Simple branched covers and planar maps: Cayley vs Catalan

Gilles Schaeffer CNRS & École Polytechnique

AofA 2014, Paris

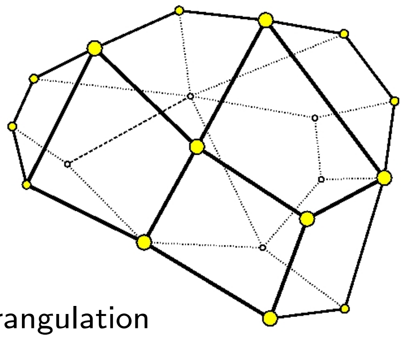
include recent work with E. Duchi and D. Poulalhon

Introduction

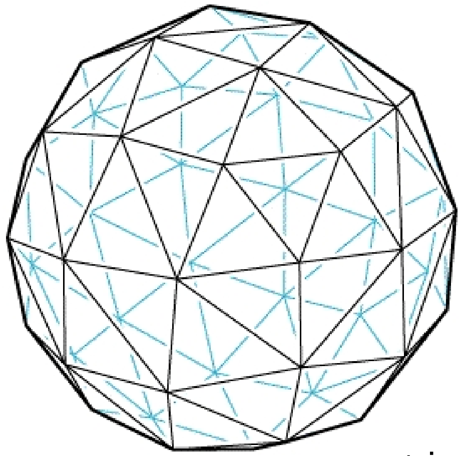
Planar maps

A planar map is a graph embedded on the sphere

and considered up to homeomorphisms (deformations) of the sphere



a quadrangulation



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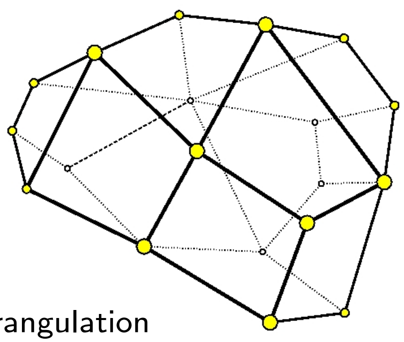

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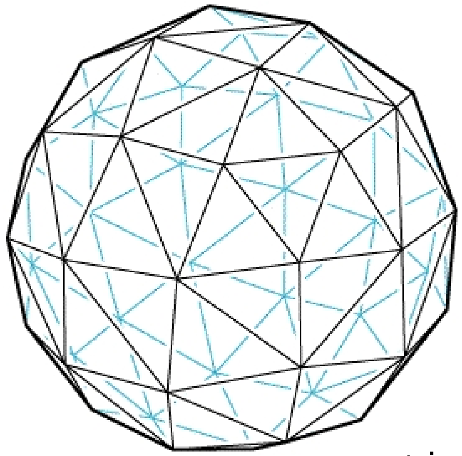

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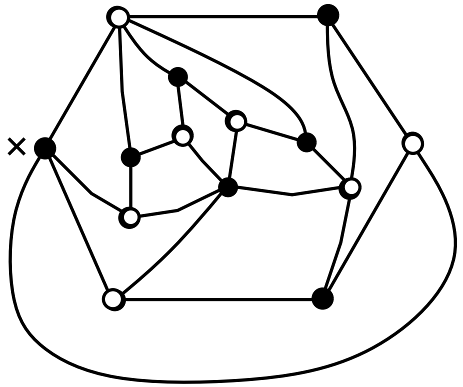
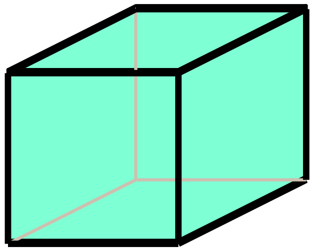


It is easier to make pictures in the plane...

For counting purpose planar maps
are rooted: a corner is marked

Planar maps

The most famous map is the skeleton of the cube...



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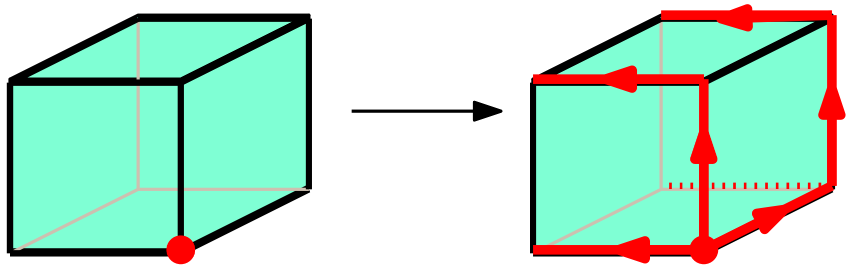

Explore the map along a spanning tree

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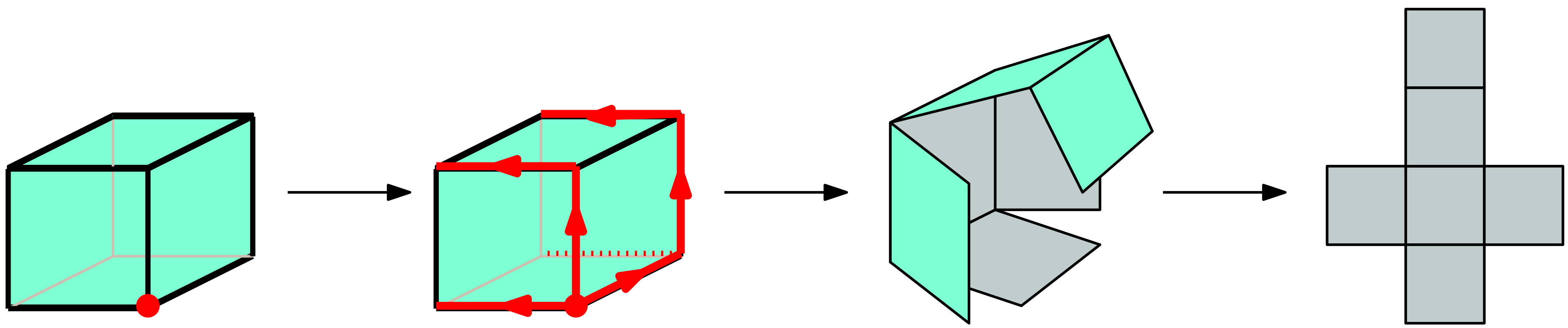

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and cut the surface to obtain a polygonal net (Dürer, 1525)

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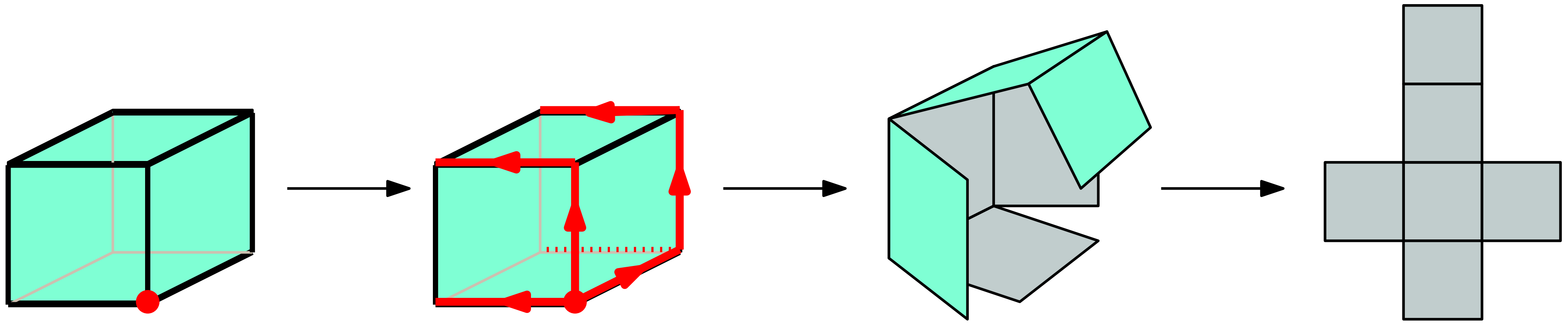

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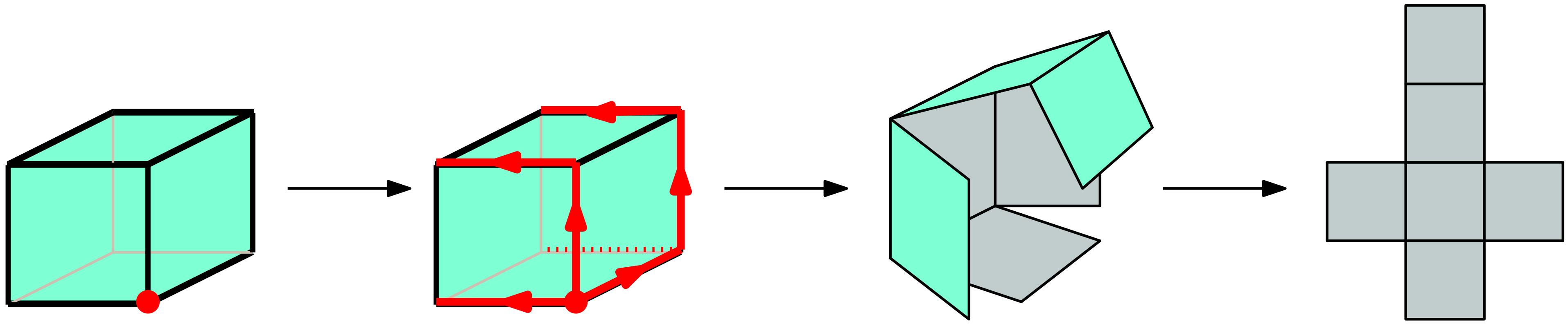

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f faces $\Rightarrow$ $f - 1$ gluing edges

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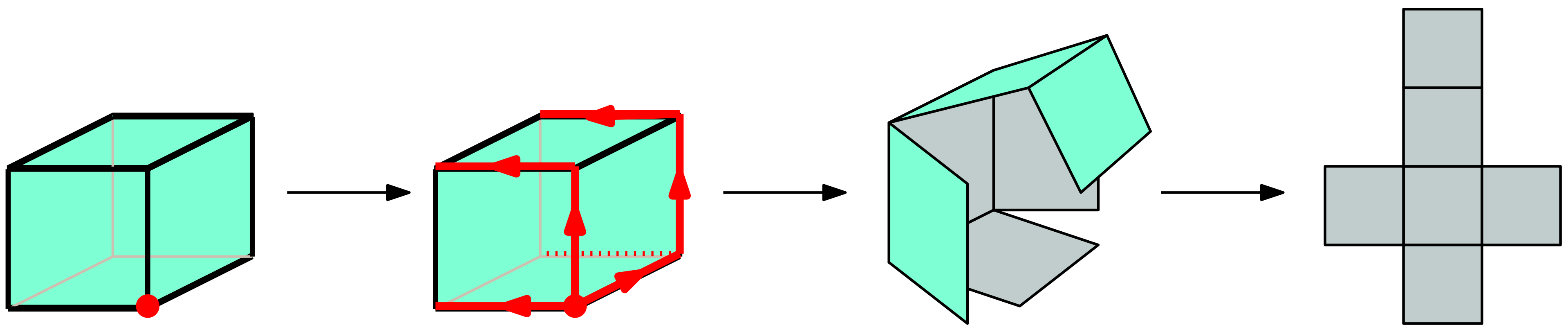

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Theorem. For any planar map: $v+f=e+2$ (Euler, 1752)

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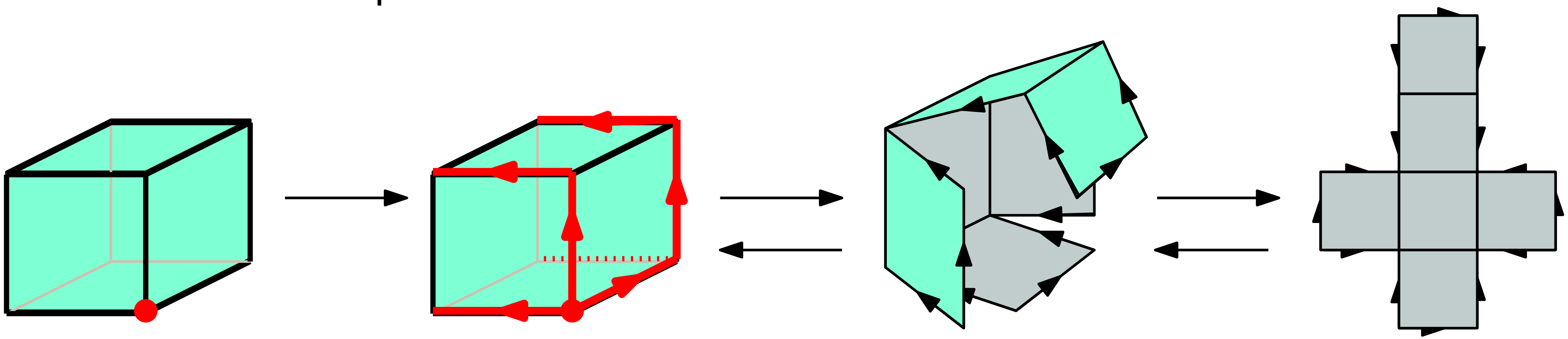

The boundary of the polygonal net is a cycle.

To allow reconstruction of the surface, the polygonal net must record the orientations of cuts: to reconstruct, glue together successive edges that form a sink and iterate.

With these *decorations*, the polygonal net encodes the original map.

Planar maps

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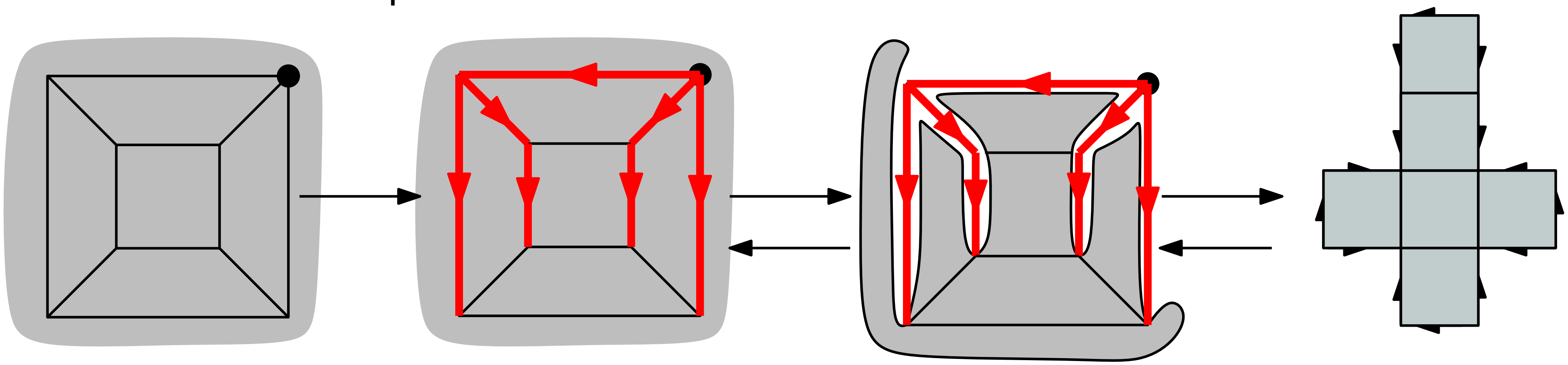

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Theorem (Tutte, 1963): Let $\mathcal{Q}_n = \{$rooted quadrangulations with n faces$\}$

and let $Q(t) = \sum_{q \in \mathcal{Q}_n} t^{|q|}$ be the gf where $|q| = \#$faces of q.

$$Q(t) = 1 + 2t + 9t^2 + \ldots$$

Then $Q(t)$ is solution of the system $\begin{cases} Q(t) = R(t) - tR(t)^3 \\ R(t) = 1 + 3tR(t)^2 \end{cases}$

so that $Q(t) = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}$ and $|\mathcal{Q}_n| = \frac{2}{n+2} \frac{3^n}{n+1} \binom{2n}{n}$.

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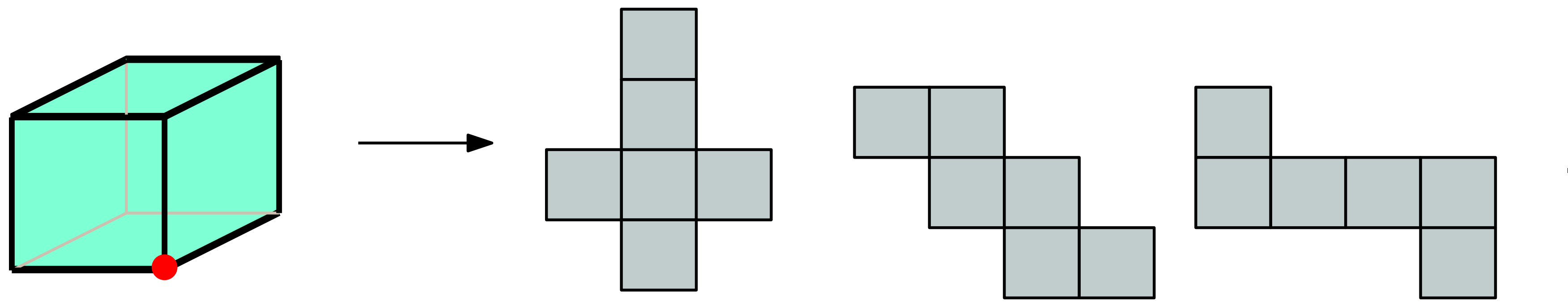
- Cori, Vauquelin *et al.* (70/80's $\rightarrow$ 2014, bijections with trees)

  to *explain* the nice formulas and algebraicness

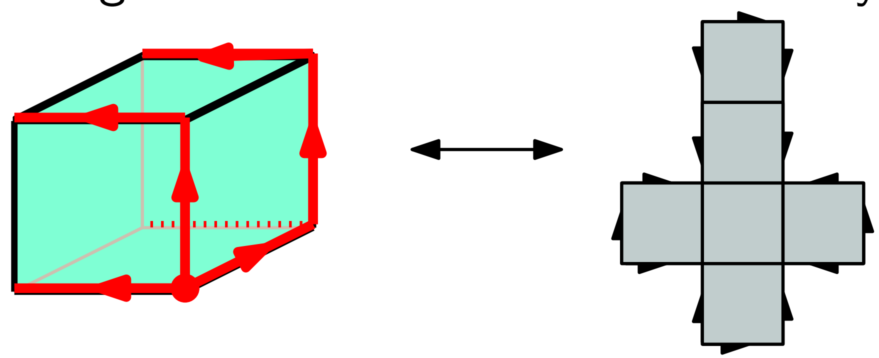
The bijective approach

A strategy to prove counting formulas?

