Fighting Fish:

enumerative properties

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LIX, CNRS and École Polytechnique

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Summary of the talk

Fighting fish, a new combinatorial model
of discrete branching surfaces

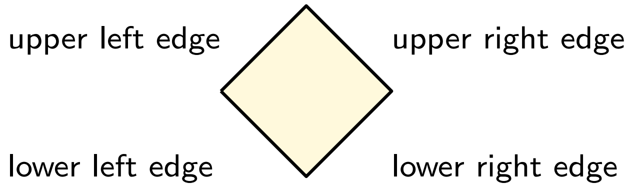
Exact counting formulas for fighting fish
with a glipse of the proof

Fighting fish VS classical combinatorial structures
a bijective challenge...

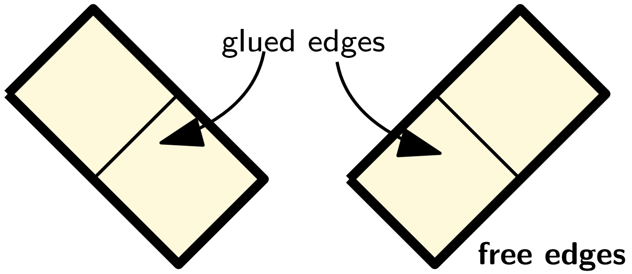
Fighting fish, definition

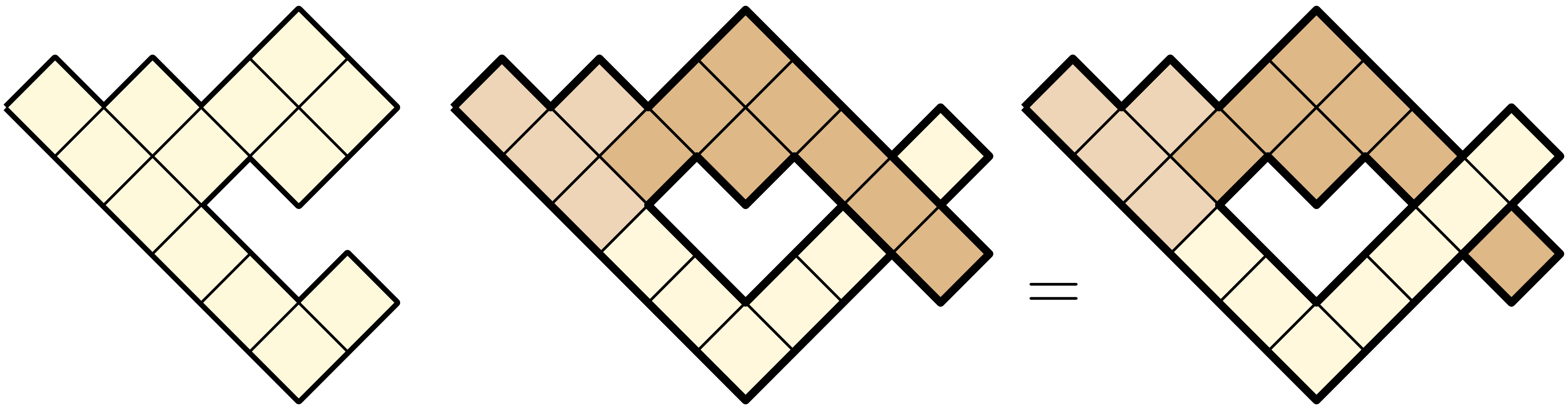
Cells

45^o tilted unit square
(of thin paper or cloth)



Build surface by gluing cells along edges in a coherent way: upper left with lower right or lower left with upper right.

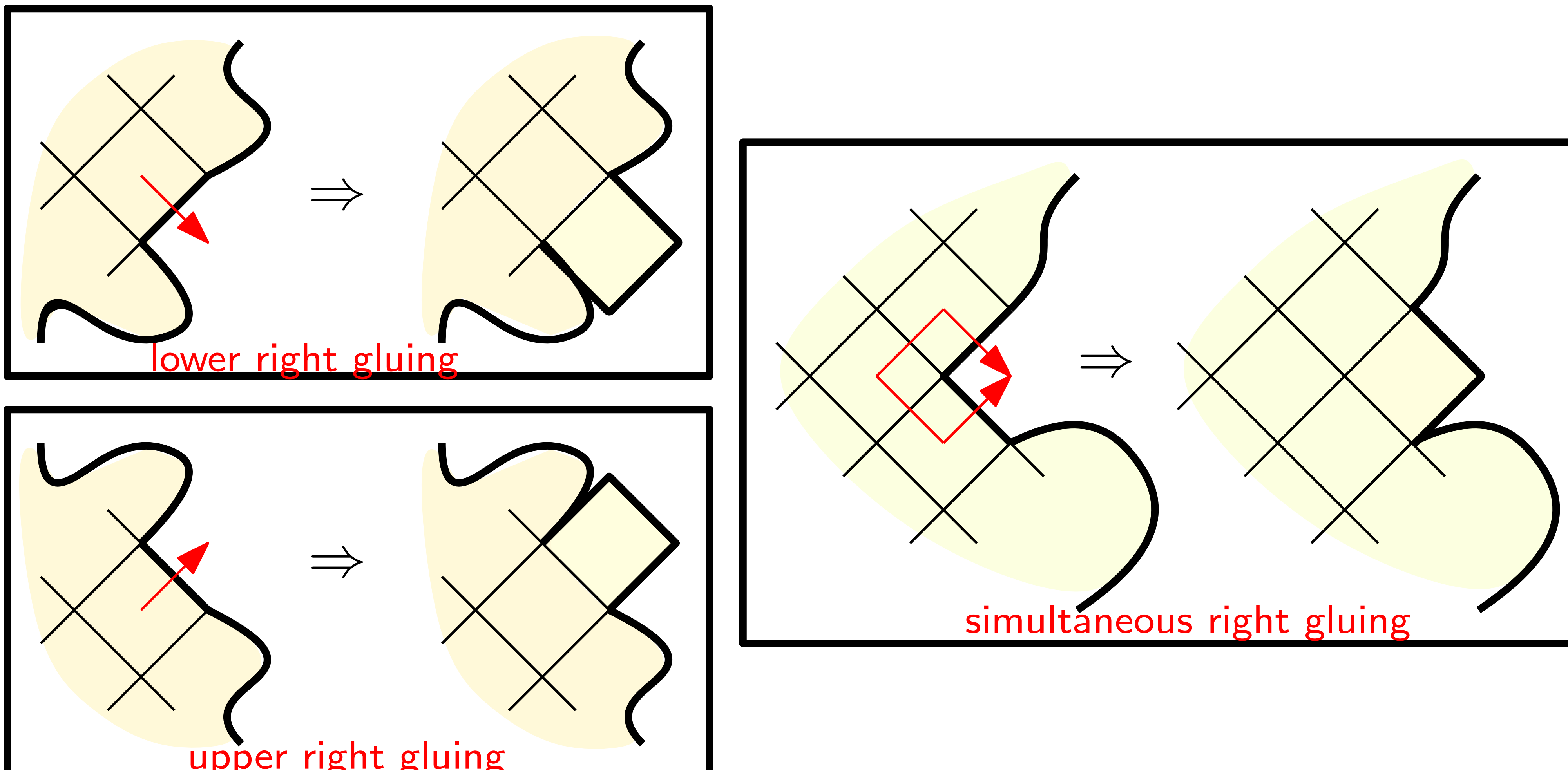




These objects do not necessarily fit in the plane so my pictures are projections of the actual surfaces: Apparently overlapping cells are in fact independant.

Fighting fish, definition

Directed cell aggregation. Restrict to only three legal ways to add cells: by lower right gluing, upper right gluing, or simultaneous lower and upper right gluings from adjacent free edges.

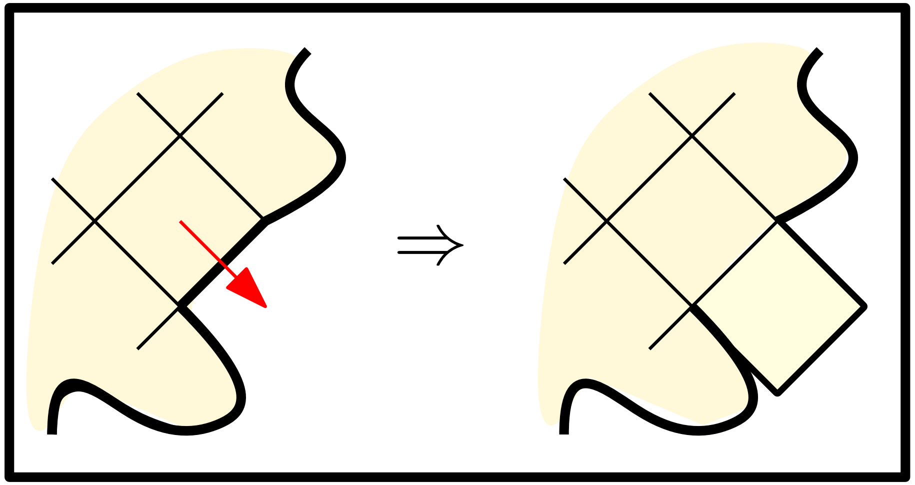


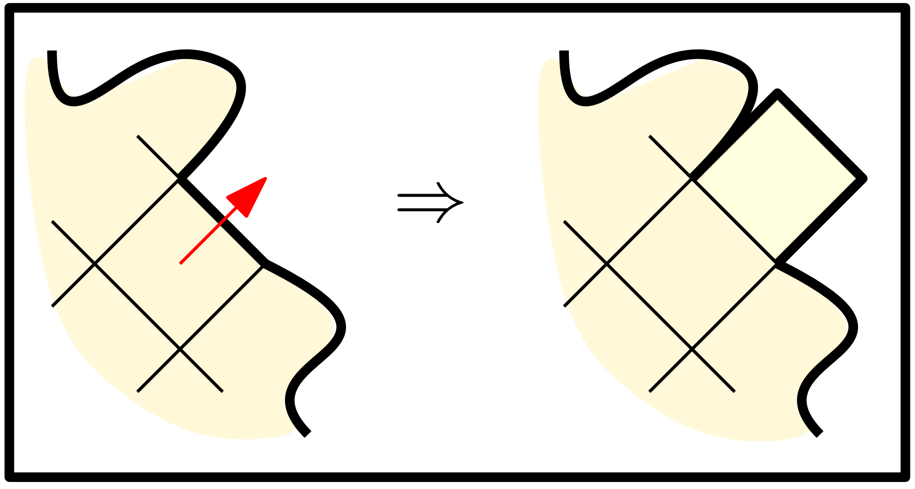
Fighting fish, definition

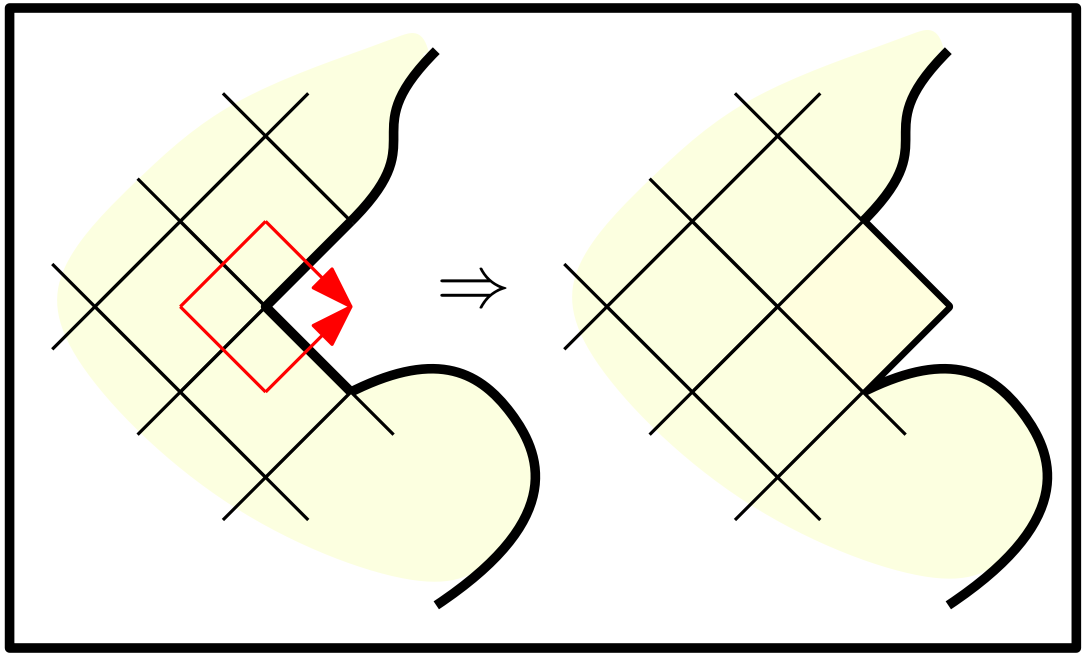
Lemma. Single cell + aggregations
$\Rightarrow$ a simply connected surface

Proposition. Such surfaces can be recovered from their boundary walk.

