Random triangulations,

planar maps,

and a Brownian snake

PART I

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An overview of the talk

A combinatorial model

Planar maps and triangulations

Random planar maps

as a discrete model of random geometries

Encoding the distance

From quadrangulations to embedded trees

Quadrangulations and Brownian snakes

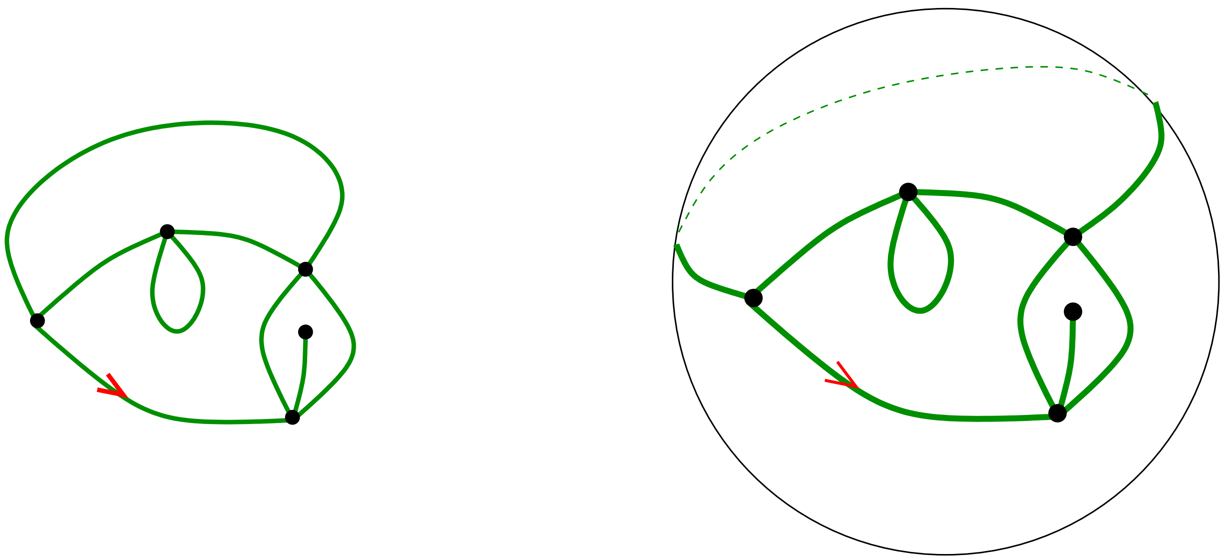
Toward a continuum random map ?

A combinatorial model

Planar maps and triangulations

Planar maps. Definition

planar map = proper embedding of a connected graph in the sphere, considered up to homeomorphisms of the sphere.

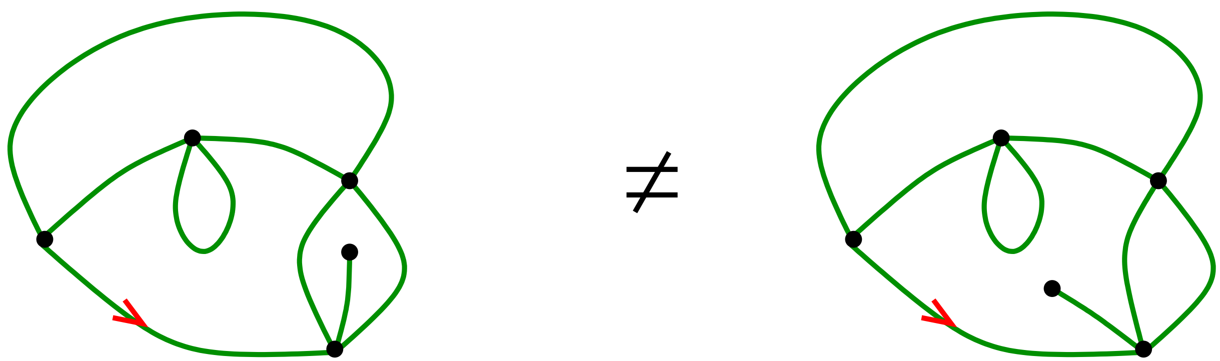


Rooted planar map = one edge is distinguished and oriented.

When making pictures in the plane we can always choose the infinite face so that the root goes counterclockwise around the map.

Planar maps. Maps vs graphs

Distinct planar maps may share the same underlying planar graph.

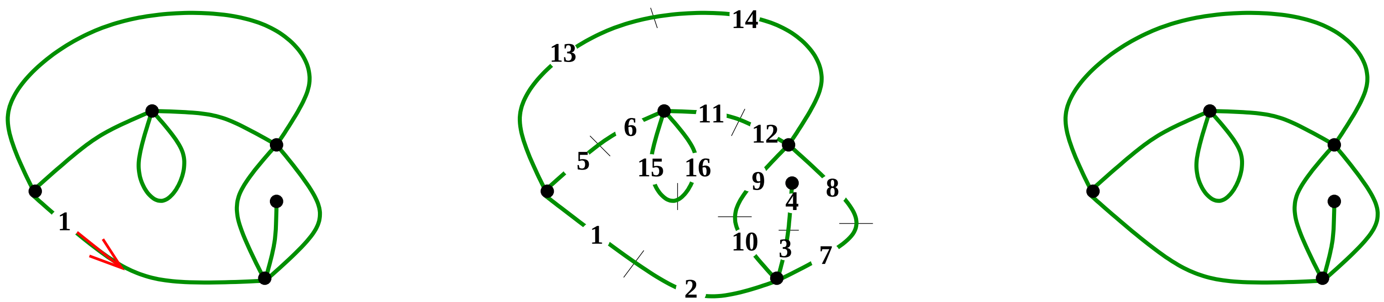


planar map = planar graph + cyclic order around vertices.

– Planar maps are combinatorial objects.
– The number of planar maps with n edges is finite.

Unlike graphs, rooted maps are trivial to test for isomorphisms:
one can decide if $M_1 = M_2$ in linear time in the size.

Planar maps. Labels and root.



$\mathcal{R}_n$: rooted; $\mathcal{L}_n$: labelled; $\mathcal{M}_n$: unrooted.

Rooted $\equiv$ labelled:

– A *rooted* map with n edges has $(2n-1)!$ distinct $\frac{1}{2}$-edge labellings.

Rooted $\approx$ unrooted:

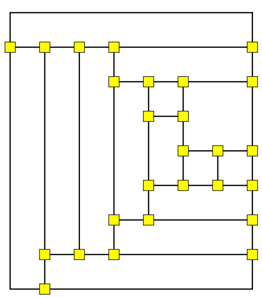
– A planar map M with n edges has $\frac{2n}{\mathrm{Aut}(M)}$ possible roots.

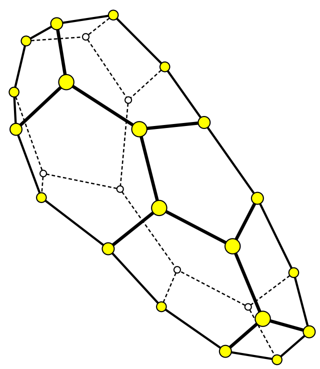
In other terms

$$|\mathcal{R}_n| = \frac{1}{(2n-1)!}|\mathcal{L}_n| = \sum_{M\in\mathcal{M}_n} \frac{2n}{\mathrm{Aut}(M)}$$

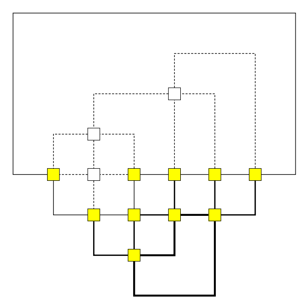
Planar maps. Example of subfamilies

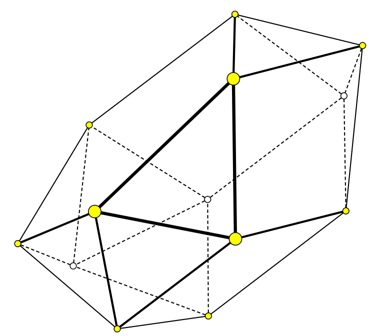
3-regular (or cubic) maps



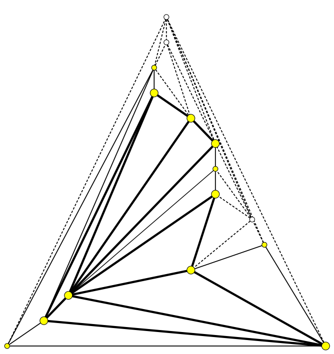


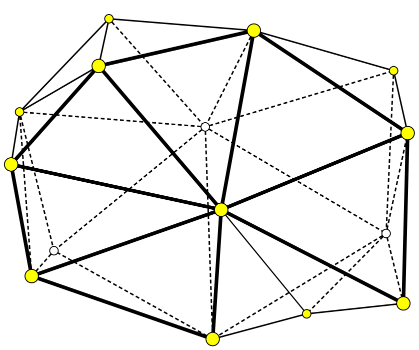
4-regular maps



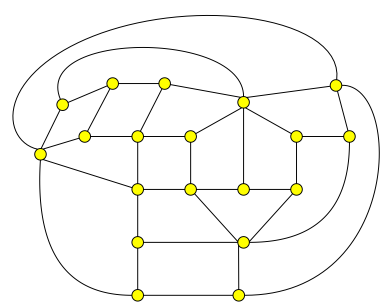


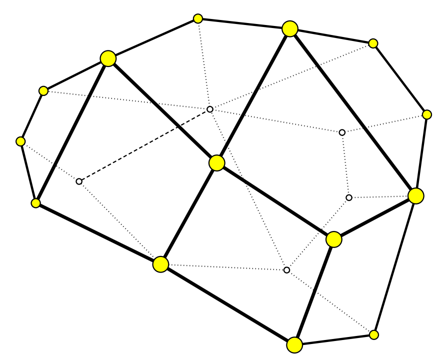
Triangulations





Quadrangulations





Enumerative questions.

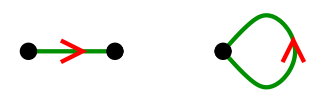
– What is the number of rooted planar maps with n edges ?

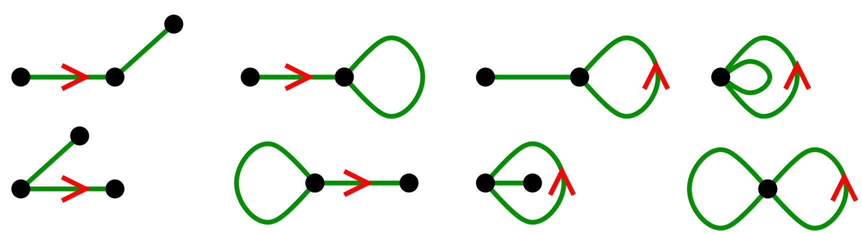
$$1,\ \ 2,\ \ 9,\ \ 54,\ \ 378, \ldots$$

– What is the number of rooted triangulations with n triangles ?

$$1,\ \ 3,\ \ 13,\ \ 68,\ \ 399,\ \ldots$$

Questions raised by Tutte (60's) in relation with the four color theorem.