To a map are associated several polyhedral nets.



But a given algorithm associate one net to every map.

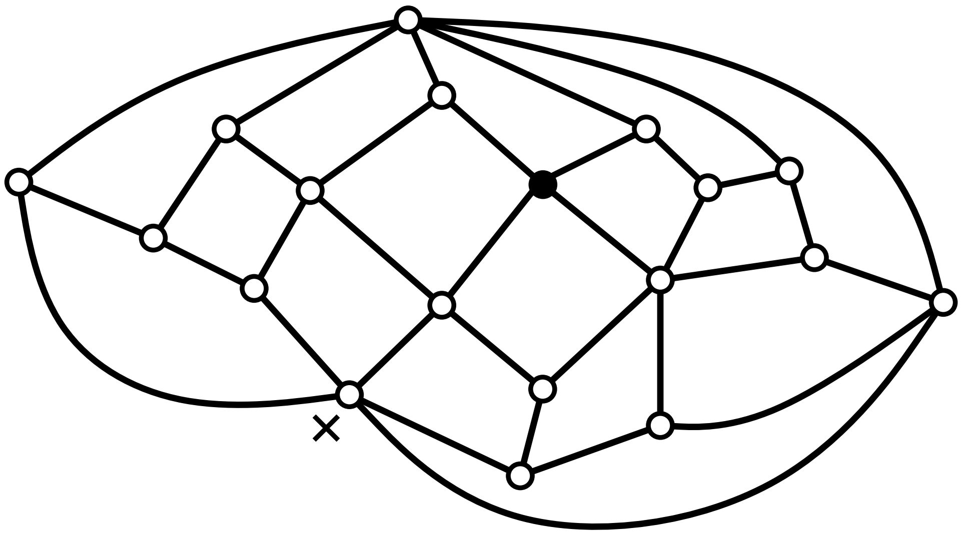


To each exploration algorithms corresponds a set of *valid polyhedral nets*

Tutte's formula suggests to look for an algorithm whose set of valid nets is clearly counted by $3^n \frac{1}{n+1}\binom{2n}{n}$

(Once again) Cori-Vauquelin's bijection revisited

Perform breadth first search from a marked vertex: at round i visit vertices at distance i.



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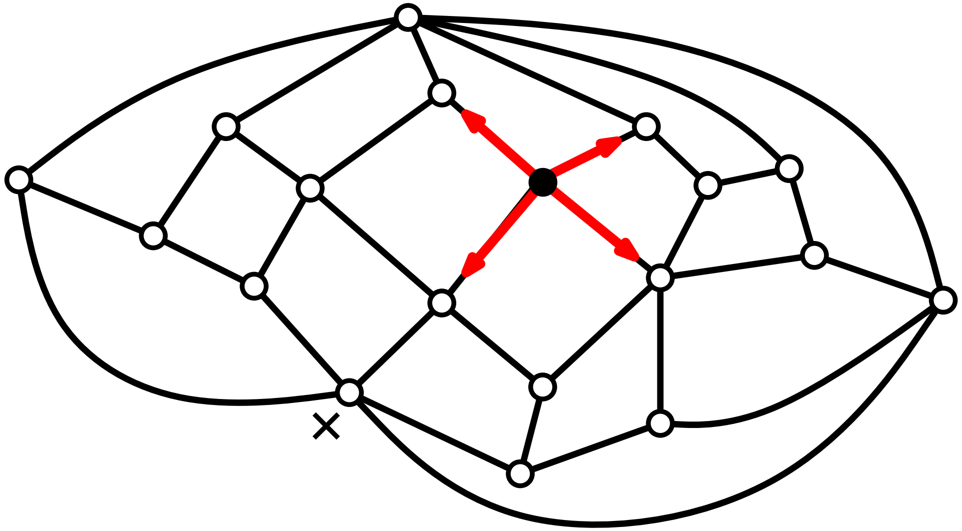

Around each vertex apply a symmetric priority rule

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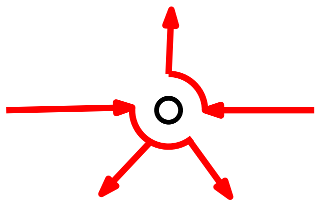

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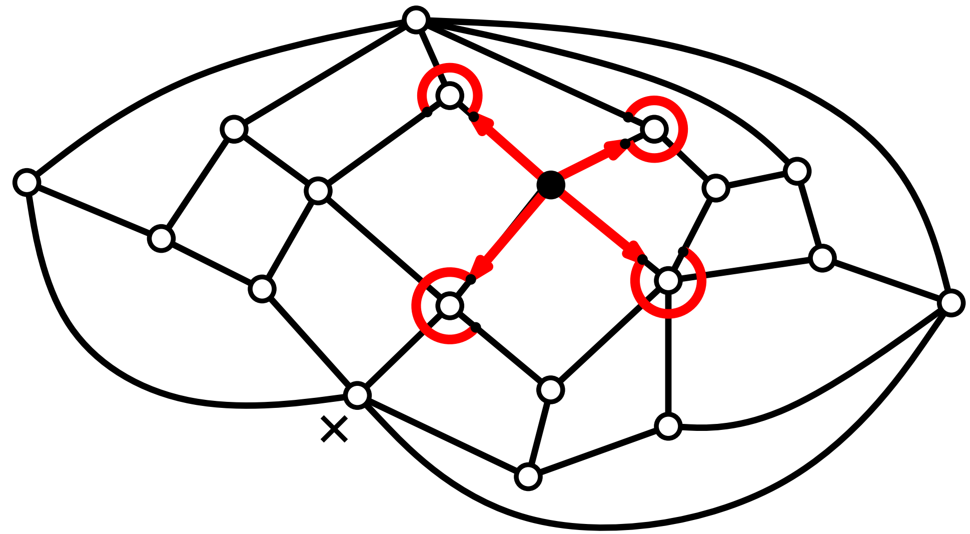

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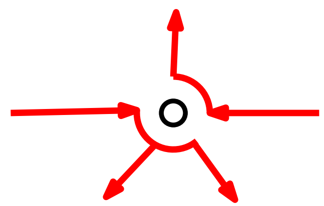

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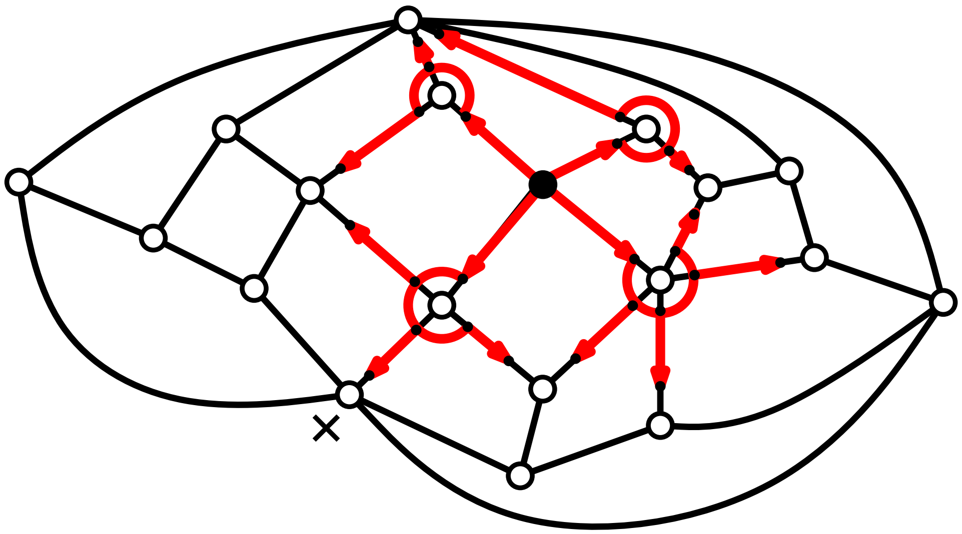

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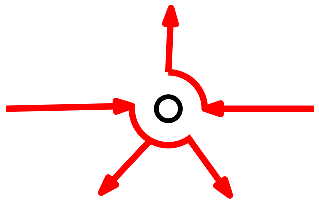

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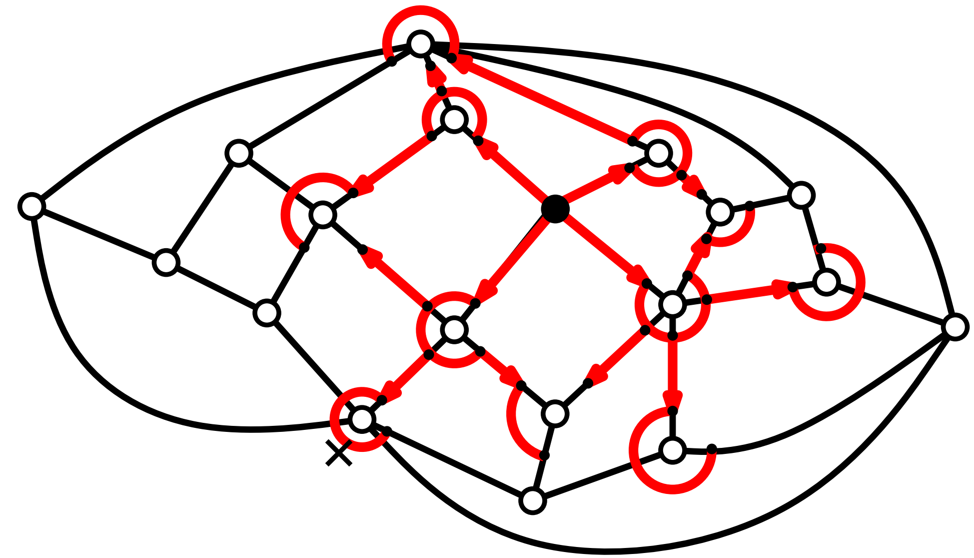

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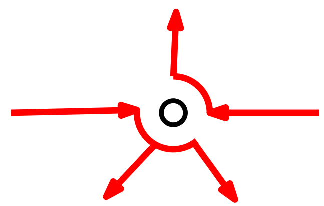

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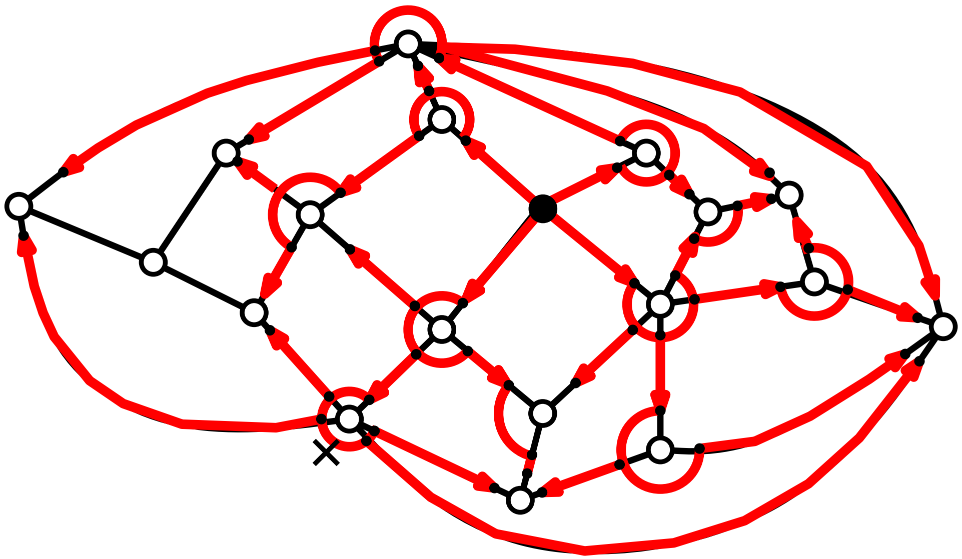

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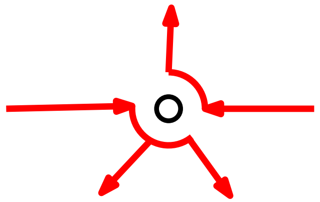

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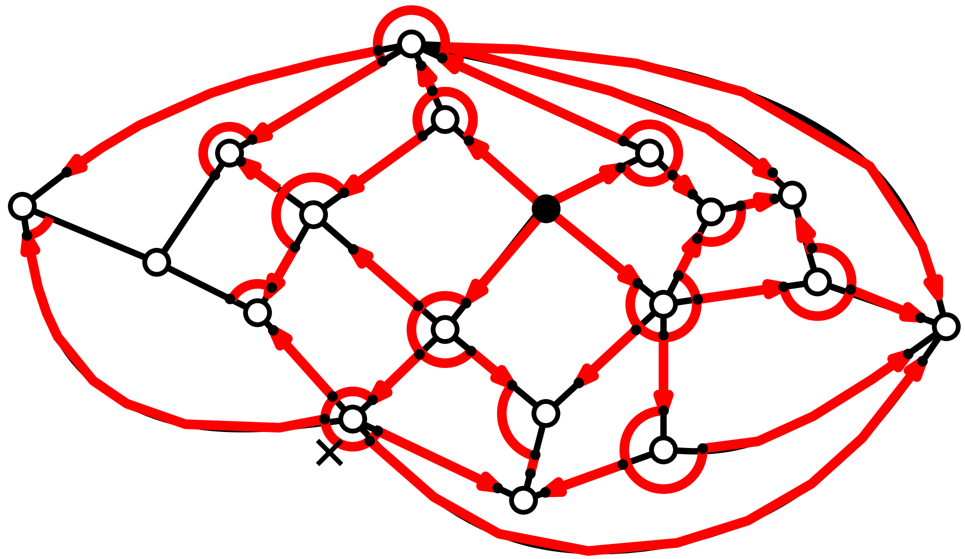

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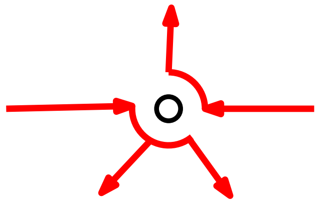

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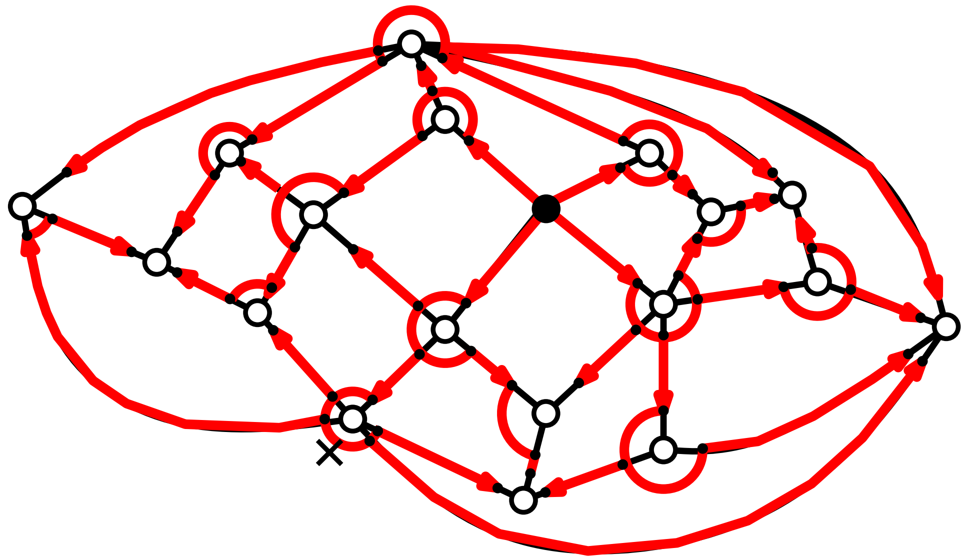

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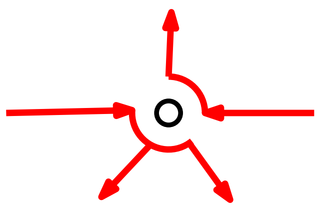

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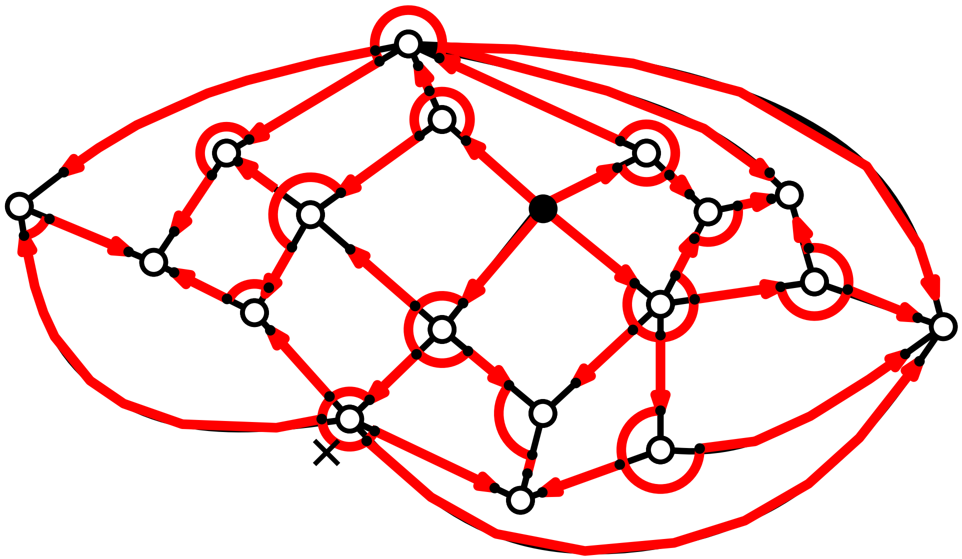

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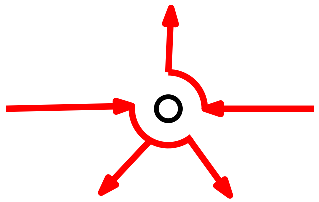

