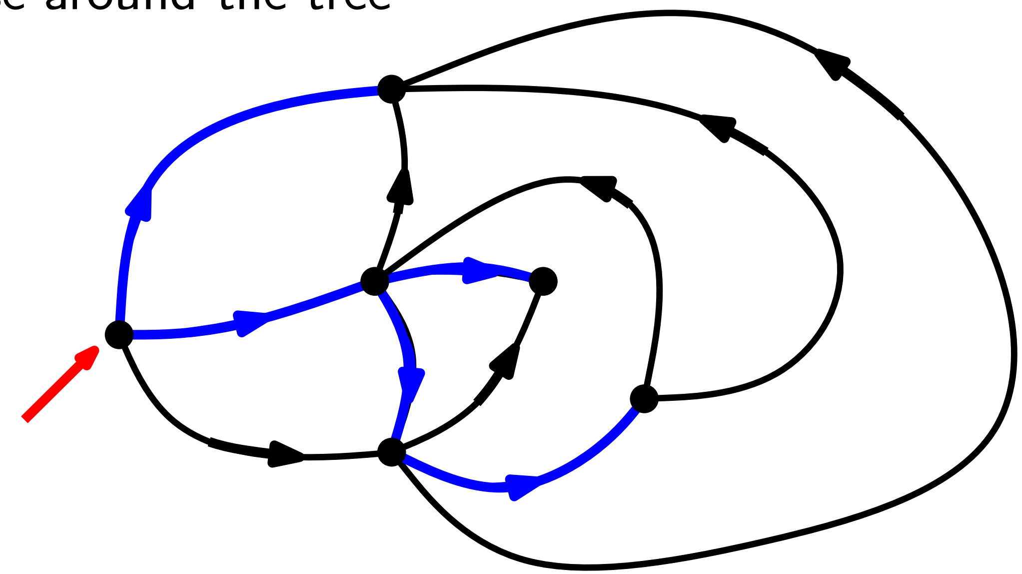

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A oriented map is accessible if every vertex can be reach by an oriented path from the root.

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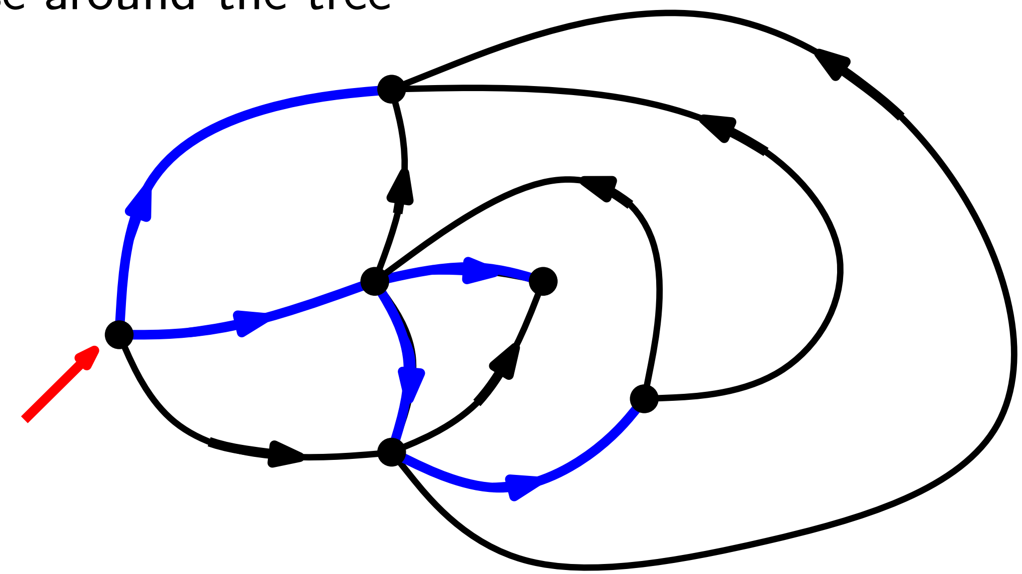

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Theorem (Bernardi 2005) This is a bijection between tree-rooted maps with n edges and minimum accessible maps with n edges

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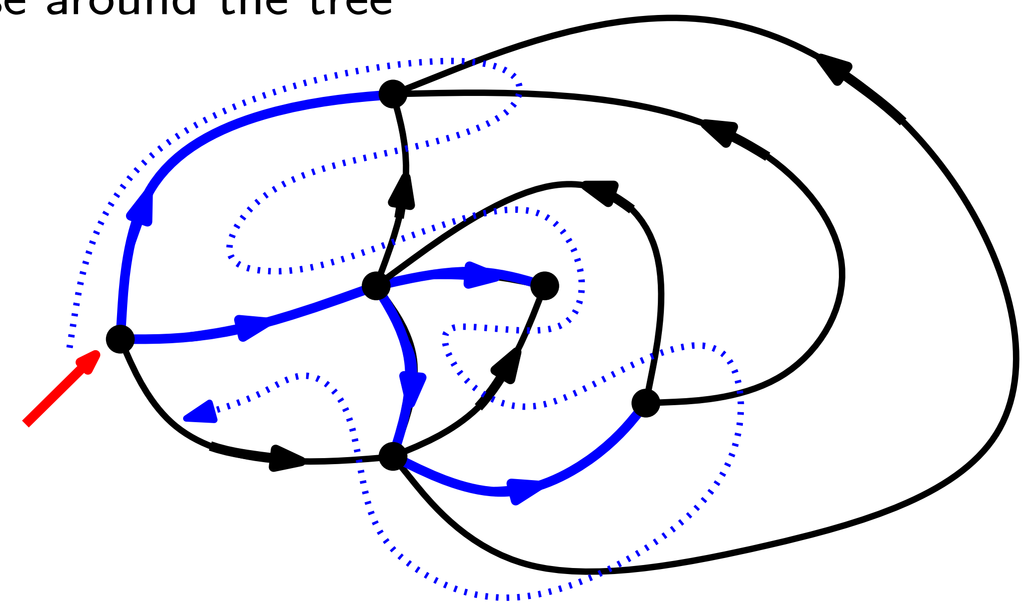

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The tree is recovered by reconstructing its contour .

Minimal orientations and canonical spanning trees

Idea:

Choose a minimal accessible orientation to get a spanning tree

Our pb becomes:

How to choose a canonical accessible minimal orientation?

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A function $\alpha : V \rightarrow \mathbb{N}$ is feasible on a plane map M if there exists an orientation of M such that for each vertex v the outdegree of v is $f(v)$.

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A function $\alpha : V \rightarrow \mathbb{N}$ is feasible on a plane map M if there exists an orientation of M such that for each vertex v the outdegree of v is $f(v)$.

Theorem (Felsner 2004). Let α be a feasible function on a plane map M. Then the map M has a unique minimal α-orientation.

Minimal orientations and canonical spanning trees

Idea:

Choose a minimal accessible orientation to get a spanning tree

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Our pb becomes: How to choose a canonical α? (and check accessibility)

Fact: For many subclasses $\mathcal{F}$ of planar maps, there exists an $\alpha_{\mathcal{F}}$ s.t.:

A planar map is in $\mathcal{F}$ if and only if it admits an $\alpha_{\mathcal{F}}$-orientation.

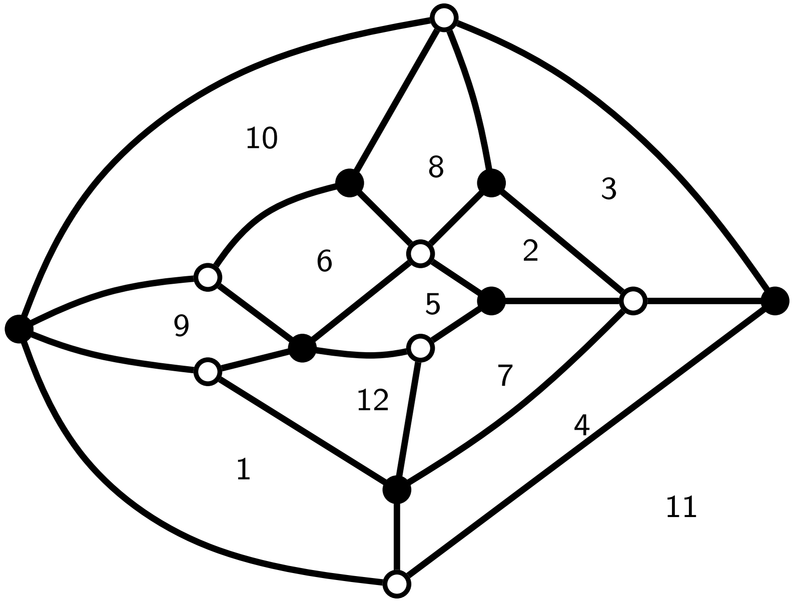
α-orientations for increasing quadrangulations

Recall increasing quadrangulations are planar maps with faces of degree 4 such that:

– faces have labels in $\{1, \ldots, 2n-2\}$

– around labeled vertices, face labels increase in ccw order

– around white vertices, face labels increase in cw order



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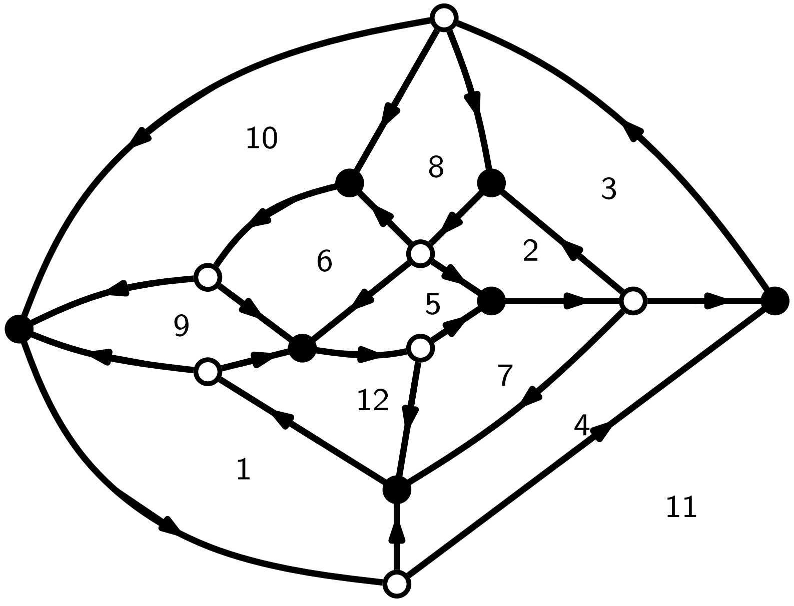

Orient each edge so that the minimum incident label is on the left

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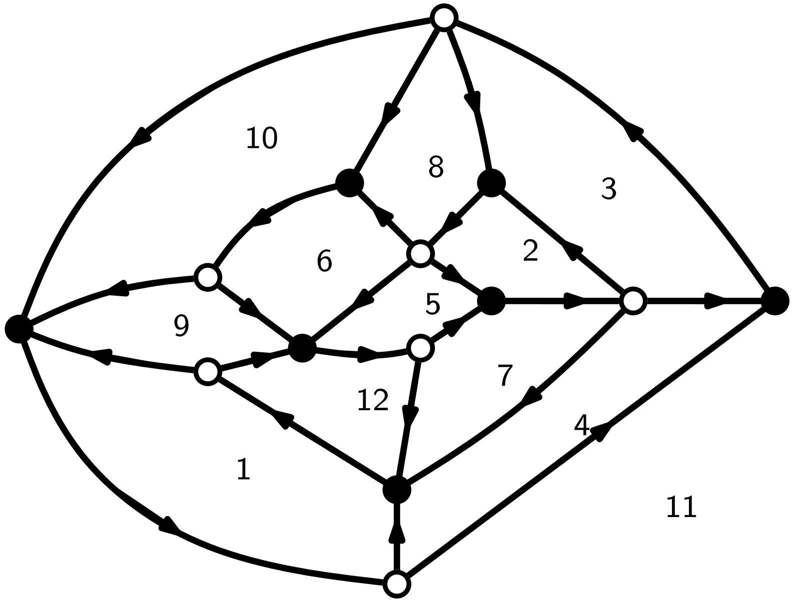

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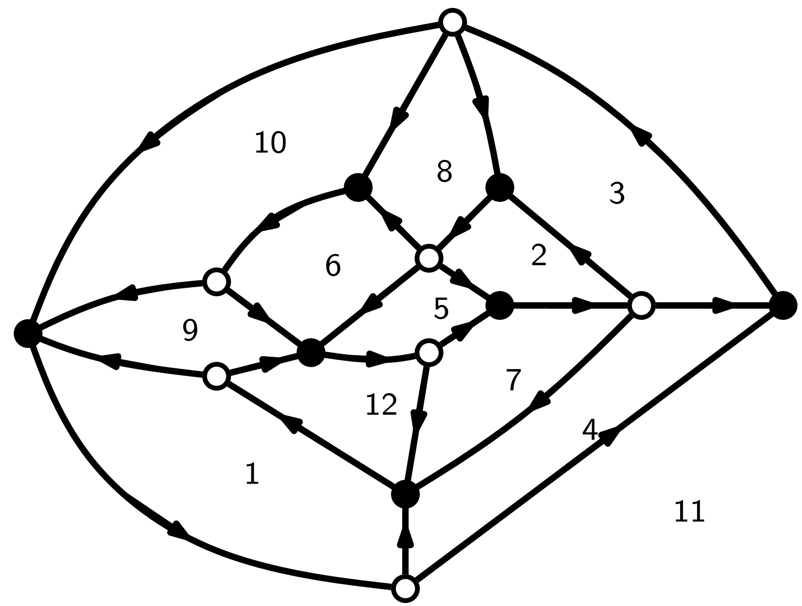

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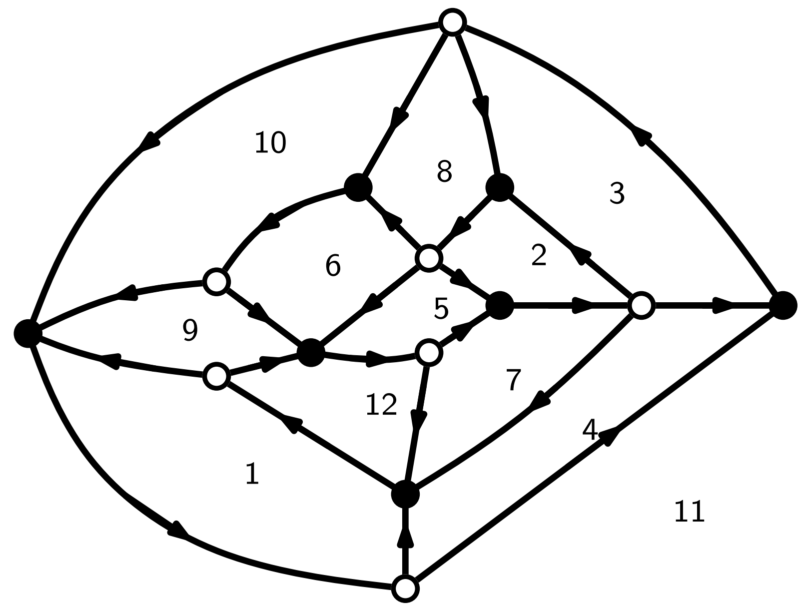