The smallest maps: $\mathcal{R}_0 = \{ \bullet \}$, $\mathcal{R}_1 = \{$  $\}$

$\mathcal{R}_3 = \{$  $\}$

Enumerative questions. "Universal" asymptotic results

Theorem (Tutte'62, Bender–Canfield'92...)

Let D be a set of acceptable vertex degrees and $\mathcal{F}_n$ the set of rooted maps with n edges and vertex degrees in D.

Then, (for coherent values of n)

$$|\mathcal{F}_n| \underset{n\to\infty}{\sim} c_0\,\rho^n n^{-5/2},$$

for c_0 and ρ some positive constants that depend on D.

Examples:

– $D = \{3\}$, cubic maps (or triangulations by duality)

– $D = \{4\}$, 4-regular maps (or quadrangulations by duality)

Enumerative questions. "Universal" asymptotic results

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Compare to similar results for trees:
plane trees, binary trees, Cayley trees with n nodes $\to \rho^n n^{-3/2}$
(# shapes of GW trees conditioned to total progeny of size n)

Enumerative questions. Some exact results

Theorem (Tutte'62)

$$\#\{\text{triangulations, } 2n \text{ faces}\} = \frac{2^{n+1}(3n)!}{(2n+2)!n!} \sim \frac{c_1}{n^{5/2}} (27/2)^n$$

$$\#\{\text{4-regular maps, } n \text{ vert.}\} = \frac{2 \cdot 3^n (2n)!}{(n+2)!n!} \sim \frac{c_2}{n^{5/2}} \; 12^n$$

Of course such nice formulas are not the rule...

However for about twenty families of maps, generating functions (partition functions) are known and algebraic.

→ *Planar constellations.* Bousquet-Mélou & S. '99.

→ *5-connected triangulations.* Gao & Wormald '01.

Enumeration. Terminology in Combinatorics / Physics

Up to differencies of terminology, many results of Tutte were rediscovered in physics via "perturbative expansion of matrix integrals" *à la Feynman*.

In combinatorics. Planar maps, or: fat graphs, hypermaps, rotation systems, "triangulations", "2d-polyhedra"...

In physics. Dynamical triangulations, or: "fluid lattices", discretised euclidean two dimensional quantum geometry, ϕ^4-model...

– edges, vertices, loops = links, nodes, tadpoles...

– Generating function $\Leftrightarrow$ Partition function

$$f(z) = \sum_{R \in \mathcal{R}} z^{|R|} \;\Leftrightarrow\; Z_\mu = \sum_{M \in \mathcal{M}} \frac{1}{C_M} e^{-\mu|M|}, \text{ where } C_M = \mathrm{Aut}(M).$$

Planar maps in physics ? (a naive point of view)

Consider a toy model of statistical physics (Ising) in a 2d universe...

- Conventional gravity: the universe is flat.

  $\Rightarrow$ discretised by a regular grid: Ising on $\mathbb{Z}^2$.

- Quantum gravity: a distribution of proba on possible universes.

  $\Rightarrow$ discr. by a random map: Ising on 4-reg. maps (ϕ^4).

- planar case is easier $\Rightarrow$ assume spherical topology to start with.

The emphasis is on random maps; enumeration appears as a by-product here.

Which random maps ?

What is the distribution that should be taken on maps ?

– the uniform distribution on rooted maps with n edges,

– Delaunay triangulations of a random set of points,

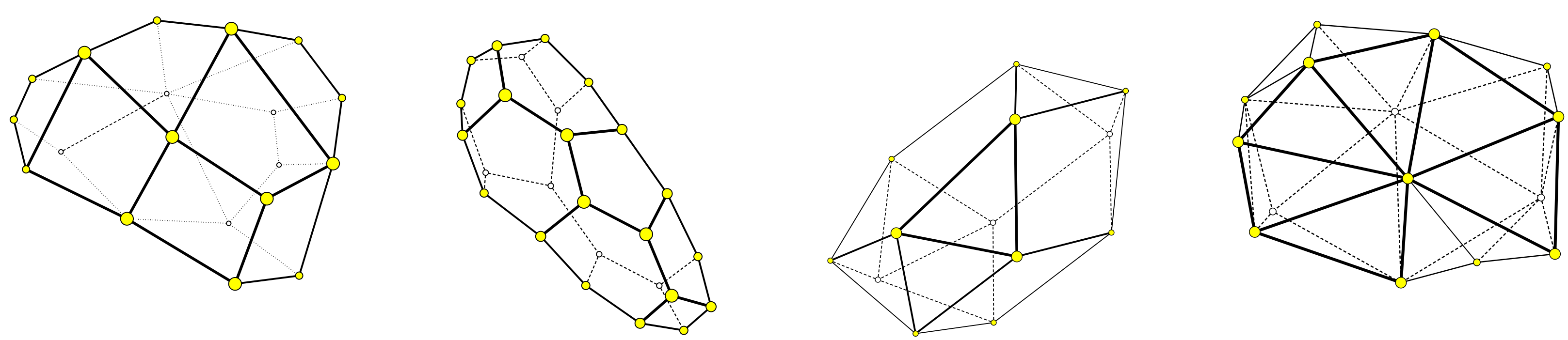
– anything else ??

We take the first distribution, which leads to the dynamical triangulations studied in the physics litterature.

There is an a posteriori "justification" of this choice:
For all reasonable toys models that are "solved" on both type of lattices, critical exponents on random maps and on $\mathbb{Z}^2$ are related according to the predictions of the KPZ relation (Knizhnik, Polyakov, Zamolodchikov).

Random planar maps

as a discrete model of random geometries



Random planar maps. Definition.

Let $\mathcal{R}_n$ be a family of rooted planar maps (say 4-regular maps with n vertices).

Consider a r.v. X_n with uniform distribution on $\mathcal{R}_n$:

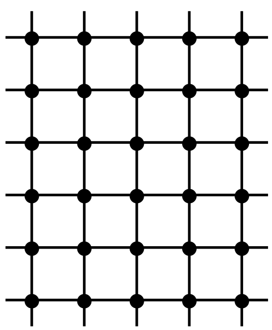
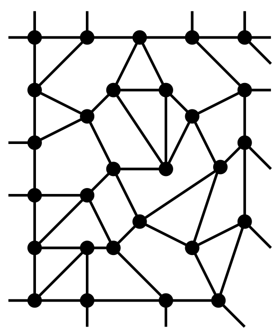
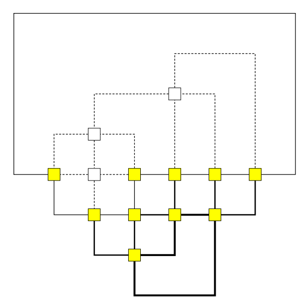
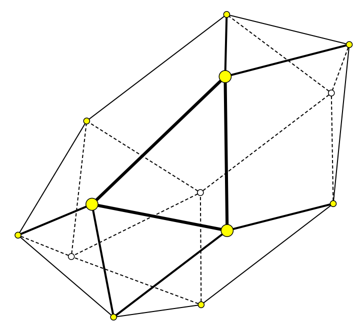
$$\Pr(X_n = R) \;=\; \frac{1}{|\mathcal{R}_n|} \;=\; \frac{(n+2)!n!}{2 \cdot 3^n (2n)!}, \qquad \text{for all } R \in \mathcal{R}_n.$$

In the terminology of physics, this corresponds to the *canonical* distribution, as opposed to the *grand canonical* distribution which is:

$$\Pr(X_\mu = R) \;=\; \frac{e^{-\mu|R|}}{Z_\mu}, \qquad \text{for all } R \in \cup_n \mathcal{R}_n,\ \mu > \mu_c.$$

Rooting correspond to the combinatorial factors $\frac{1}{C_T}$ arising from Feynman type expansion.

How do we sample a true random map ?

Random 4-regular maps.

Recall Tutte's formula

$$\#\{\text{4-regular maps, } n \text{ vert.}\} \quad = \quad \frac{2}{n+2}\,\frac{3^n}{n+1}\binom{2n}{n}.$$

– Tutte, 1962: through recursions and algebra on generating functions.

– Bessis-Itzykson-Parisi-Zuber, 1978: through perturbative expansions of matrix integral with quadratic potentials.

These two approaches lead to quadratic algorithms for sampling.
We shall consider instead a direct combinatorial approach.

Tutte's formula. A bijective proof.

$$\#\{\text{ 4-regular maps with } n \text{ vertices }\} \text{ is } \frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}.$$

Our tactic will be to construct simple combinatorial objects that are *clearly* counted successively by

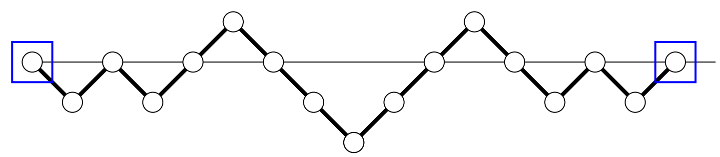
$$\begin{array}{lr} (i) & \binom{2n}{n} \\ (ii) & \frac{1}{n+1}\binom{2n}{n} \\ (iii) & \frac{3^n}{n+1}\binom{2n}{n} \\ (iv) & \frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n} \end{array}$$

Finally we shall exhibit a simple one-to-one correspondence between the latter objects and 4-regular maps with n vertices.

Tutte's formula. A bijective proof (i).

$\#\{$ 4-regular maps with n vertices $\}$ is $\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{\mathbf{2n}}{\mathbf{n}}$.

The number of bridges of length $2n$ with increments $\{+1,-1\}$ is $\binom{2n}{n}$.



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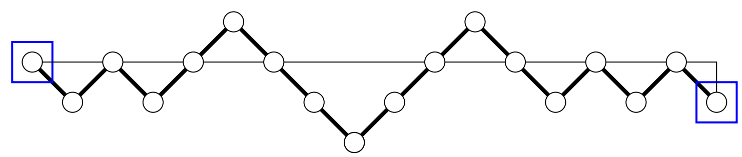
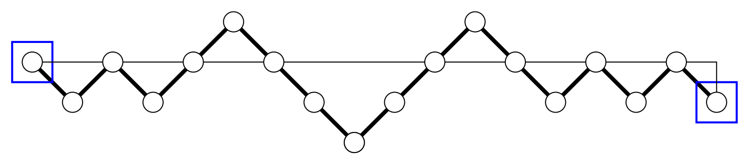


Add a last down step.