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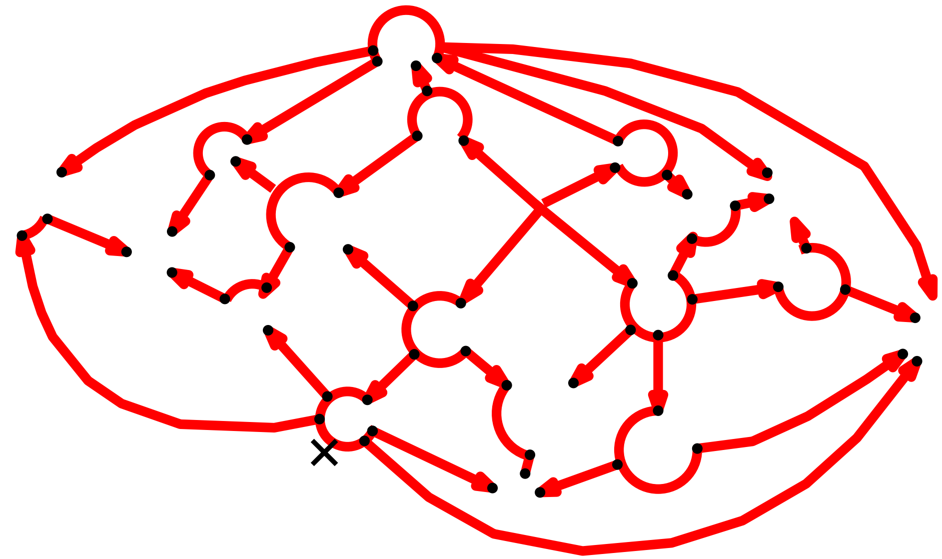

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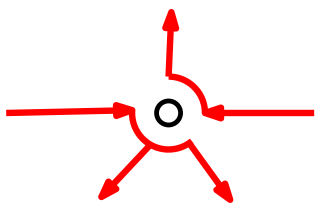

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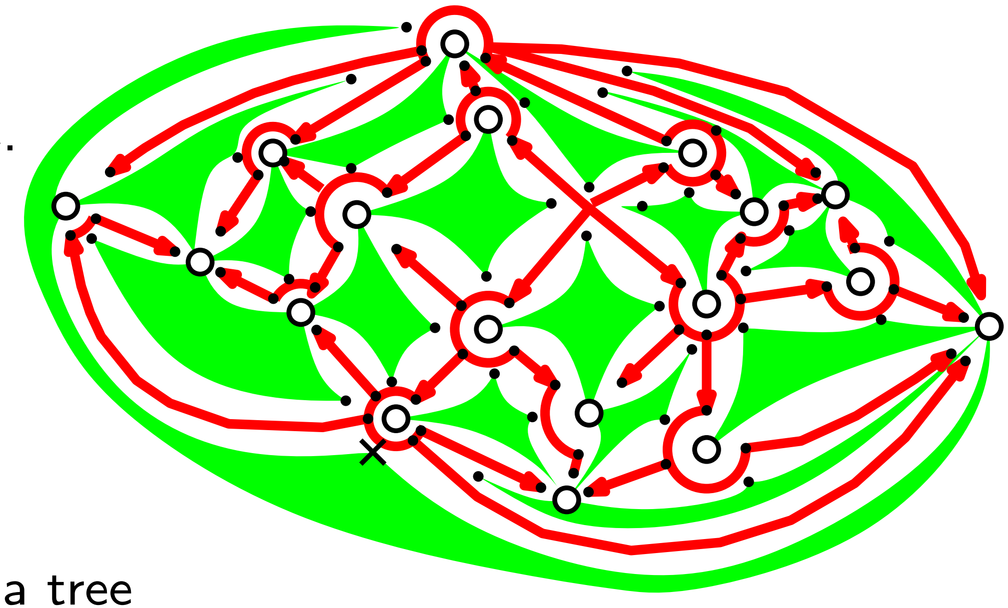

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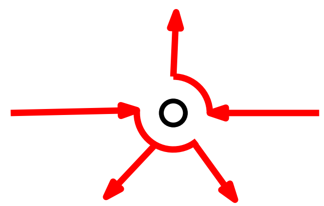

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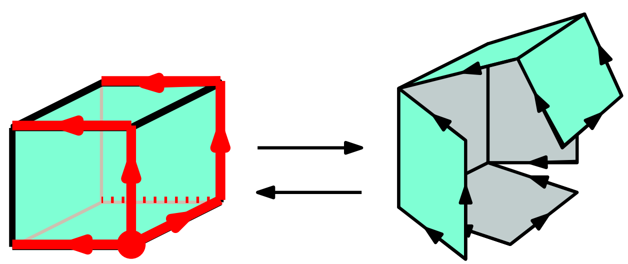
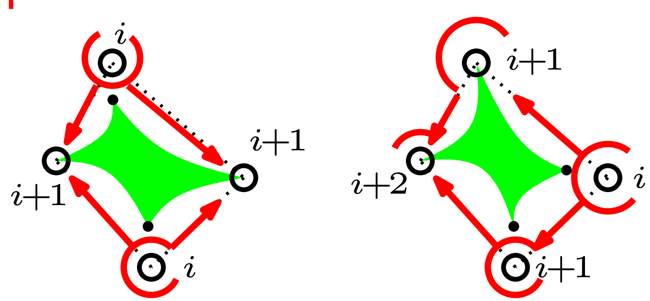
Cutting the map along this tree yields a polygonal net



with only 2 types of squares



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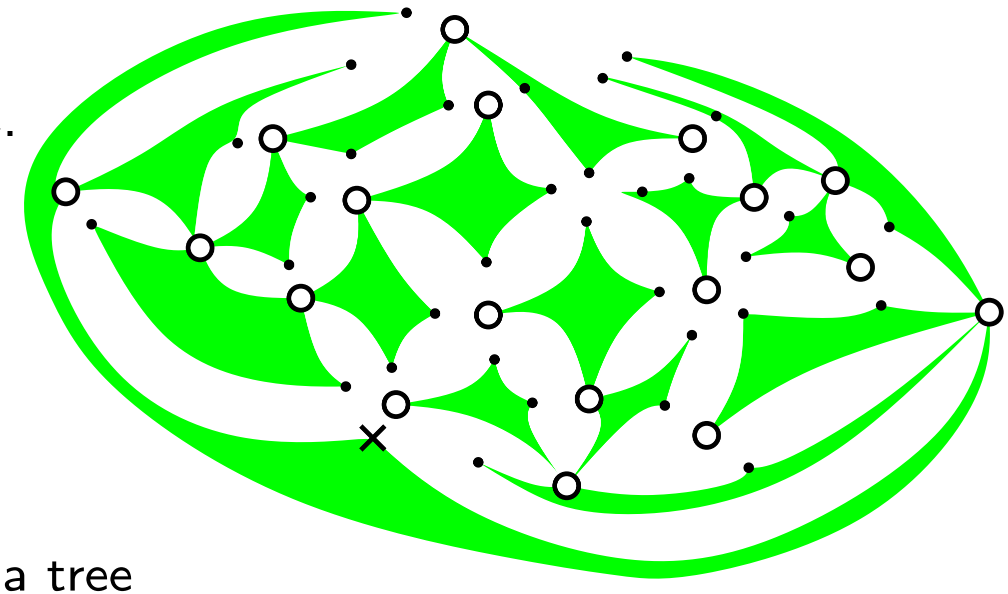

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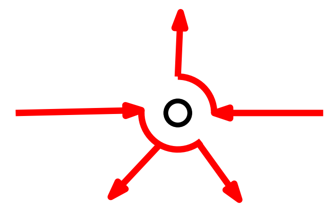

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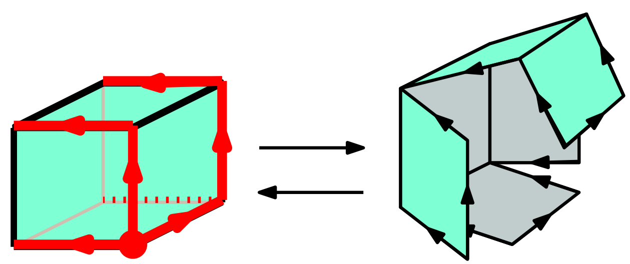

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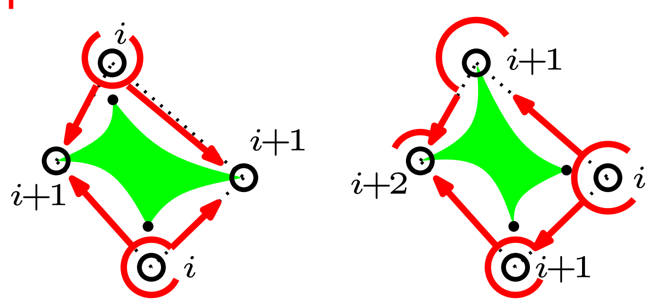

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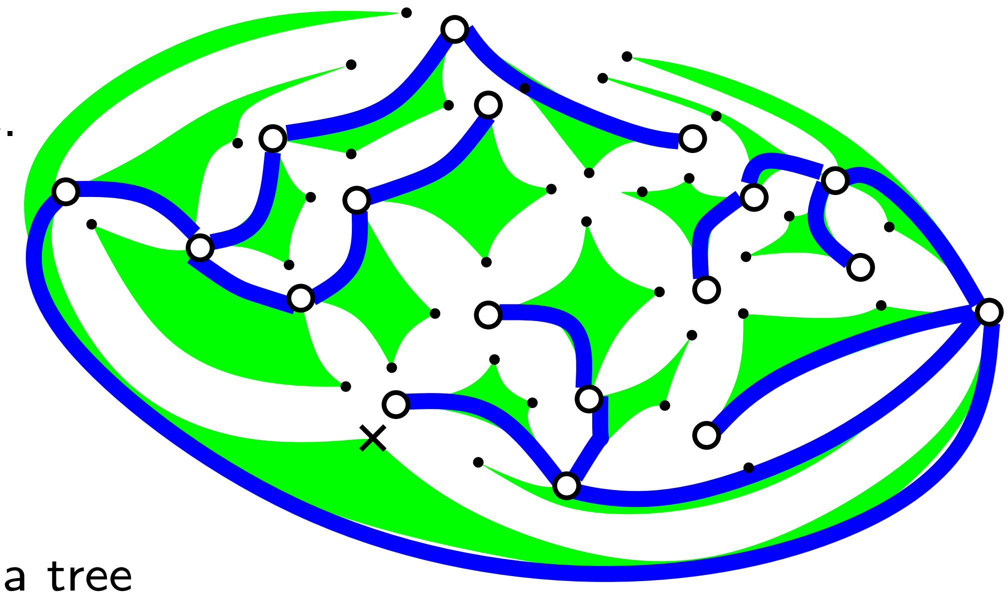

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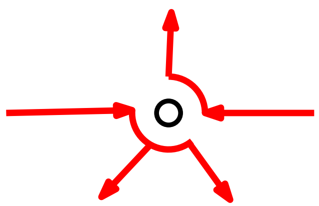

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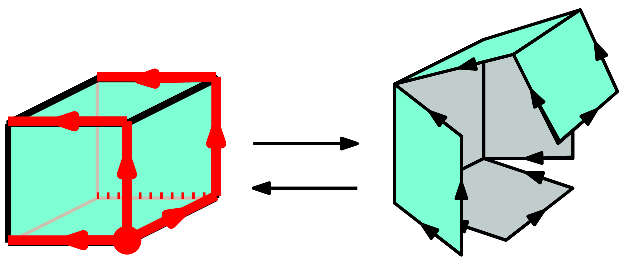

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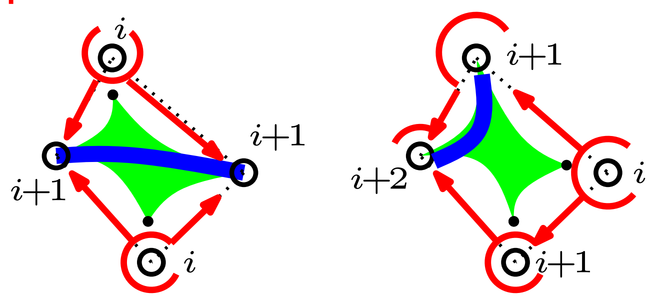
– transverse

– ou lateral



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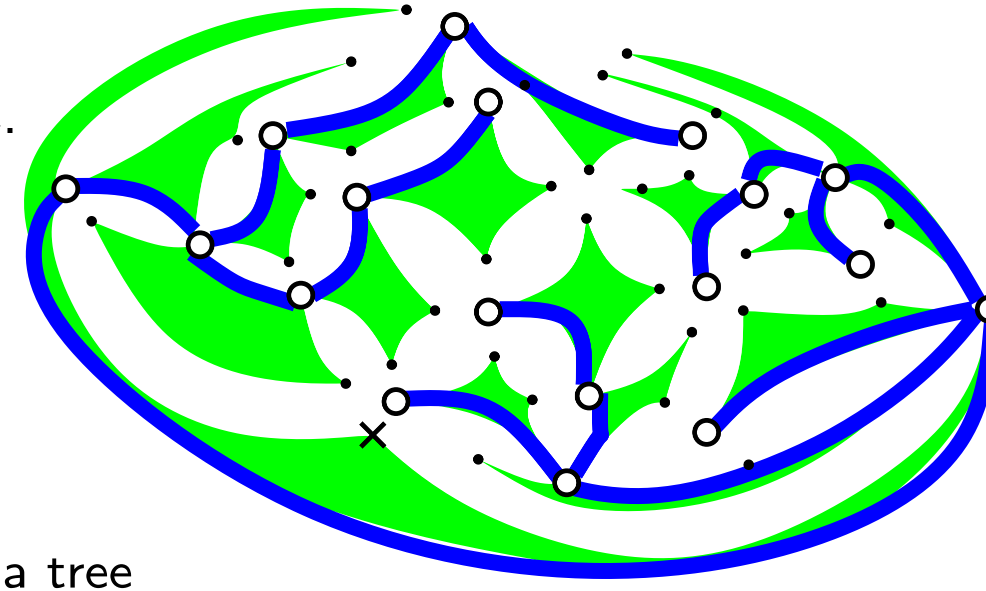

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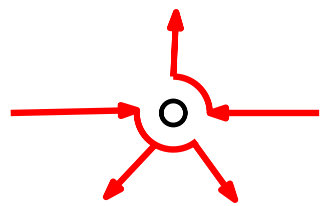

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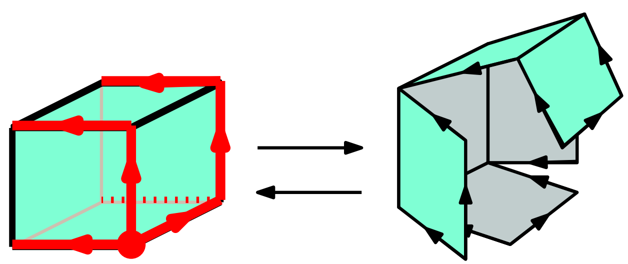
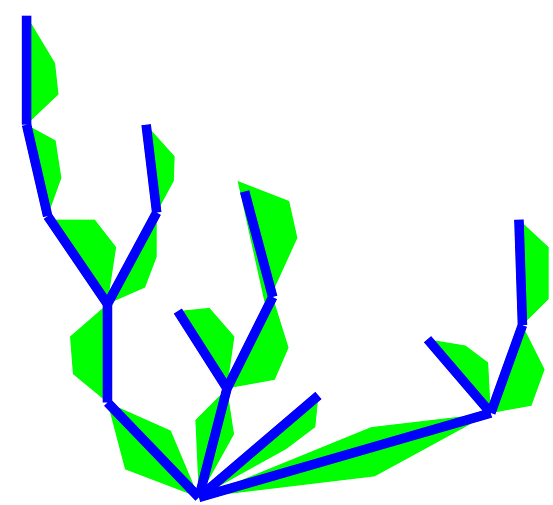
Cutting the map along this tree yields a polygonal net

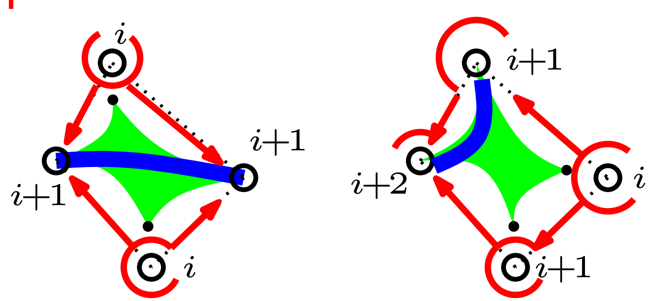


The polyhedra net redrawn as a plane tree.



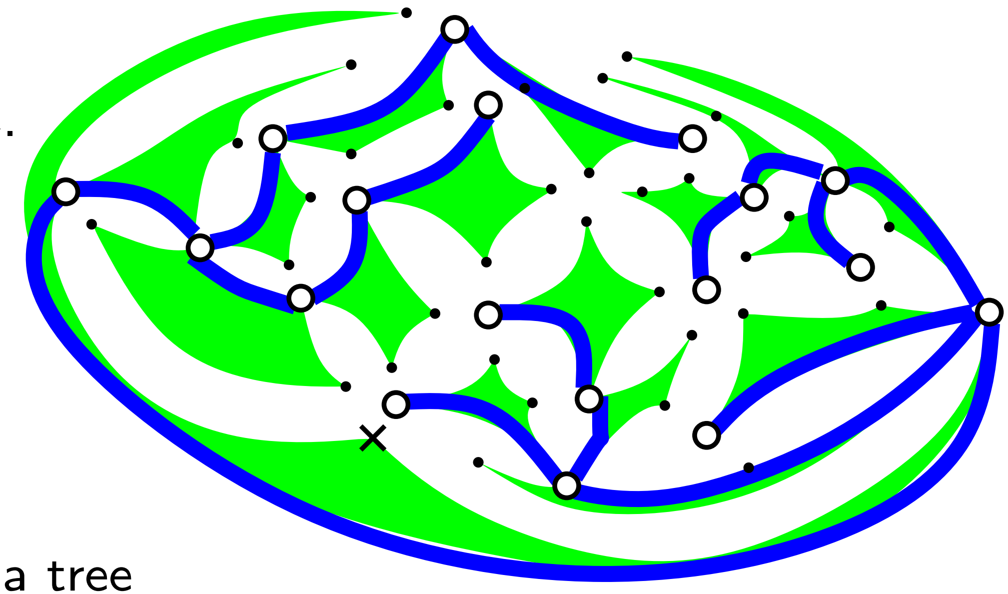
with only 2 types of squares

- transverse
- ou lateral



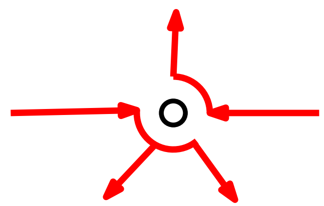
(Once again) Cori-Vauquelin's bijection revisited

Perform breadth first search from a marked vertex: at round i visit vertices at distance i.