(not used later)



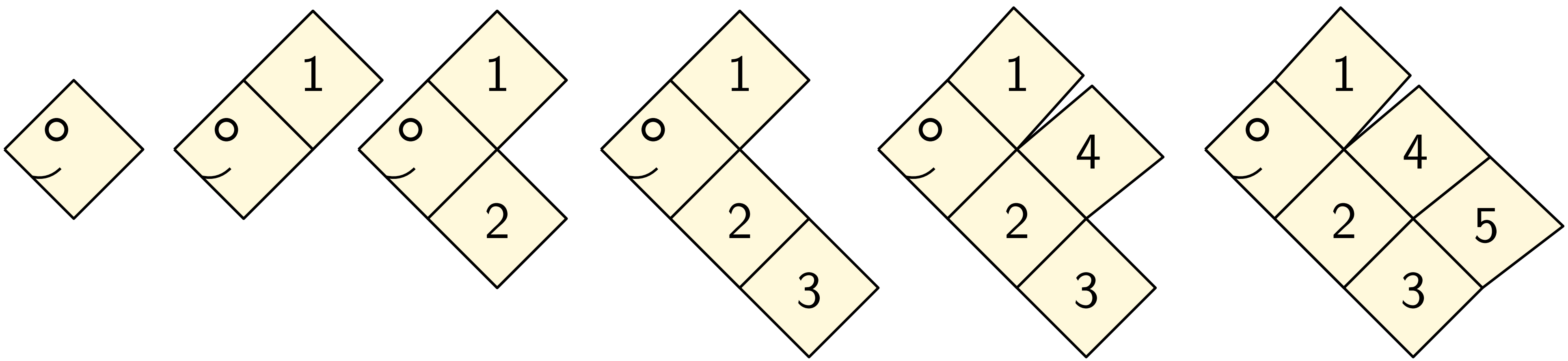




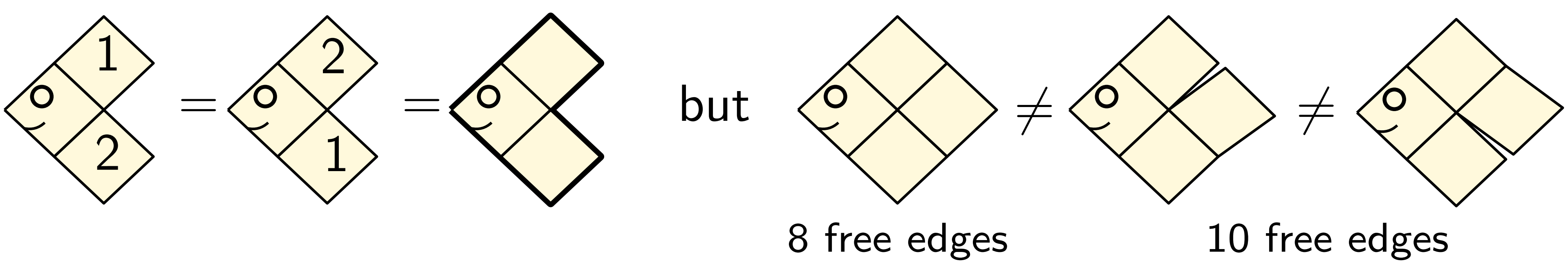
Fighting fish, definition

Fighting fish

A fighting fish is a surface that can be obtained from a single cell by a sequence of directed cell agregations.

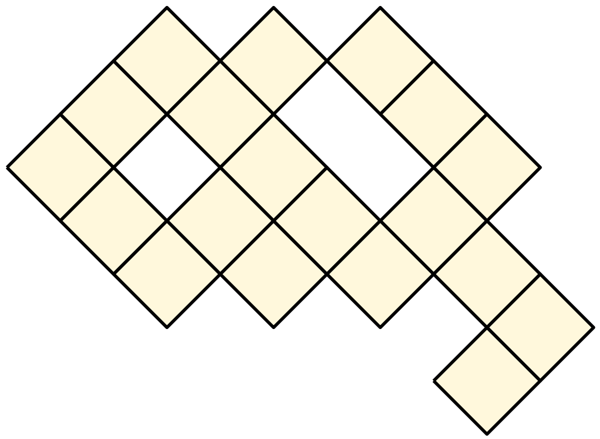


We are interested only in the resulting surface, not in the aggregation order (but type of aggregation matters)



Fighting fish versus polyominoes

Polyomino = edge-connected set of cells **of the planar square lattice**



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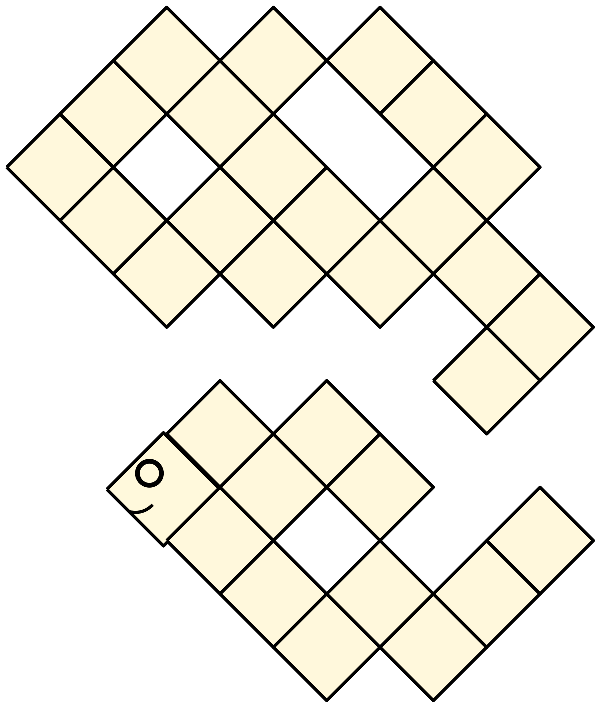
Directed polyominoes: there is a cell, the head, from which all cells can be reached by a left-to-right path.



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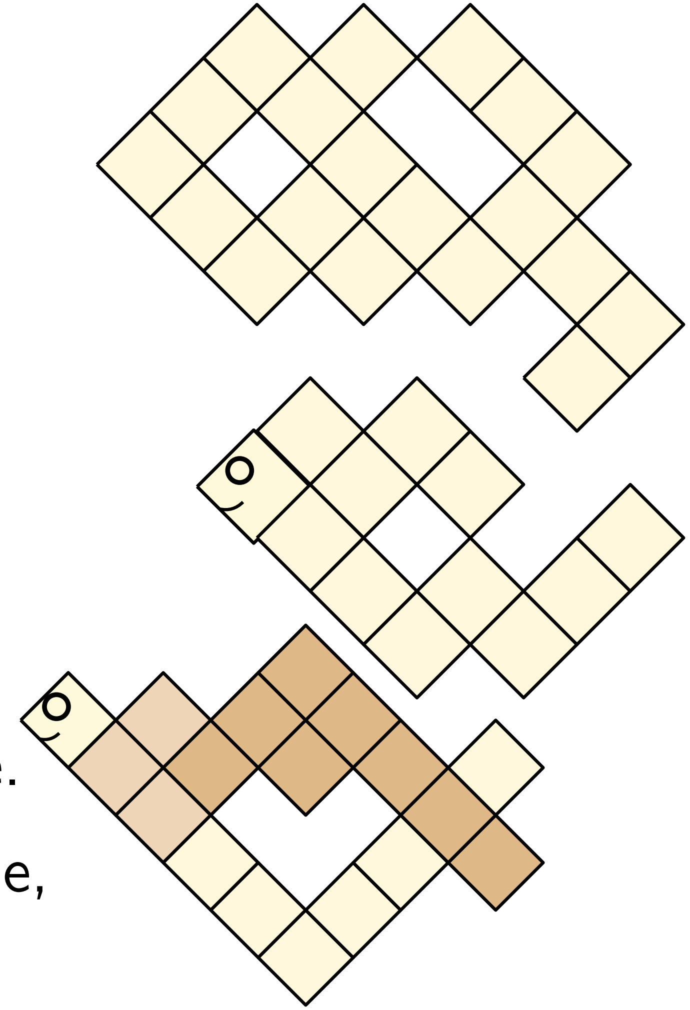
Proposition.
A fighting fish is a directed polyomino **iff** its projection in the plane is injective.

$\Rightarrow$ fighting fish do not all fit in the plane, *ie* they are not all polyominoes.



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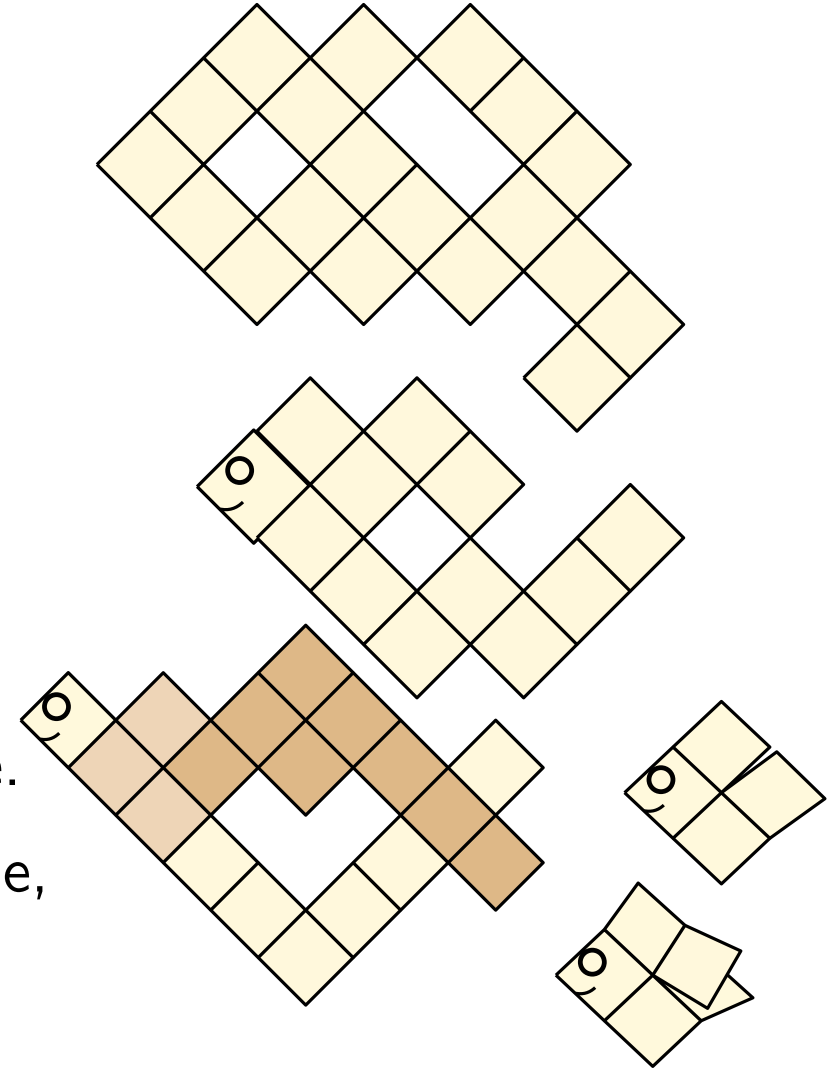