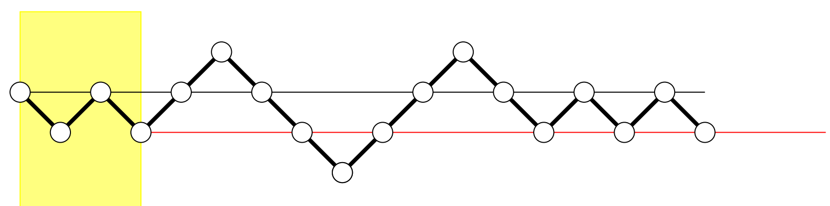
Tutte's formula. A bijective proof (i).

$\#\{$ 4-regular maps with n vertices $\}$ is $\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{\mathbf{2n}}{\mathbf{n}}$.

The number of "bridges" of length $2n+1$ is $\binom{2n}{n}$.



Cyclic shifts at down steps define *classes of* $n+1$ *conjugate paths.*



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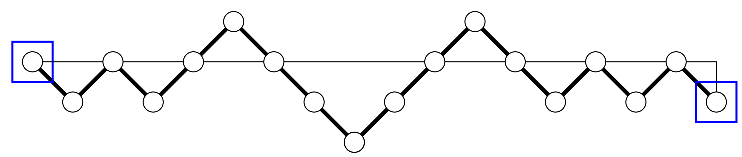

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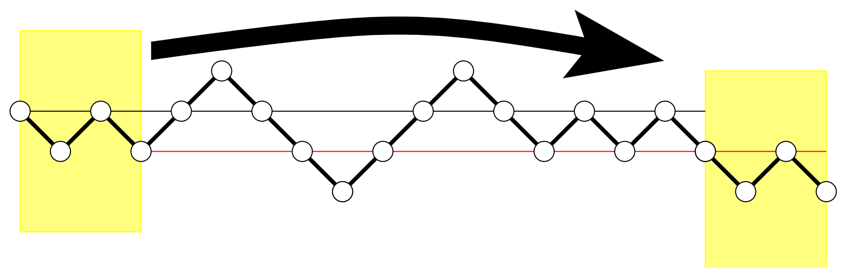

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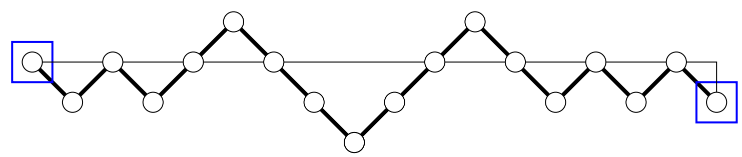

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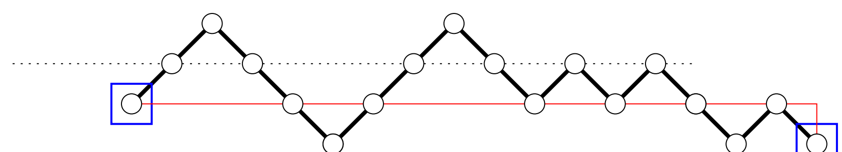

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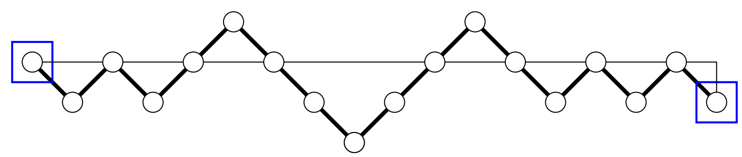
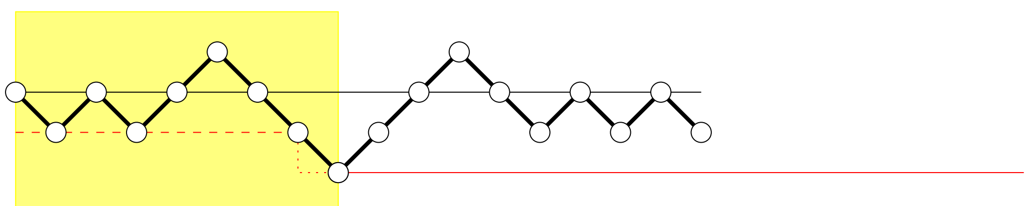


Exactly one of the $n+1$ conjugate has the *positive prefix property.*



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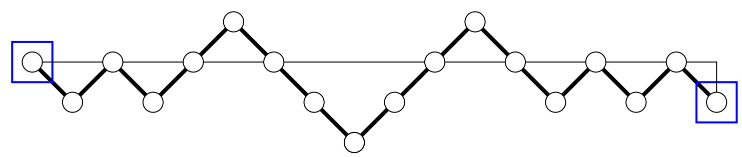

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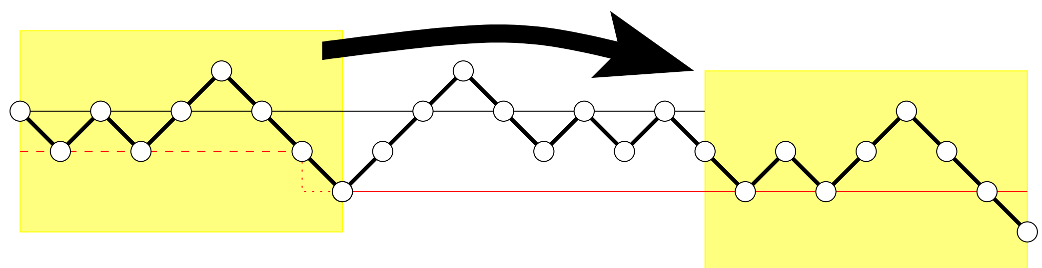

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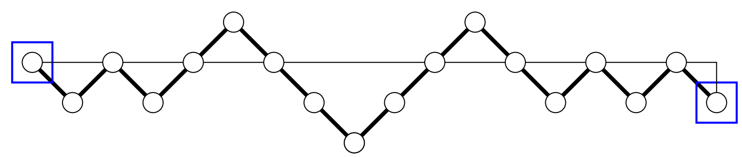

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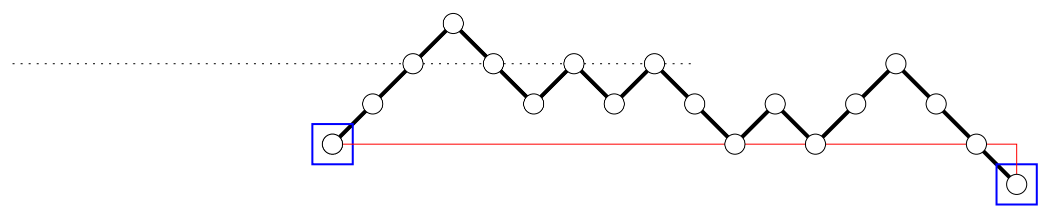


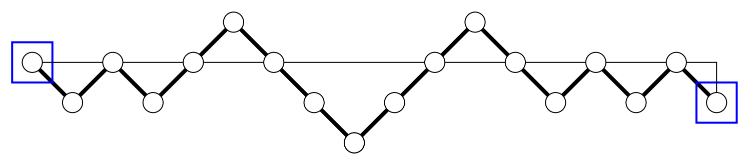
Tutte's formula. A bijective proof (i).

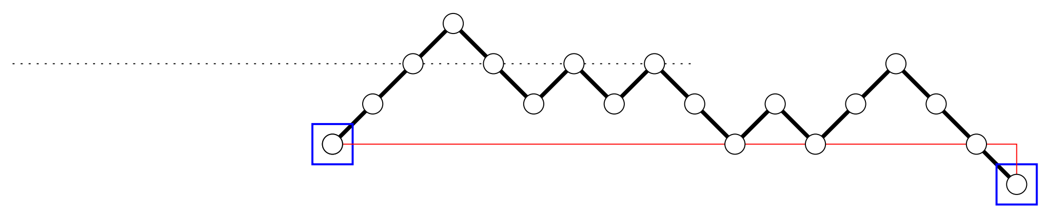
$\#\{$ 4-regular maps with n vertices $\}$ is $\frac{2}{n+2} \cdot \frac{3^n}{\mathbf{n+1}}\binom{\mathbf{2n}}{\mathbf{n}}$.

The number of *Dyck paths* of length $2n+1$ (excursion, or bridges with the positive prefix property) is

$$\frac{1}{n+1}\binom{2n}{n}.$$

(Exactly one of the $n+1$ conjugates has the *positive prefix property*.)





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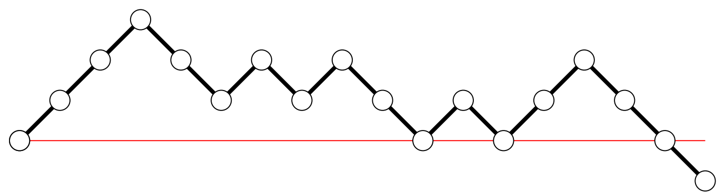
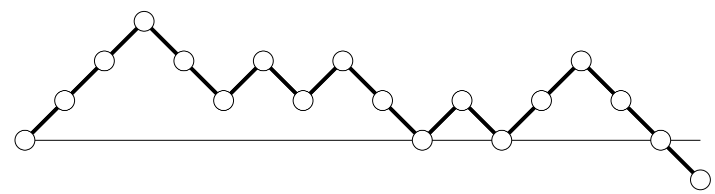


This proof of the *cyclic lemma* is due to Dvoretzky-Motzkin'47.

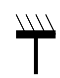
Vervaat's method to construct the brownian excursion is its continuum analog.

Tutte's formula. A bijective proof (i).

#{ 4-regular maps with n vertices } is $\frac{2}{n+2} \cdot \frac{3^n}{\mathbf{n+1}} \binom{\mathbf{2n}}{\mathbf{n}}$.



Dyck paths are prefix codes of binary trees.