Around each vertex apply a symmetric priority rule

(Cori-Vauquelin's splitting)
(Bernardi/Marckert/Miermont priority rule)



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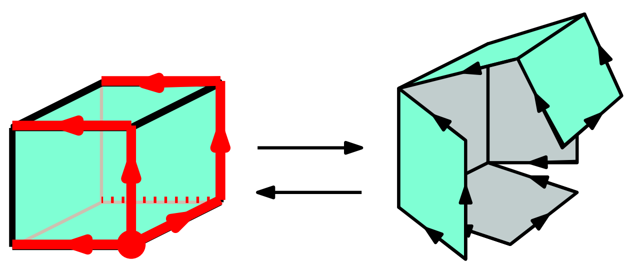

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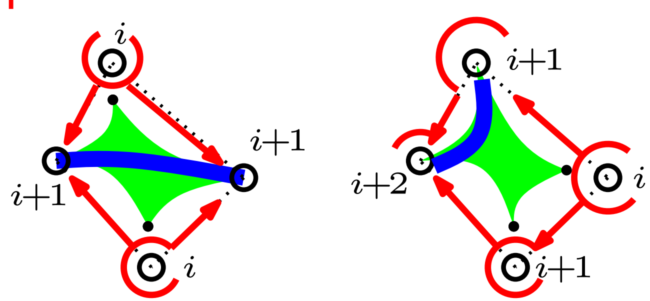
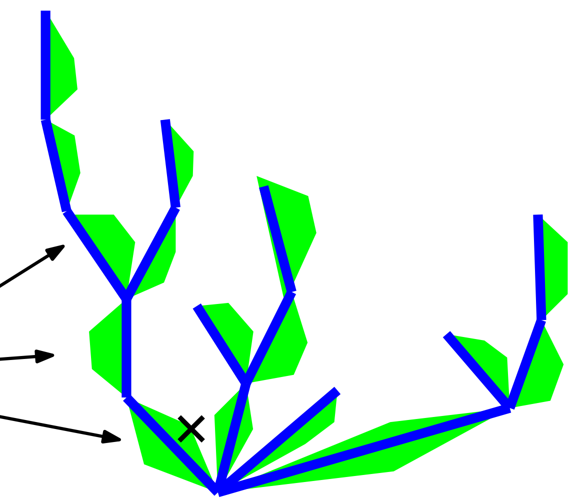


3 types of edges



$$\Rightarrow \qquad 2 \cdot 3^n \cdot \frac{1}{n+1}\binom{2n}{n}$$

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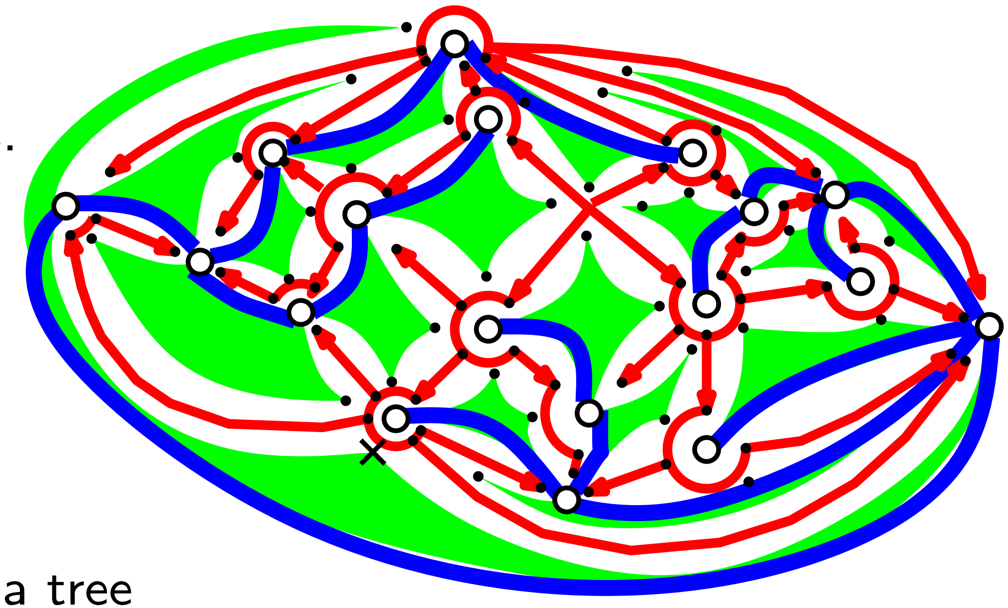

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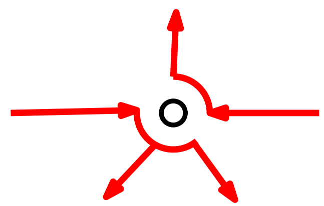

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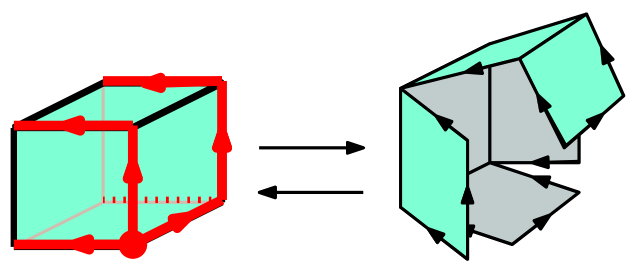

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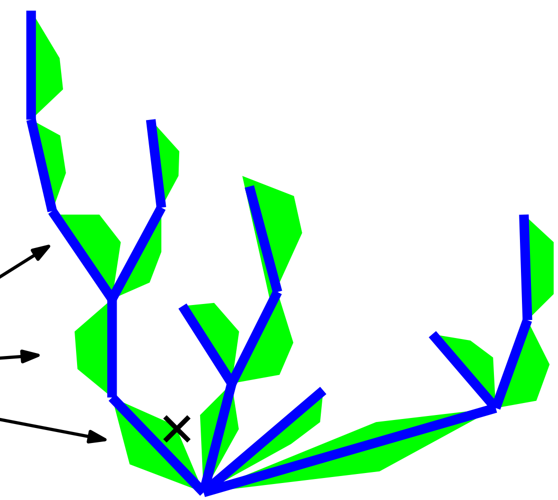

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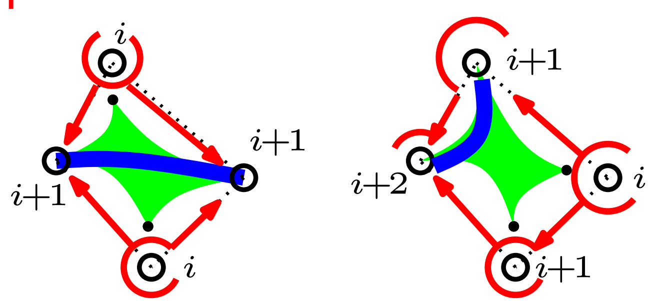

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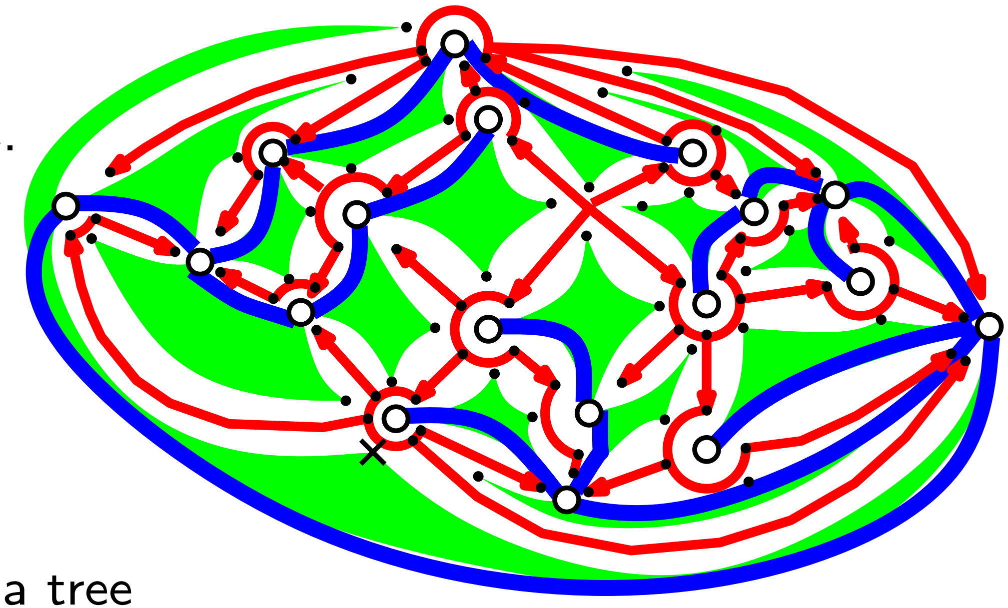

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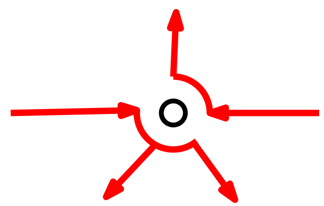

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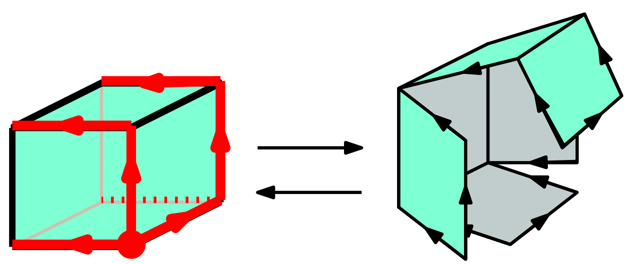

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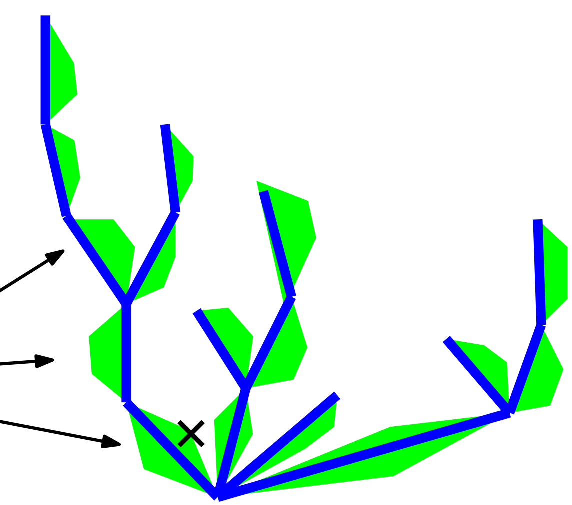

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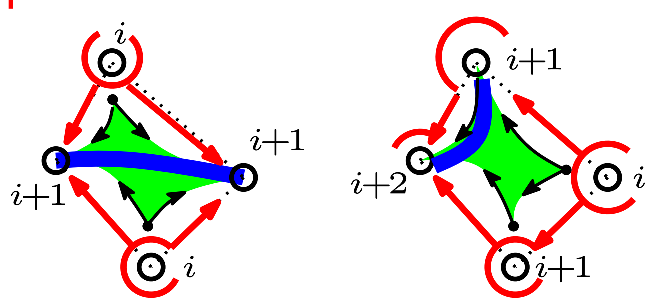

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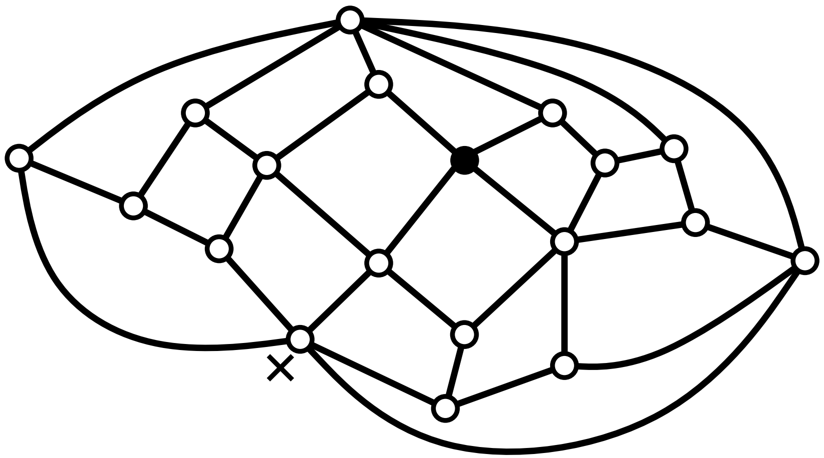
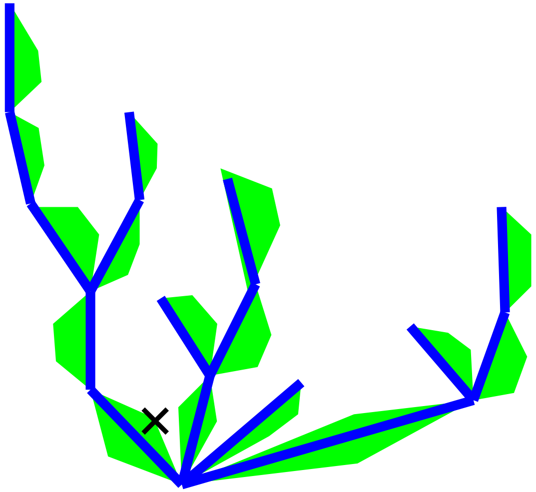
We have described a bijection between:

Pointed rooted quadrangulations with n faces & Rooted decorated trees with n edges





Distances

From the bijection to distances in random maps

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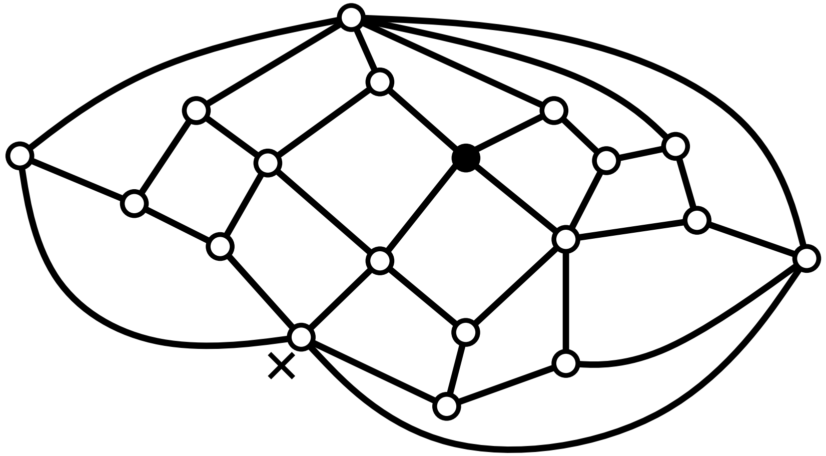

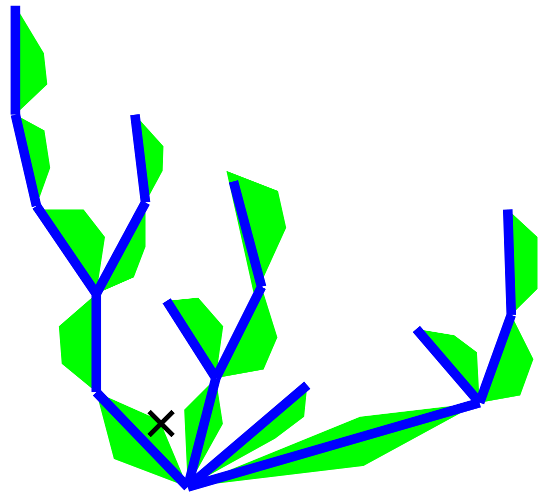

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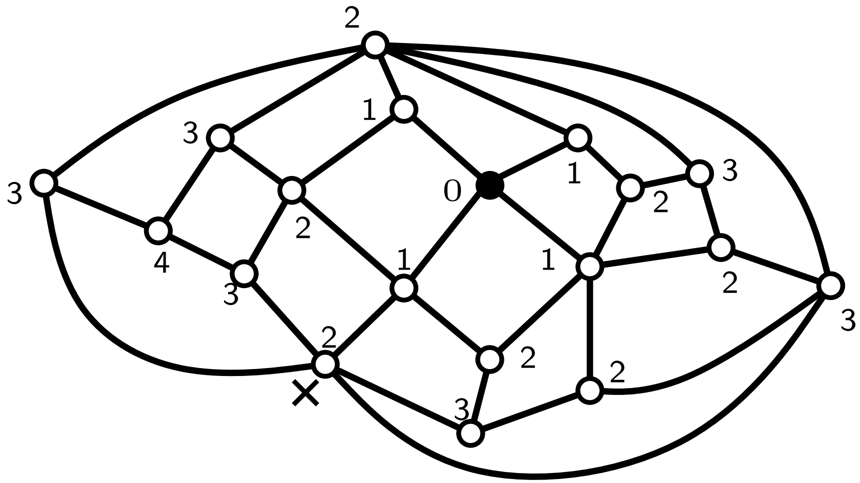

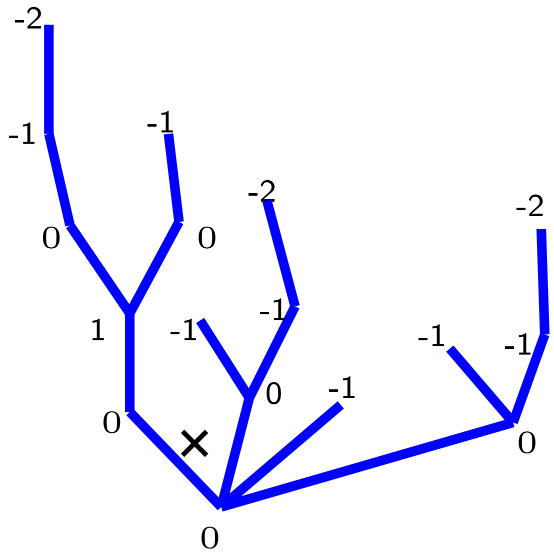

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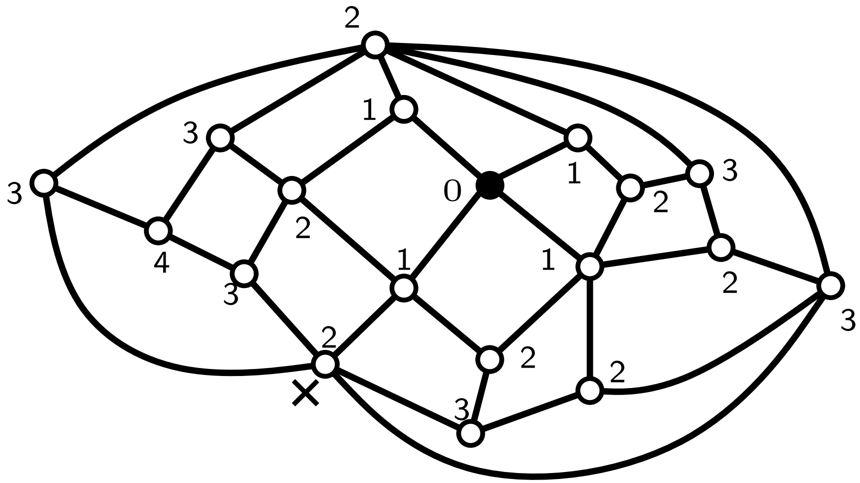