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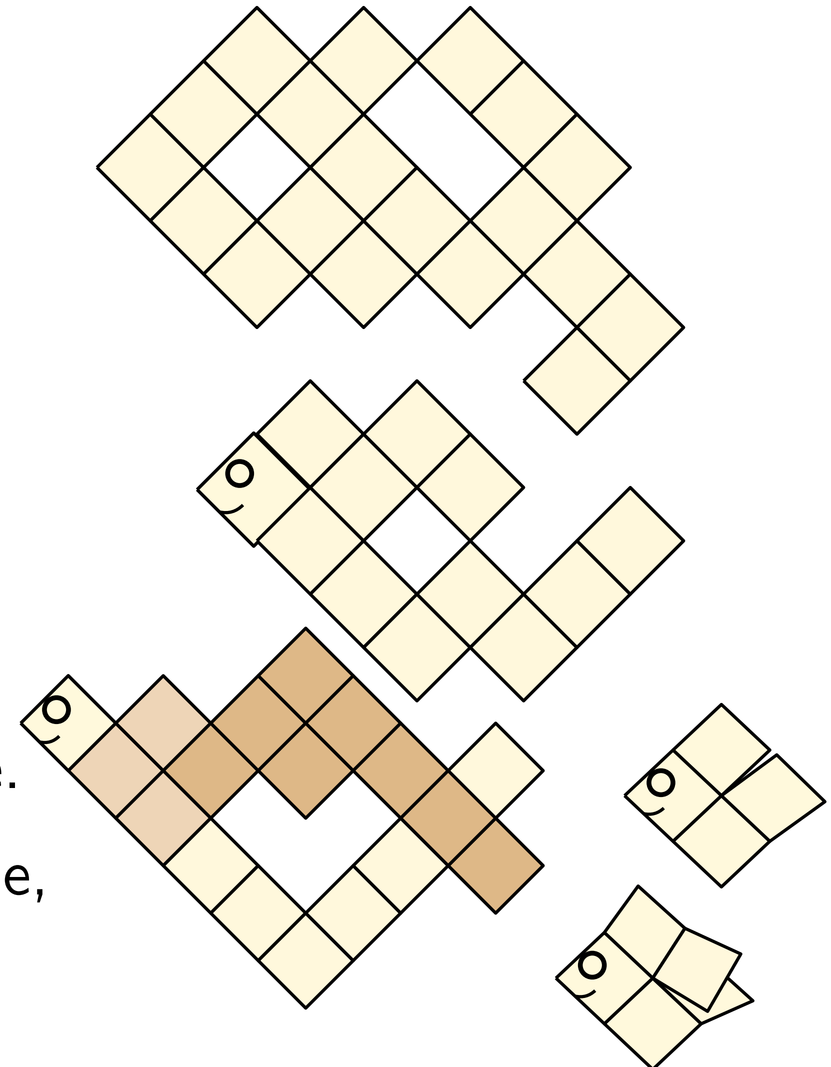

Conversely there are polyominoes that are not fighting fish:

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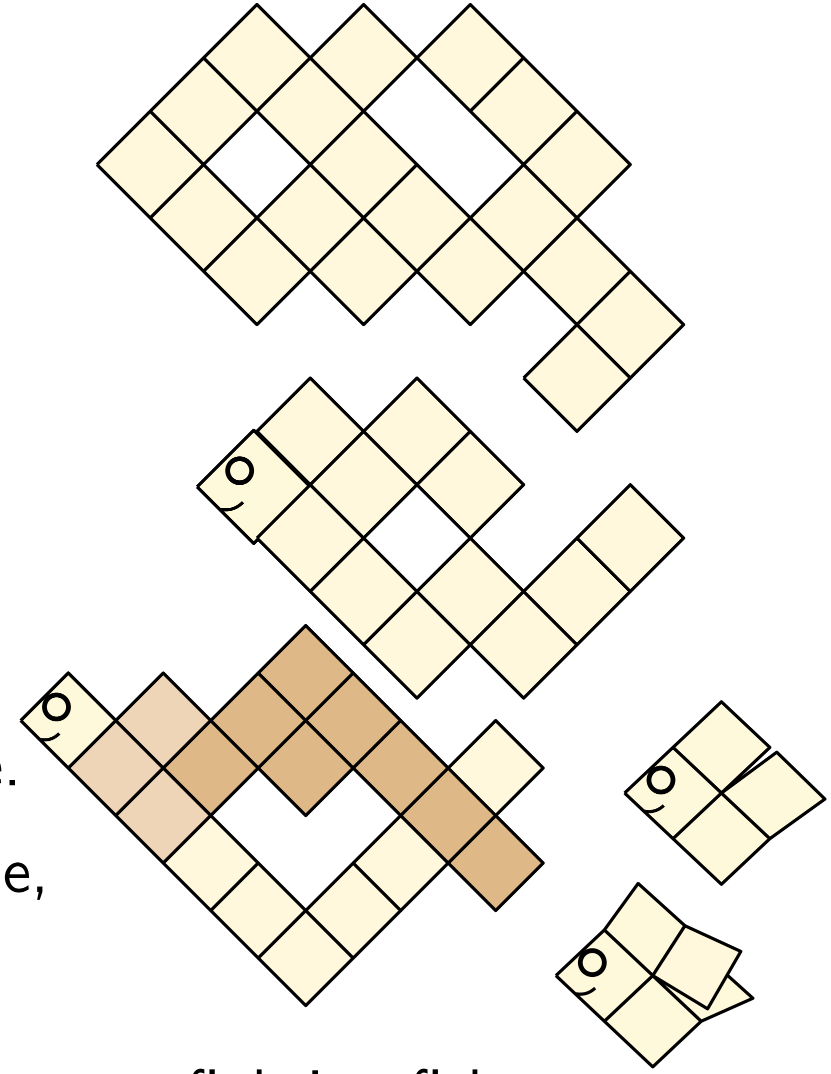

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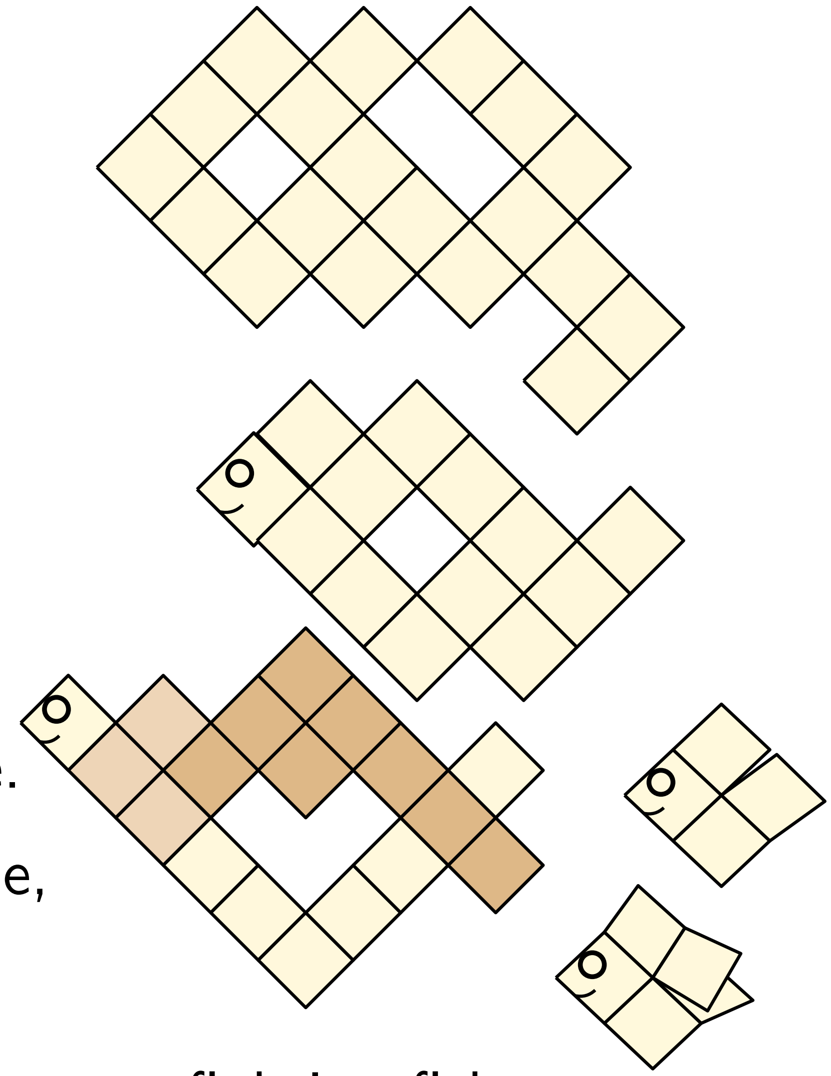

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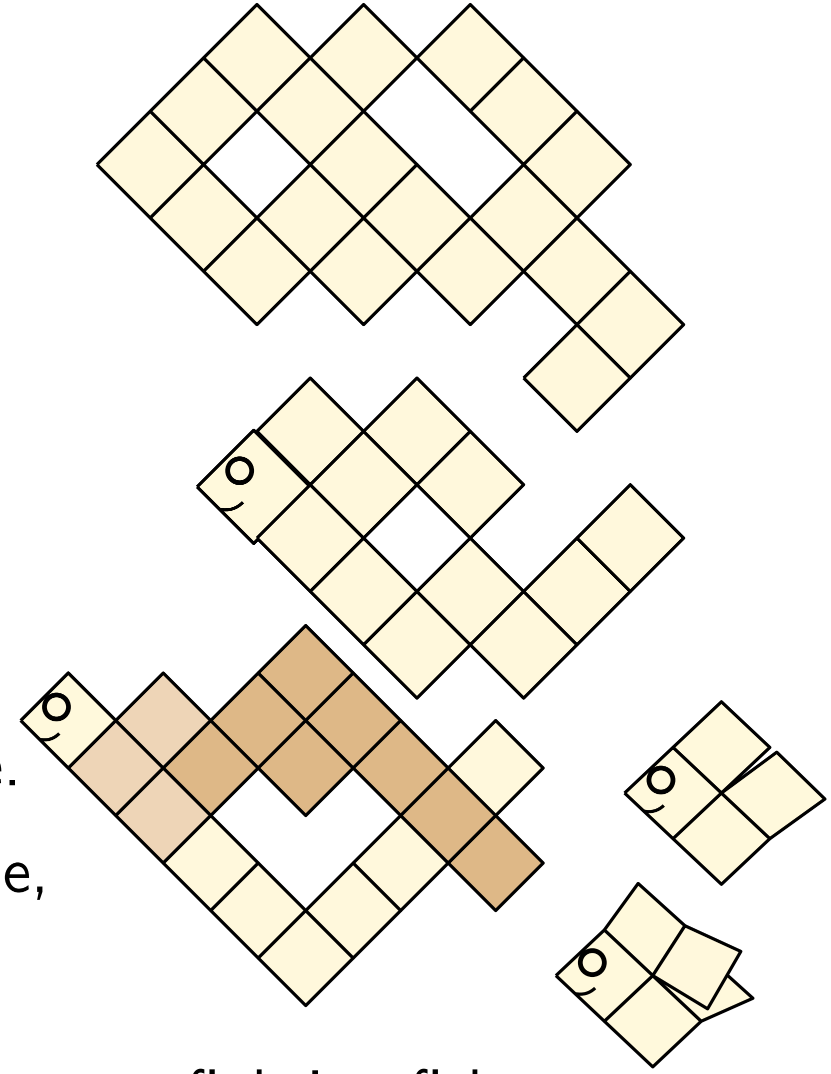

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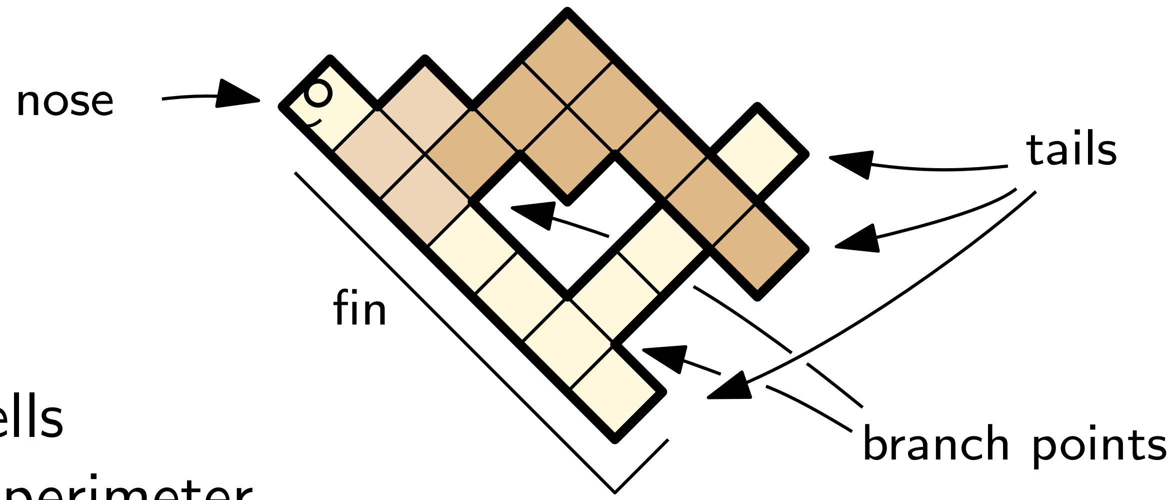
In particular all directed convex polyominoes are fighting fish.