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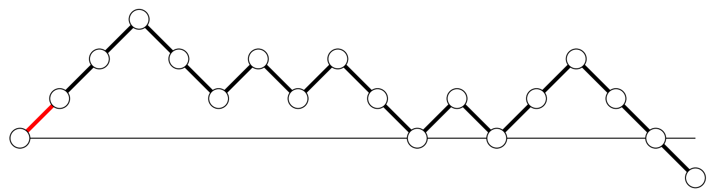

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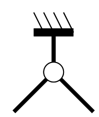

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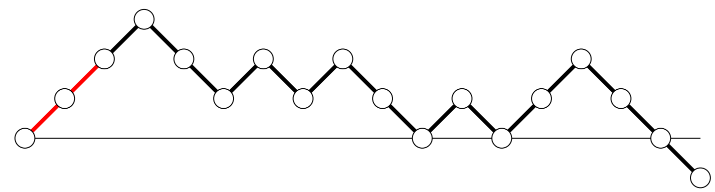

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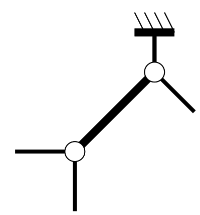

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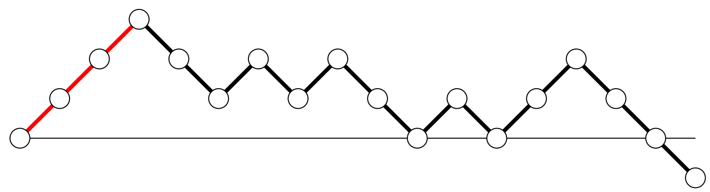
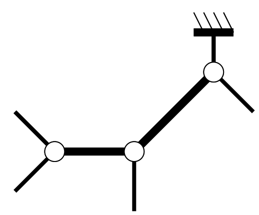
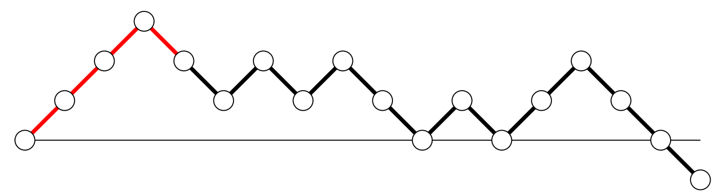


Dyck paths are prefix codes of binary trees.



Tutte's formula. A bijective proof (i).

#{ 4-regular maps with n vertices } is $\frac{2}{n+2} \cdot \frac{3^n}{\mathbf{n+1}}\binom{\mathbf{2n}}{\mathbf{n}}$.



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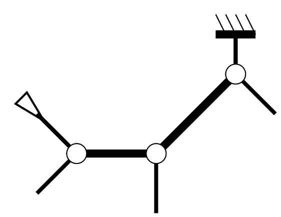

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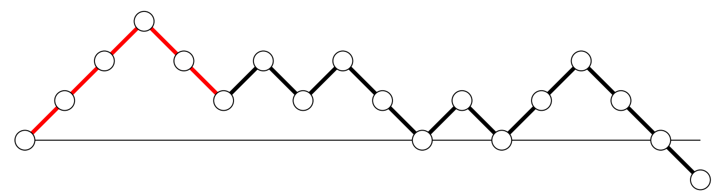

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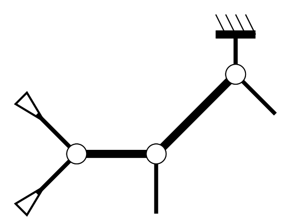

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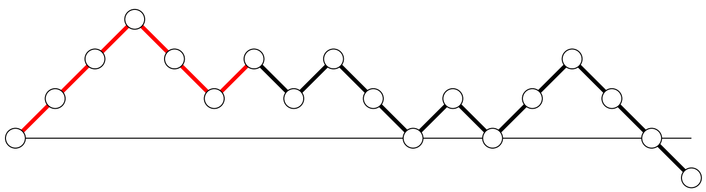

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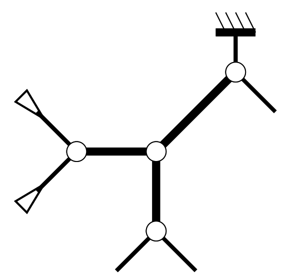

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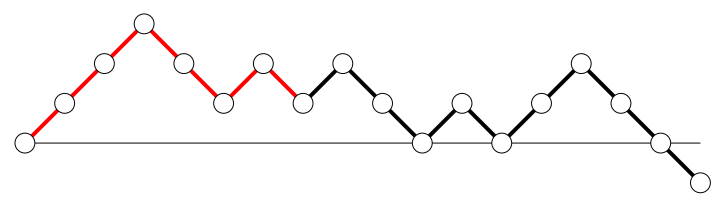

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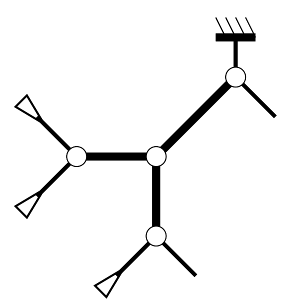

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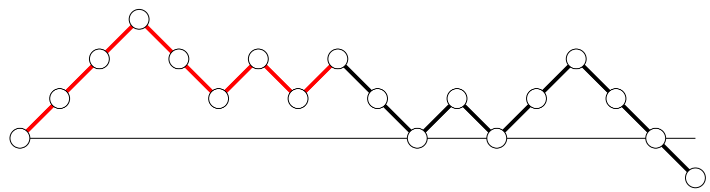

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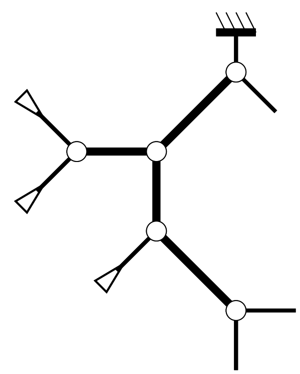

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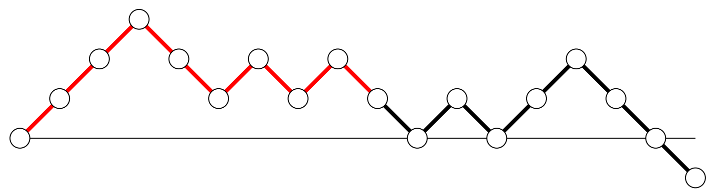

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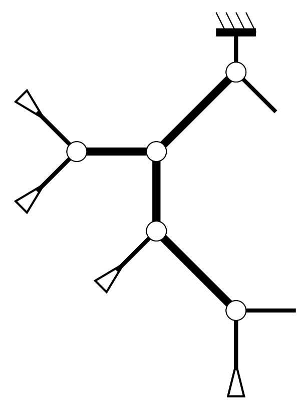

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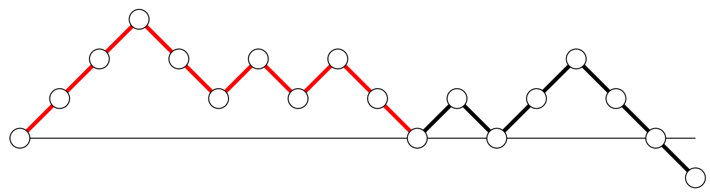

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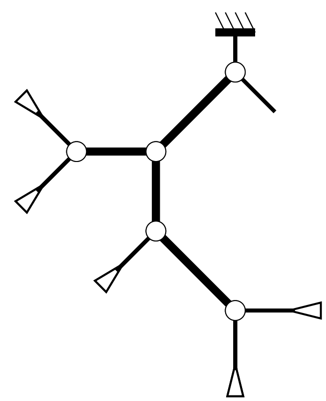

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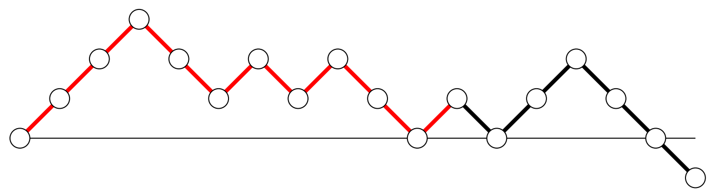

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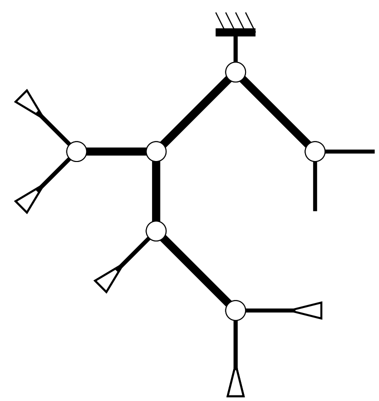

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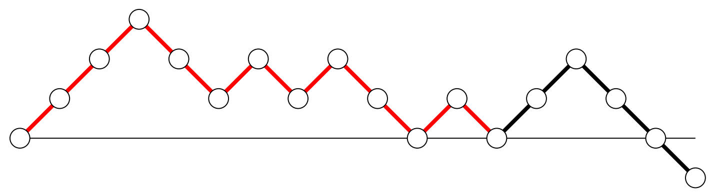

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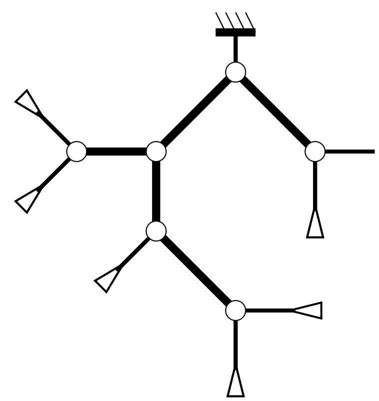

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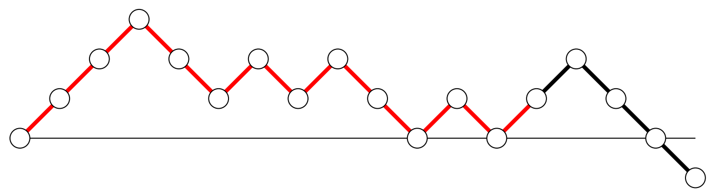

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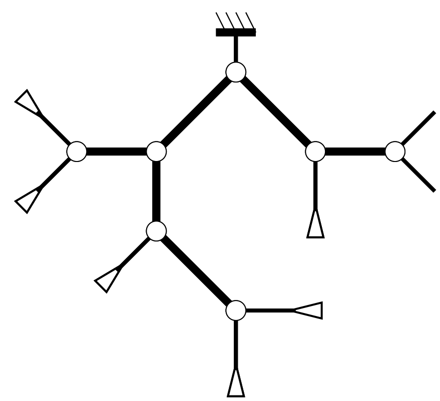

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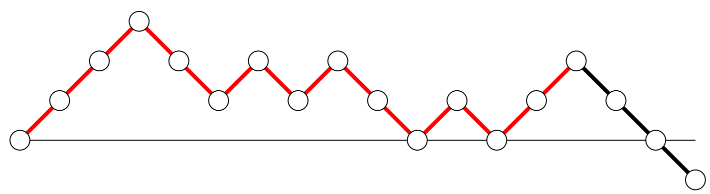