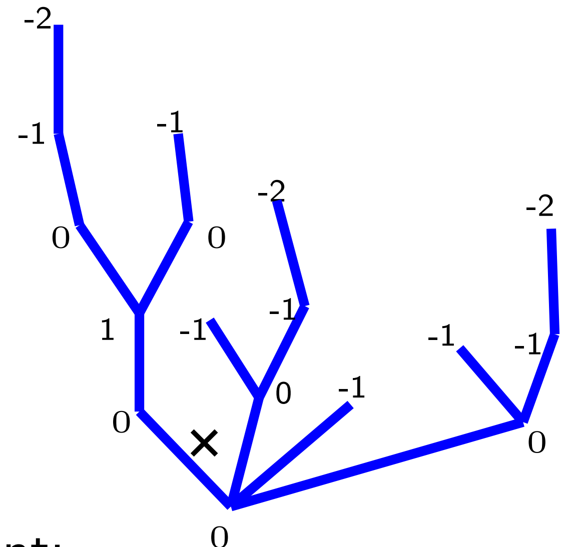

Labels of branches give distances to the marked point:

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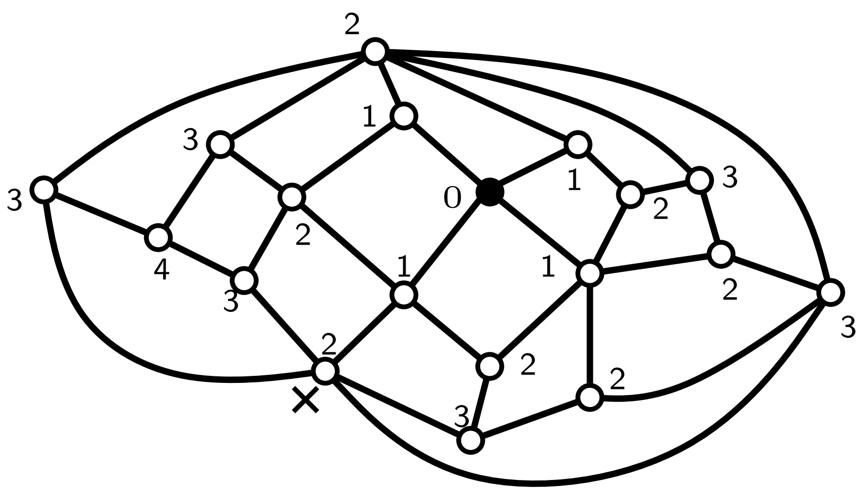

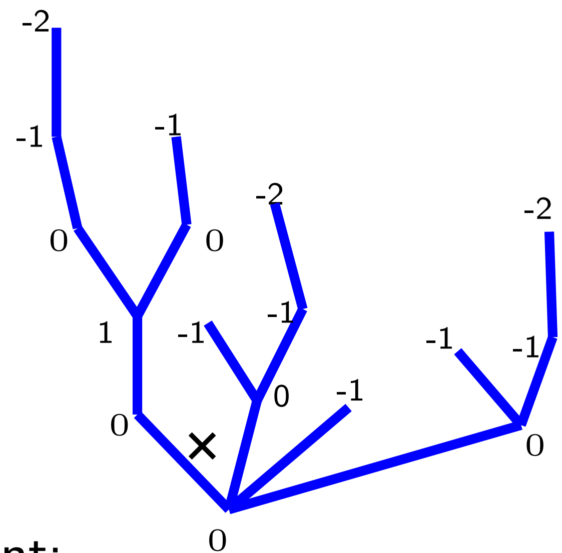

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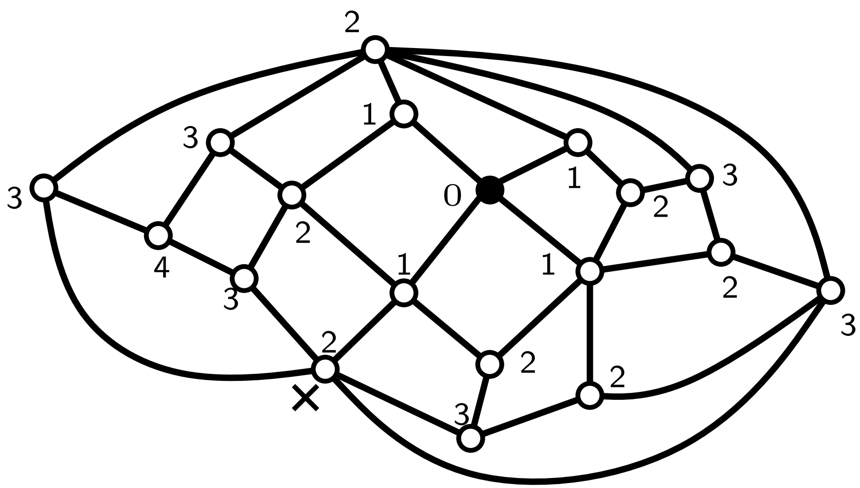

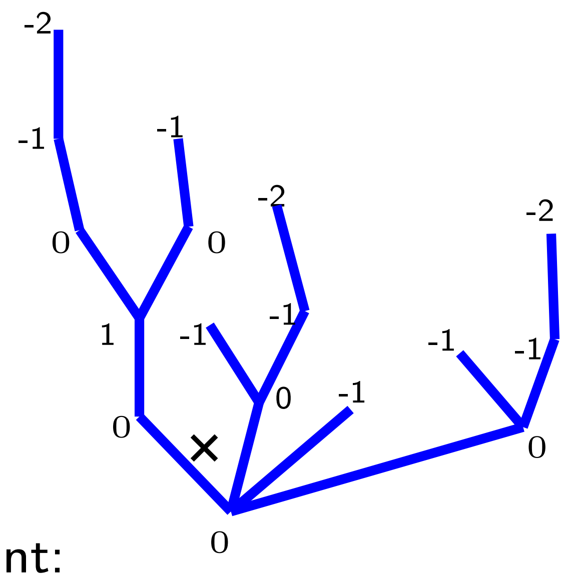

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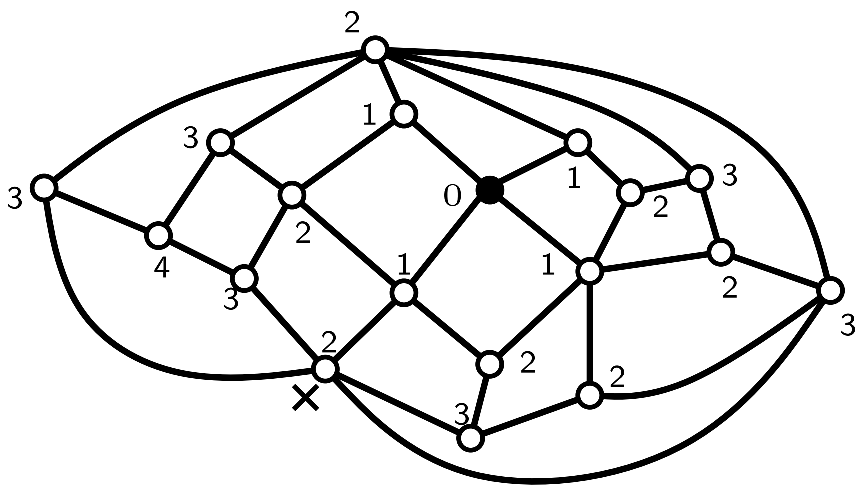

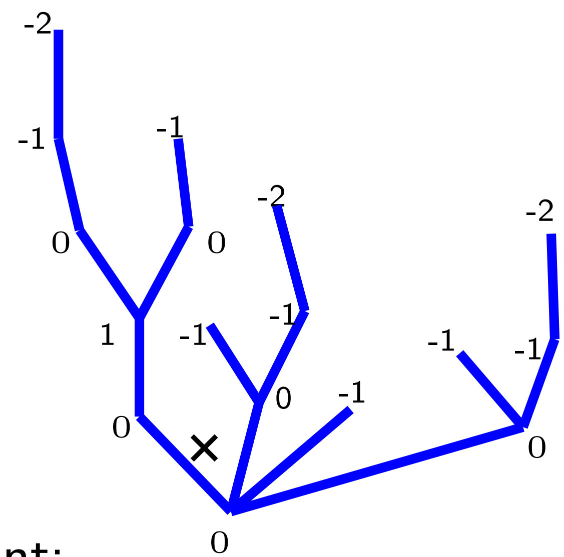

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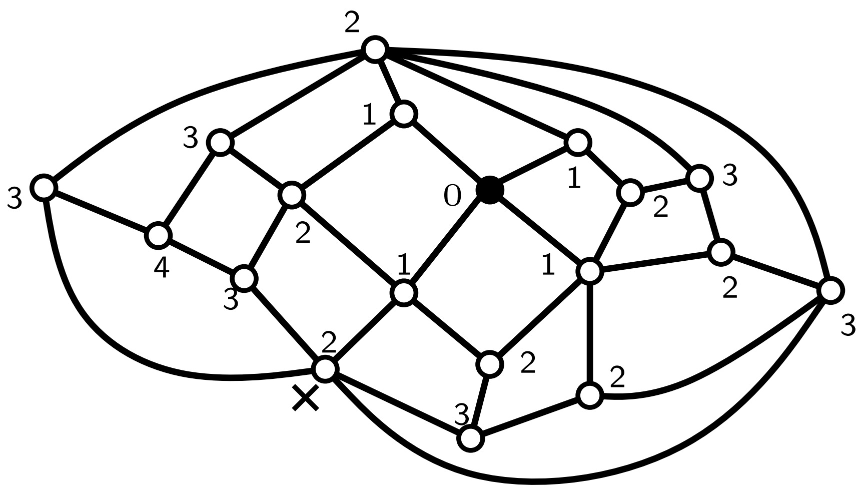

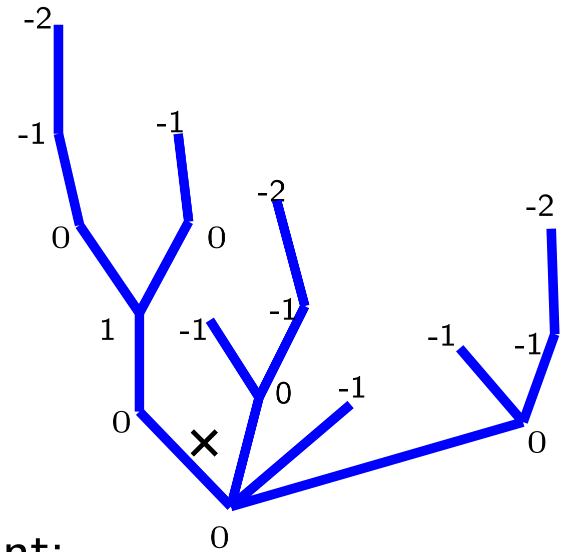

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Typical distances are of order $n^{1/4}$

(Chassaing-Schaeffer)

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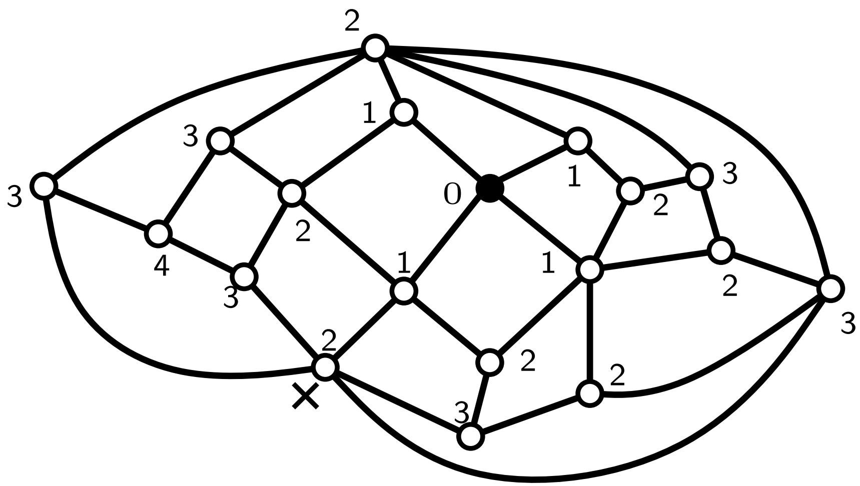

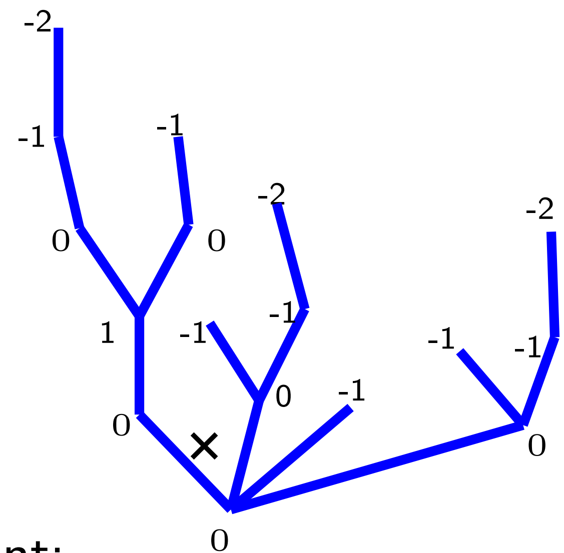

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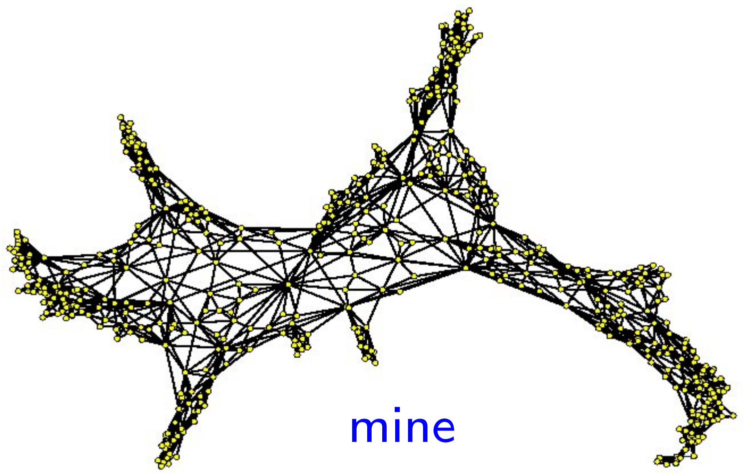
(Chassaing-Schaeffer) (Le Gall, Miermont)

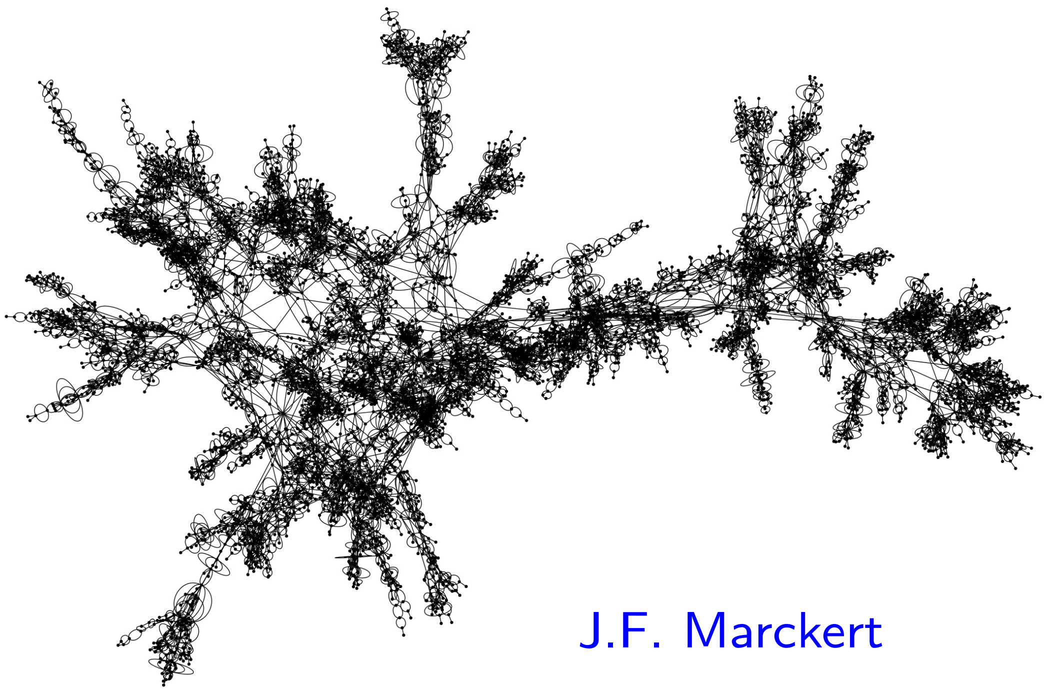
Some nice pictures

Pictures of uniform random quadrangulations and triangulations by various people...