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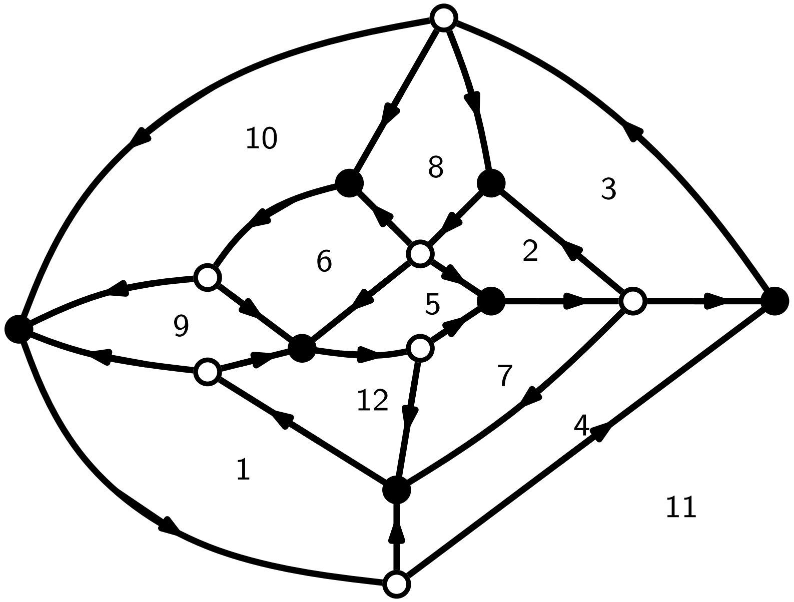
This is our choice of canonical α to decompose increasing quadrangulations.

opening of an increasing quadrangulation



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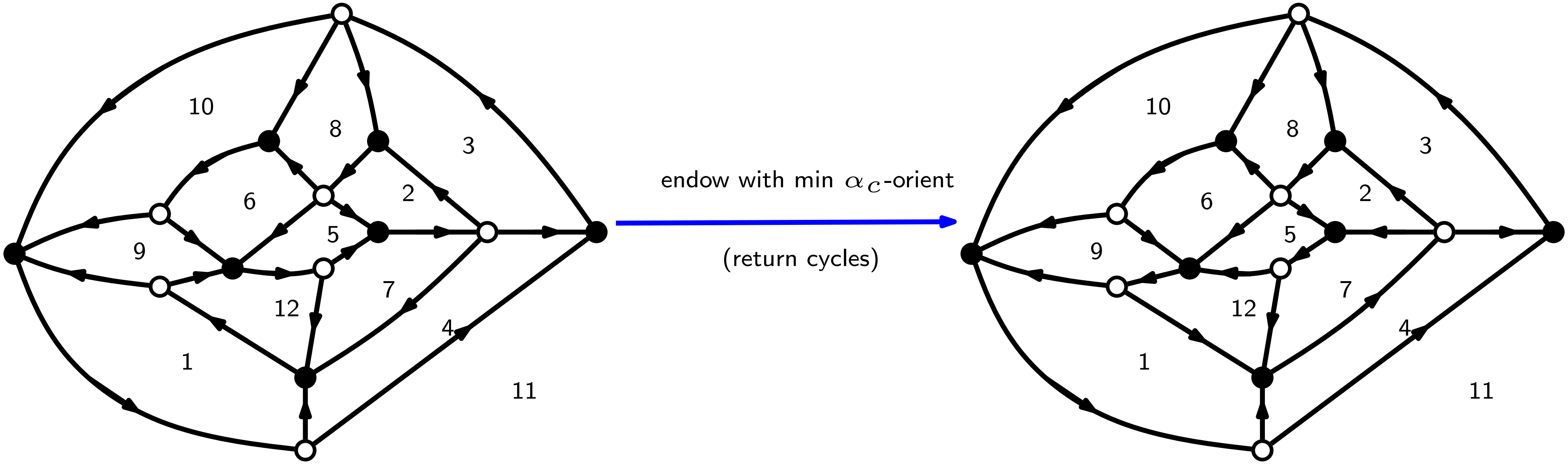

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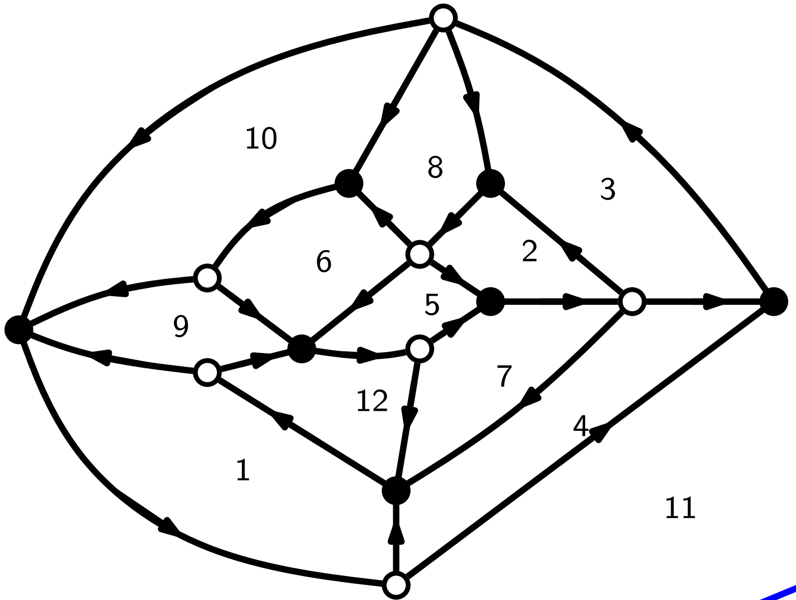
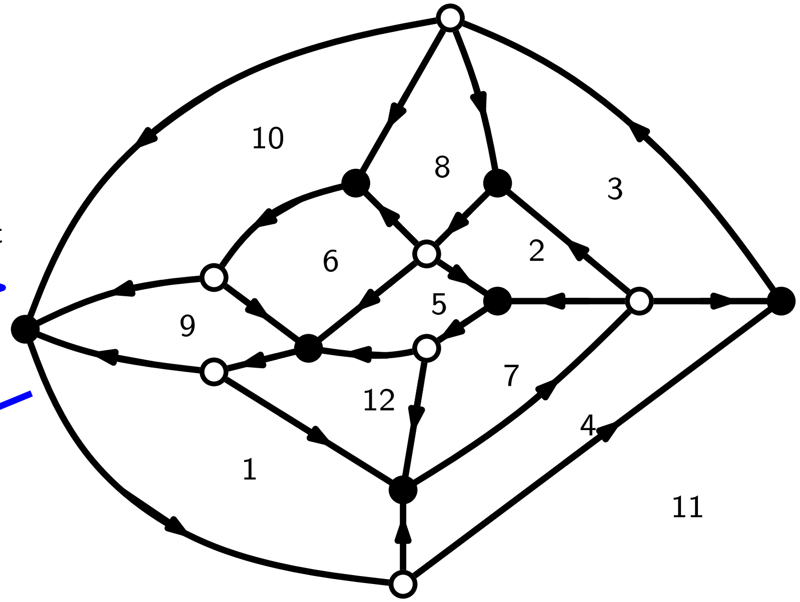


endow with min α_C-orient

(return cycles)

find spanning tree



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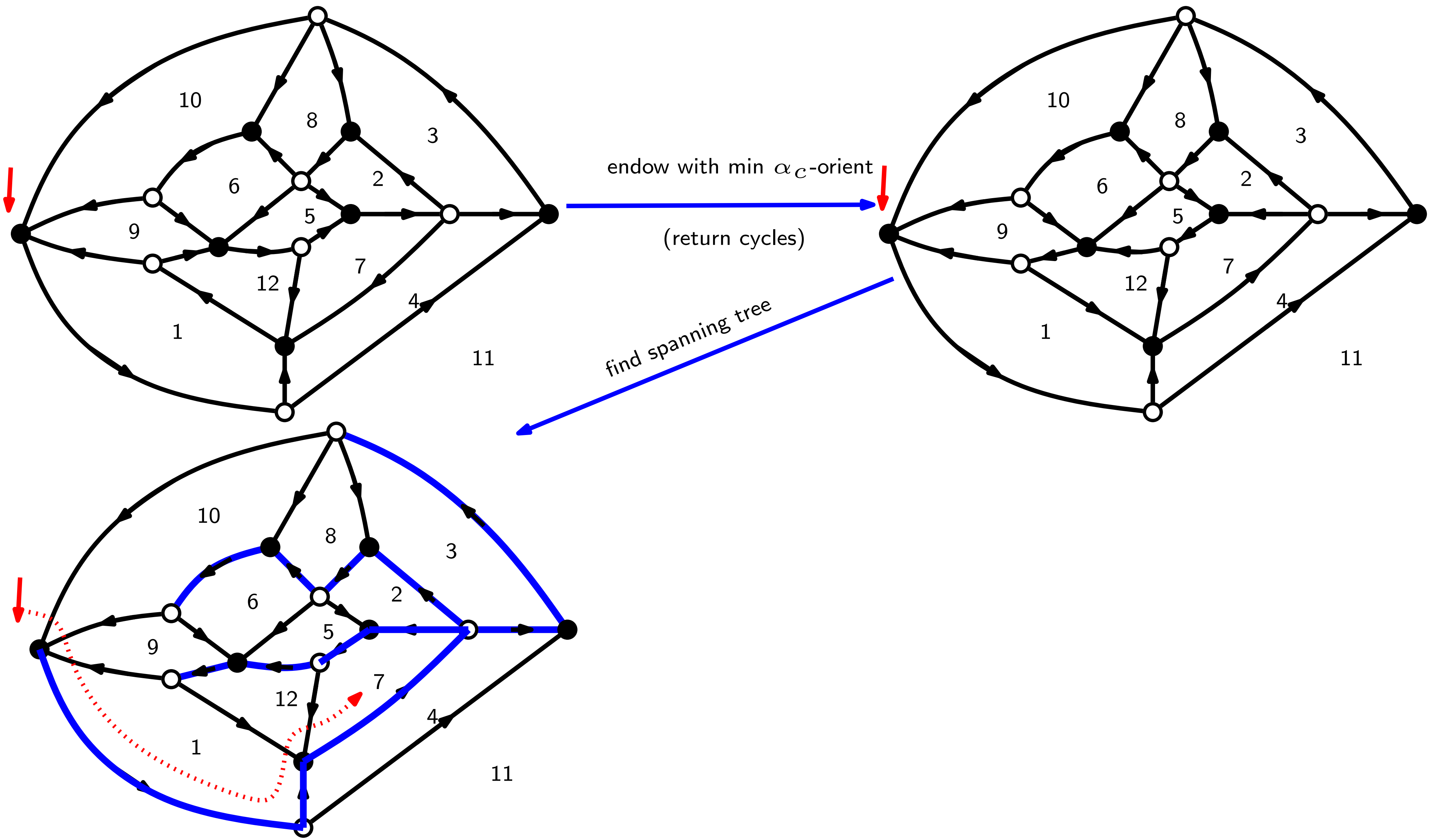

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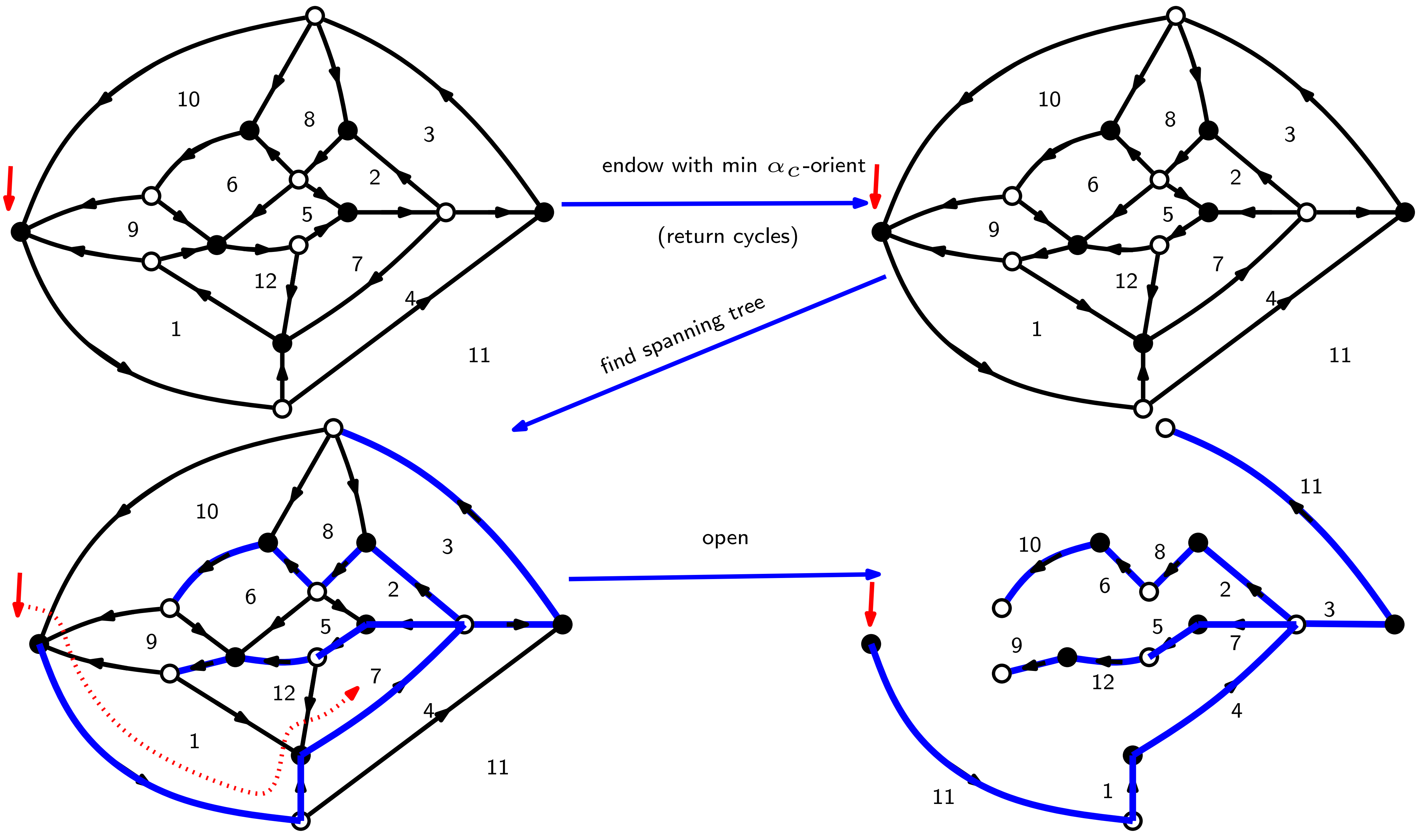