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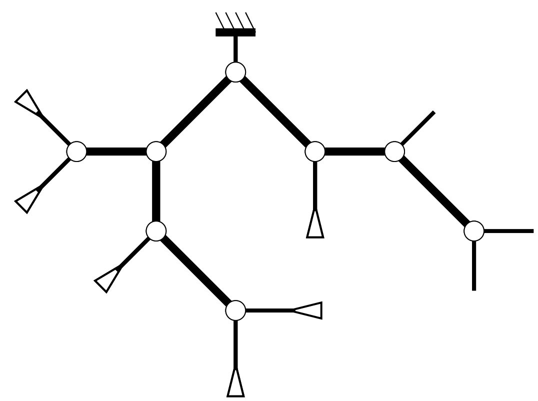

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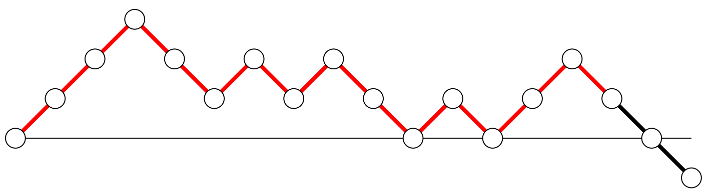

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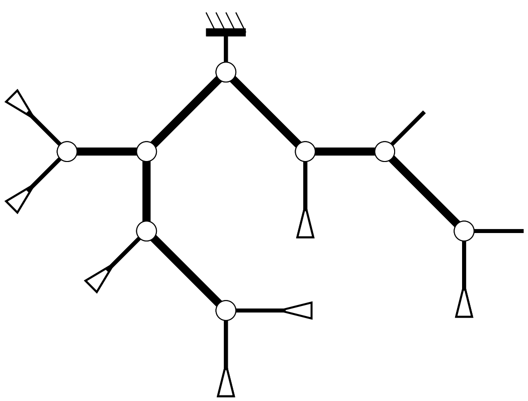

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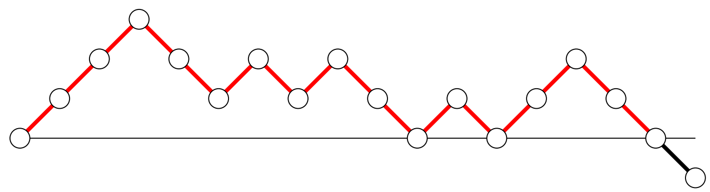

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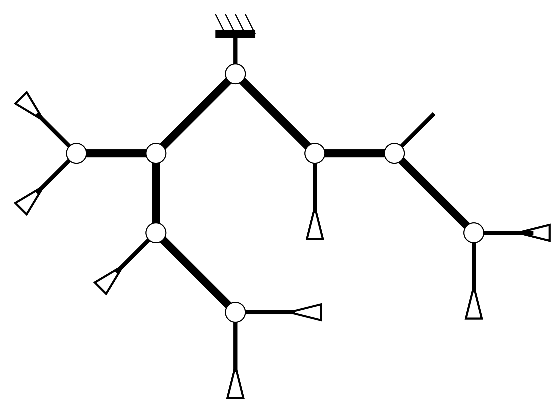

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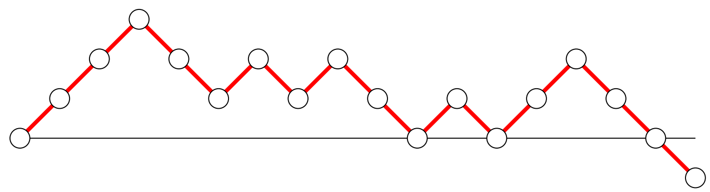

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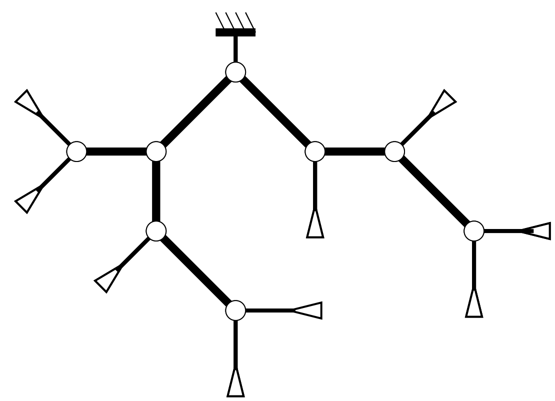

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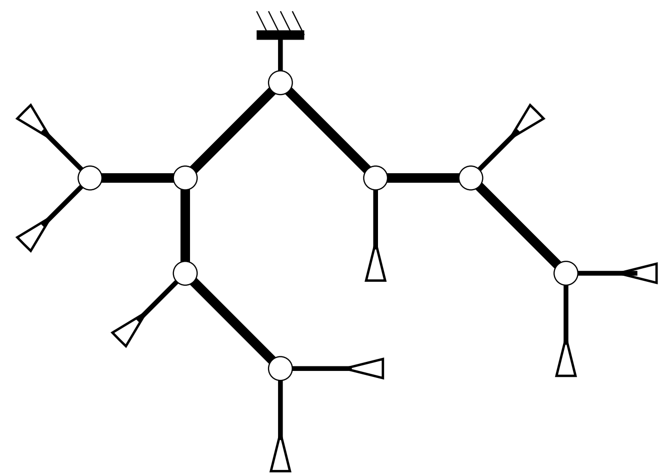
$\#\{$ 4-regular maps with n vertices $\}$ is $\frac{2}{n+2} \cdot \frac{3^n}{\mathbf{n+1}}\binom{\mathbf{2n}}{\mathbf{n}}$.

There are

$$\frac{1}{n+1}\binom{2n}{n}$$

binary trees with n nodes.



Such trees have n (internal) nodes and $n+2$ leaves (root included).

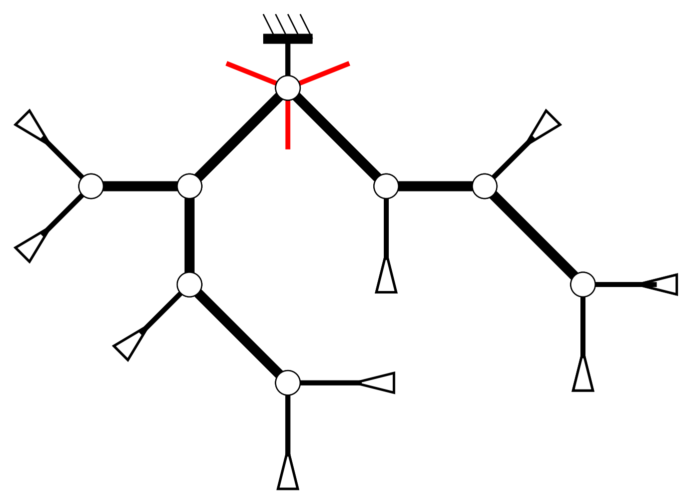
Tutte's formula. A bijective proof (ii).

#{ 4-regular maps with n vertices } is $\frac{2}{n+2} \cdot \frac{\mathbf{3^n}}{\mathbf{n+1}}\binom{\mathbf{2n}}{\mathbf{n}}$.

On each node, a bud can be added in three ways, giving rise to

$$\frac{3^n}{n+1}\binom{2n}{n}$$

blossom trees with n nodes.



Blossom trees have n buds and $n+2$ leaves around the tree.

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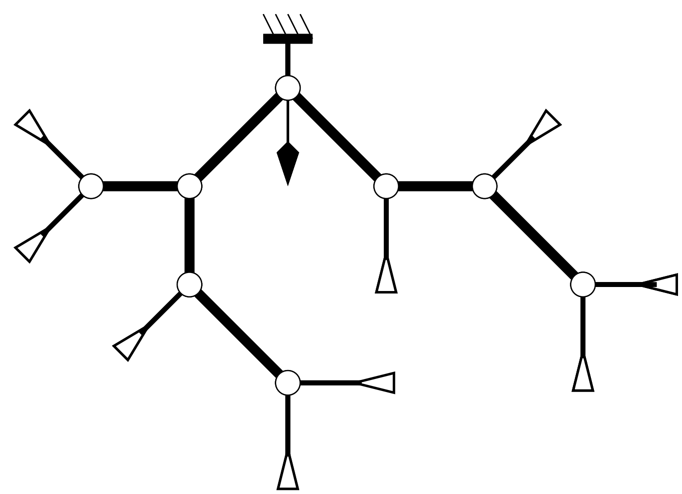

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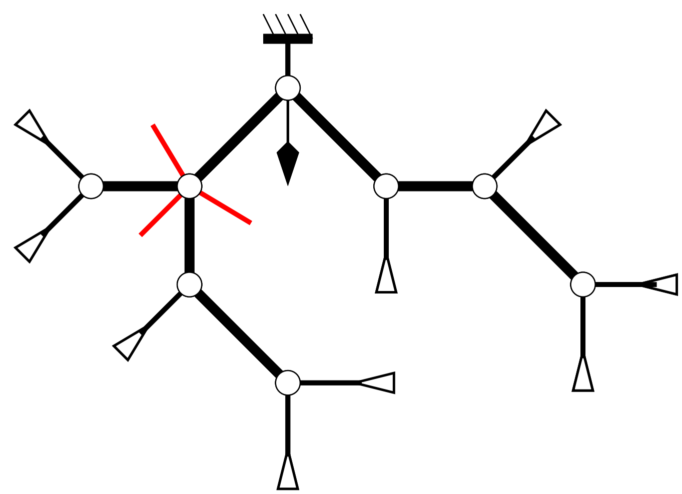

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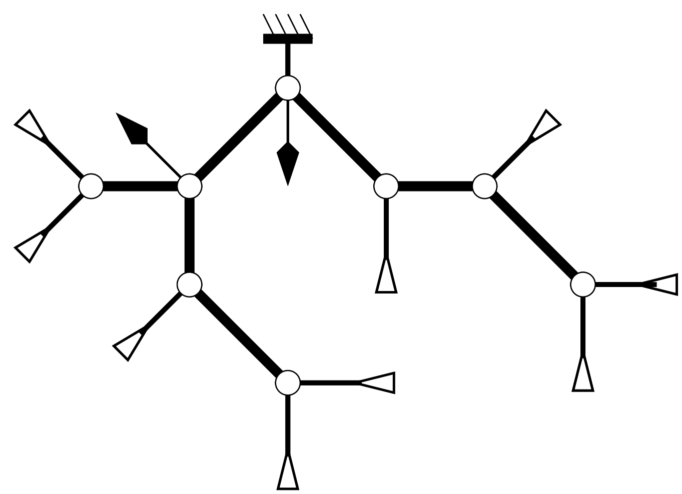