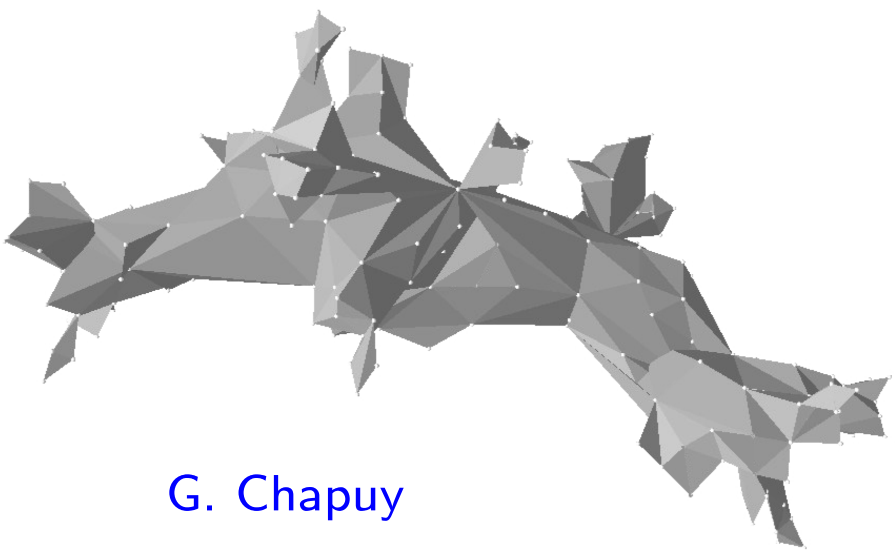
These 3d pictures are bound to be "spiky" since we have seen that random planar have Hausdorf dimension 4.



mine



J.F. Marckert



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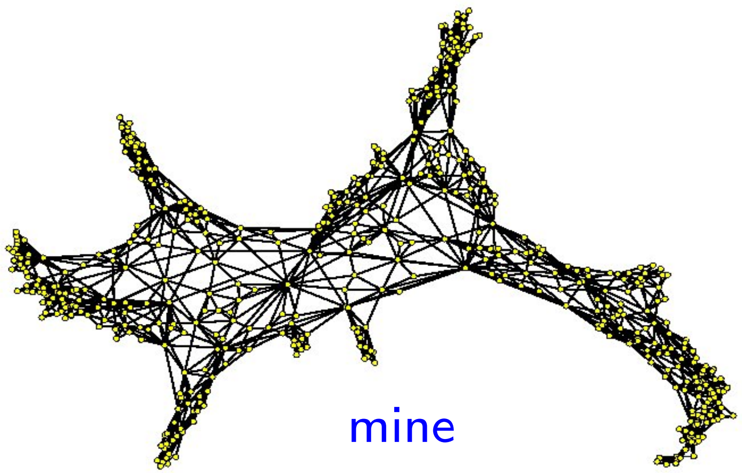

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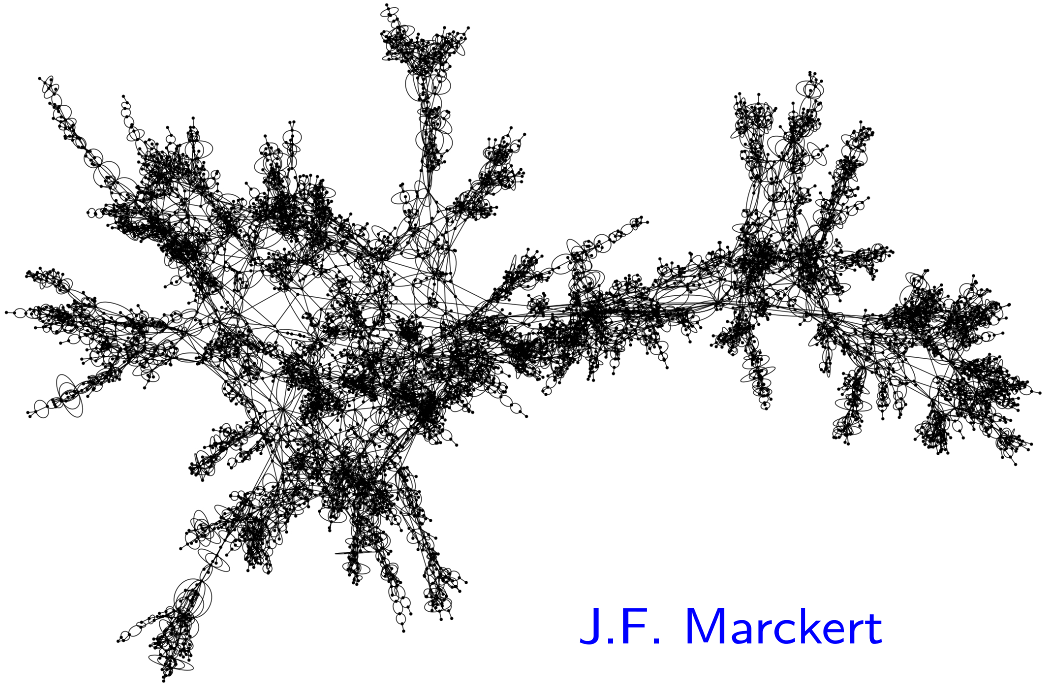

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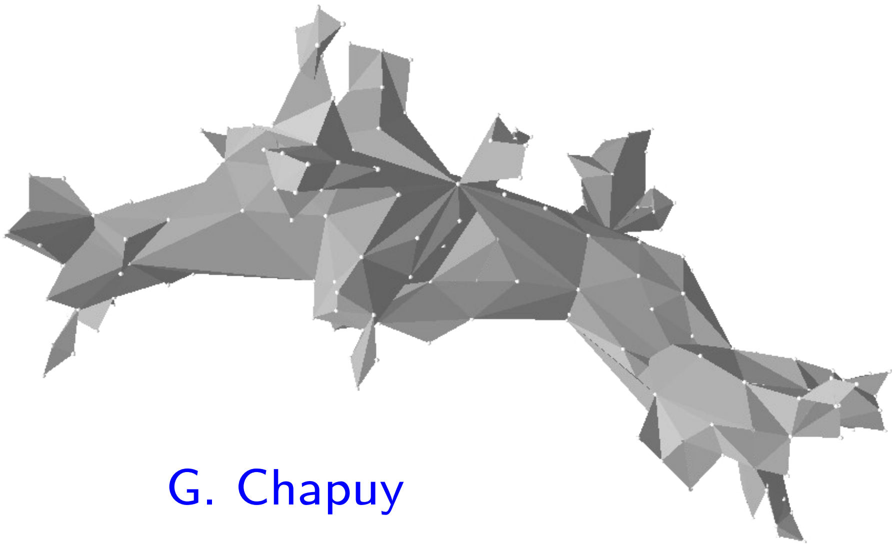


G. Chapuy

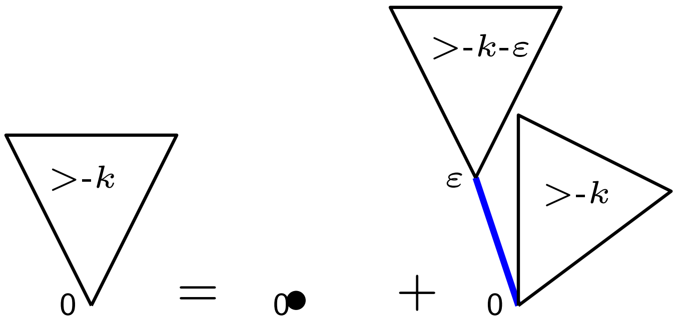
Some even nicer formulas

To study distances exactly, first thing to do is count trees with minimum label $< k$.

The OGF of trees with min label $< k$ satisfies

$T_0 = 0$ and

$T_k = 1 + z(T_{k-1} + T_k + T_{k+1})T_k$ for $k \geq 1$.



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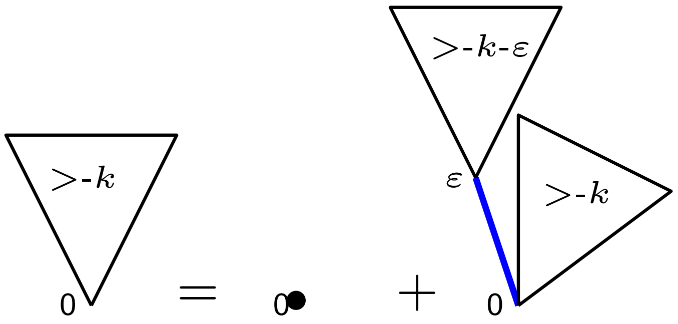

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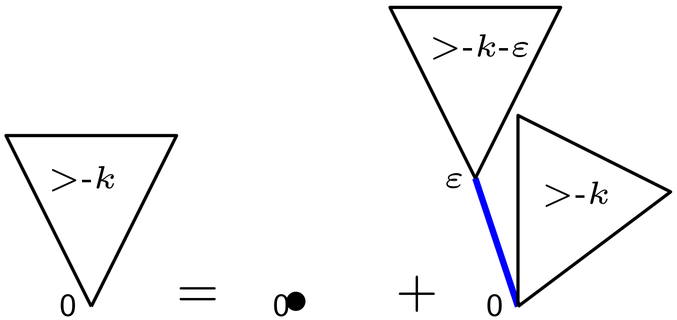

Surprisingly Bouttier, Di Francesco and Guitter have shown that this system of equations has a beautiful explicit solution:

$$T_i = T\frac{(1-Y^i)(1-Y^{i+3})}{(1-Y^{i+1})(1-Y^{i+2})} \text{ where } T = 1 + zT^2 \text{ and } Y = zT^2(1+Y+Y^2).$$

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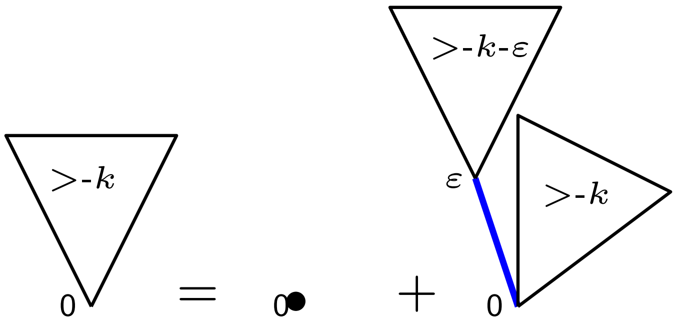

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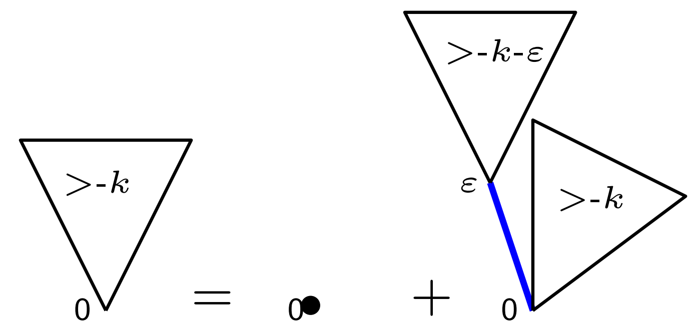

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No direct combinatorial interpretation on trees, but some interpretations via maps and continuous fractions (Bouttier-Guitter, Albenque-Bouttier)

Recently it was observed by Bouttier and Eynard that there is a systematic way to find such solutions using "Lax pairs" and "Plucker/Hirota equations"...

A pattern in enumeration and random structures

The previous bijection is only the tip of the iceberg:

Dozens of bijections between families of maps and trees have since been found...

D. Arquès, M. Marcus, M. Bousquet-Mélou, D. Poulalhon, O. Bernardi, E. Fusy, J. Bouttier, P. Di Francesco, E. Guitter, G. Chapuy, E. Vassilieva, G. Miermont, V. Feray, J. Ambjørn, T. Budd, G. Collet...

They have started to merge into Master bijections:

- Using minimal accessible orientations and blossoming trees.

  M. Albenque and D. Poulalhon (2013)

- Or left-accessible orientations and mobiles

  O. Bernardi, G. Chapuy and É. Fusy (2011-2014)

- Or geodesic orientations and pizza slices

  J. Bouttier and E. Guitter (2014)

A pattern in enumeration and random structures

What	Type of GF	Asympt	Rescaled continuum limit
$\{-1,+1\}$-Walks	OGF $= \frac{1}{1-2x}$	2^n	Brownian motion
Plane trees	$P = 1 + zP^2$	$\frac{4^n}{n^{3/2}}$	Continuum Random Trees (CRT)
Well labeled trees	$T = 1 + 3zT^2$	$\frac{12^n}{n^{3/2}}$	Brownian snake (ISE)
Planar maps	(OGF)$' = T$	$\frac{12^n}{n^{5/2}}$	Brownian map

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Simple walks	$\mathbb{N}$-rational OGF	ρ^n	Universality (proved)
Plane trees	$P = 1 + zP^2$	$\frac{4^n}{n^{3/2}}$	Continuum Random Trees (CRT)
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Well labeled trees	$T = 1 + 3zT^2$	$\frac{12^n}{n^{3/2}}$	Brownian snake (ISE)
Decorated plane trees	$\mathbb{N}$-algebraic OGF	$\frac{\rho^n}{n^{3/2}}$	Universality (proved)
Planar maps	(OGF)$' = T$	$\frac{12^n}{n^{5/2}}$	Brownian map
Planar maps with degree constraints	$\mathbb{N}$-algebraic (OGF)'	$\frac{\rho^n}{n^{5/2}}$	Universality (many examples)

A pattern in enumeration and random structures

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A pattern in enumeration and random structures

What	Type of GF	Asympt	Rescaled continuum limit
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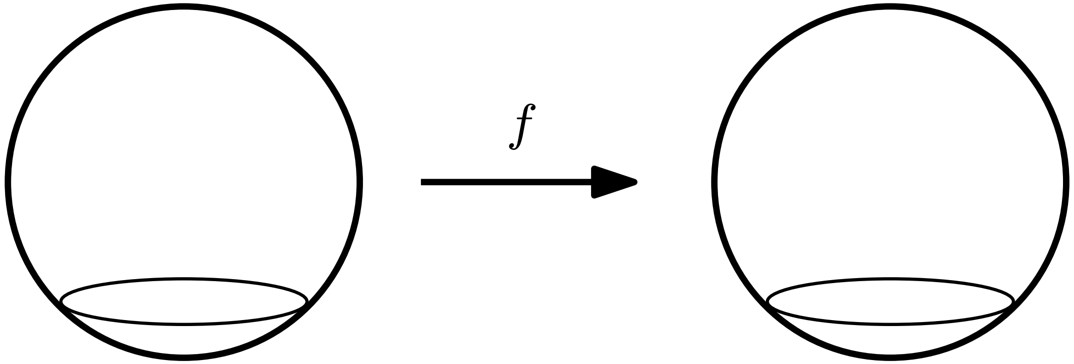
?	EGF' = H	$\frac{n!\rho^n}{n^{5/2}}$	Universality (?)

Branched covers

Branched covers and Hurwitz numbers

Consider a rational function, $f : \mathbb{S} \to \mathbb{S}$,



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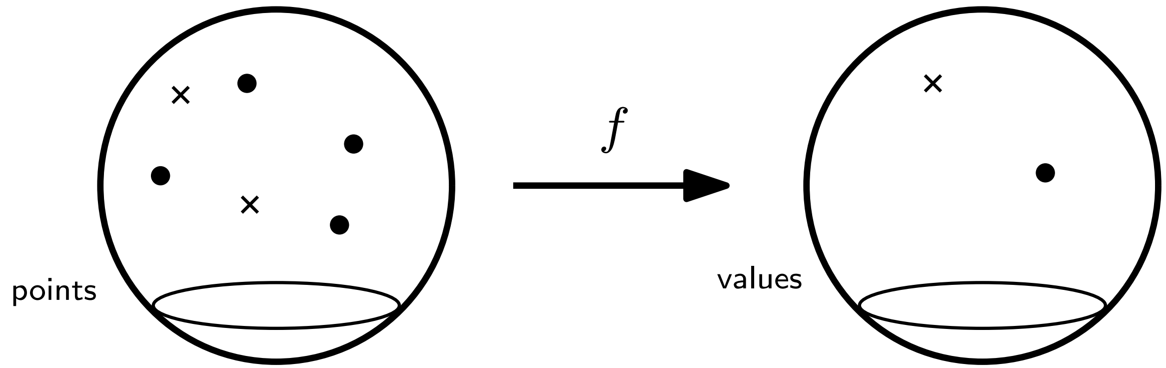

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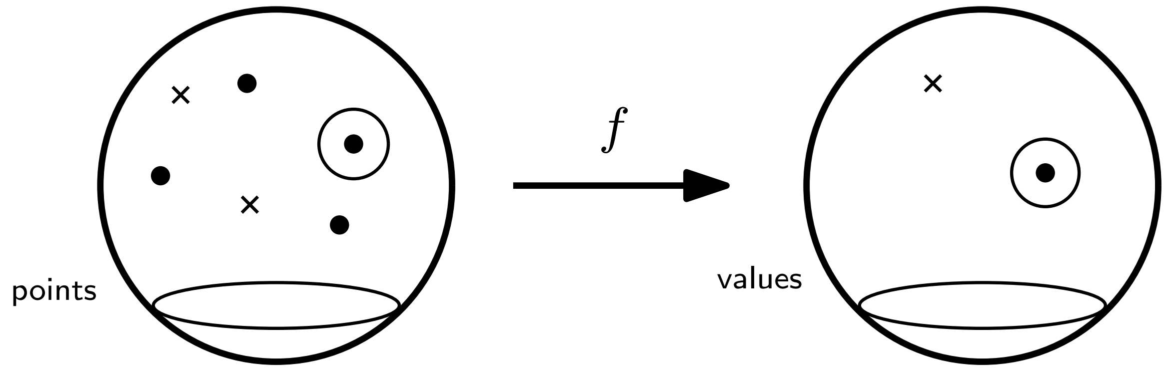

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in other terms, f is locally homeorphic to the map $y \to y$

This is the case for all preimages of regular values.