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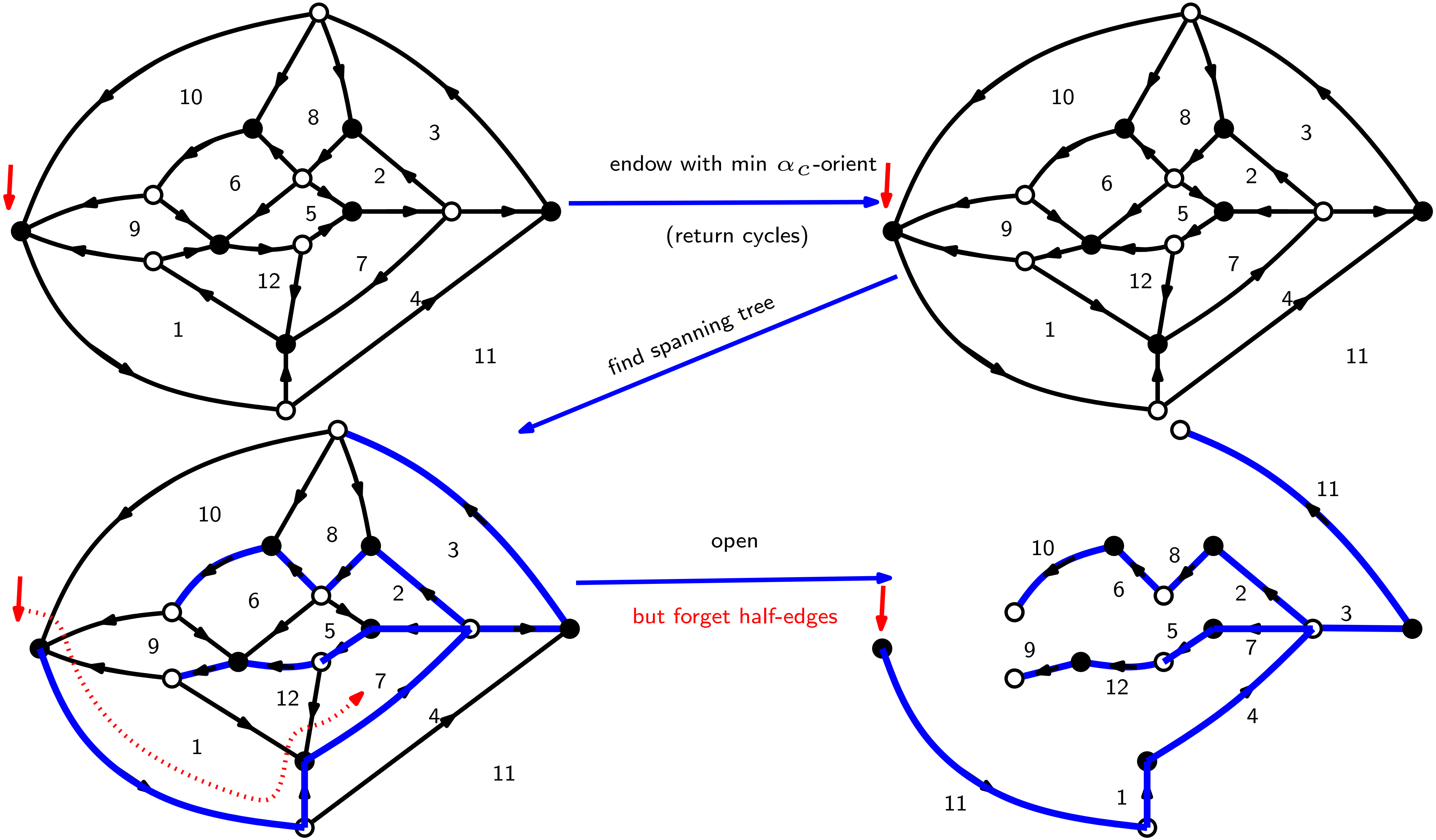

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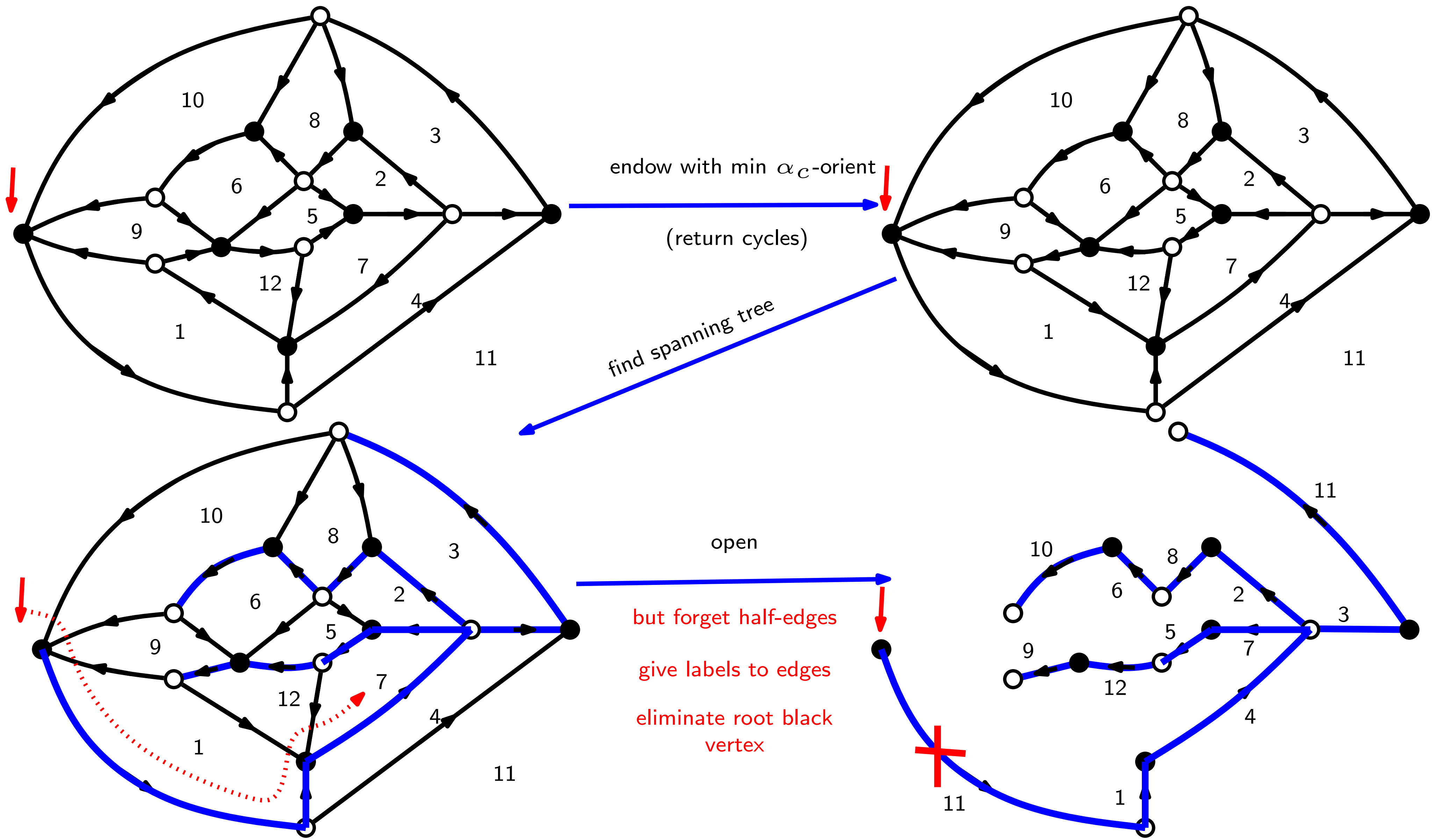

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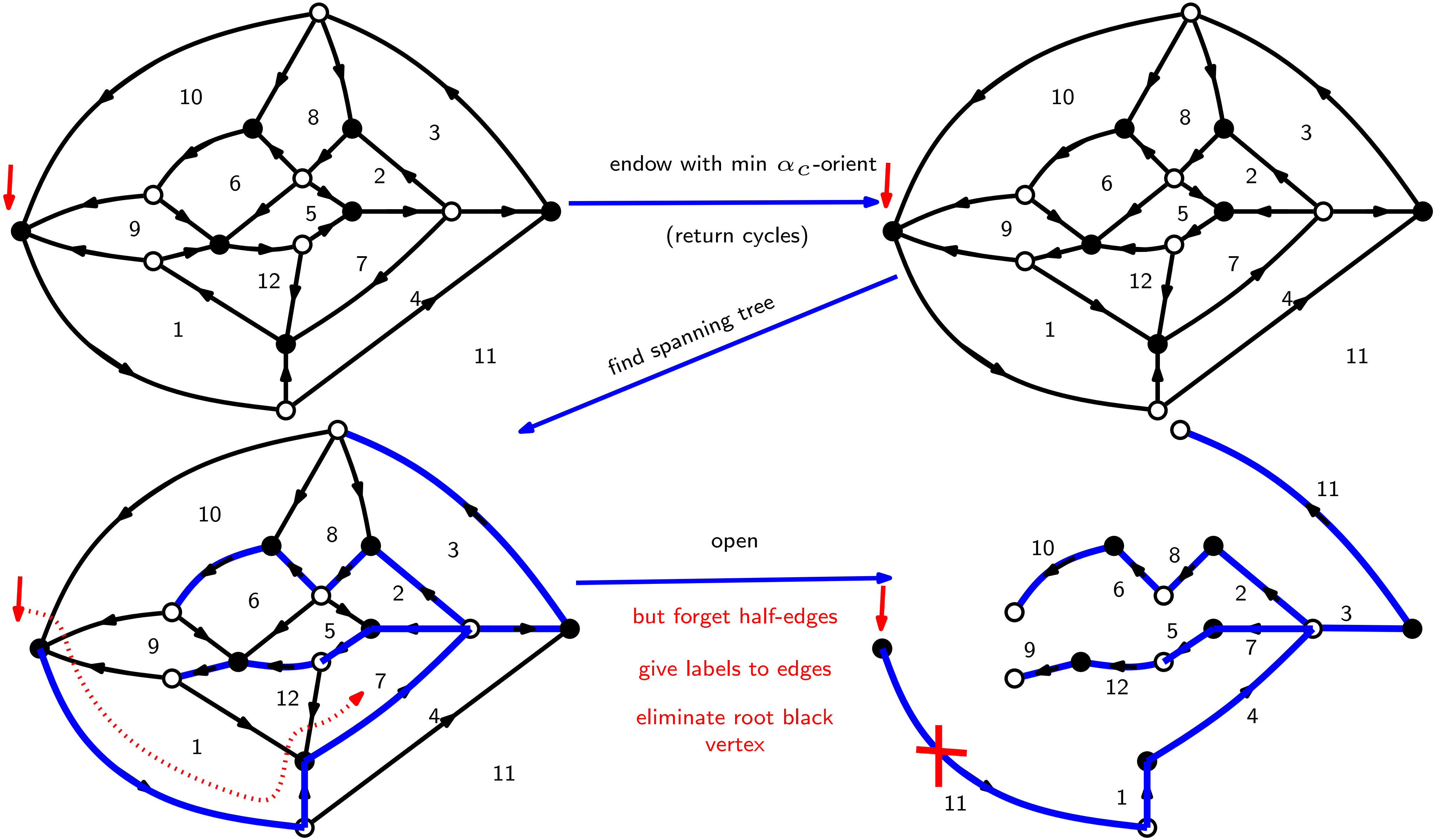
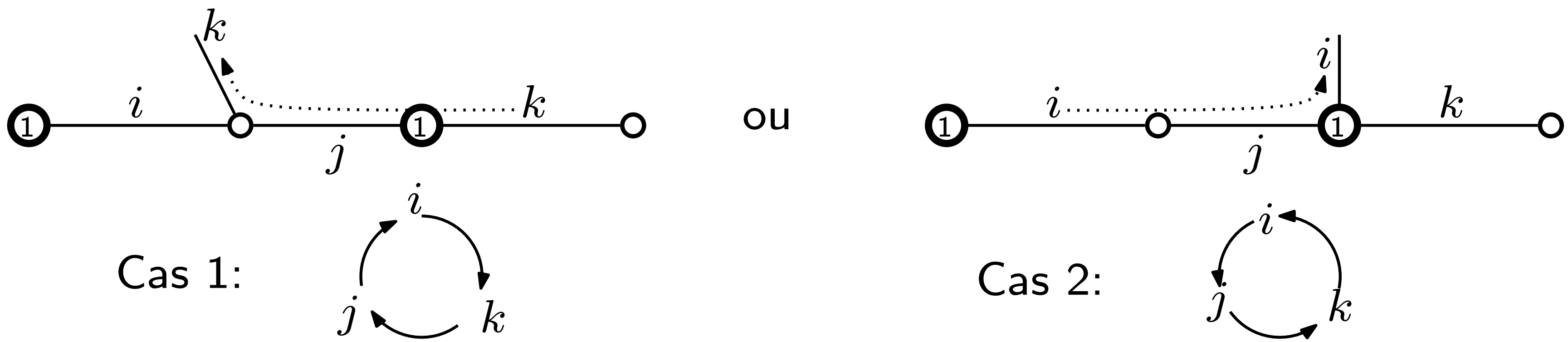


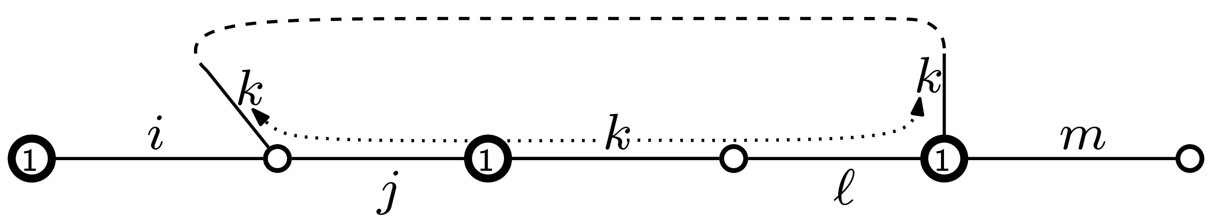
Proposition. The resulting simple Hurwitz trees has n unlabelled vertices, $n-1$ labeled vertices of degree 2, $2n-2$ edges that increase ccw around labeled vertices.

From simple Hurwitz trees to increasing quadrangulations

A local rule to create increasing half edges

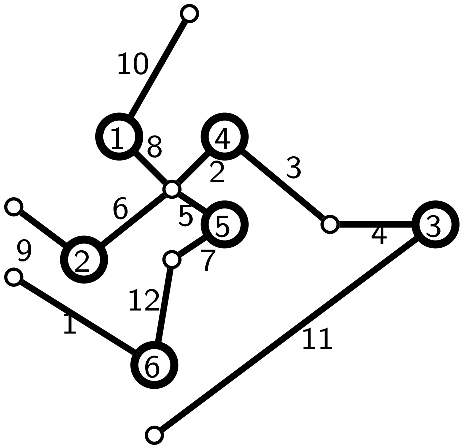


Two half-edges with same label $\Rightarrow$ edge and face of degree 4



Iterate the local rules as long as possible...