Parameters of fighting fish



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Size = semi-perimeter

= #{upper free edges}

= #{upper left free edges} + #{upper right free edges}

The fin length = #{ lower free edges from head to first tail }

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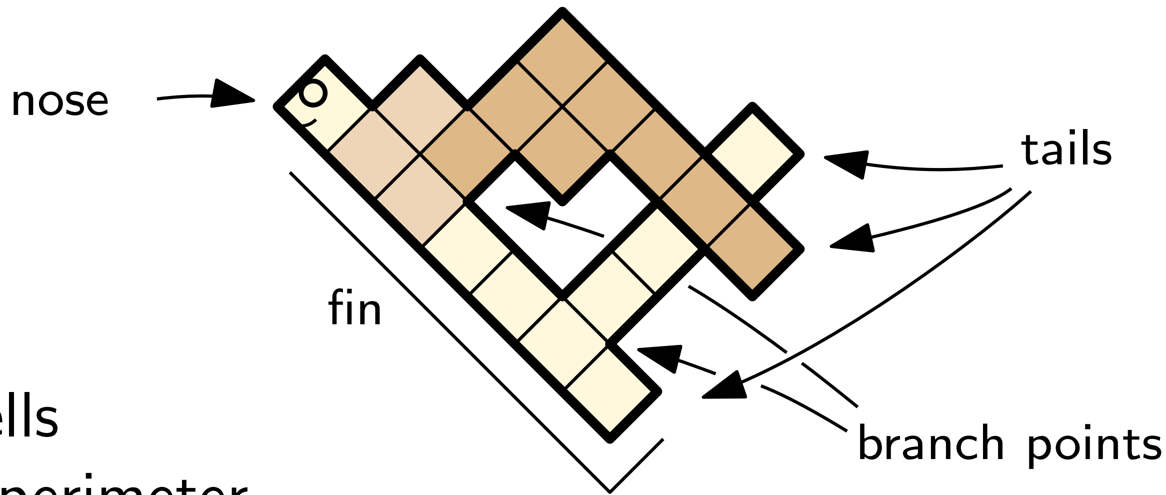

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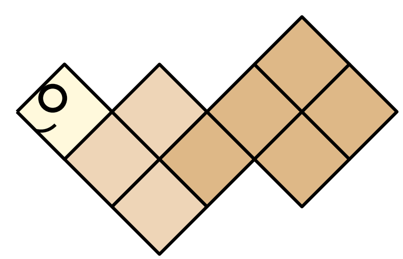
Fighting fish with exactly 1 tail



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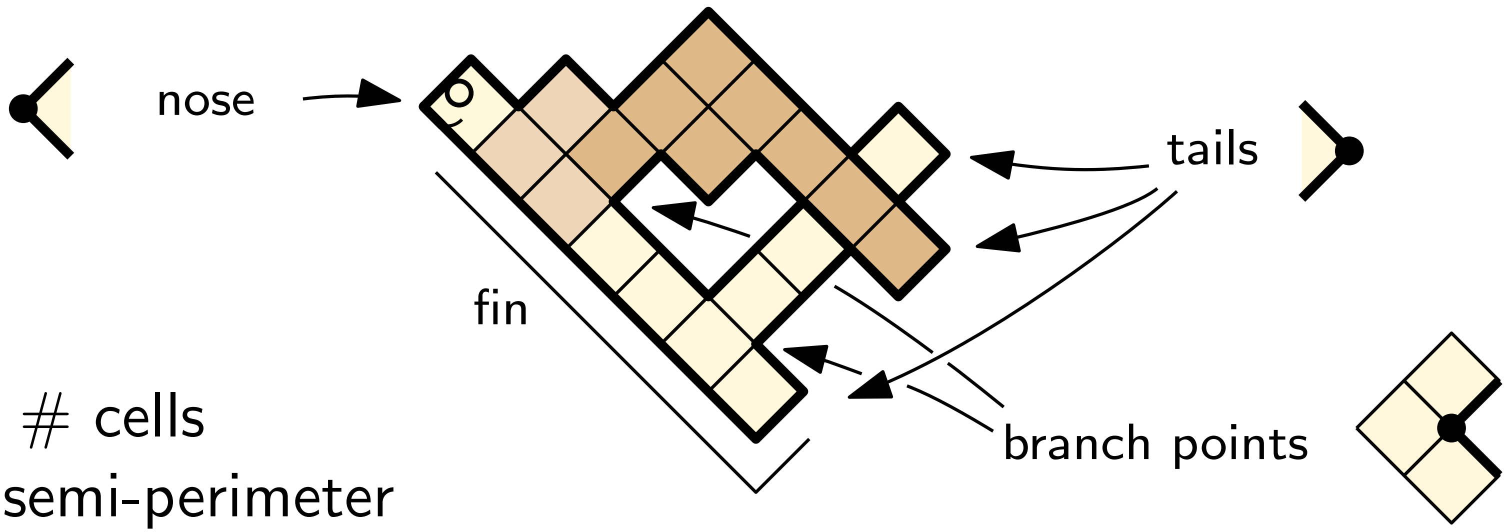

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Size $=$ semi-perimeter

$=$ #{upper free edges}

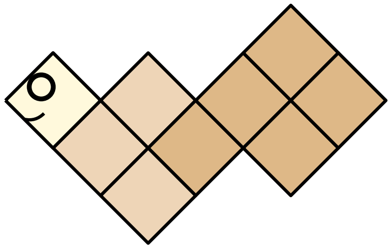
$=$ #{upper left free edges} $+$ #{upper right free edges}

The fin length $=$ #{ lower free edges from head to first tail }

Fighting fish with exactly 1 tail

$=$ parallelogram polyominoes

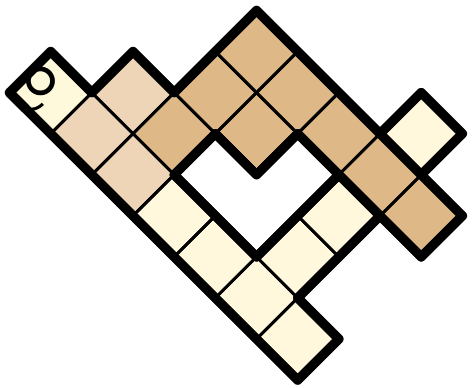
aka staircase polygons



in this case, fin length $=$ semi-perimeter

Fighting fish as random branching surfaces

Let F_n be a fighting fish taken uniformly at random among all fighting fish of size n. (F_n is called a URF of size n)



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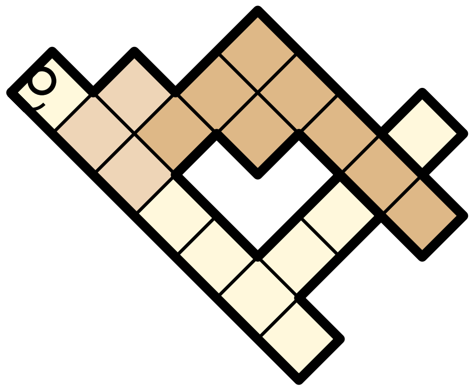

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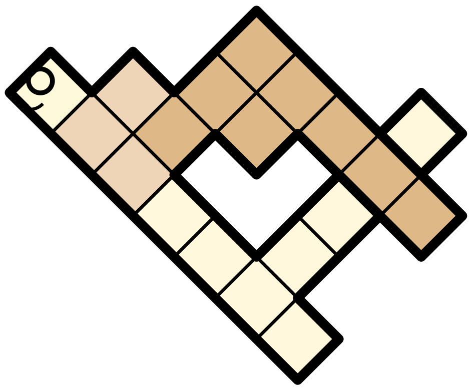

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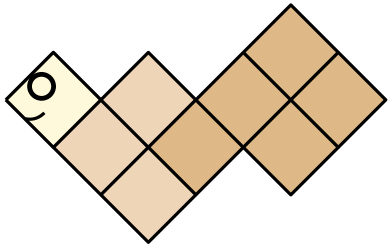

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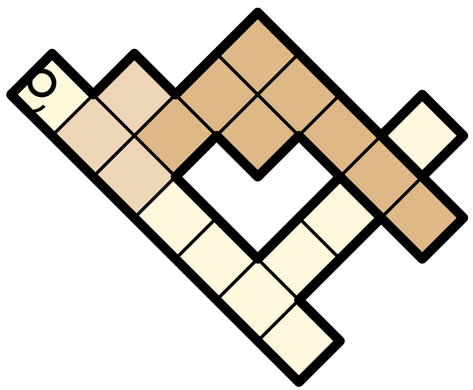

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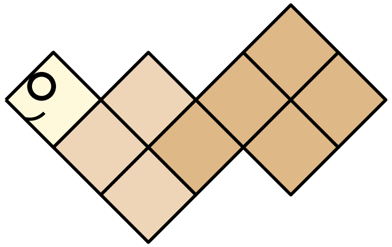

The Uniform Random fighting Fish of size n (URF) yields a new model of random branching surfaces with original features.

Enumerative results

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Theorem (folklore)

$$\#\left\{\begin{array}{c}\text{parallelogram polyominos}\\ \text{with semi-perimeter } n+1\end{array}\right\} = \frac{1}{2n+1}\binom{2n}{n}$$

$$\#\left\{\begin{array}{c}\text{parallelogram polyominos with}\\ i \text{ top left and } j \text{ top right edges}\end{array}\right\} = \tfrac{1}{i+j-1}\tbinom{i+j-1}{i}\tbinom{i+j-1}{j}$$

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fighting fish with 1 tail

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Theorem (Duchi, Guerrini, Rinaldi, S. 2016)

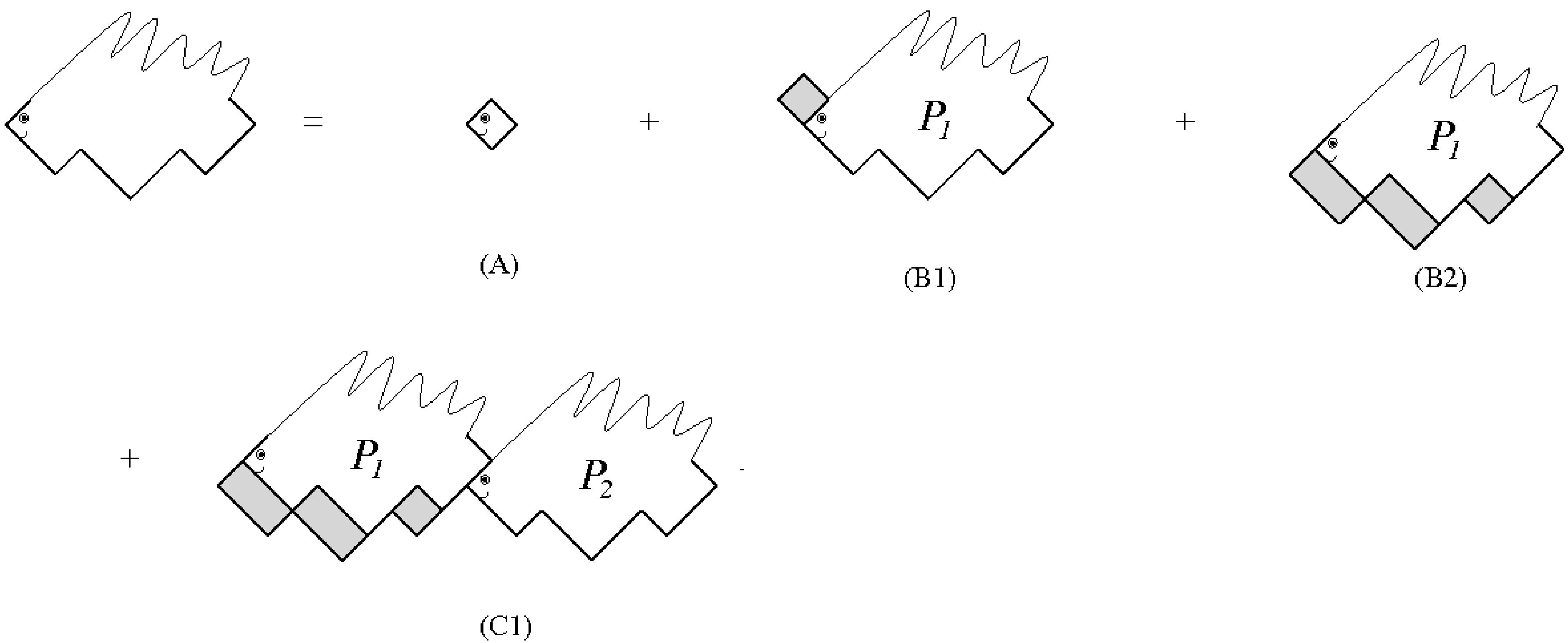
$$\#\left\{\begin{array}{c}\text{fighting fish}\\ \text{with semi-perimeter } n+1\end{array}\right\} = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

$$\#\left\{\begin{array}{c}\text{fighting fish with}\\ i \text{ top left and } j \text{ top right edges}\end{array}\right\} = \frac{1}{(2j+j-1)(2j+i-1)}\binom{2i+j-1}{i}\binom{2j+i-1}{j}$$