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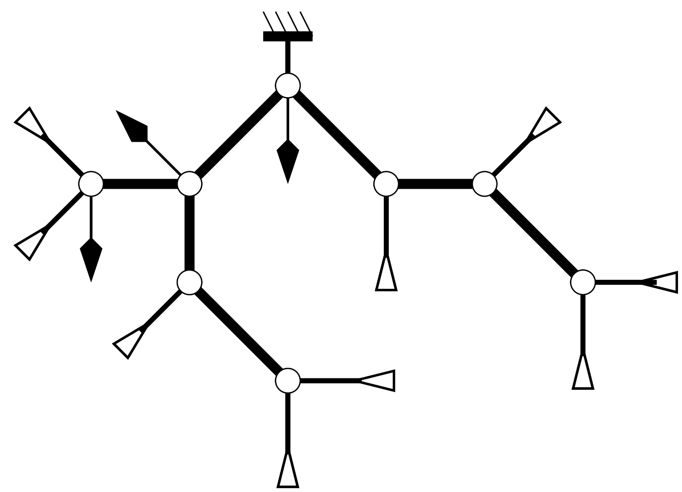

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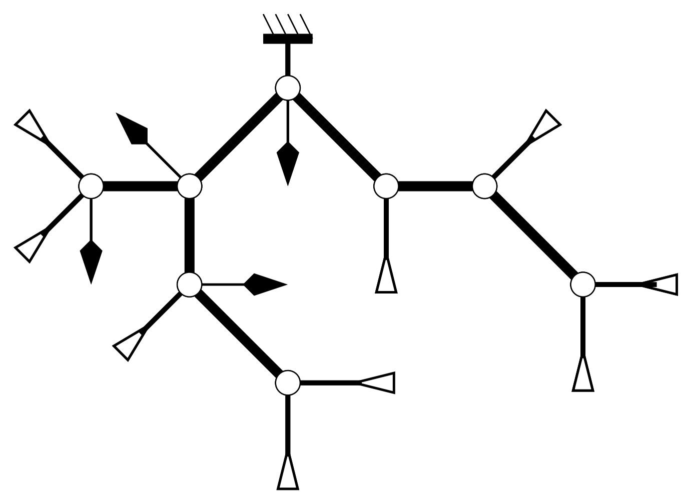

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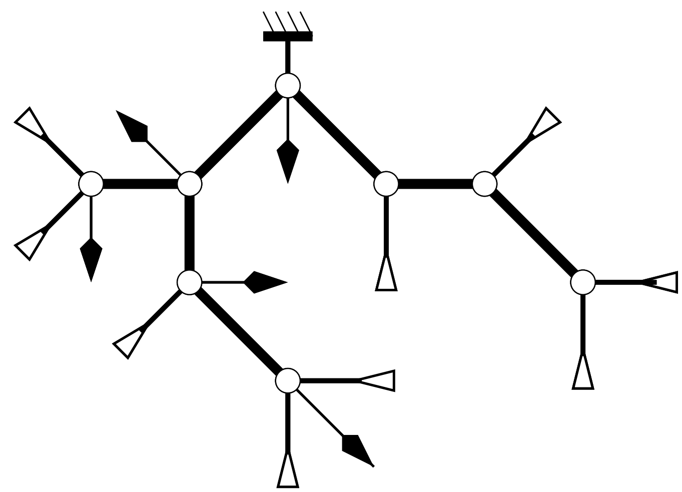

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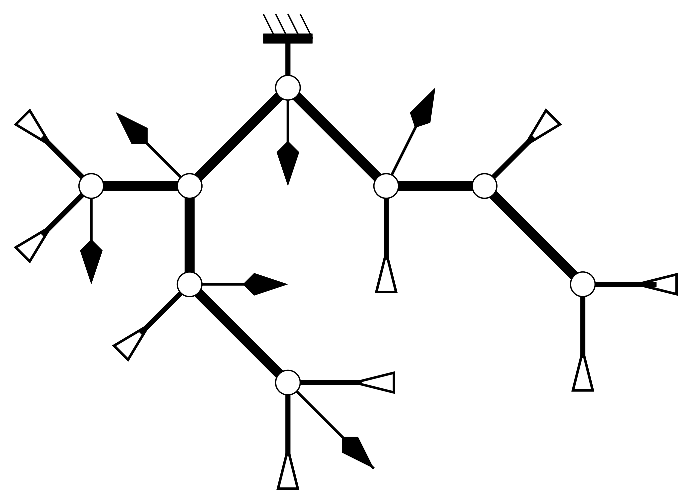

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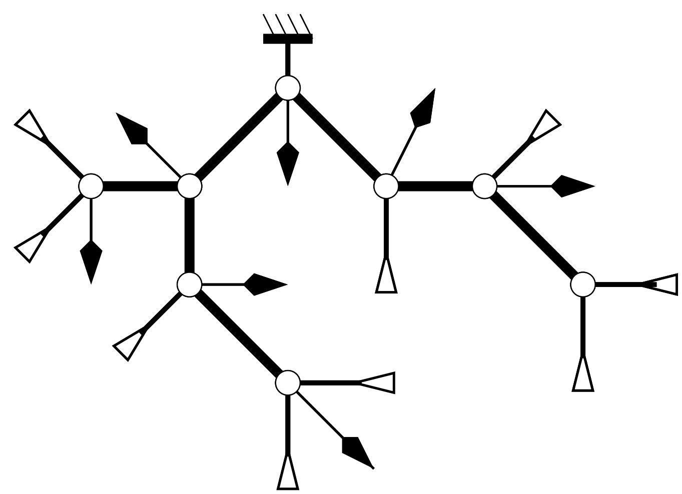

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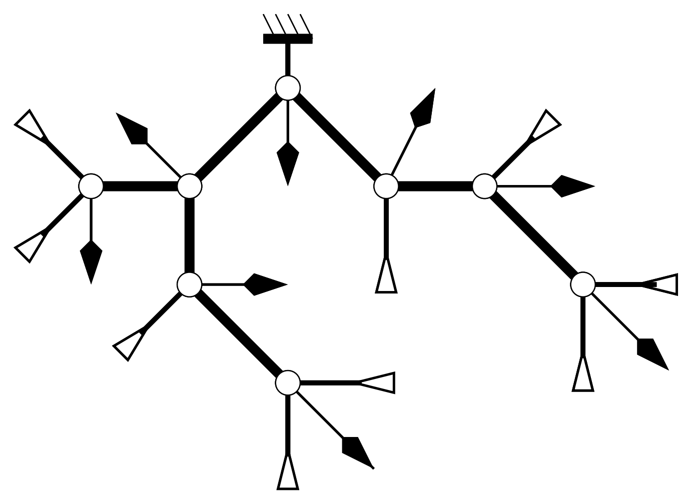

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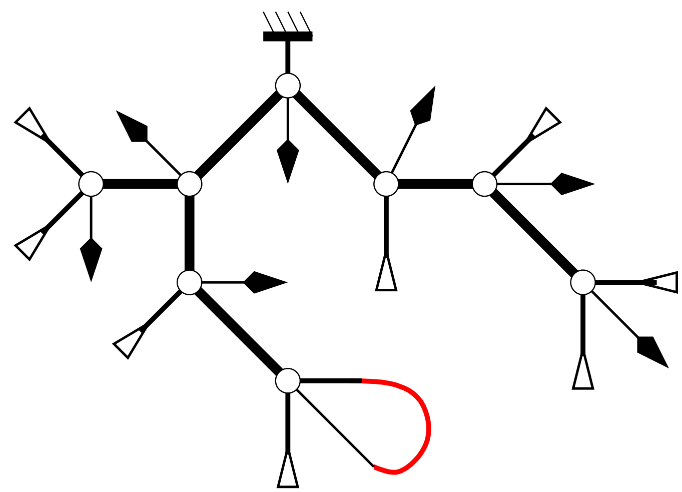

Blossom trees have n buds and $n+2$ leaves around the tree.
Upon matching them counterclockwise, two leaves remain *unmatched*.

Tutte's formula. A bijective proof (ii).

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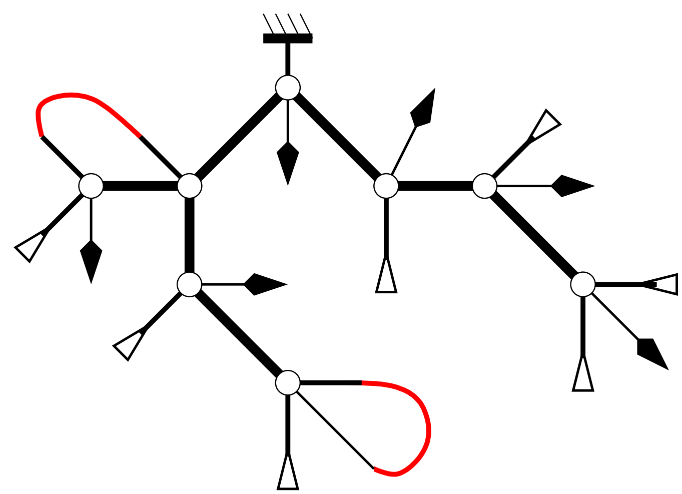

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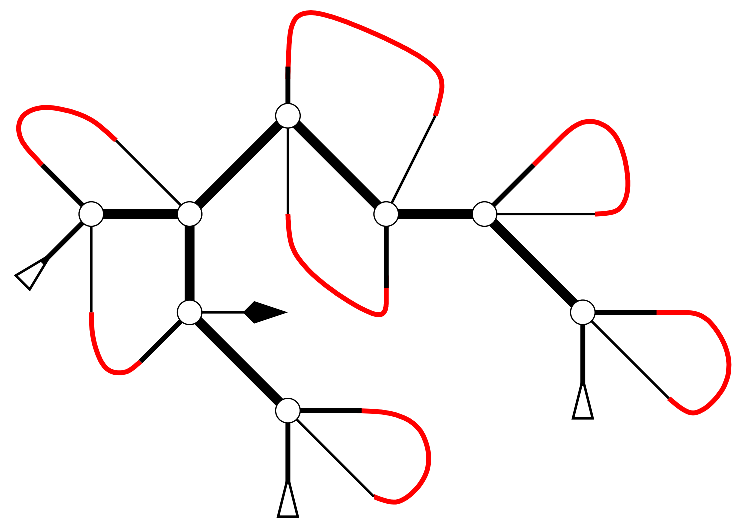