From simple Hurwitz trees to factorizations



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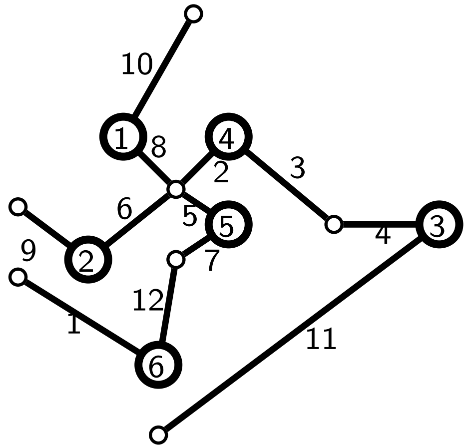
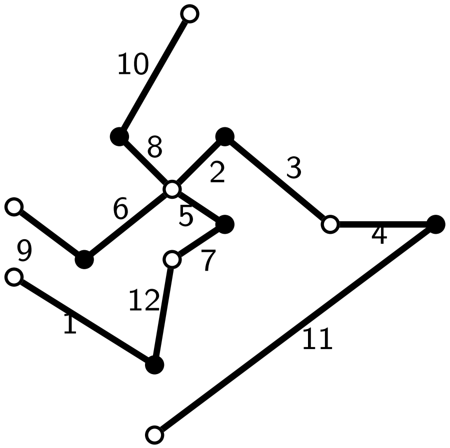


vertex label are useless
for the bijection

From simple Hurwitz trees to factorizations



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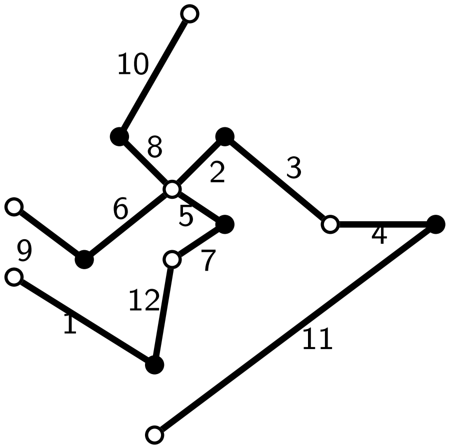

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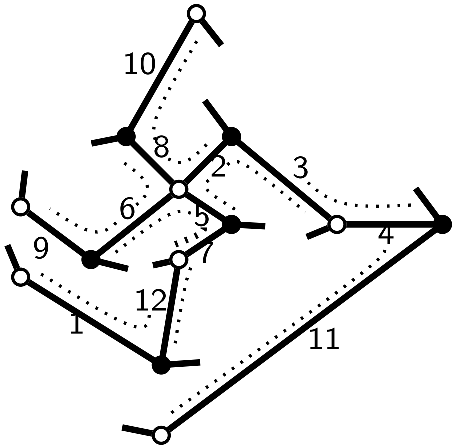

adding buds

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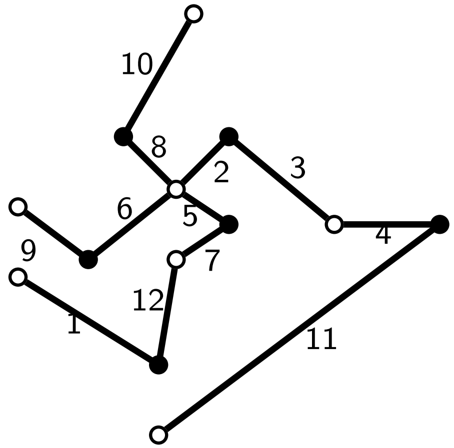

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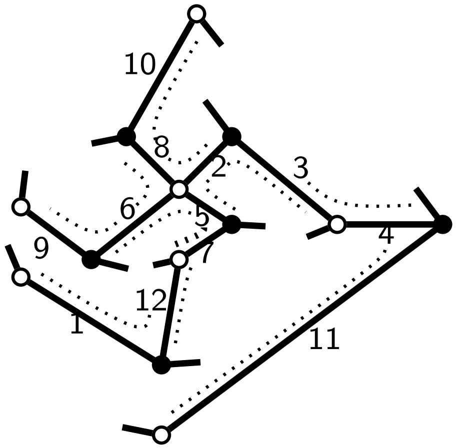

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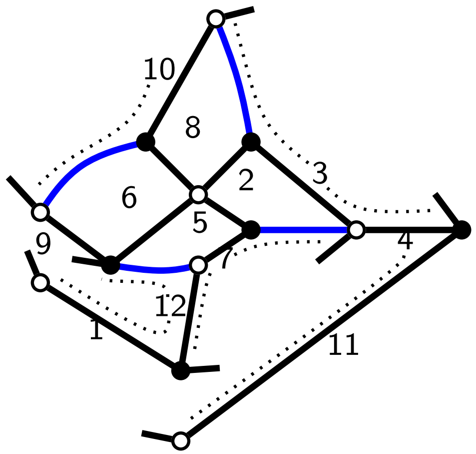
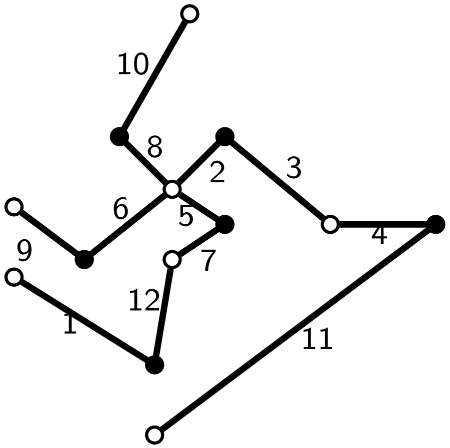


Parings and adding buds again

From simple Hurwitz trees to factorizations



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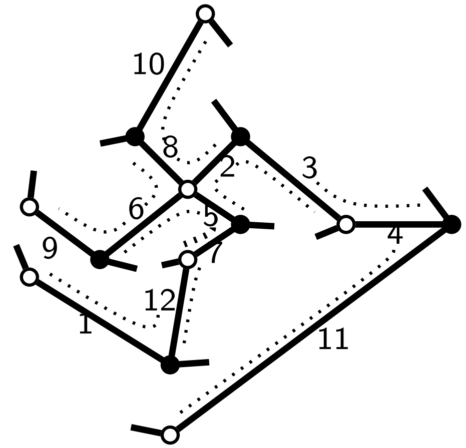

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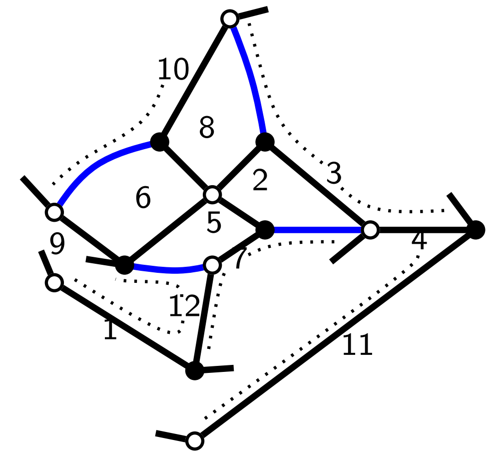

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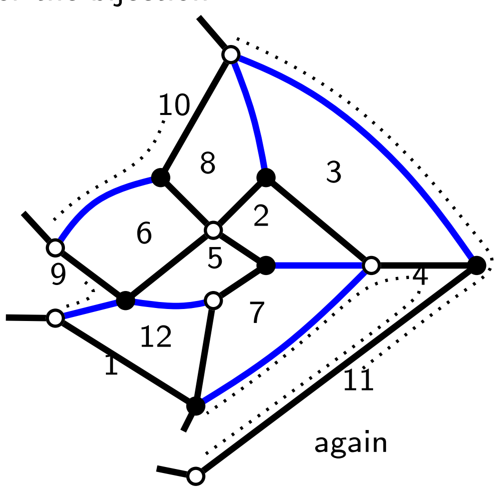

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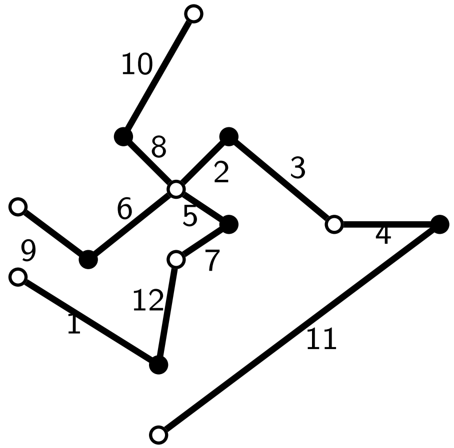

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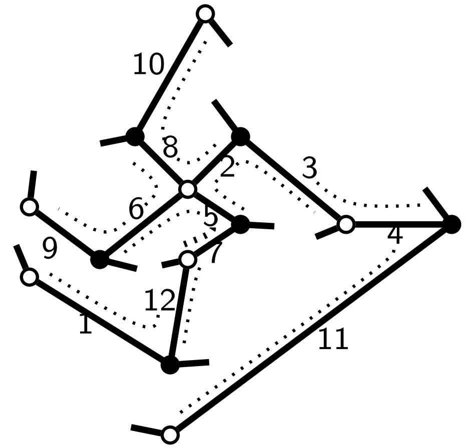

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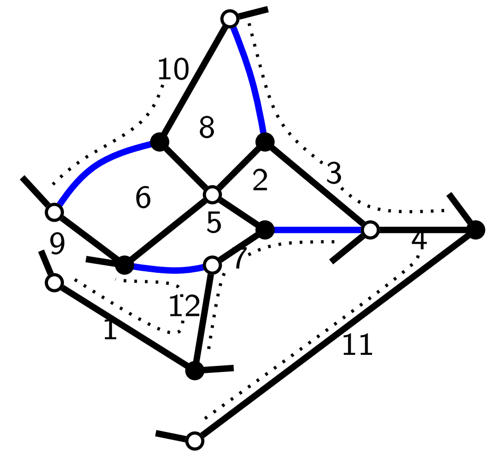

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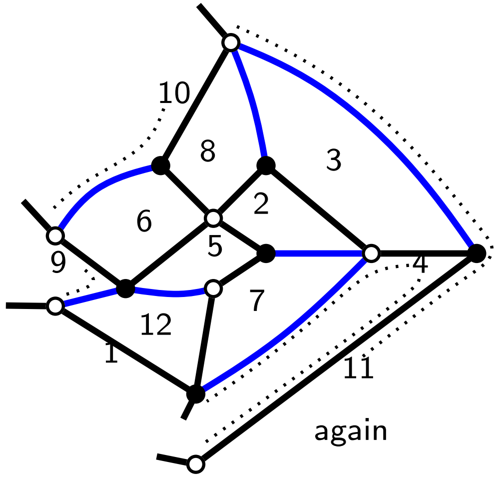

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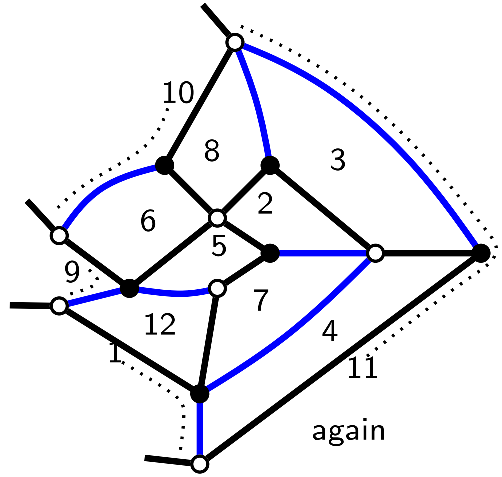

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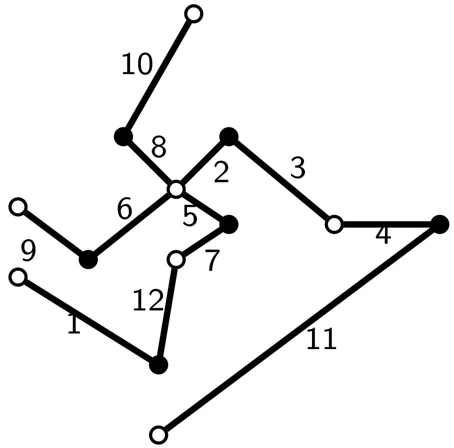

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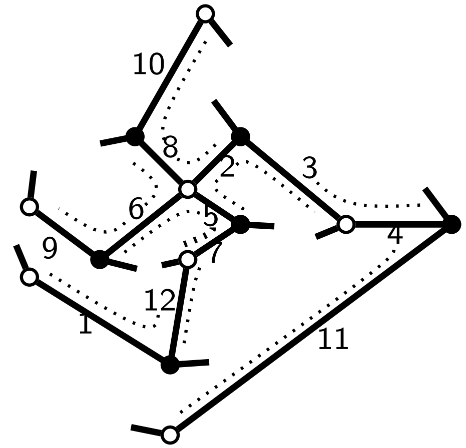

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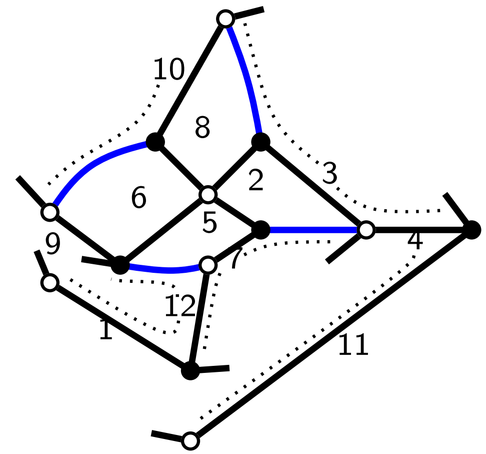

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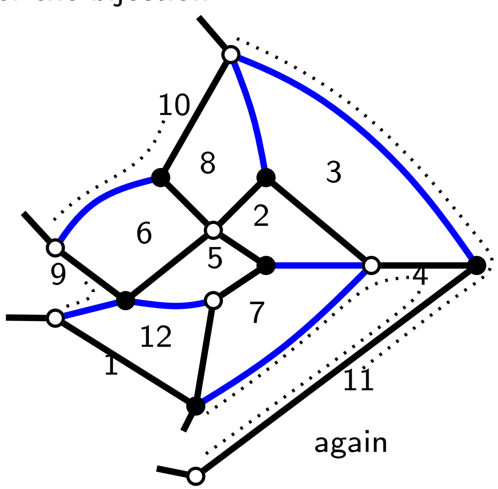