A glipse of the proof

Extend the *wasp-waist decomposition* of parallelogram polyominoes:

remove one cell at the bottom of each diagonal, from left to right along the fin, until this creates a cut



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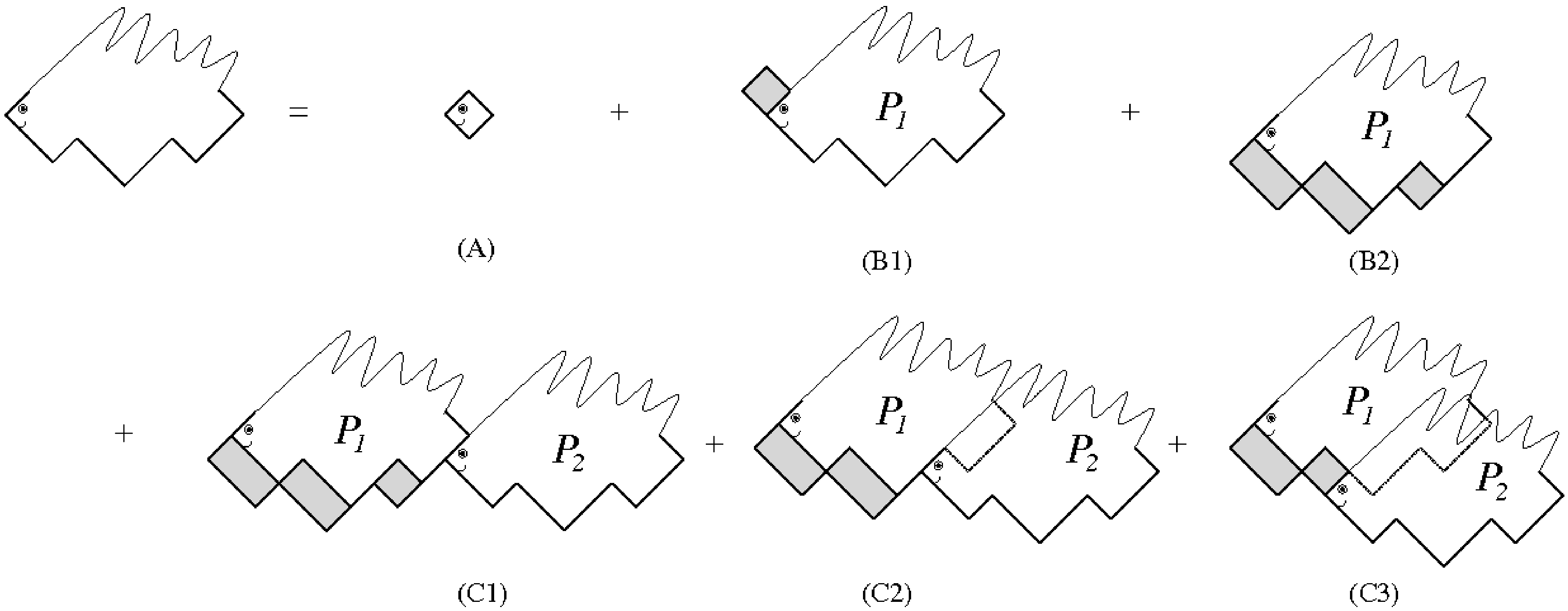

Two more cases must be considered for fighting fish...

A glipse of the proof

Let $P(u) = \sum_f t^{|f|} u^{\mathrm{fin}(f)} y^{\mathrm{tail}(f)}$ be the GF of fighting fish according to the size, fin length and number of extra tails. Then

$$P(u) = tu(1 + P(u))^2 + ytuP(u)\frac{P(1) - P(u)}{1 - u}$$

.

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Case $y = 0$. Fish with one tail, *ie* parallelogram polyominoes:
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General case. A polynomial equation with one catalytic variable:
easily solved using Bousquet-Mélou-Jehanne approach.

$\Rightarrow$ an algebraic equation that generalizes the equation for parallelogram polyominoes to an arbitrary number of tails.

Corollary (DGRS 2016). The gf of fighting fish with k tails for any fixed k is a rational function of the Catalan GF.

Bijections and parameter equidistributions?

Sloane's OEIS...

$$\#\left\{\begin{array}{c}\text{fighting fish}\\ \text{with semi-perimeter } n+1\end{array}\right\} = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

1, 2, 6, 91, 408, 1938...

This integer sequence was already in Sloane's !

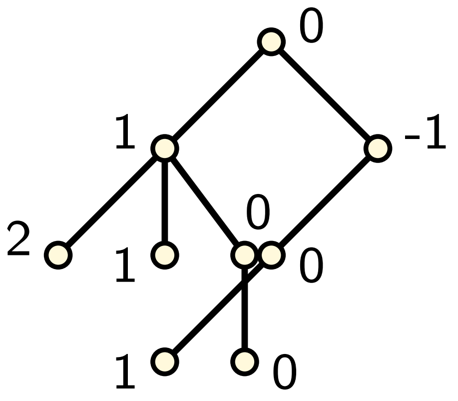
The number of fighting fish of size $n+1$ (with i left and j down top edges) is equal to the number of:

- Two-stack sortable permutations of $\{1,\ldots,n\}$ (i ascending and j descending runs) (West, Zeilberger, Bona, 90's)
- Rooted non separable planar maps with n edges ($i+1$ vertices, $j+1$ faces) (Tutte, Mullin and Schellenberg, 60's)
- Left ternary trees with n edges ($i+1$ even, j odd vertices) (Del Lungo, Del Ristoro, Penaud, late XXth century)

Left ternary trees and further equidistributions

Natural embedding of a ternary tree:

- root vertex has label 0
- vertex with label $i \Rightarrow$ left child $i-1$,
  central child i, right child $i+1$.



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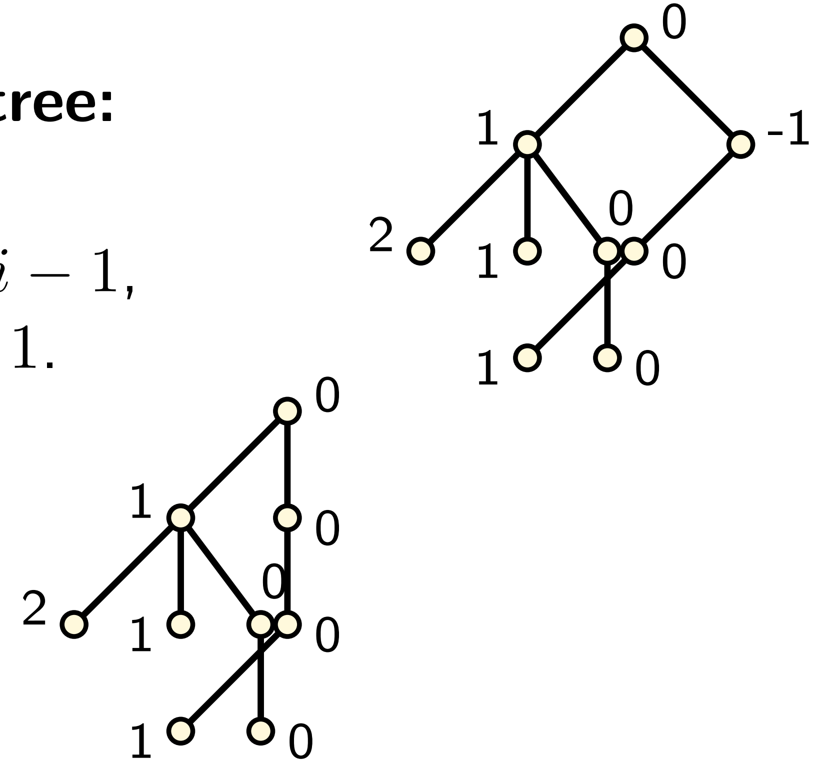

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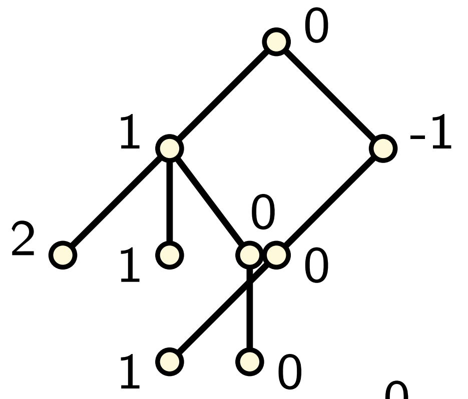

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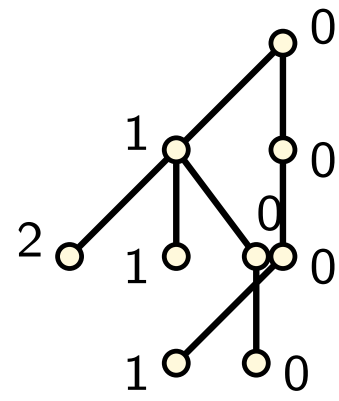
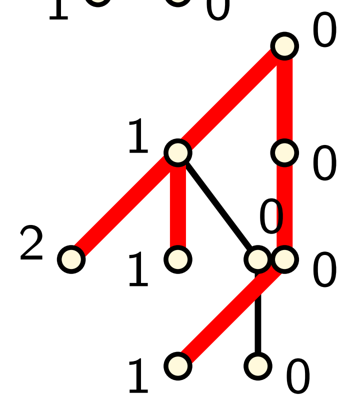
Core $=$ binary subtree of the root after pruning all right edges





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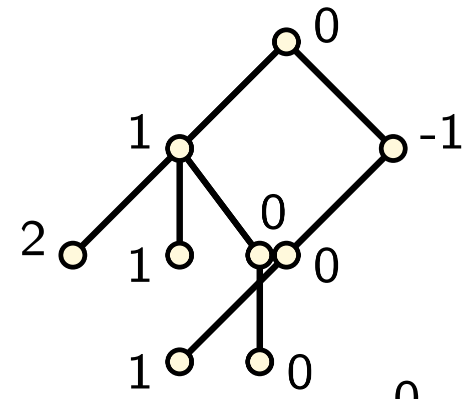

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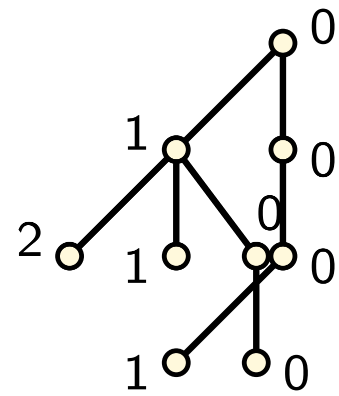
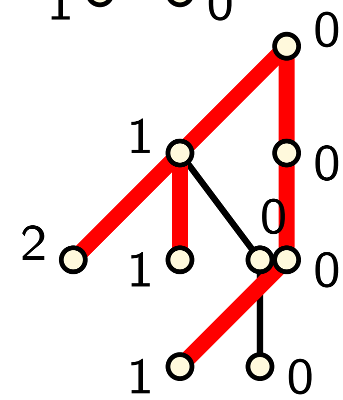
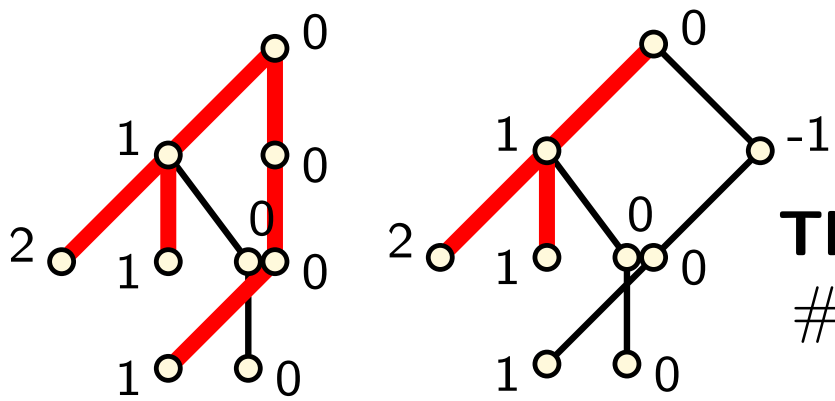




Theorem (DGRS 2016): The number of fighting fish with size $n+1$ and fin length k equals the number of left ternary trees with n nodes and core size k.