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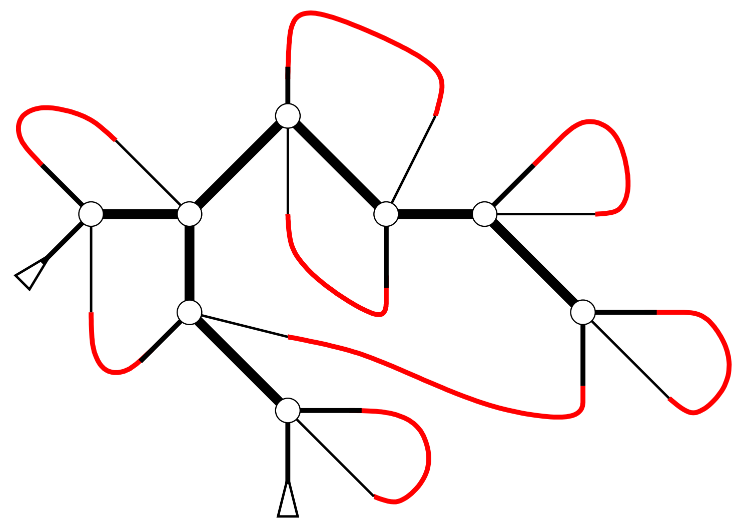

Blossom trees have n buds and $n+2$ leaves around the tree.
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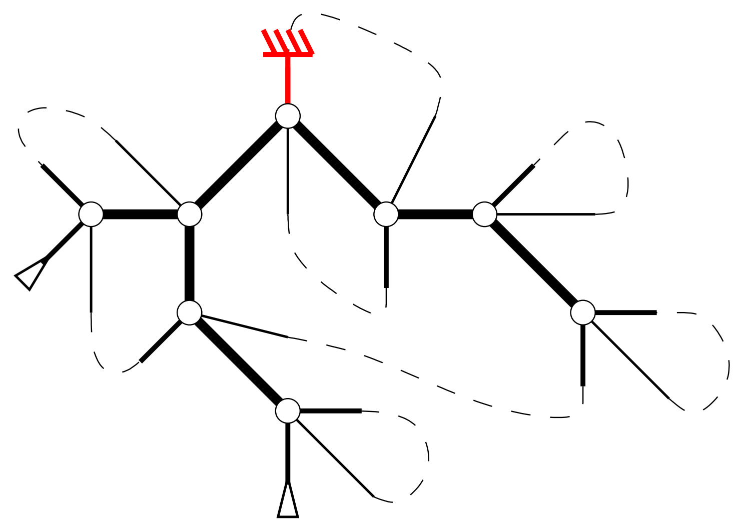
#{ 4-regular maps with n vertices } is $\frac{\mathbf{2}}{\mathbf{n+2}} \cdot \frac{\mathbf{3^n}}{\mathbf{n+1}} \binom{\mathbf{2n}}{\mathbf{n}}$.

A blossom tree is *unbalanced* if its root is matched.
It is *balanced* if its root remains unmatched.

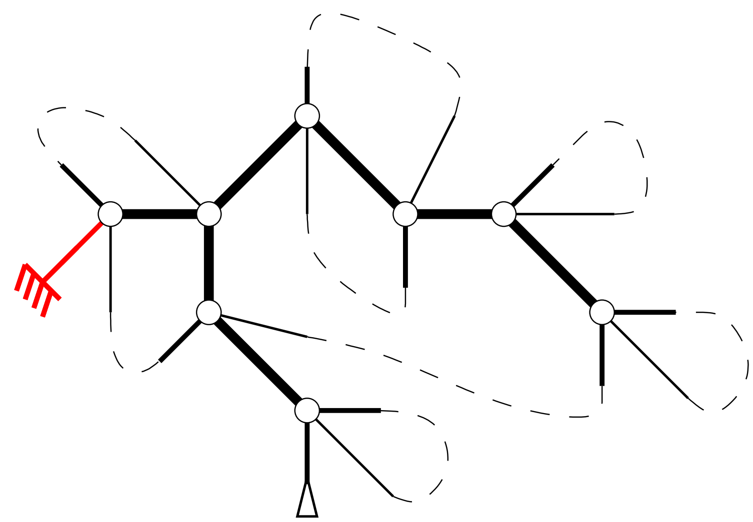


Let us call *conjugacy class* an equivalence class of blossom trees obtained one from another by moving the root.

Tutte's formula. A bijective proof (iii).

#{ 4-regular maps with n vertices } is $\frac{\mathbf{2}}{\mathbf{n+2}} \cdot \frac{\mathbf{3^n}}{\mathbf{n+1}}\binom{\mathbf{2n}}{\mathbf{n}}$.

A blossom tree is *unbalanced* if its root is matched.
It is *balanced* if its root remains unmatched.



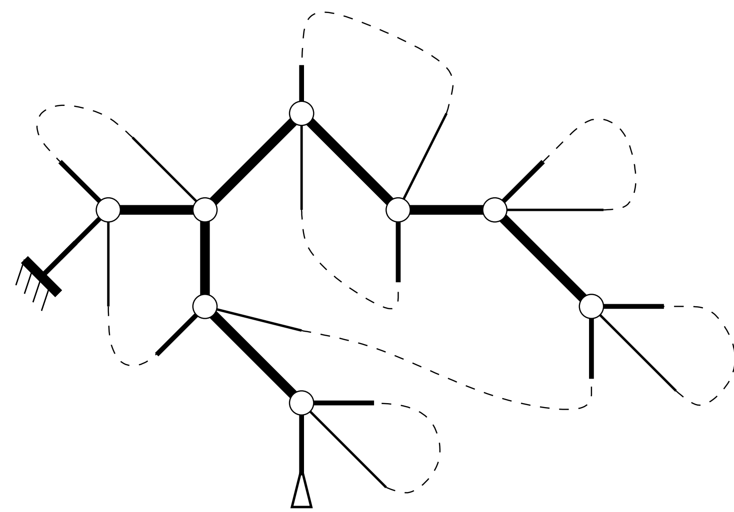
Let us call *conjugacy class* an equivalence class of blossom trees obtained one from another by moving the root.

Tutte's formula. A bijective proof (iii).

#{ 4-regular maps with n vertices } is $\mathbf{\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}}$.

The number of balanced blossom trees with n nodes is

$$\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}.$$



Indeed each *conjugacy class of blossom trees* contains $n+2$ blossom trees, 2 of which are balanced.

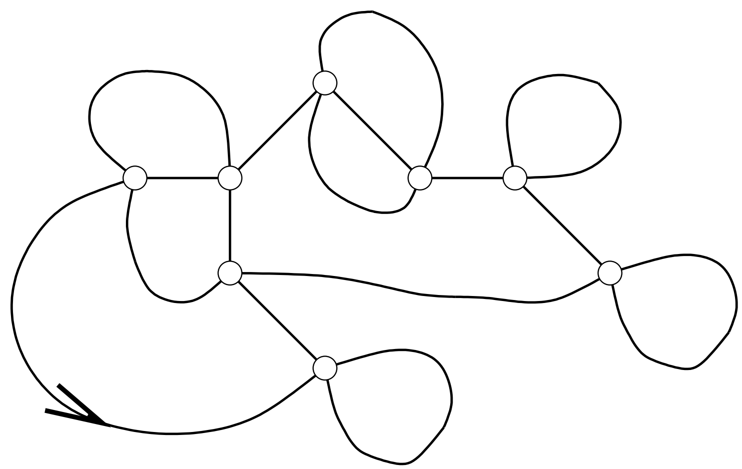
Tutte's formula. A bijective proof (iv).

$\#\{$ 4-regular maps with n vertices $\}$ is $\frac{\mathbf{2}}{\mathbf{n+2}} \cdot \frac{\mathbf{3^n}}{\mathbf{n+1}}\binom{\mathbf{2n}}{\mathbf{n}}$.

Theorem (S. 1998):

This closure is one-to-one between

- balanced blossom trees with n nodes
- and rooted 4-regular maps with n vertices.



The converse bijection is based on a bfs traversal of the dual graph.

Random sampling for X_n

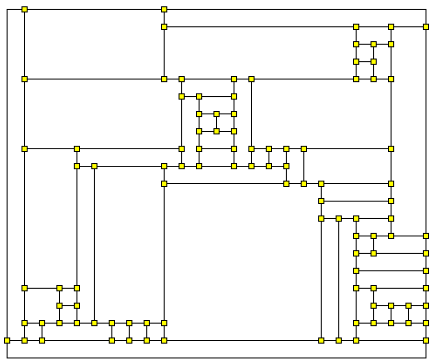
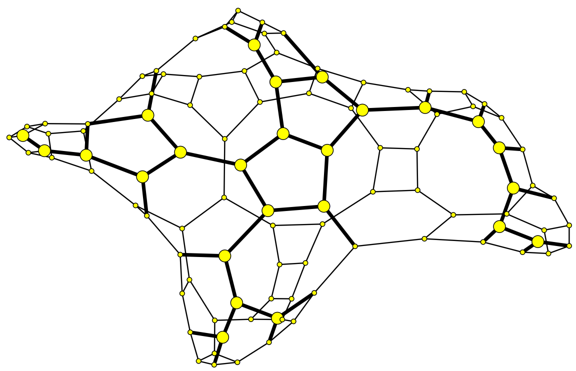
To generate a random 4-regular map with n vertices,

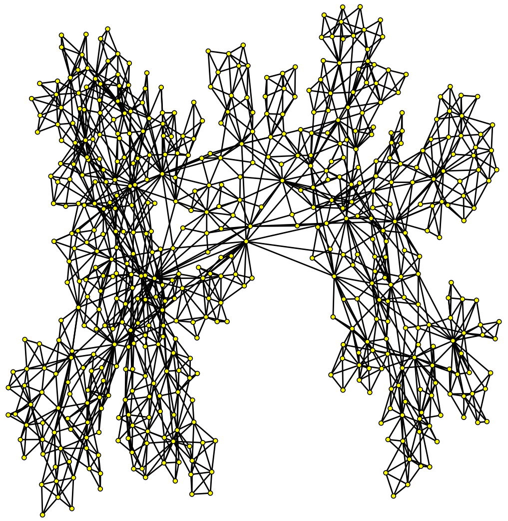
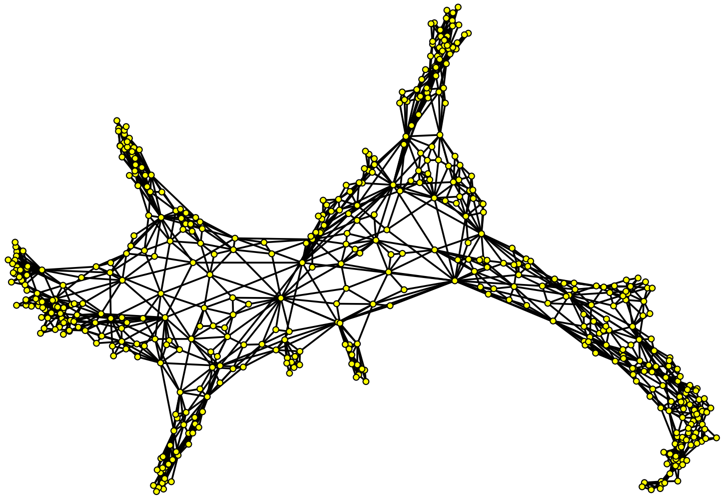
1. sample a uniform "bridge" with $2n + 1$ steps,
   conjugation yields a uniform random tree with n nodes,
2. add independantly random buds on all nodes,
3. perform the matching and add the root,
   conjugation yields a uniform random 4-reg. map with n vert.