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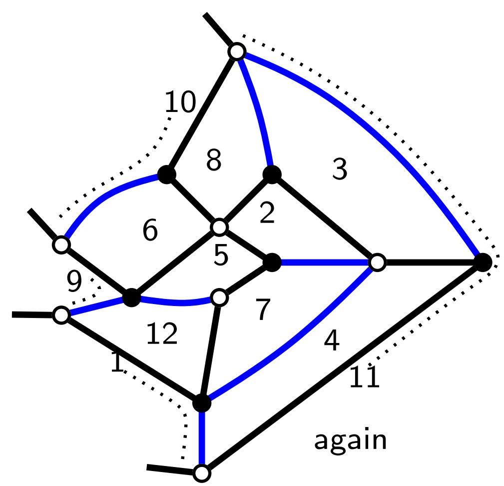
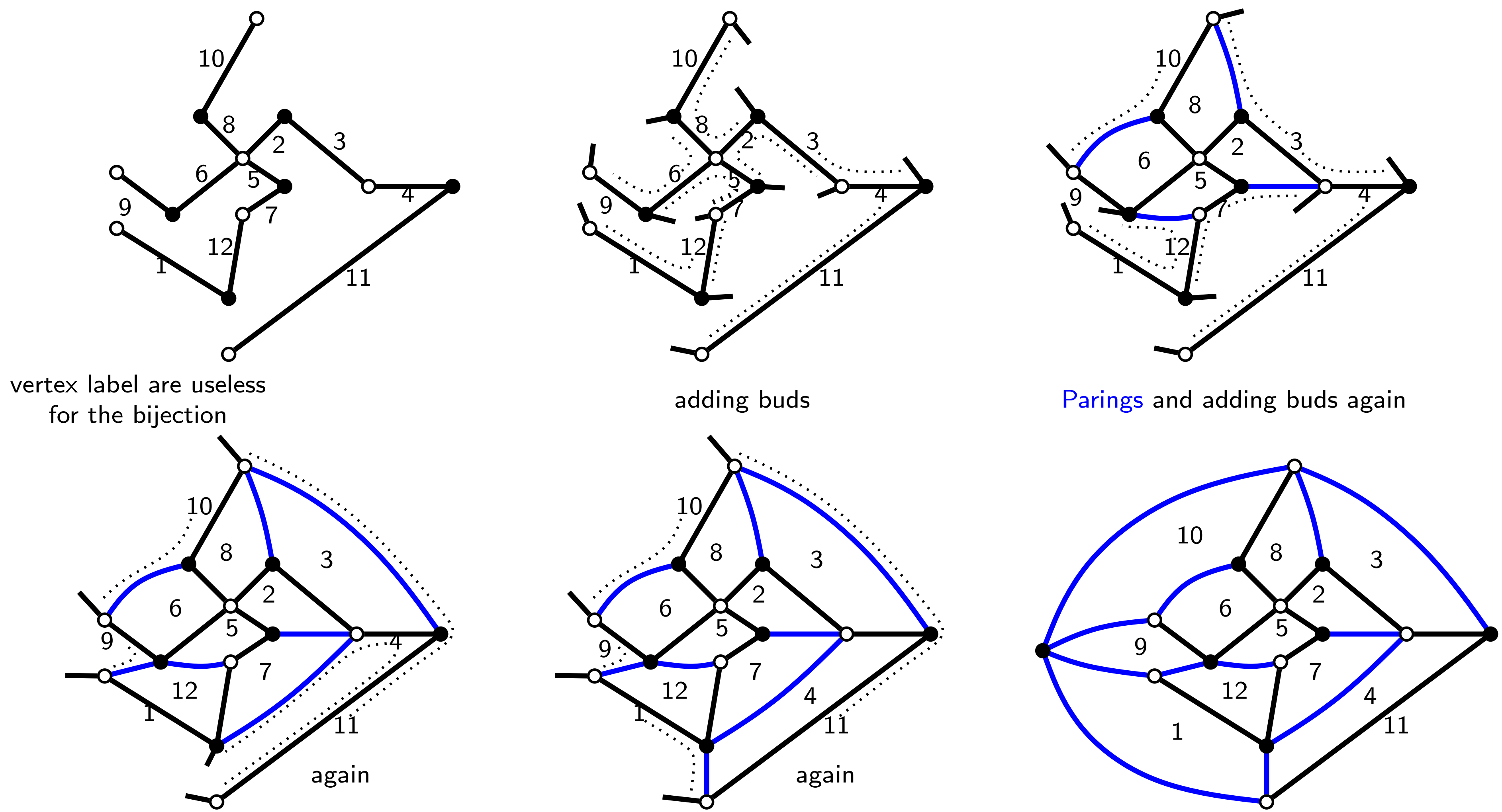


again

Lemma. When it stops, there are only white half-edges left.

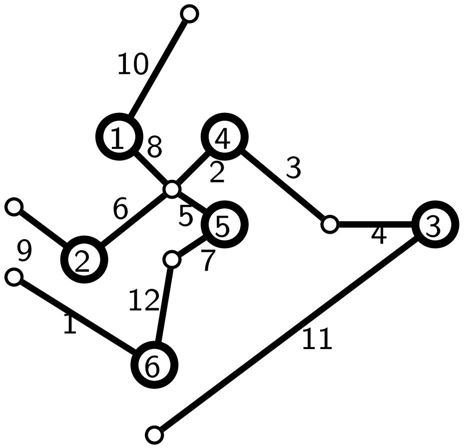
From simple Hurwitz trees to factorizations



Lemma. When it stops, there are only white half-edges left.

We connect them to a new black vertex and reload labels.

From simple Hurwitz trees to factorizations



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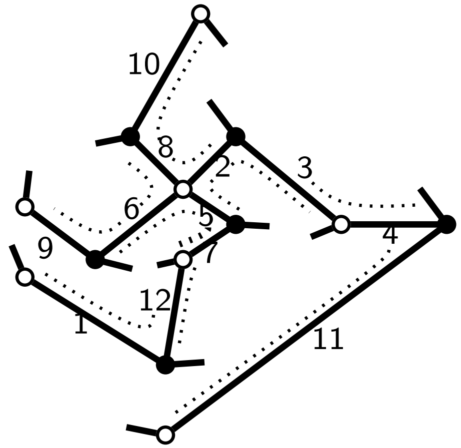

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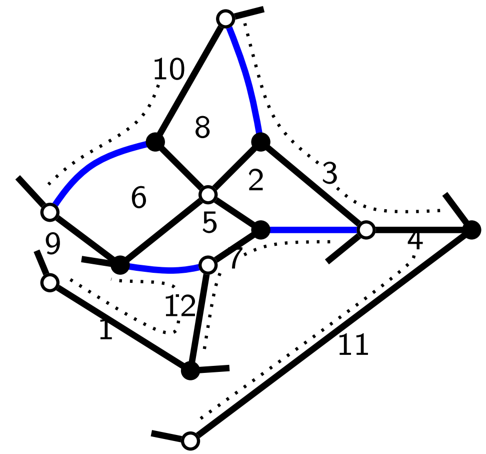

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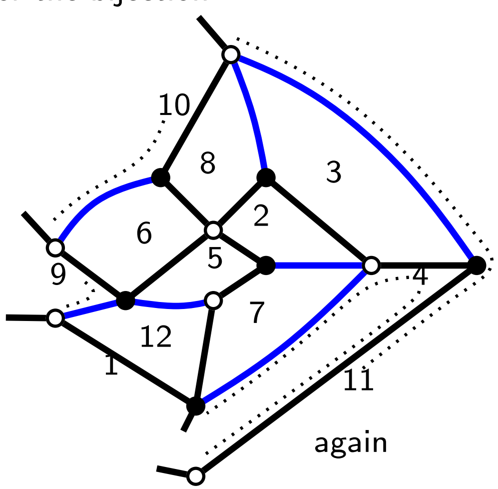

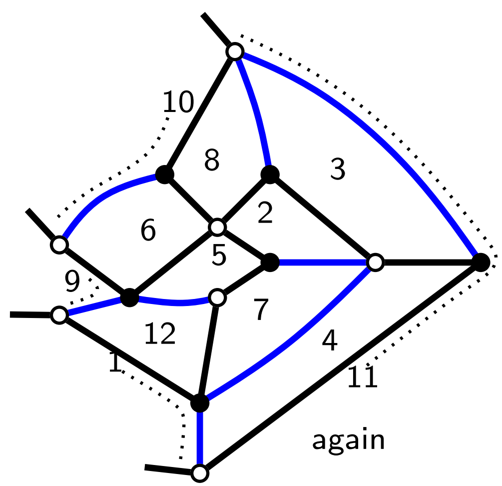

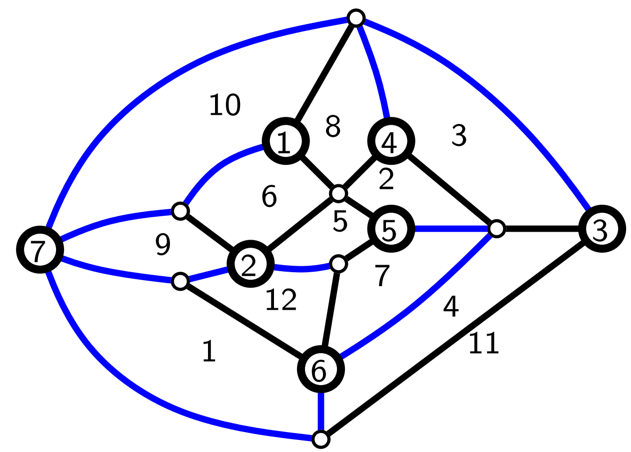

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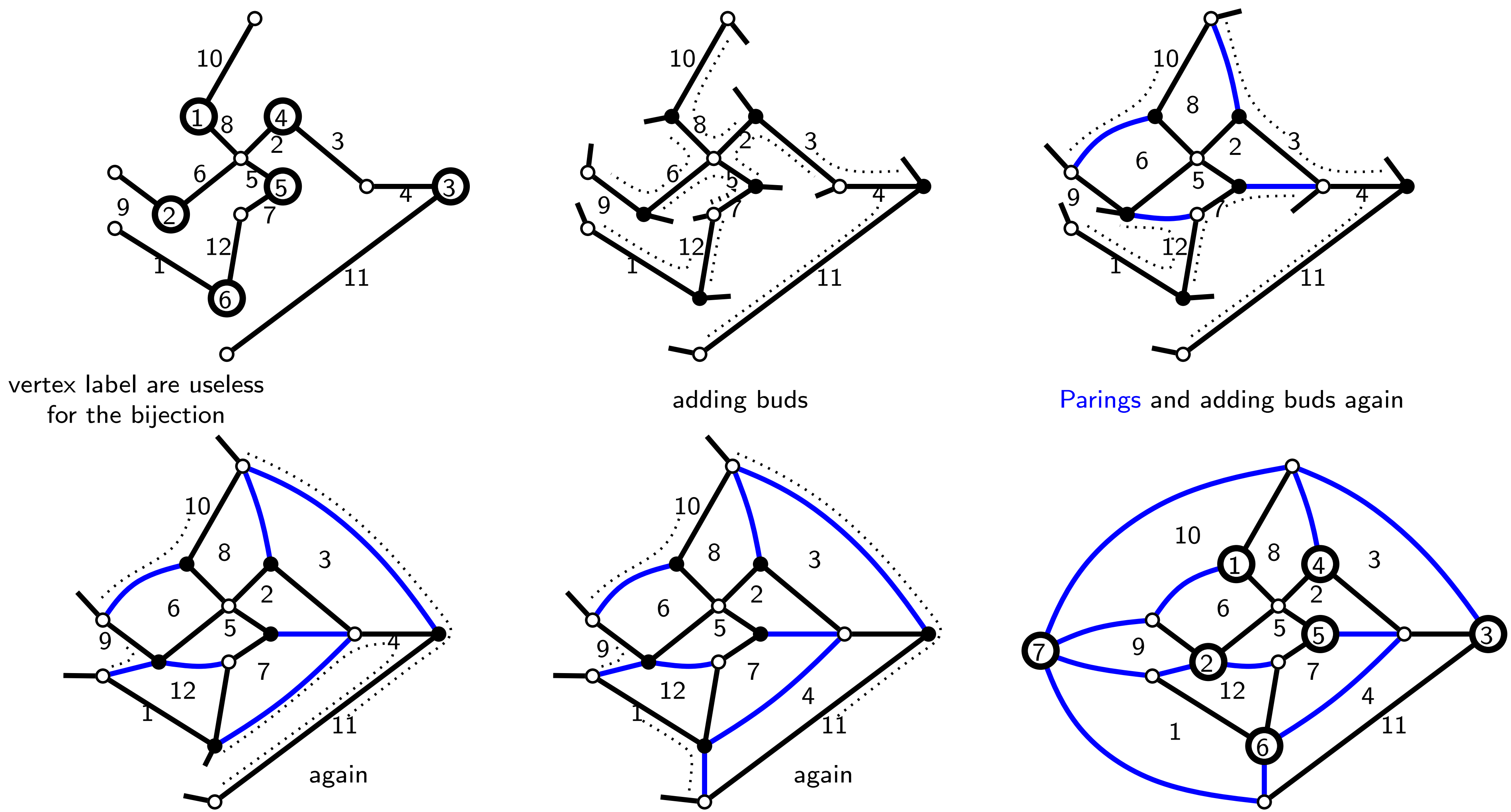