Left ternary trees and further equidistributions



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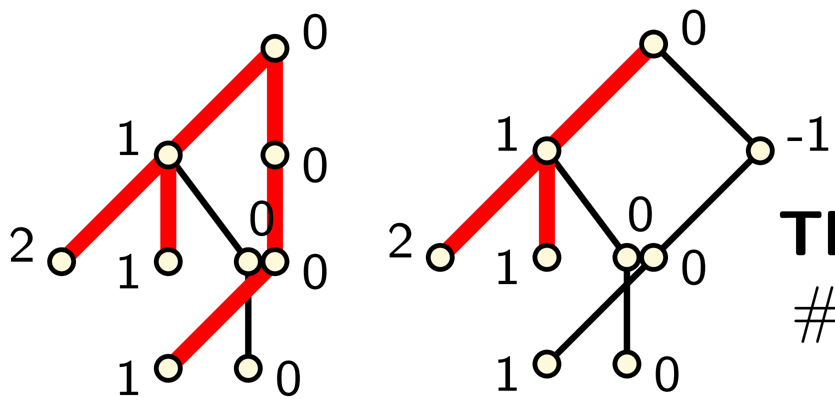

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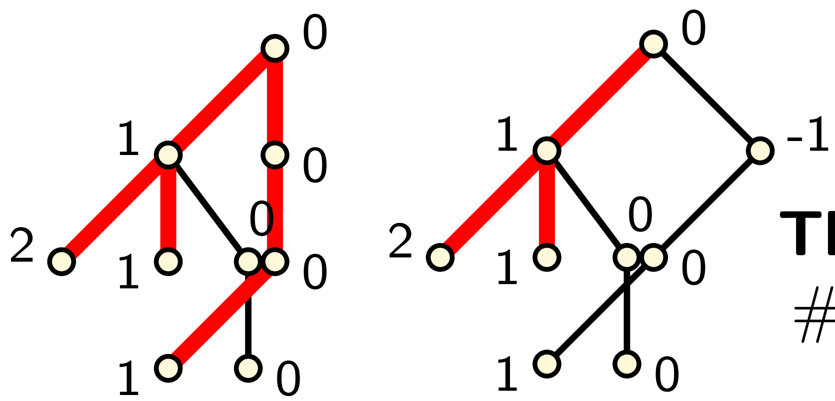
Proof by an explicit guess and check *à la* Di Francesco for the tree GF:

Left ternary trees and further equidistributions



Core $=$ binary subtree of the root after pruning all right edges

Theorem (DGRS 16)
$\#\{$ fighting fish, size $n+1$, fin length k $\}$
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Theorem (Di Francesco 05, Kuba 11) The size GF of ternary trees with label at least $-i$ is

$$\tau_j = \tau \frac{(1-X^{j+5})}{(1-X^{j+4})} \frac{(1-X^{j+2})}{(1-X^{j+3})} \quad \text{where} \left\{ \begin{array}{ccc} \tau & = & 1 + t\tau^3 \\ X & = & (1+X+X^2)\frac{\tau-1}{\tau} \end{array} \right. .$$

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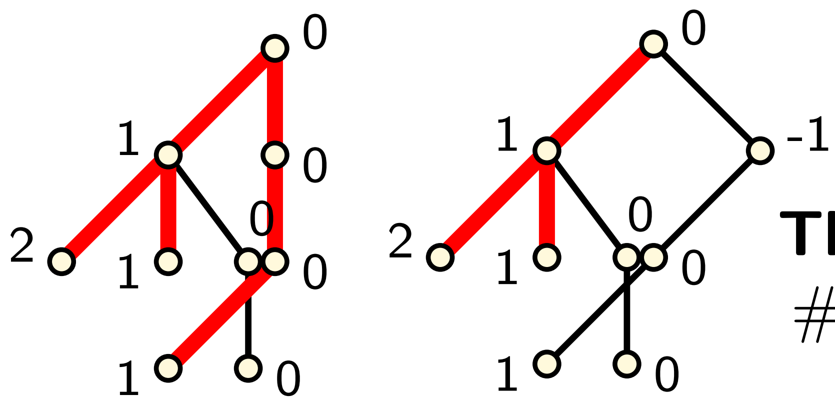

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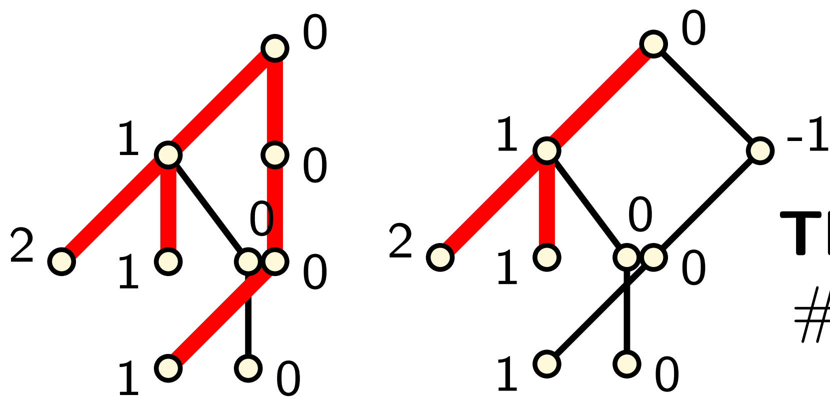

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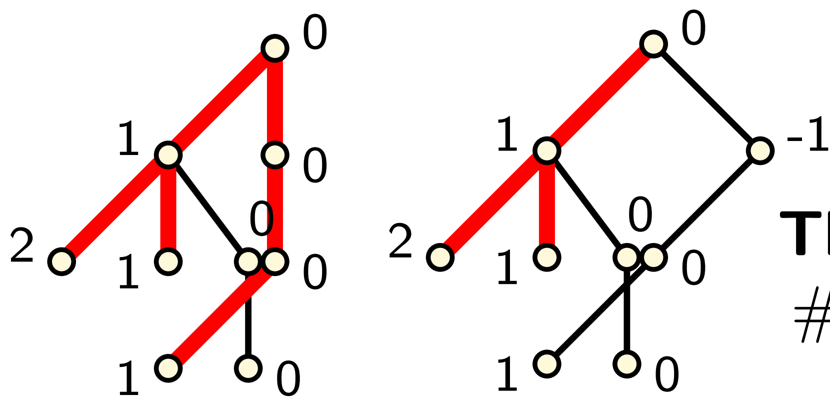
Theorem (DGRS16) The bivariate size and core size GF of ternary trees with label at least $-i$ is

$$T_j(u) = T(u) \frac{H_j(u)}{H_{j-1}(u)} \frac{1-X^{j+2}}{1-X^{j+3}} \text{ where } \left\{ \begin{array}{rl} T(u) & = 1+tuT(u)^3\tau \\ H_j(u) & = (1-X^{j+1})XT(u) \\ & \quad -(1+X)(1-X^{j+2}) \end{array} \right. .$$

Left ternary trees and further equidistributions



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Conjecture (DGRS 2016): The previous computation can be refined to prove joined equidistribution of:

fin length $\leftrightarrow$ core size
number of tails $\leftrightarrow$ number of right branches
number of left/right free edges $\leftrightarrow$ number of even/odd labels

Bijections ?

fighting fish

2SS-permutations

left ternary trees

ns planar maps

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Tutte

recursive decomposition + GF

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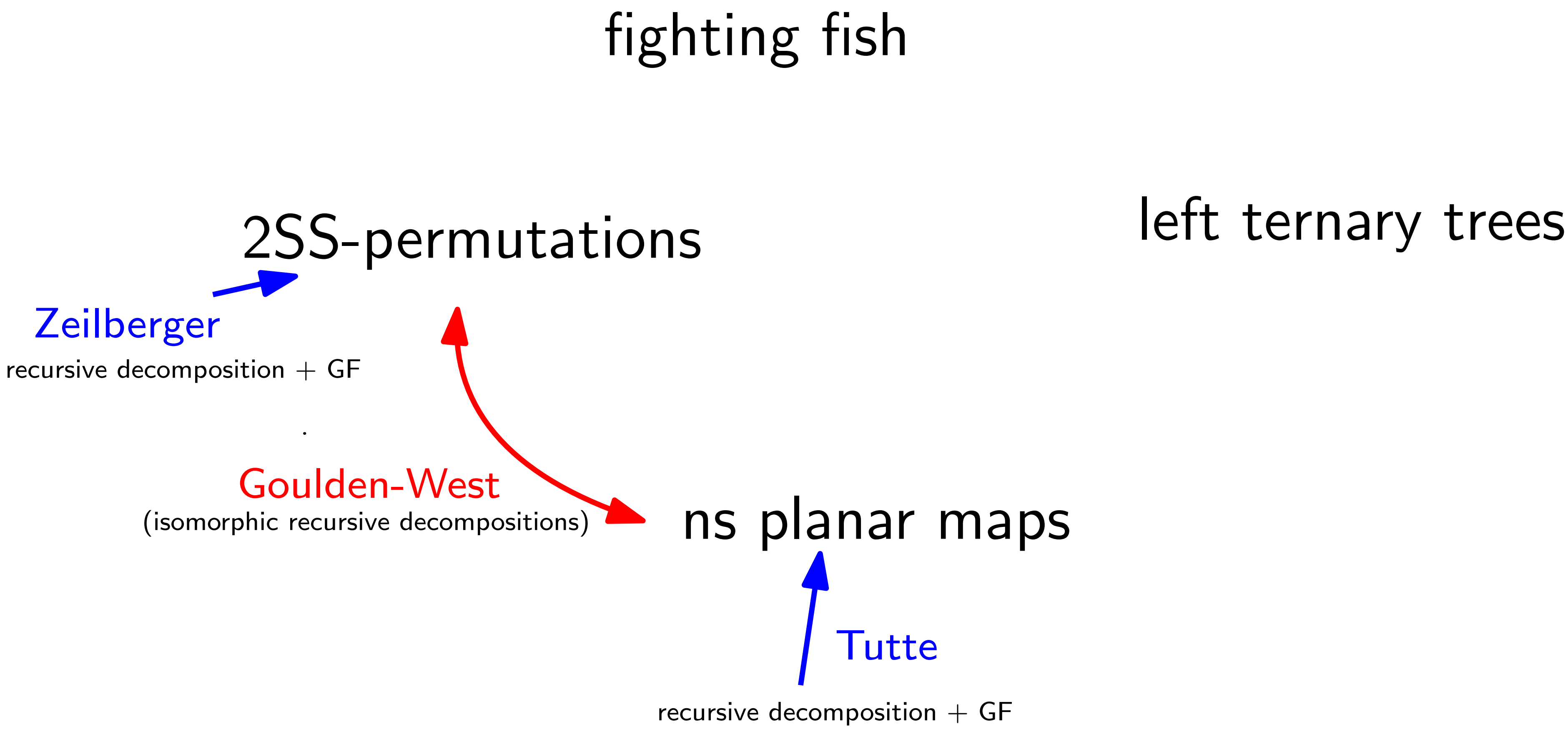