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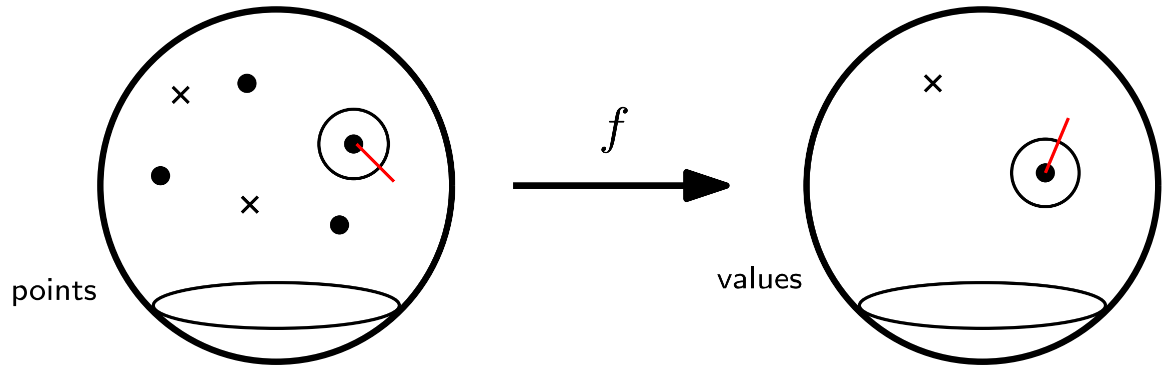

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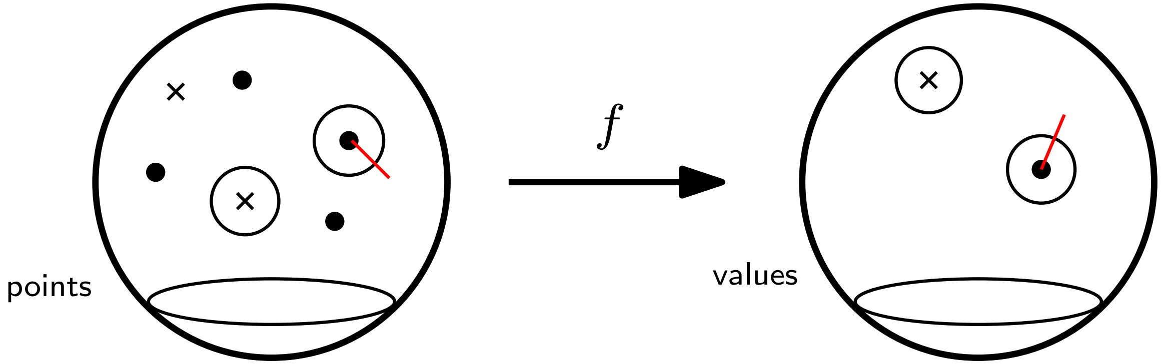

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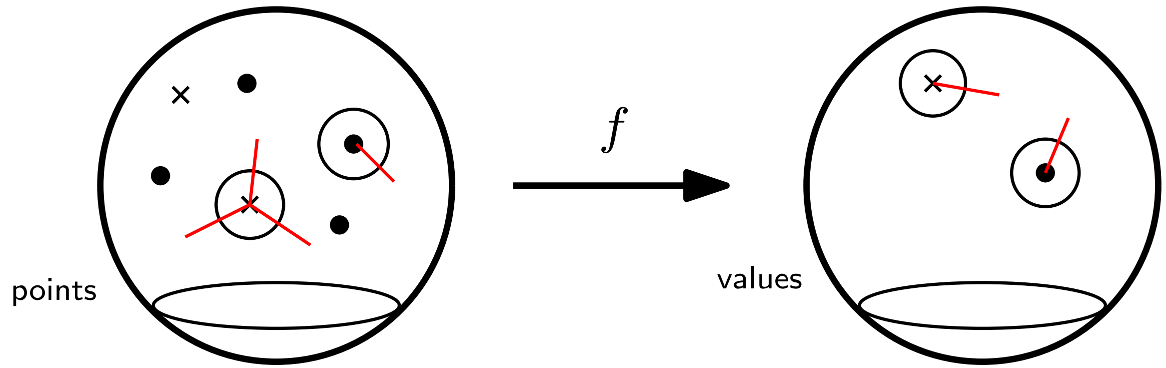

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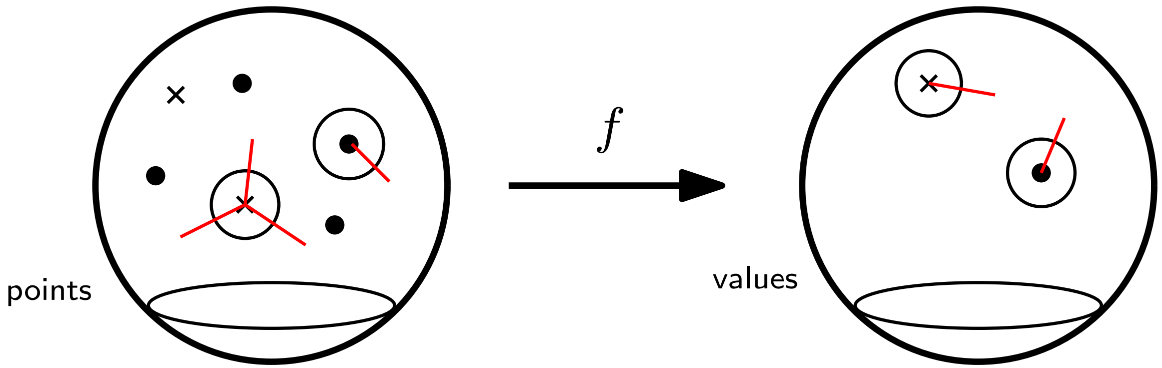

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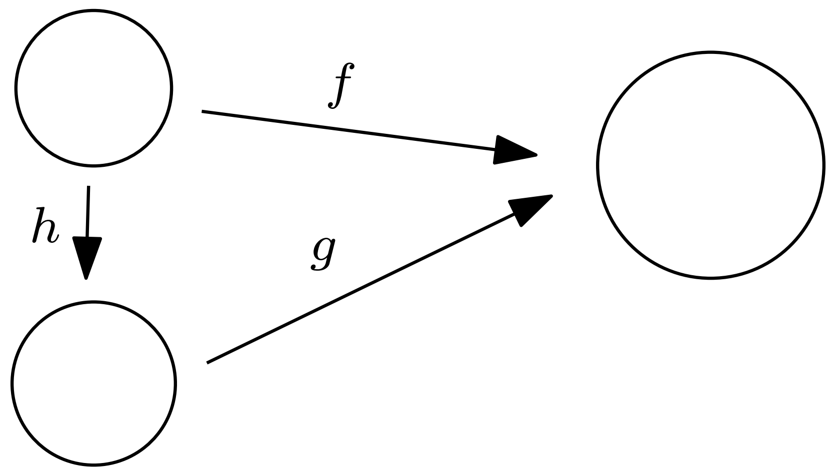
If there are $n - 1$ preimages (and one has $k = 2$), the critical value is called simple

Branched covers and Hurwitz numbers

Rational functions f and g are equivalent if $g = f \circ h$ with h homeomorphism.



Since the global complex structure is lost we are in fact talking about equivalence classes of branched covers instead of rational function.

Hurwitz counting problem is to count branched covers with respect to the number and type of critical values and in particular those with almost only simple critical values.

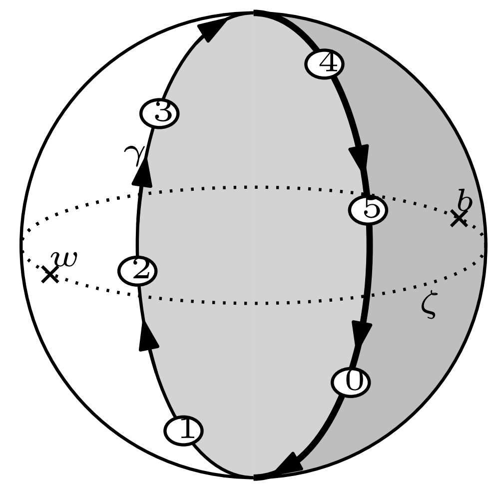
Branched covers and maps

Consider a branched cover, $f : \mathbb{S} \to \mathbb{S}$ with $m + 2$ critical points.

draw a halving circle through the first m critical values, labeled $1, \dots, m$

and label 0 a regular value on the circle.

$\mathbb{S} \quad \longrightarrow$



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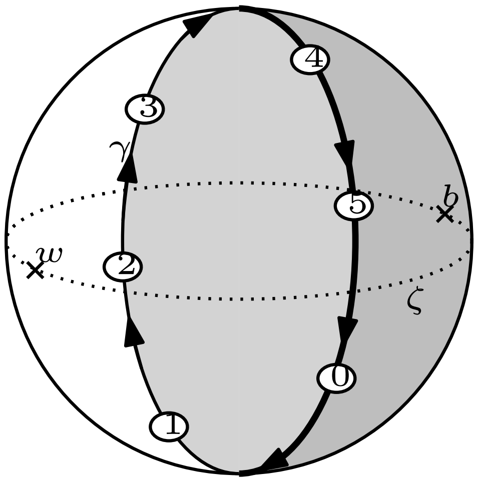

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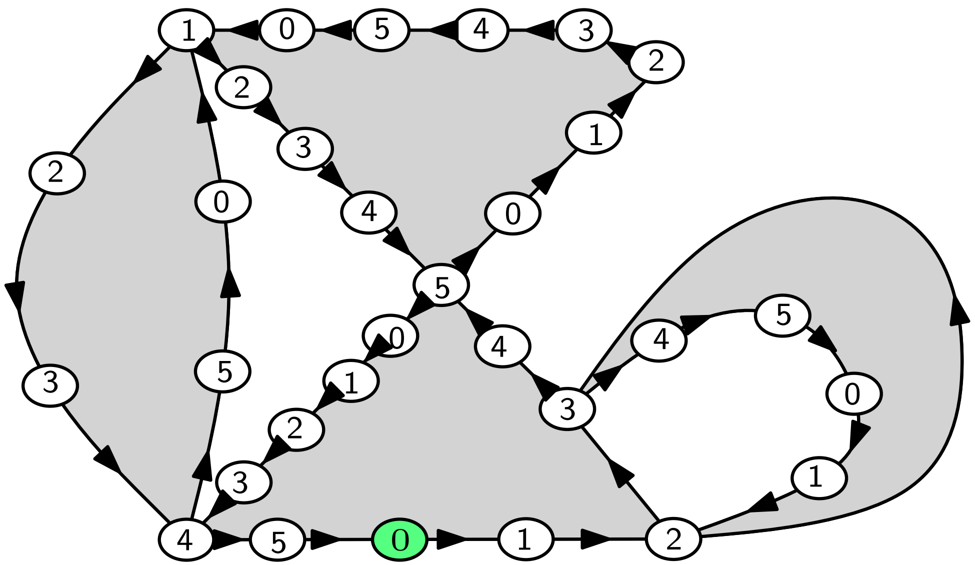

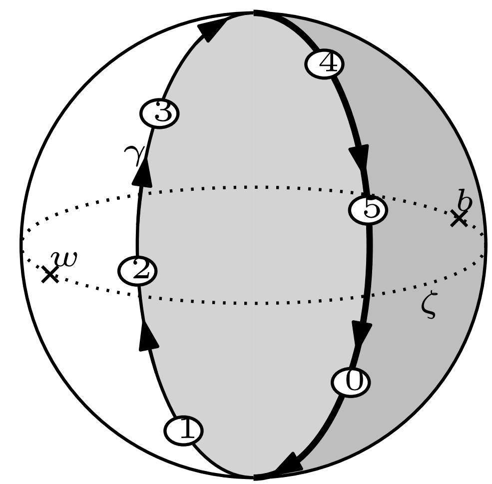

The preimage of the circle forms a bicolored map, called a galaxy.

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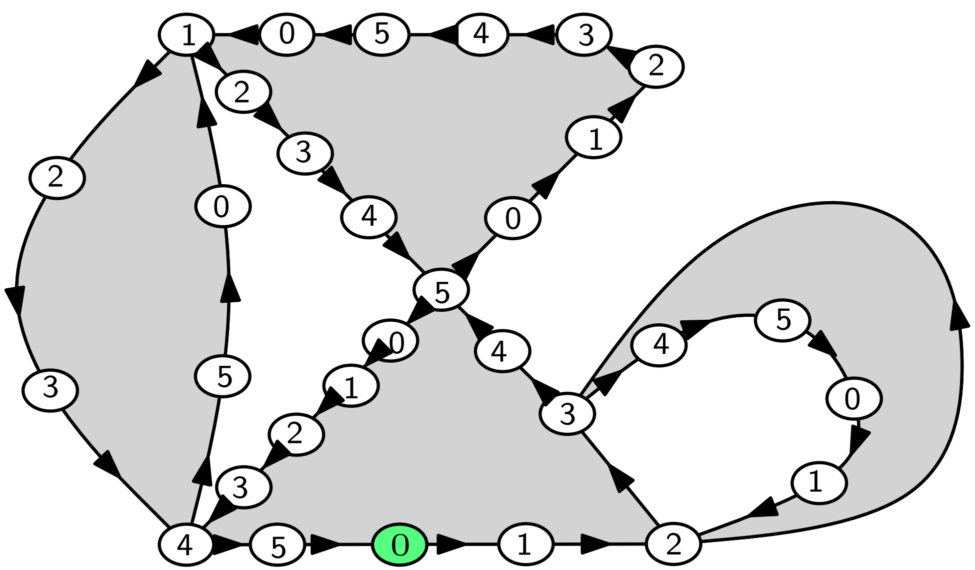

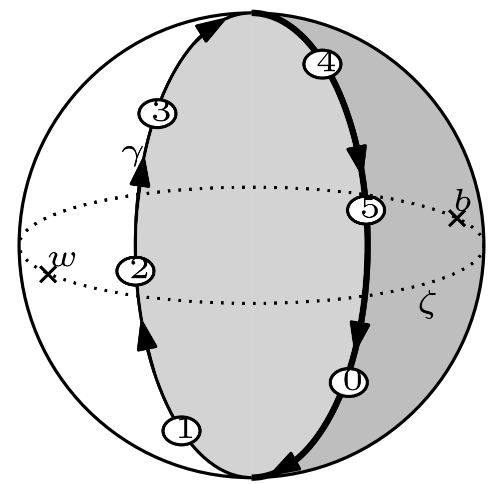

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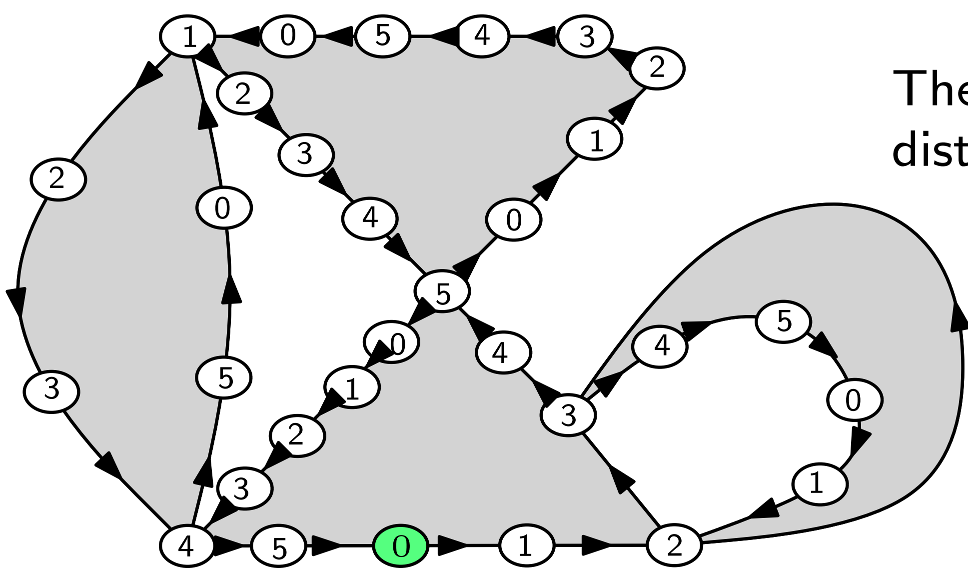
Claim. Galaxies are in correspondence with equivalence classes of branched covers.

Branched covers and maps

Through the pullback, Hurwitz problem is rephrased as a counting problem for planar galaxies: more precisely

a planar m-galaxies is a bicolored map with black and white faces of degree multiple of $m+1$. Its vertices can be labeled $0, 1, \ldots, m$ in counterclockwise order around black faces.

it is simple if exactly one vertex of color i has out degree 2, all other out degree 1.



The type (λ, μ) of a galaxy is the degree distribution of black and white faces.

Let $G(\lambda, \mu)$ be the number of simple planar m-galaxies of type (λ, μ).
The $h(\lambda, \mu) = G(\lambda, \mu)/(m+1)$ are called double Hurwitz numbers.

Branched covers and maps

Hurwitz (1892) proved:

For 0 non-simple critical points, and $2n-2$ simple ones: $h(1^n, 1^n) = n^{n-3}(2n-2)!$
($=$ Galaxies with all faces of degree $m = 2n-2$)

For 1 non-simple critical point of type $\lambda = 1^{\ell_1} \dots n^{\ell_n}$ and $m = n + \ell - 2$ simple ones:

$$h(\lambda, 1^n) = n^{\ell-3} \cdot m! \cdot n! \cdot \prod_{i \geq 1} \frac{1}{\ell_i!} \left(\frac{i^i}{i!} \right)^{\ell_i}$$

($=$ Galaxies with all black faces of degree $m = 2n-2$, and white degrees given by λ)

For 2 non-simple critical point of type λ and μ and $\ell(\lambda) + \ell(\mu) - 2$ simple ones:

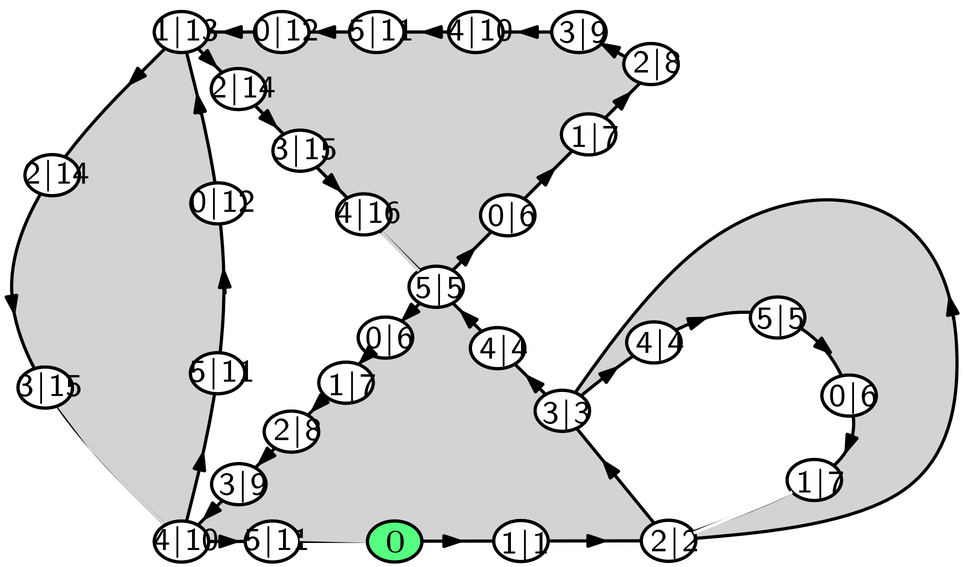
for $h(\lambda, \mu)$ Hurwitz had no formulas...

But several polynomiality properties have been observed in the last 15 years.