Theorem[Duchi-Poulalhon-S. 2012] Closure is the reverse bijection between

– simple Hurwitz trees of size n, and

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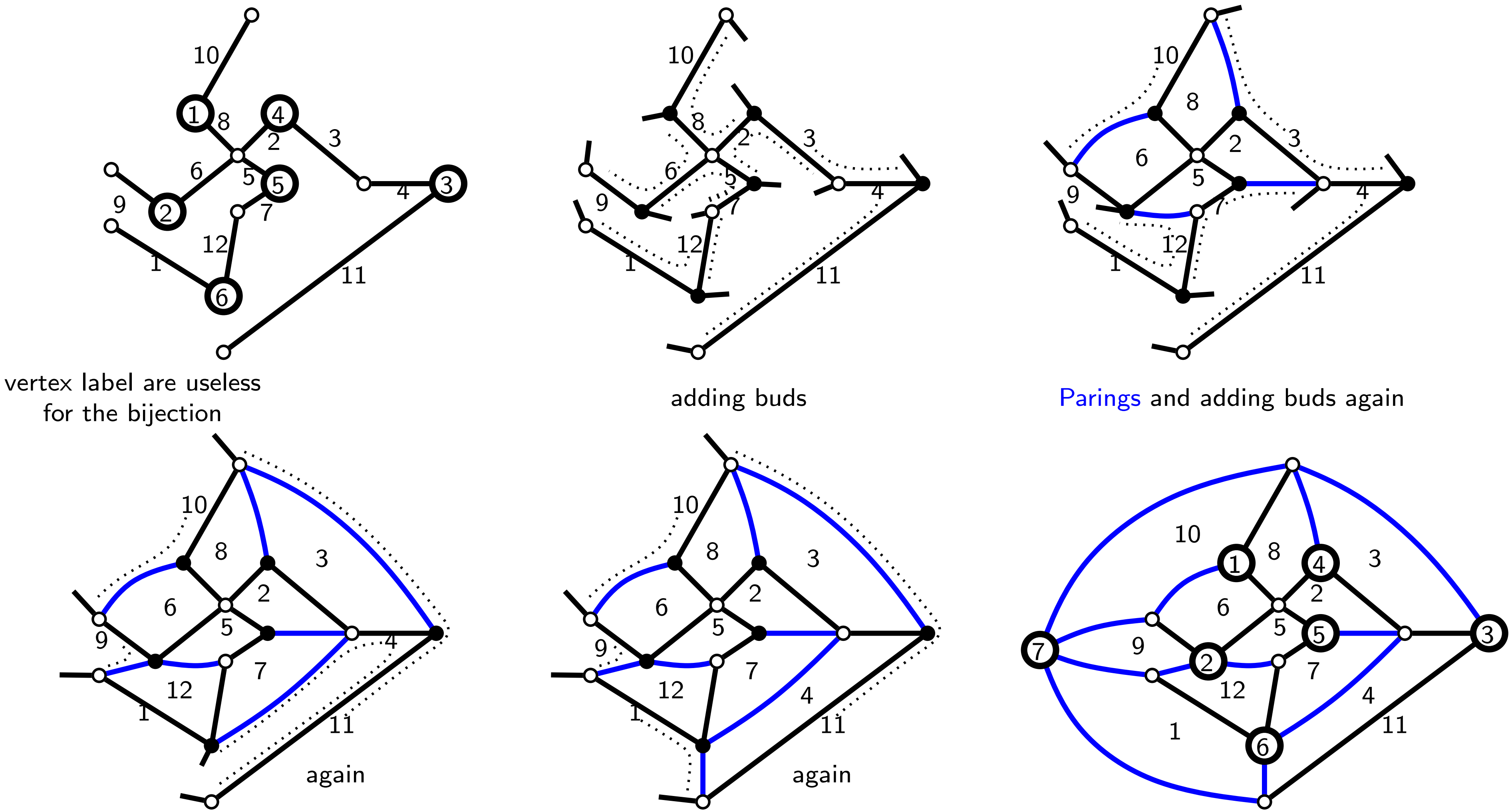
– simple ramified covers of $\mathbb{S}$ by itself with $m = 2n - 2$ critical values.

From simple Hurwitz trees to factorizations



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Résumé des 2 premiers épisodes

Compter des classes d'équivalence de revêtements ramifiés

$\Updownarrow$

compter certaines plongements de graphes

$\Updownarrow$

compter certains arbres

Plan de l'exposé

Plan de l'exposé

Revêtements ramifiés et cartes

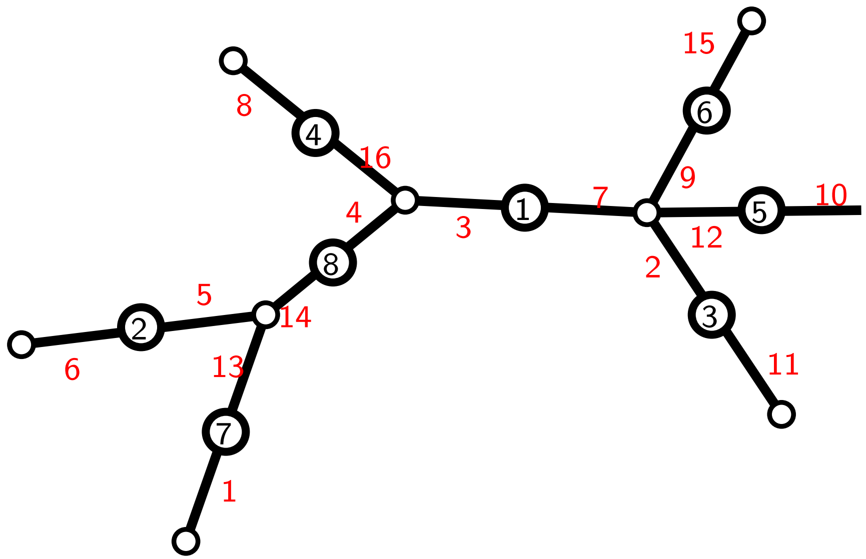
Cartes et arbres

Énumération d'arbres et formule d'Hurwitz

Revêtements et cartes aléatoires

Hurwitz formula for increasing quadrangulations

Theorem[Duchi-Poulalhon-S. 2012] Increasing quadrangulations (size n) are in bijection with simple Hurwitz trees having n unlabelled vertices, $n-1$ labeled vertices of degree 2, $2n-2$ edges that increase ccw around labeled vertices.



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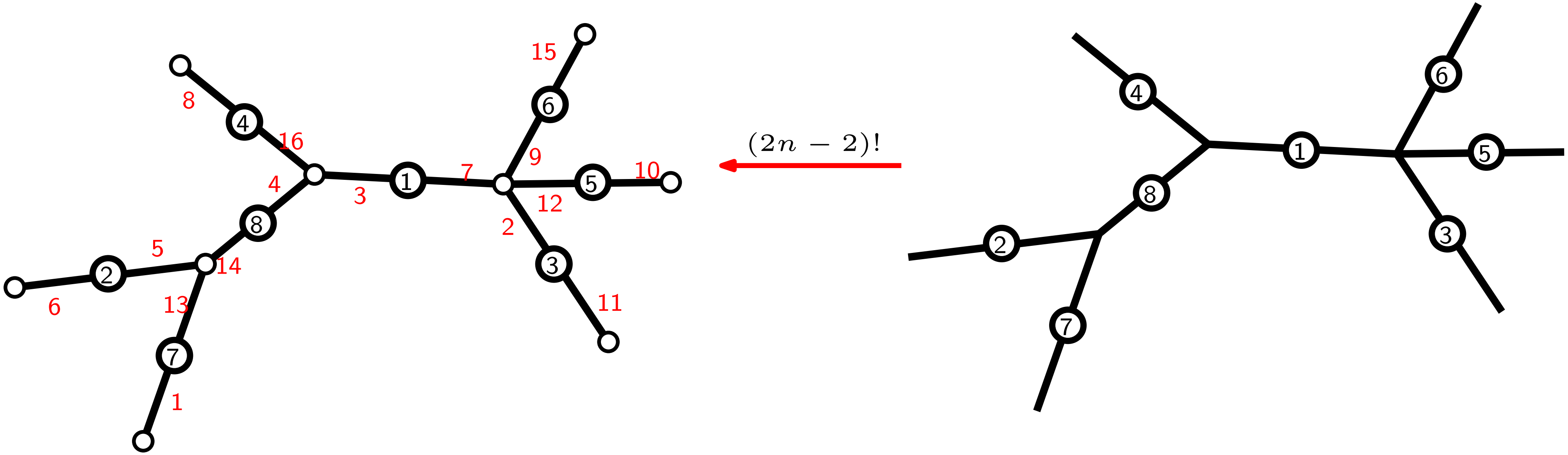

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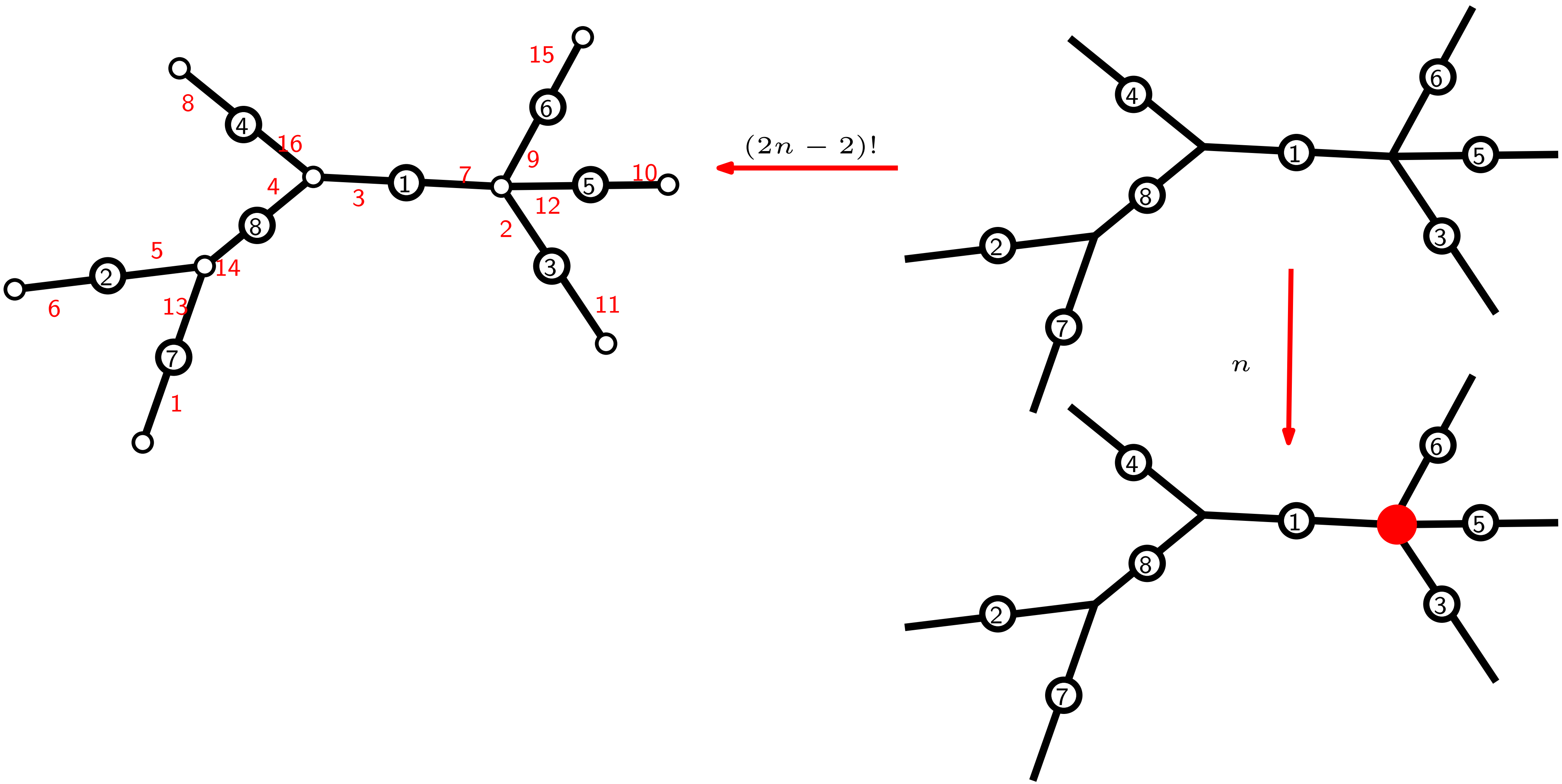

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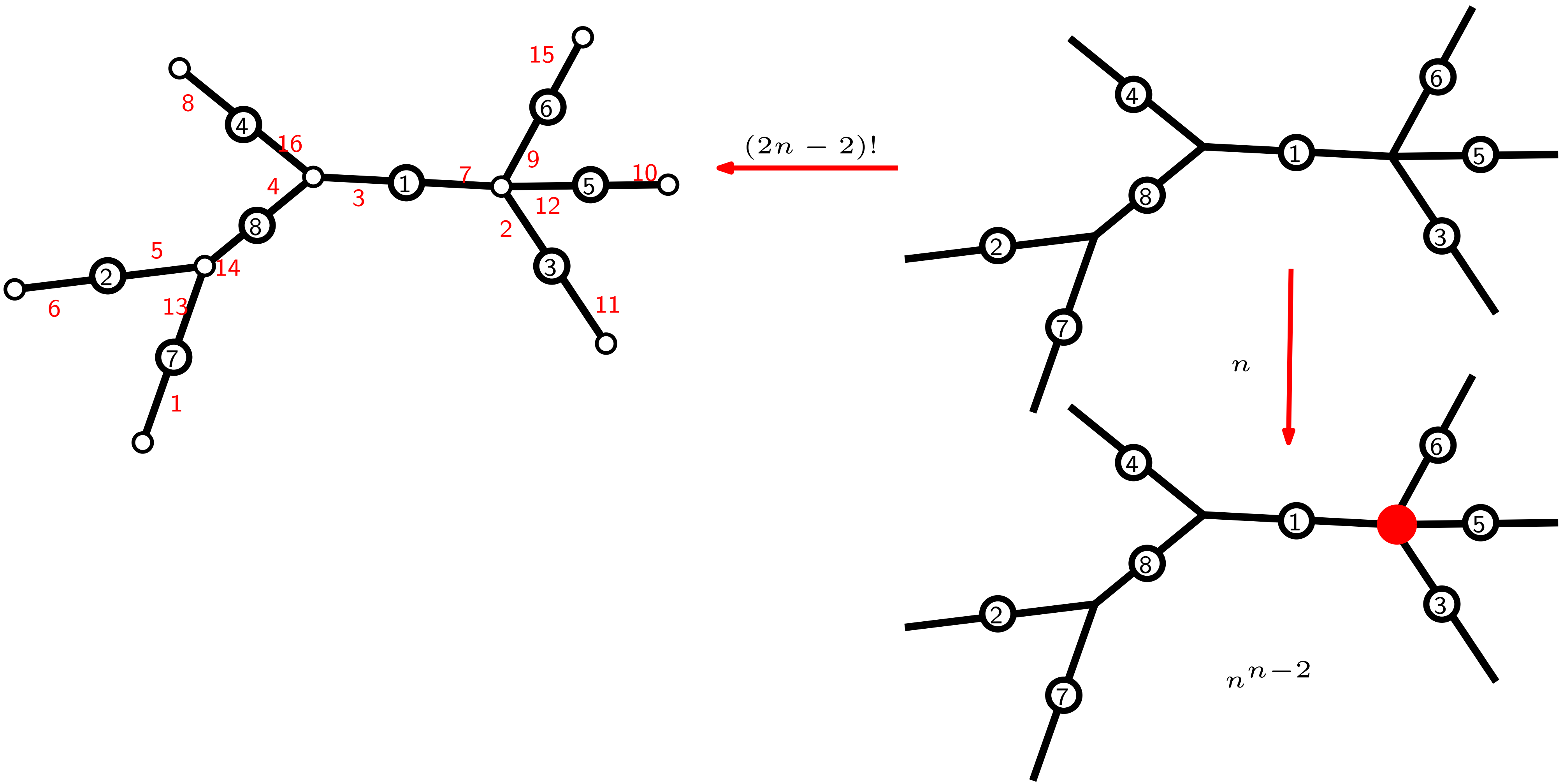