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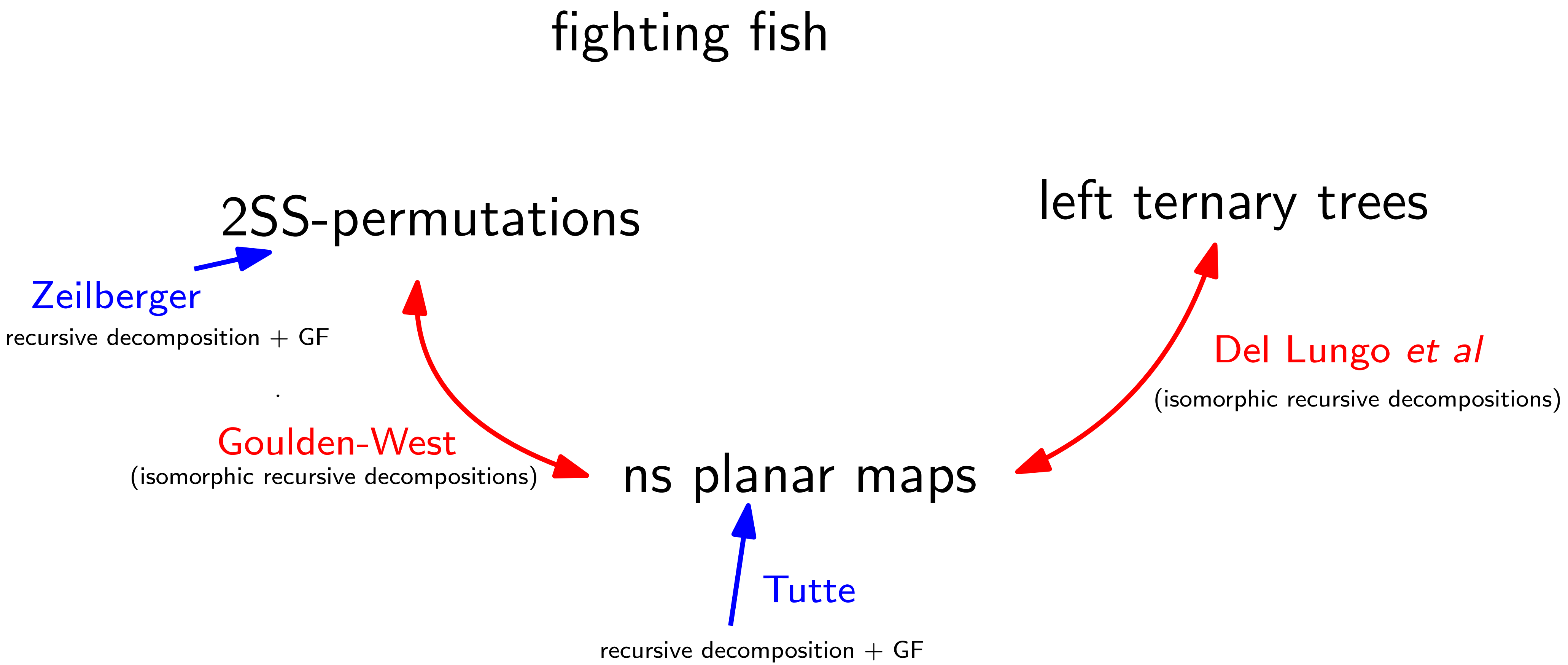
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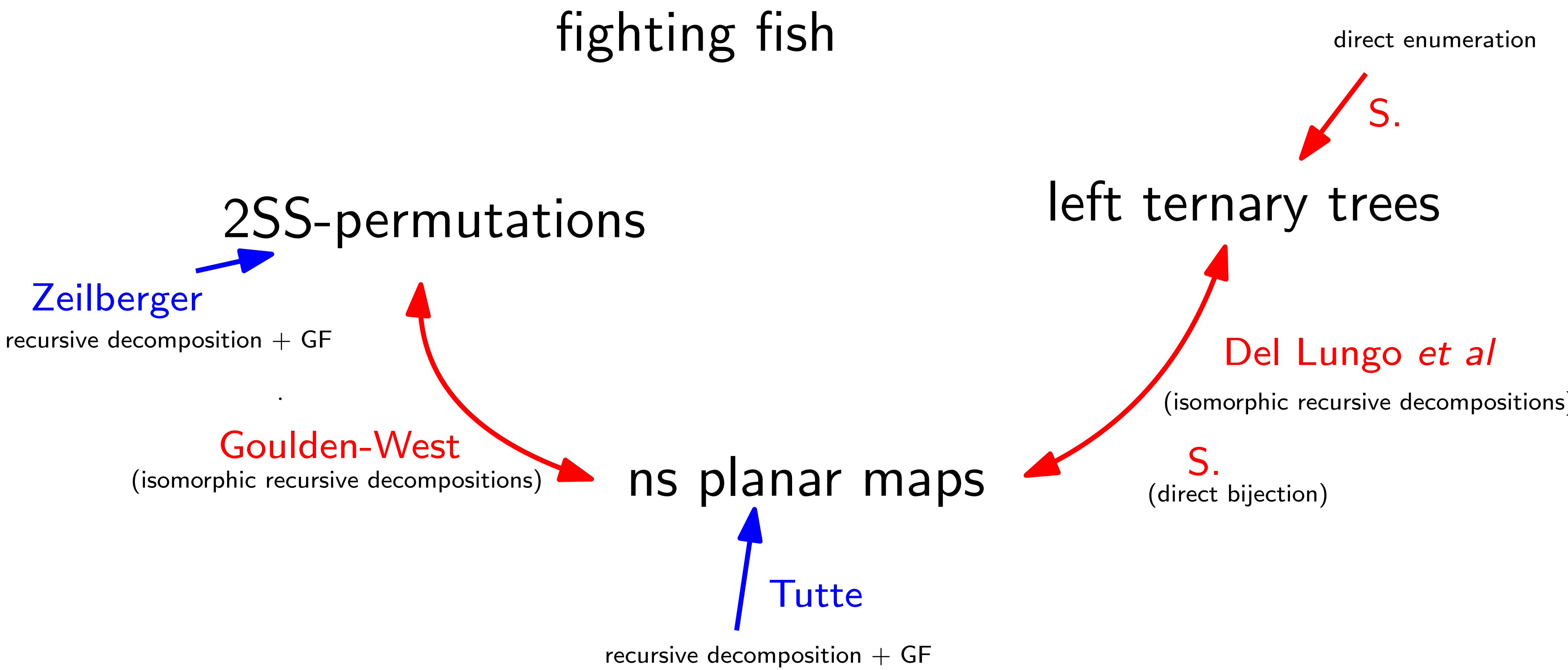
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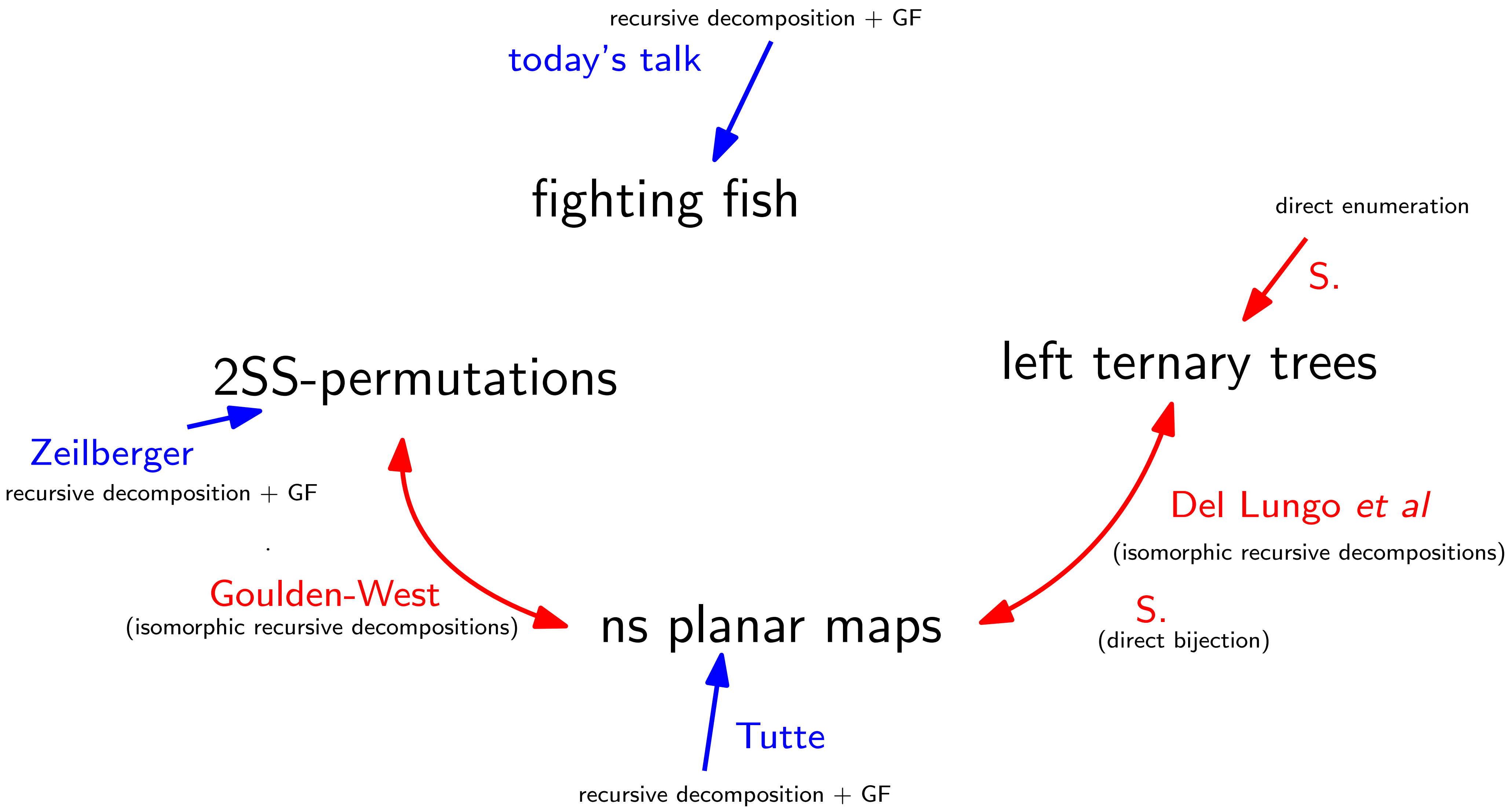
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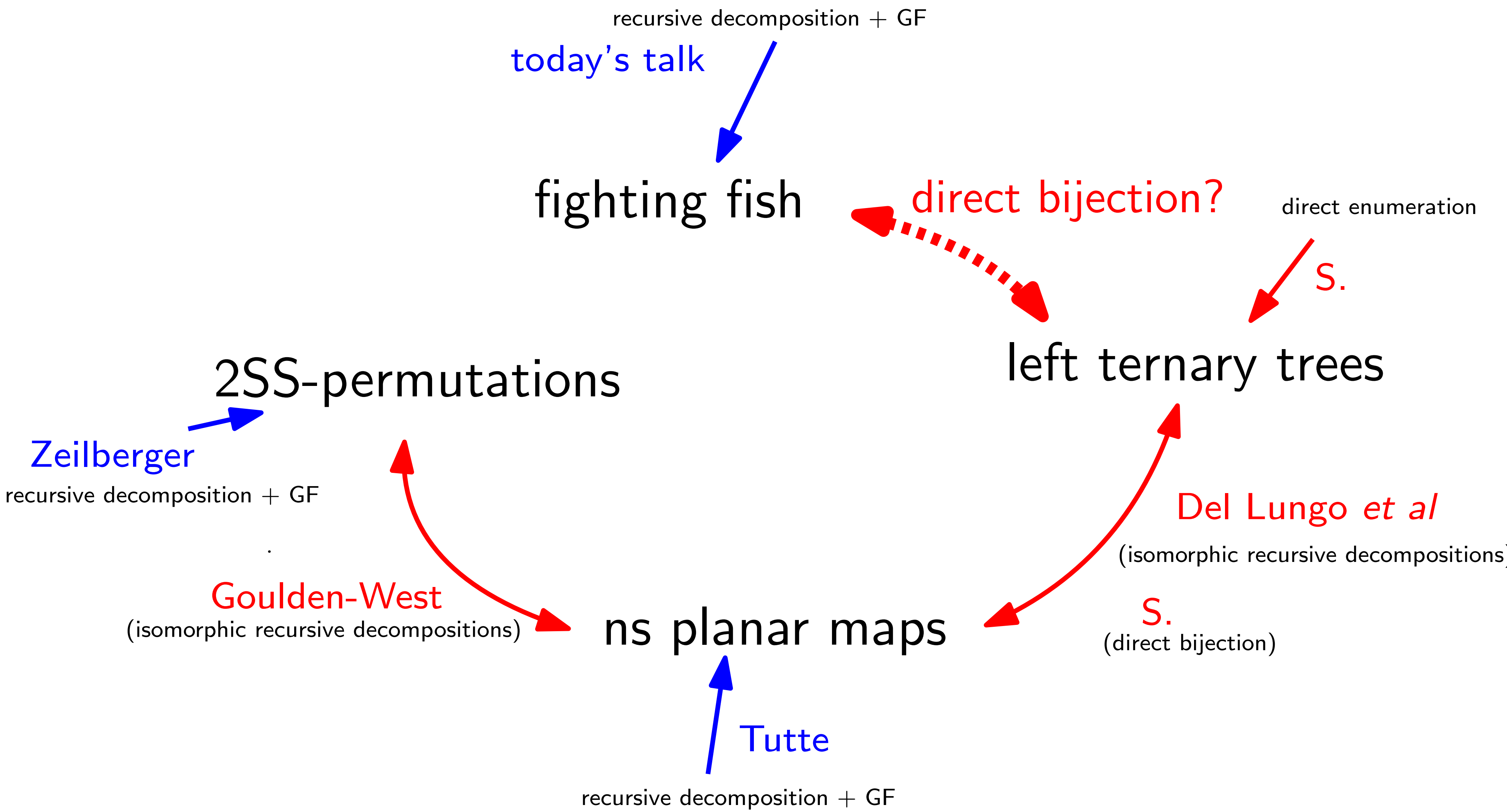