The number of operations is linear in the size n

$\rightarrow$ generation speed $\approx$ 100.000 vertices per second.

A gallery of random maps

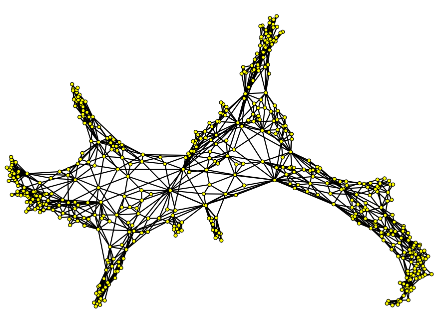
First summary and observations

– A model of random surfaces with spherical topology (fixed genus).

– Various techniques are available to study it analytically.

– Large random instances can be effectively generated.

Experimentally these random surfaces appear quite different from regular lattices.



– fat tree structure ?

– branchings into baby universes ?

– Hausdorf dimension ?

– short separators ?

What is the typical geometry of a random map ?

Some results from the combinatorial litterature.

The random planar map. Vertex or face degree.

Let $\zeta_k = \zeta_k(n)$ be the number of vertices of degree k in a random triangulation with n vertices.

Tutte *et al.* (60's). For fixed k,

$$\mathbb{E}(\zeta_k) \underset{n\infty}{\sim} n\mu_k, \qquad \text{with } \mu_k = \frac{8(k-2)(-3/4)^k}{4k^2-1}\binom{-3/2}{k} \sim \frac{4(3/4)^k}{\sqrt{k\pi}}.$$

(Similar results hold for other families).

The random planar map. Maximal degree.

Let Δ_n be the maximal degree in a random n-triangulation.

Gao-Wormald (2000). For any $\Omega(n) \to \infty$,

$$\Pr\left(\left|\Delta_n - \frac{\log n - \frac{1}{2}\log\log n}{\log 4/3}\right| \leq \Omega(n) \right) \to 1.$$

For $k > \gamma_n$ (= near max degree), $\zeta_k, \zeta_{k+1}, \ldots$ "are" independant Poisson variables of mean $\alpha^{k+i-\gamma_n}$.

(Proven for two families.)

The random planar map. Submaps and 0-1 laws.

Let T_0 be a fixed triangulation and $\eta_n(T_0)$ be the number of copies of T_0 in a random n-triangulation.

Gao-Wormald (2000). For a fixed T_0, $\eta_n(T_0)$ is sharply concentrated around cn for some constant c.

Bender-Compton-Richmond (1999). First order sentences on planar maps or triangulations have 0-1 laws.

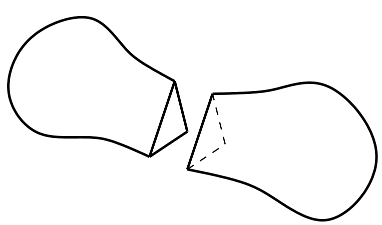
Any property expressible in first order logic on maps (with edges, vertices and faces variables, and adjacency relations) is true with probability tending either to 0 or to 1 as the size of maps goes to infinity.

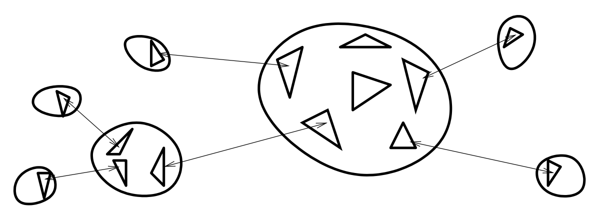
The random planar map. Minimal cycles.

A cycle is **separating** if it cuts the map in two.

Minimal separating cycles (for the length) play a special role:
they allow "baby universe decomposition".

For instance separating triangles in triangulations:





Separating triangles + spherical topology ⇒ Tree decomposition.
Nodes are triangulations without separating triangles.

The random planar map. Minimal cycles.

Remark. A random triangulation has $\Theta(n)$ separating triangles
$\Rightarrow$ a linear number of "branchings into baby universe".

Bender-Richmond (1995) "baby universes" are of size at most $n^{2/3}$ and there is one "mother universe" which is a.s. of size cn.

Banderier-Flajolet-S.-Soria (2000) In the scaling $cn + n^{2/3}x$, the size of the "mother universe" converges to a stable law of index $3/2$.

The random planar map. Separators.

Fixed length cycles a.s. separate only $n^{2/3}$ vertices.

What if the cycles get larger ? When can we get true separators ?

The average number of separator of length k in a n-triangulation

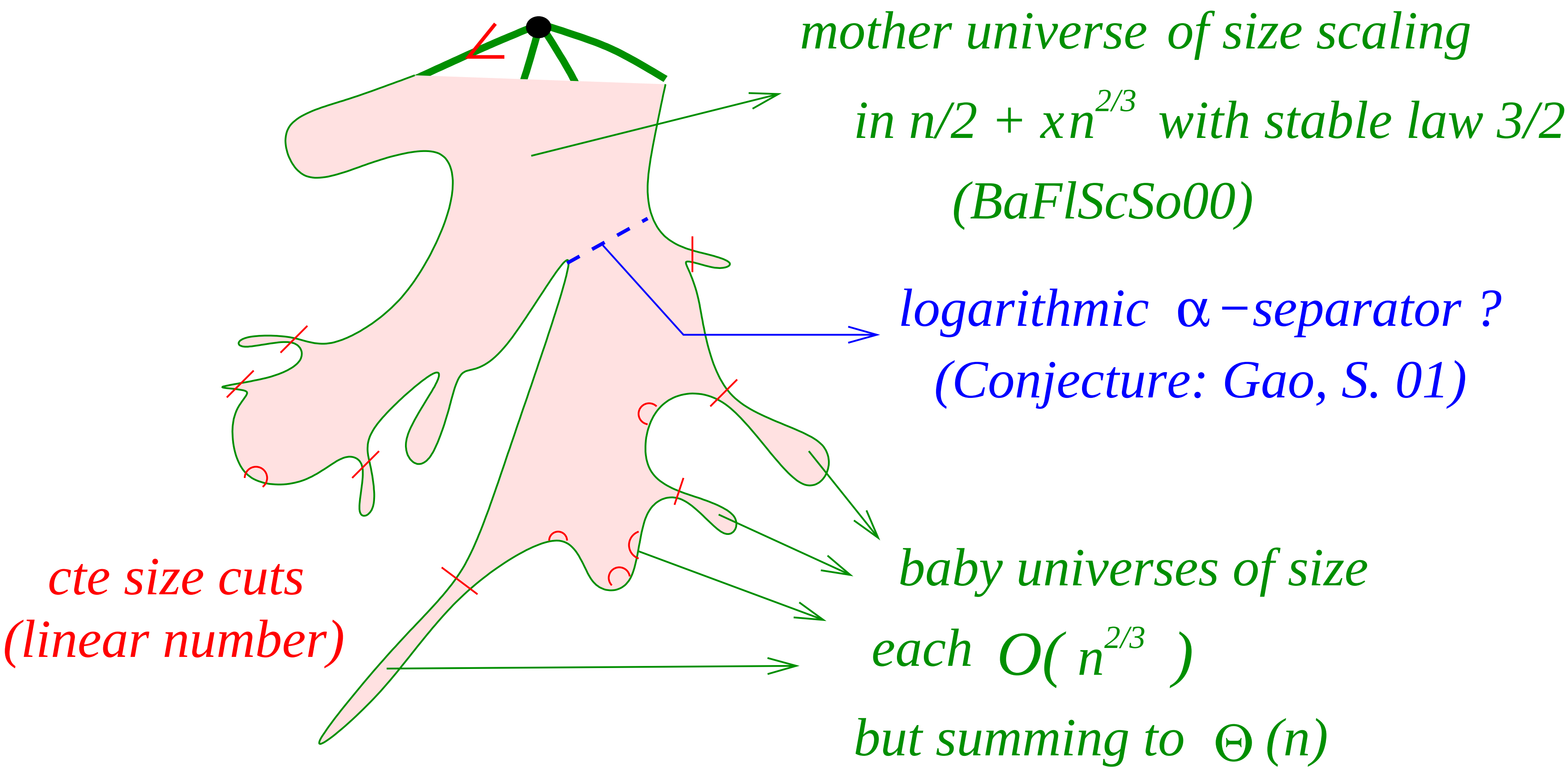
- tends to 0 if $k \ll c_1 \log n$.
- tends to infinity if $k \gg c_1 \log n$.

Maybe false conjecture (Gao-S., 2001):

for $c' > c$, there exists a. s. separators of length at most $c' \log n$.

(This would have interesting algorithmic consequences)

The random planar map. A tentative picture of cuts.



A second summary

- The random planar maps model $\mathcal{R}_n$ has many variants (triangulations, regular maps, convex polyhedra,...)

- But all *knows* results satisfy some "universality": critical exponents agree for different families; for a functional ϕ,

$$\text{if} \quad \mathbb{E}(\phi(\mathcal{R}_n)) \sim c_1 n^{\alpha}, \quad \text{then} \quad \mathbb{E}(\phi(\mathcal{T}_n)) \sim c_2 n^{\alpha}.$$

- This situation is reminiscent, *e.g.* of
  – simple random walks: universality with respect to increments,
  – Galton-Watson trees: universality w.r.t. progeny distribution.

A continuum random map ?

What is hidden behind these universalities ?

- "regardless" of increments:
  normalised simple random walk $\bar{W}_n(t) \longrightarrow$ Brownian motion W_t,
  $$\forall \phi, \qquad \phi(\bar{W}_n) \to \phi(W_t).$$

- "regardless" of progeny distribution:
  normalised simple trees $\bar{T}_n \longrightarrow$ Aldous' Continuum Random Tree $\mathcal{S}$,
  $$\forall \phi, \qquad \phi(\bar{T}_n) \to \phi(\mathcal{S}).$$

A central (open) question is thus that of a Continuum Random Map $\mathcal{R}$

$$\text{such that,} \quad \forall \phi, \qquad \phi(\bar{R}_n) \to \phi(\mathcal{R}).$$