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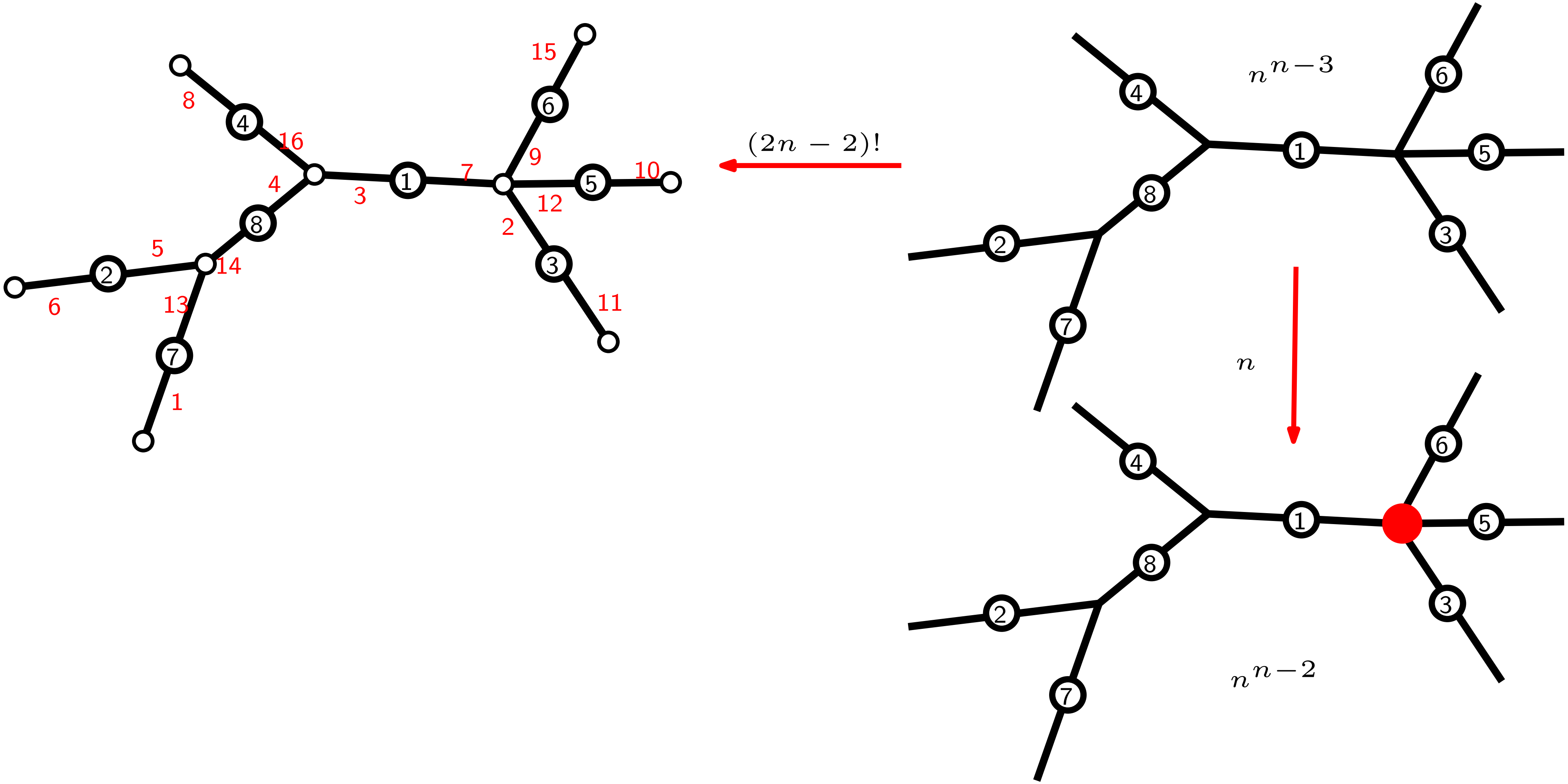

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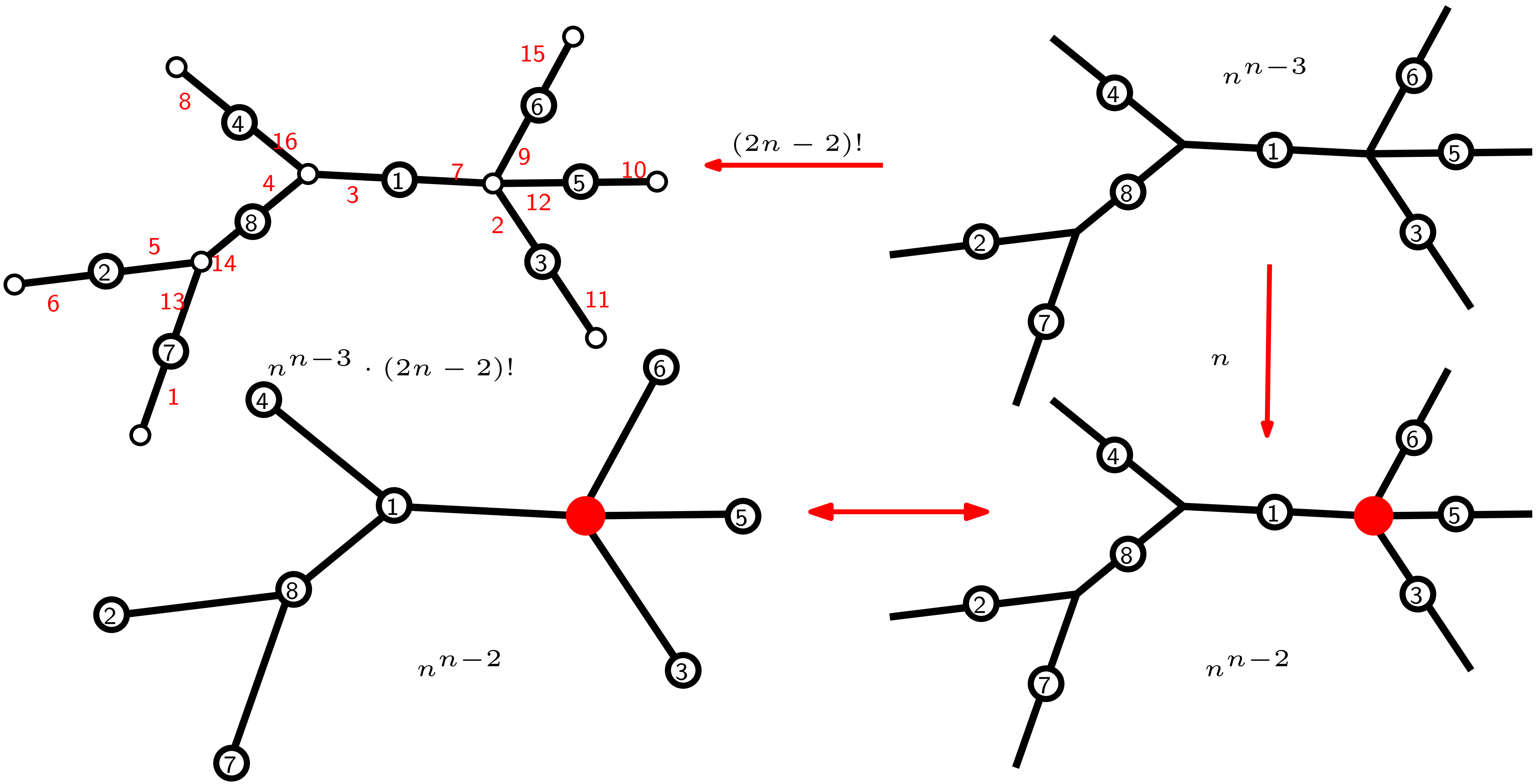

The number of simple ramified cover of $\mathbb{S}$ by itself with $m = 2n-2$ critical points is $n^{n-3}(2n-2)!$.

Hurwitz formula for factorizations in transpositions

Theorem. Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition n, and $\ell = \sum_i \ell_i$.
The number of m-uples of transpositions $(\tau_1, \ldots, \tau_m)$ such that

- (product cycle type) $\tau_1 \cdots \tau_m = \sigma$ has cycle type λ
- (transitivity) the associated graph is connected
- (minimality) the number of factors is $m = n + \ell - 2$

is

$$n^{\ell-3} \cdot m! \cdot n! \cdot \prod_{i \geq 1} \frac{1}{\ell_i!} \left(\frac{i^i}{i!}\right)^{\ell_i}$$

Proofs:

(Hurwitz 1891, Strehl 1996) (Goulden–Jackson 1997) (Lando–Zvonkine 1999) (Bousquet-Mélou–Schaeffer 2000)
(recurrences, Abel identities) (gfs and differential eqns) (geometry of LL mapping) (bijection + inclusion/exclusion)

$\lambda = n$, factorizations of n-cycles: $n^{n-2} \cdot (n-1)!$

$\lambda = 1^n$, factorizations of the identity: $n^{n-3} \cdot (2n-2)!$

A formula for general factorizations [BMS00]

Theorem. Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition of n, and $\ell = \sum_i \ell_i$.
The number of m-uple of permutations $(\sigma_1, \ldots, \sigma_m)$ such that

- (factorization) $\sigma_1 \cdots \sigma_m = \sigma$ with cycle type λ
- (transitivity) $\langle \sigma_1, \ldots, \sigma_m \rangle$ acts transitively on $\{1, \ldots, n\}$
- (minimality) the total rank of factors is $\sum_i r(\sigma_i) = n + \ell - 2$

is

$$m \frac{((m-1)n-1)!}{(mn-(n+\ell-2))!} \cdot n! \cdot \prod_i \frac{1}{\ell_i!} \binom{mi-1}{i}^{\ell_i}$$

Proofs:

(Bousquet-Mélou–Schaeffer 2000) (Goulden–Serrano 2009)

(bijection + inclusion/exclusion)(gfs and differential eqns)

$\lambda = n$, factorizations of n-cycles: $\frac{1}{(mn+1)} \binom{mn+1}{n} \cdot (n-1)!$

$\lambda = 1^n$, identity factorizations: $\frac{m}{(m-2)n+2} \frac{(m-1)^{n-1}}{(m-2)n+1} \binom{(m-1)n}{n} \cdot (n-1)!$

Résumé des 3 premiers épisodes

Compter des classes d'équivalence de revêtements ramifiés

$\Updownarrow$

compter certaines plongements de graphes

$\Updownarrow$

compter certains arbres

les formules simples appellent des preuves constructives

Plan de l'exposé

Plan de l'exposé

Revêtements ramifiés et cartes

Cartes et arbres

Énumération d'arbres et formule d'Hurwitz

Revêtements et cartes aléatoires

Quadrangulations croissantes aléatoires uniformes

$\bar{\mathcal{Q}}_n = \{$quadrangulations croissantes à n faces$\}$.

Quadrangulation croissante uniforme = variable aléatoire Q_n à valeur dans $\bar{\mathcal{Q}}_n$ avec

$$\Pr(Q_n = q) = \frac{1}{|\bar{\mathcal{Q}}_n|} = \frac{1}{n^{n-3}(2n-2)!} \qquad \text{pour tout } q \in \bar{\mathcal{Q}}_n$$

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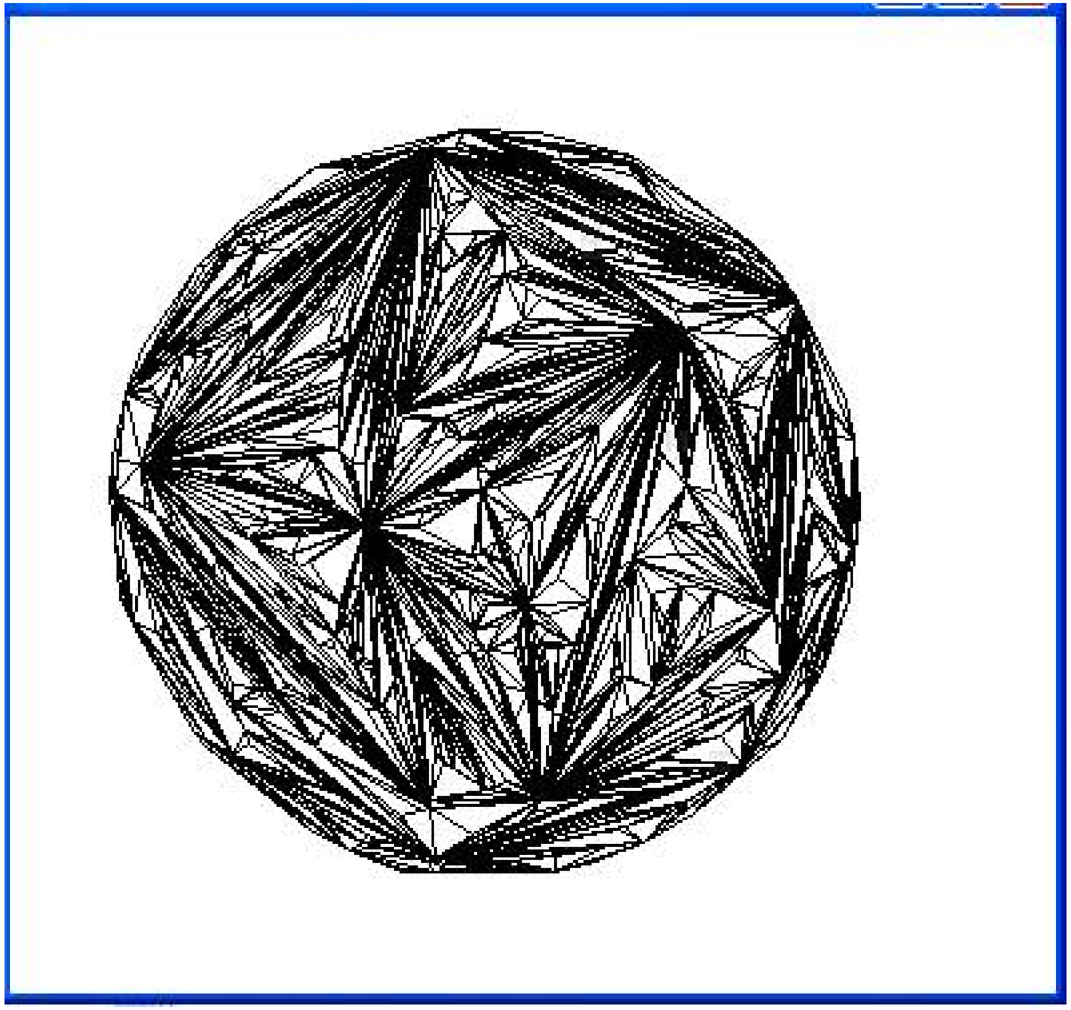
- le choix de la distribution uniforme combinatoire est le plus immédiat