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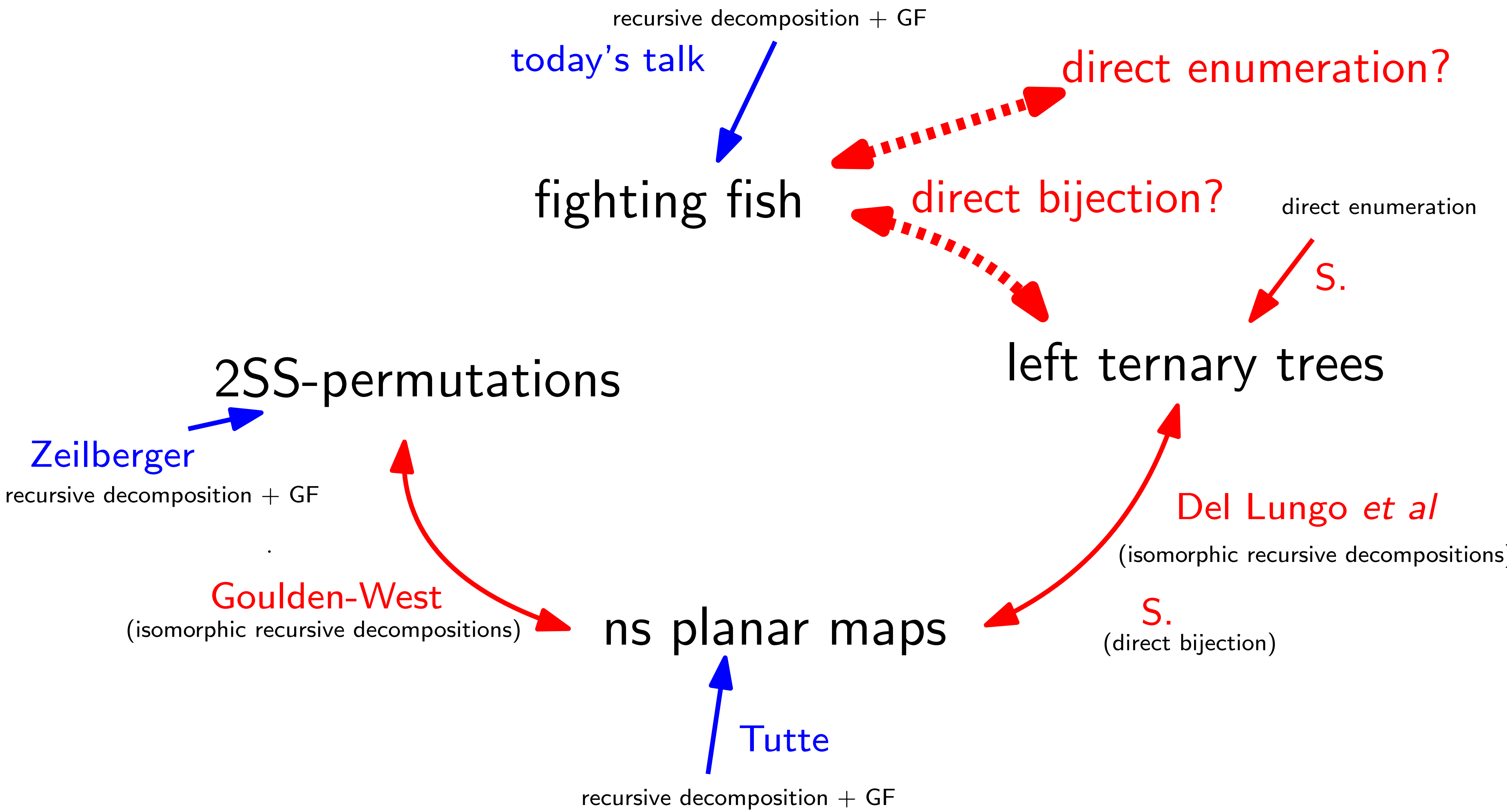
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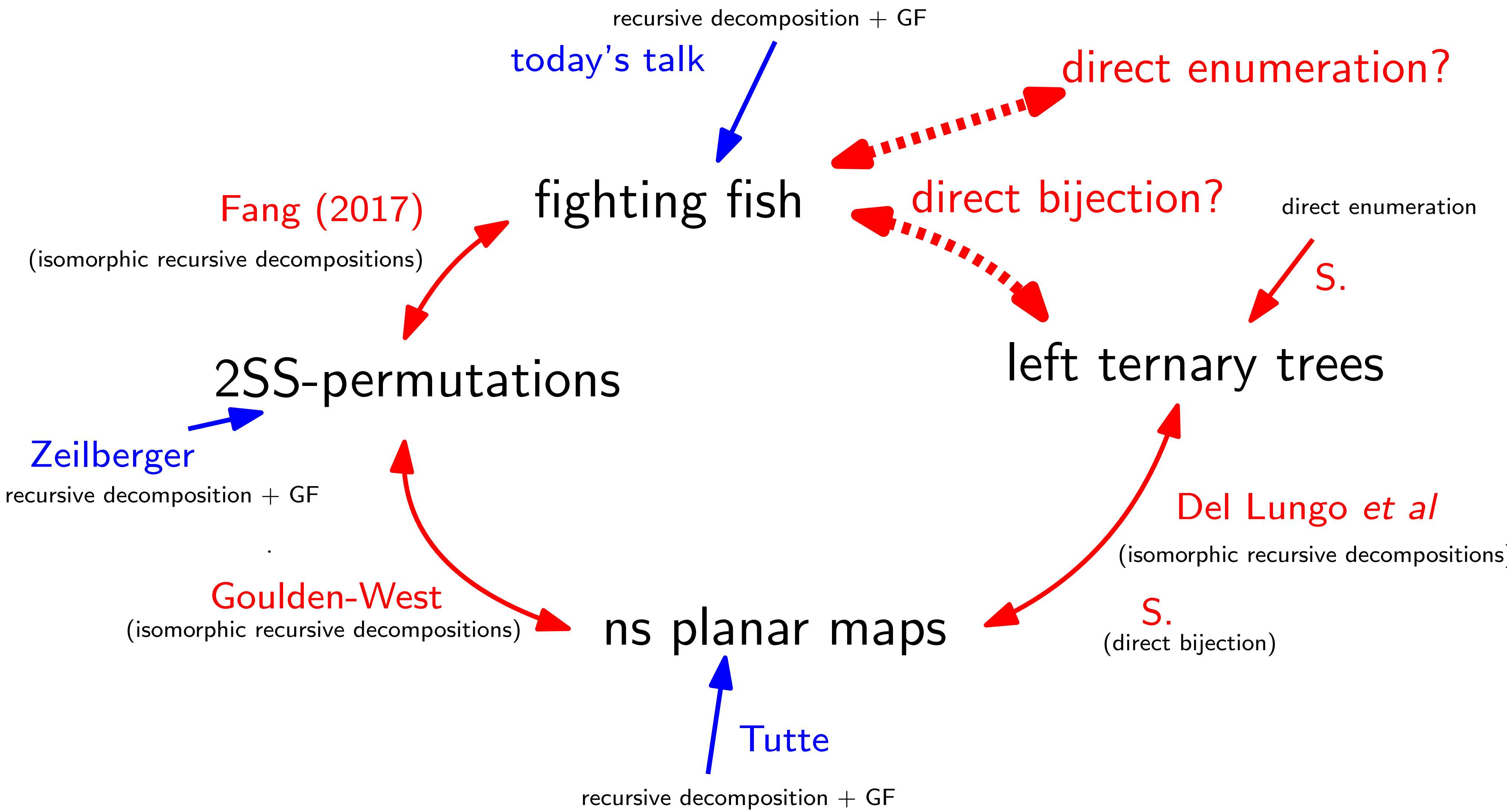
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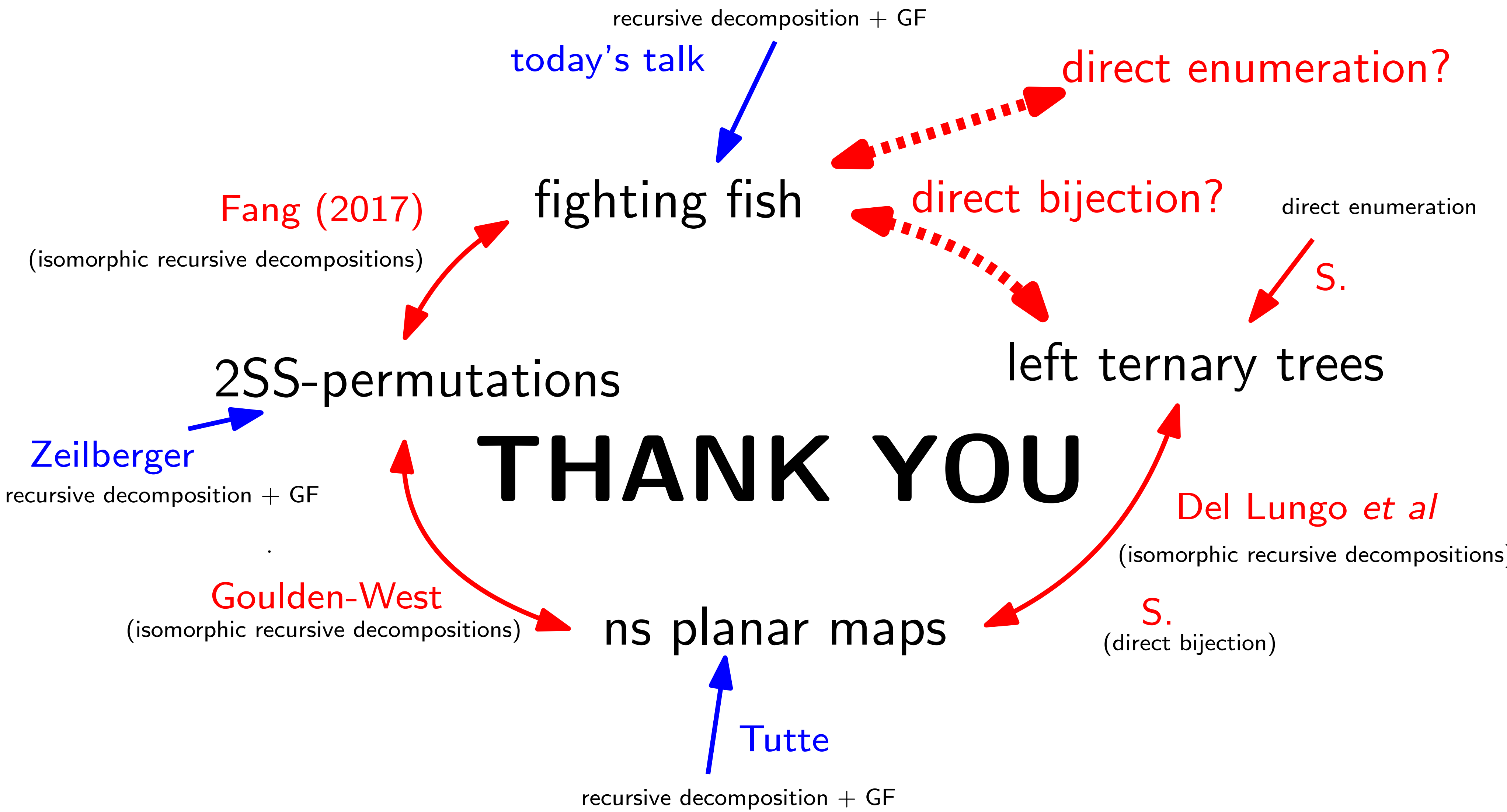

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