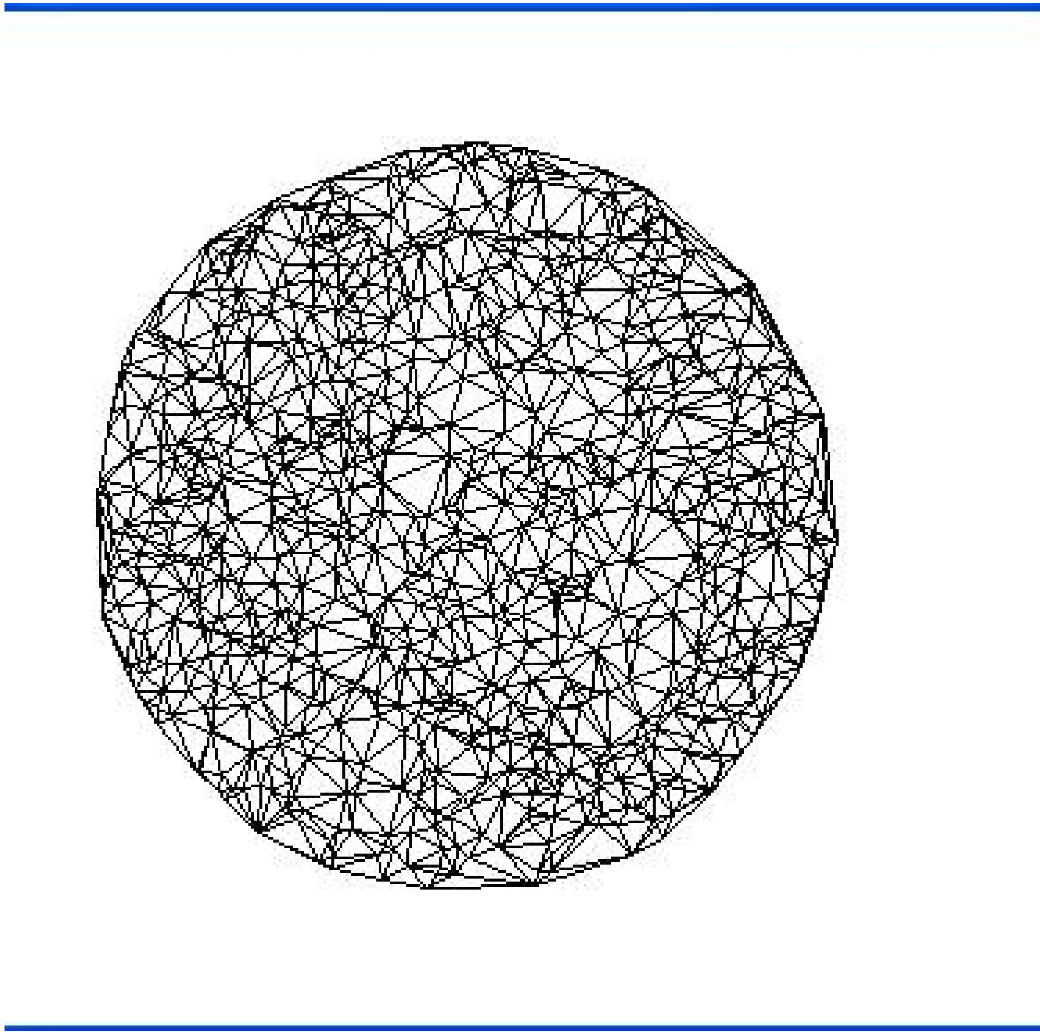
Parallèle naturel avec la distribution uniforme sur les quadrangulations enracinées:

$$\Pr(\vec{Q}_n = q) = \frac{1}{|\vec{\mathcal{Q}}_n|} = \frac{1}{\frac{2\cdot 3^n (2n)!}{(n+2)!n!}} \qquad \text{pour tout } q \in \vec{\mathcal{Q}}_n$$

Comment étudier Q_n ?

Propriétés des cartes aléatoires uniformes ?





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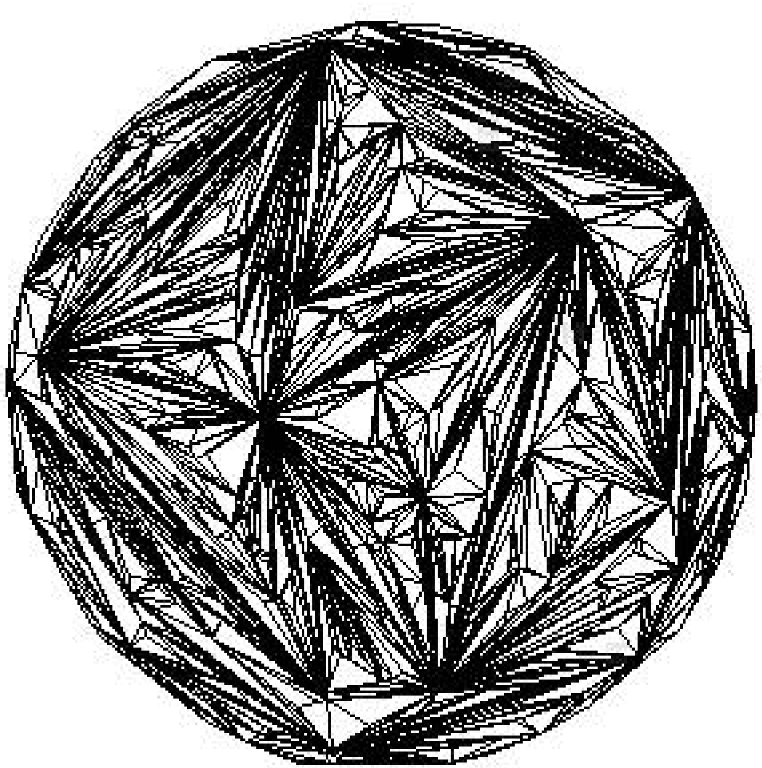
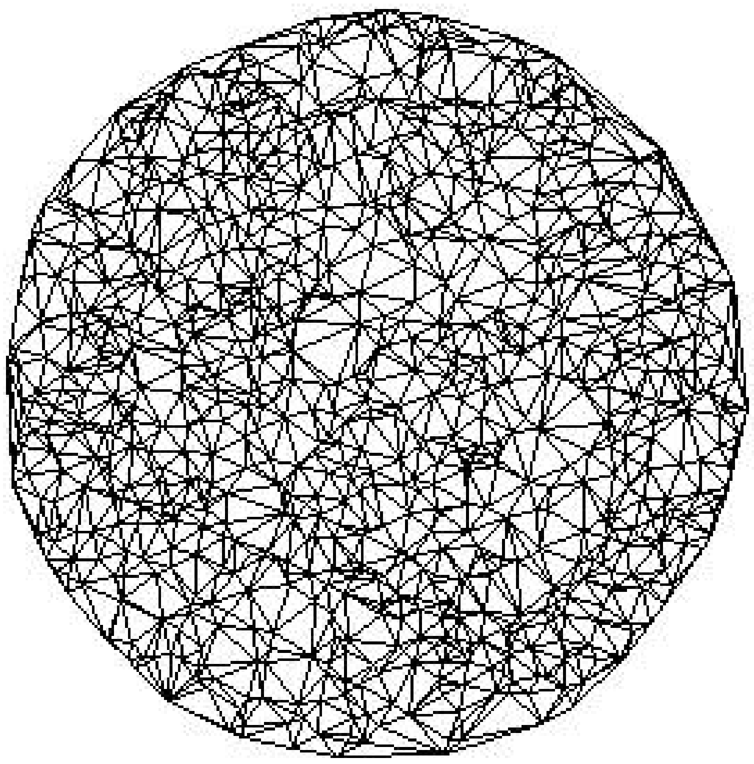


Triangulation uniforme aléatoire d'un disque



Delaunay de points aléatoires dans un disque

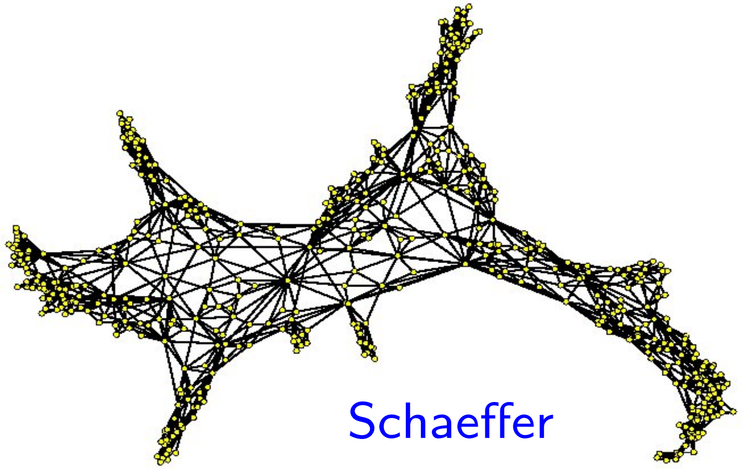
on est loin d'une discrétisation aléatoire d'une géométrie euclidienne

en physique on lie cela à la modélisation discrète de la gravité quantique

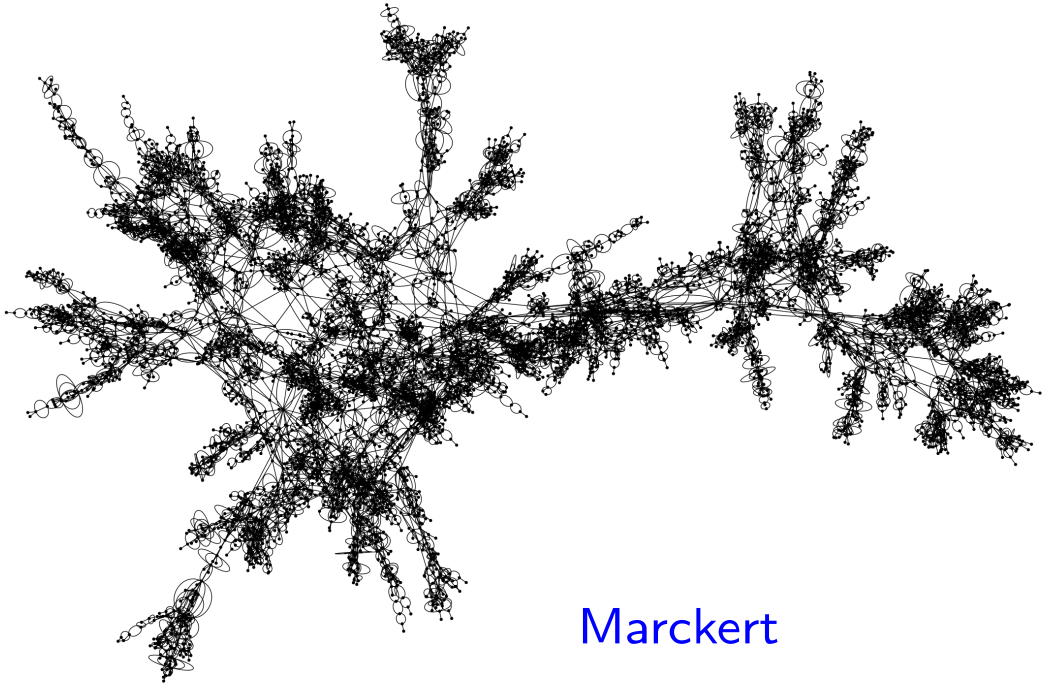
Quadrangulations uniformes comme surfaces aléatoires

L'allure d'une sphère aléatoire dépend un peu de qui dessine...

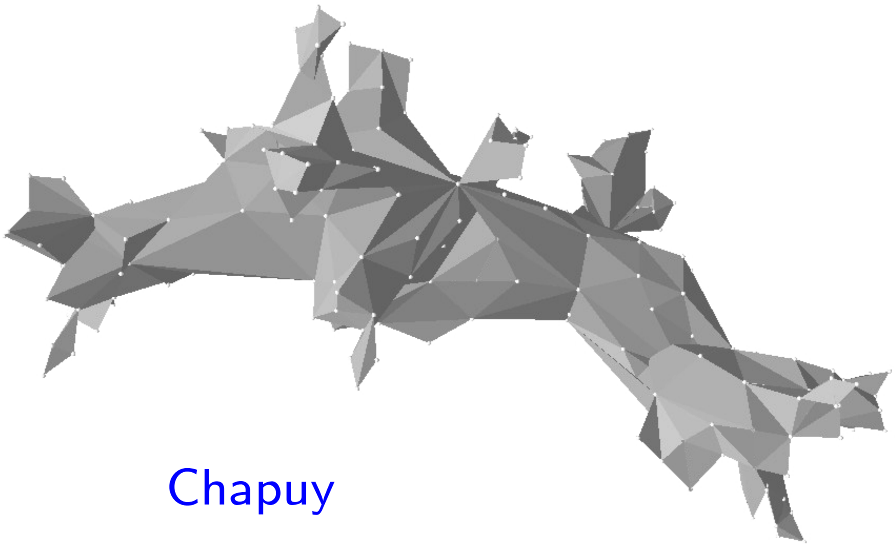
Objectif: Choisir une métrique intrinsèque et décrire les surfaces ainsi obtenues



Schaeffer



Marckert



Chapuy

Étudier les quadrangulations aléatoires uniformes

Distribution uniforme sur les quadrangulations à n faces, pour n grand

1ère approche: Étudier le comportement asymptotique de paramètres:

- degré d'un sommet aléatoire
- loi 0-1 pour les propriétés locales
- distance entre 2 sommets aléatoires
- longueur d'un plus petit cycle diviseur

$\Rightarrow$ espérance, moments, lois limites discrètes ou continues, qd $n \longrightarrow \infty$

Bender, Canfield *et al* (90's $\longrightarrow$) en combinatoire

Ambjørn, Watabiki *et al* (90's $\longrightarrow$) en physique

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Exemple: $\Delta_n =$ distance entre 2 sommets aléatoires uniformes de Q_n

Théorème (Chassaing-S. 2004) $\quad \mathbb{E}(\Delta_n) \sim c \cdot n^{1/4}$

$$(n^{-1/4}\Delta_n) \xrightarrow{d} \max \text{ (serpent Brownien)}$$

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$\Rightarrow$ la carte Brownienne Marckert, Mokkadem, Le Gall, Miermont, ...

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$\Rightarrow$ la carte Brownienne Marckert, Mokkadem, Le Gall, Miermont, ...

puis Weill, Curien, Benjamini,...

– convergence vers une limite infinie discrète

$\Rightarrow$ la quadrangulation infinie uniforme (UIPQ) Angel, Schramm, ...

puis Durhus, Chassaing, Krikun, Bettinelli,...

Conclusions

– L'excursion Brownienne décrit la limite d'échelle de toute sorte d'excursions aléatoires discrètes plus ou moins complexes.

– L'arbre continu aléatoire est limite d'échelle de toute sorte d'arbres aléatoires discrets plus ou moins complexes.

$\Rightarrow$ On pense qu'il en est de même de la carte Brownienne.

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On dispose d'un cadre bijectif très général pour la construction de cartes par recollements d'arbres (Bernardi-Chapuy-Fusy 2011, Albenque-Poulalhon 2012)
On obtient ainsi en particulier un codage d'arbres pour les revêtements...
Il reste à utiliser ces constructions pour passer à la limite...