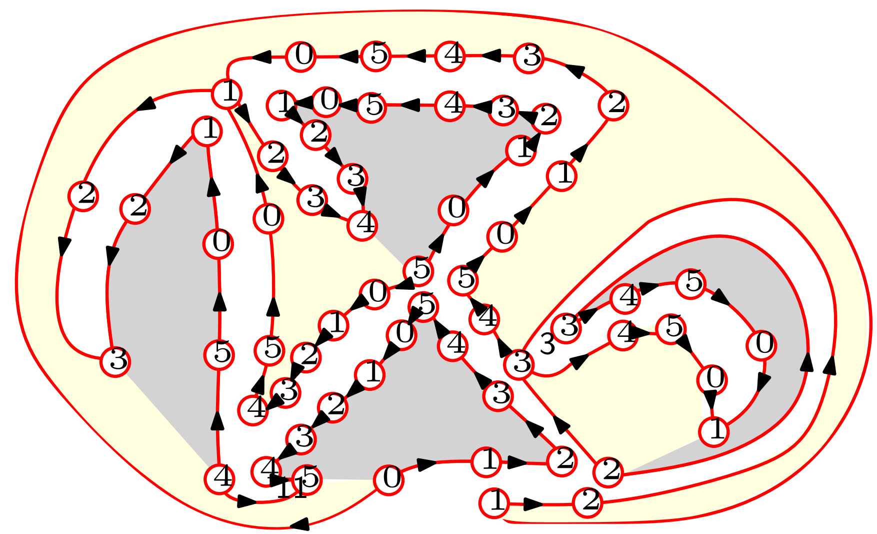
The bijective approach for branched covers

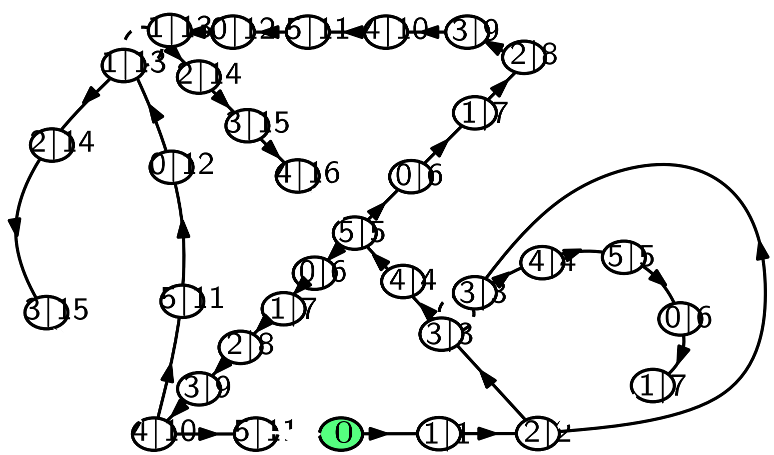
Let's apply the decomposition to galaxies

The geodesic cutting strategy on bipartite maps: BDFG bijection.



The polygonal net has bicolored faces of degree multiple of m, keeping track of the original face degrees.





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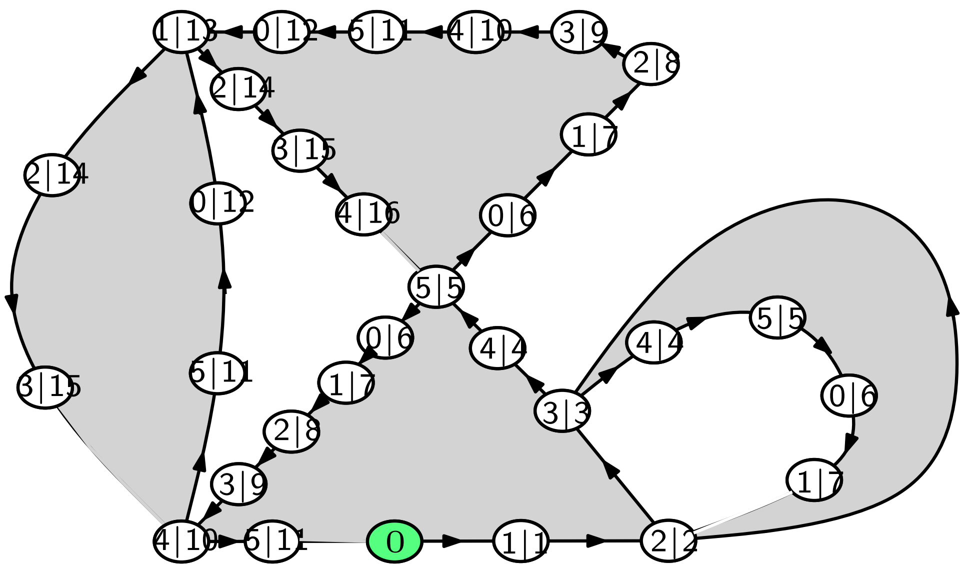

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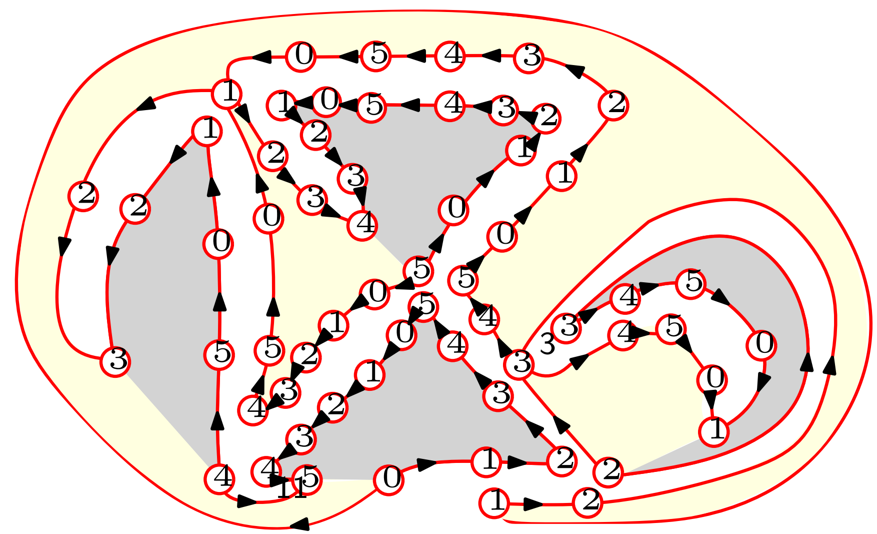

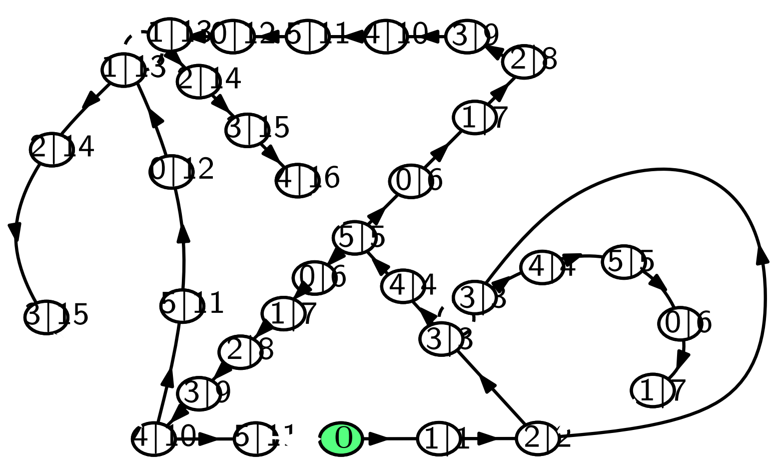

The only new trick is to find a way to record the simplicity condition.

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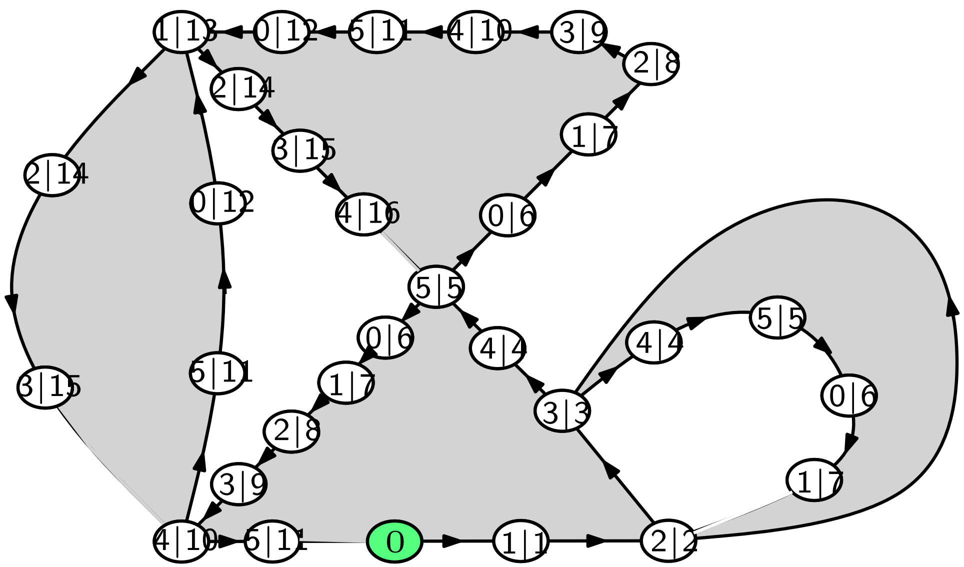

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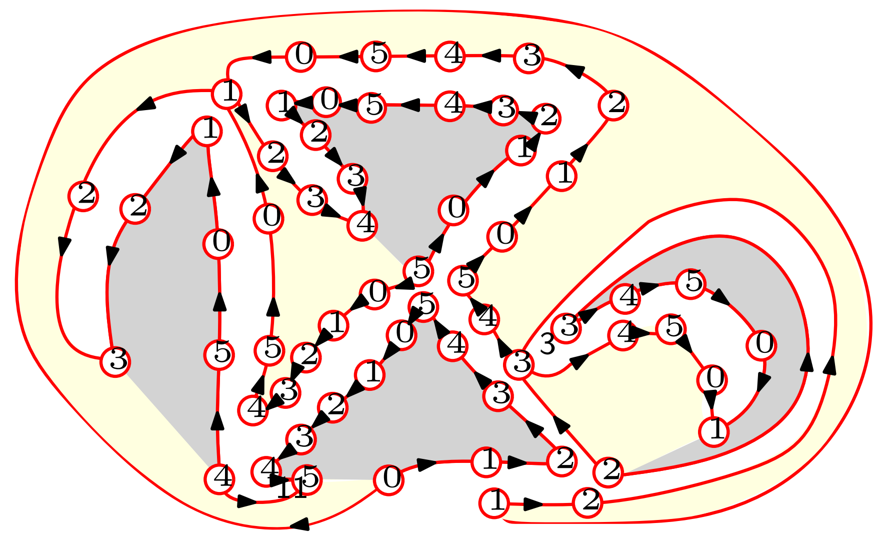
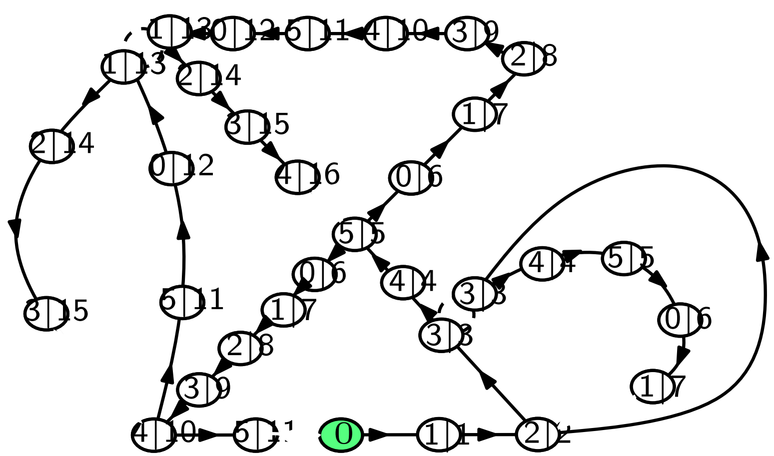


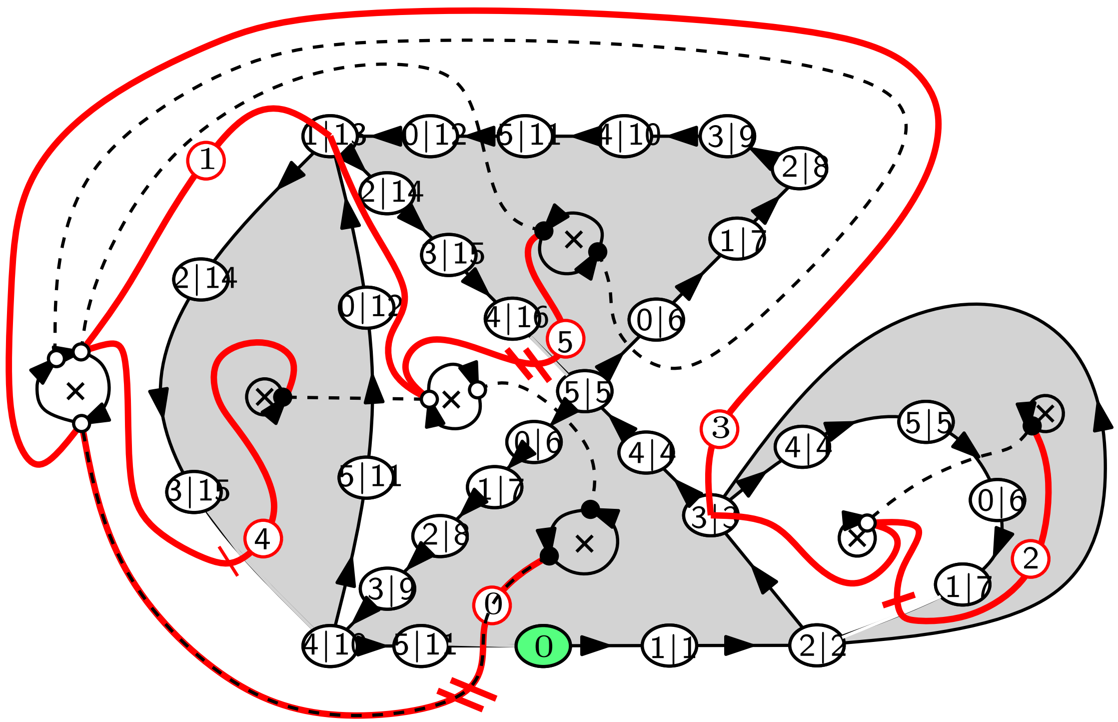


The only new trick is to find a way to record the simplicity condition.

Lemma: Vertices of degree 2 of galaxy correspond to cut points and inner edges of the net.

Let's apply the decomposition to galaxies

Again the local configurations in polygons can be classified and the polygonal net can be encoded by some labeled cactus.



Hurwitz mobile

Double Hurwitz numbers $h(\lambda, \mu)$ are thus numbers of Hurwitz mobile of type $\lambda.\mu$:

A Hurwitz cactus is a graph (not map) made of:

λ oriented cycles of white vertices (the white polygons)

μ oriented cycles of black vertices (the black polygons)

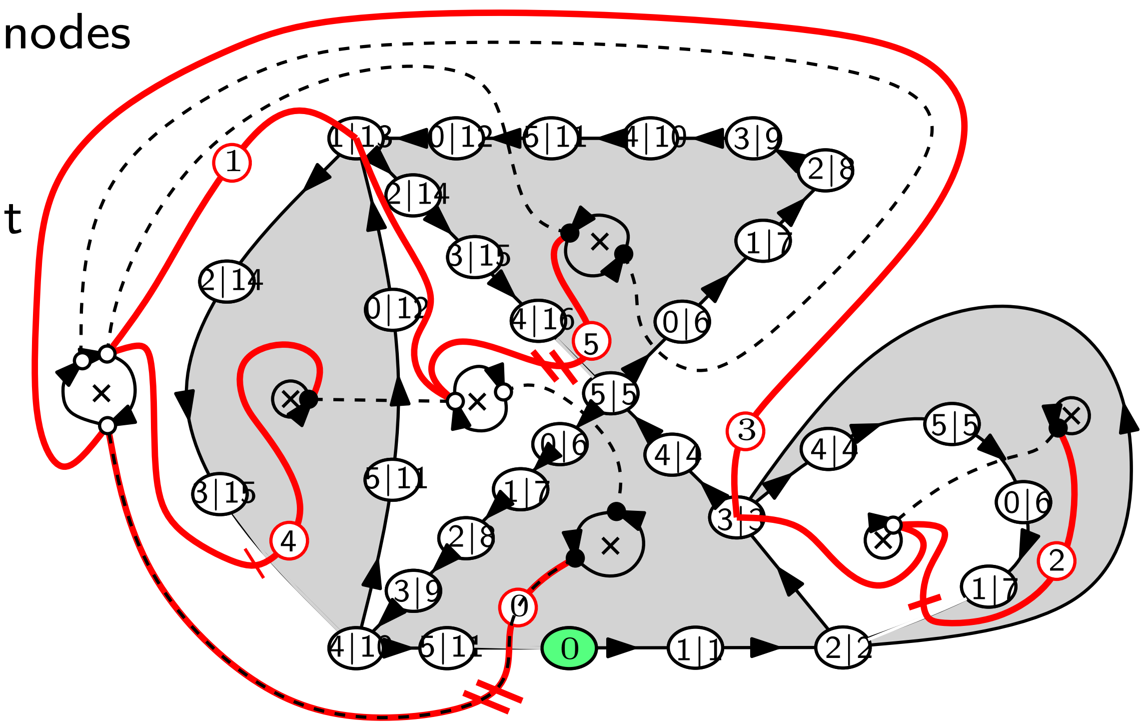
$\ell(\lambda) + \ell(\mu) - 1$ edges with labels $\{0, 1, \ldots, m\}$ and non negative weights (bars)

such that:

– edges with weight 0 connect white nodes

– edges with positive weight connect a black to a white nodes

– the sum of weights of edges incident to a i-gon equals i.



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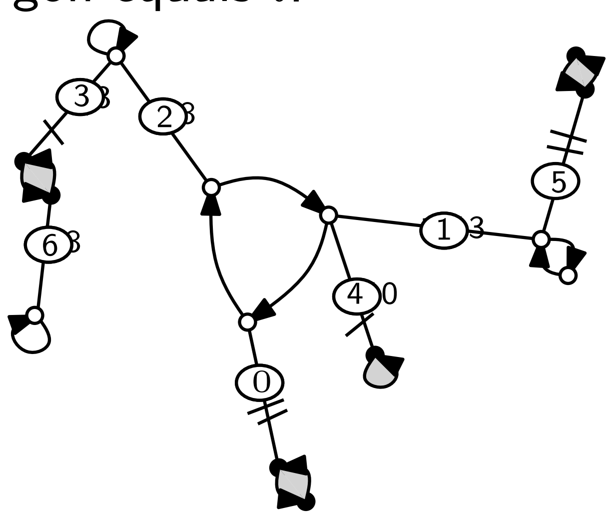
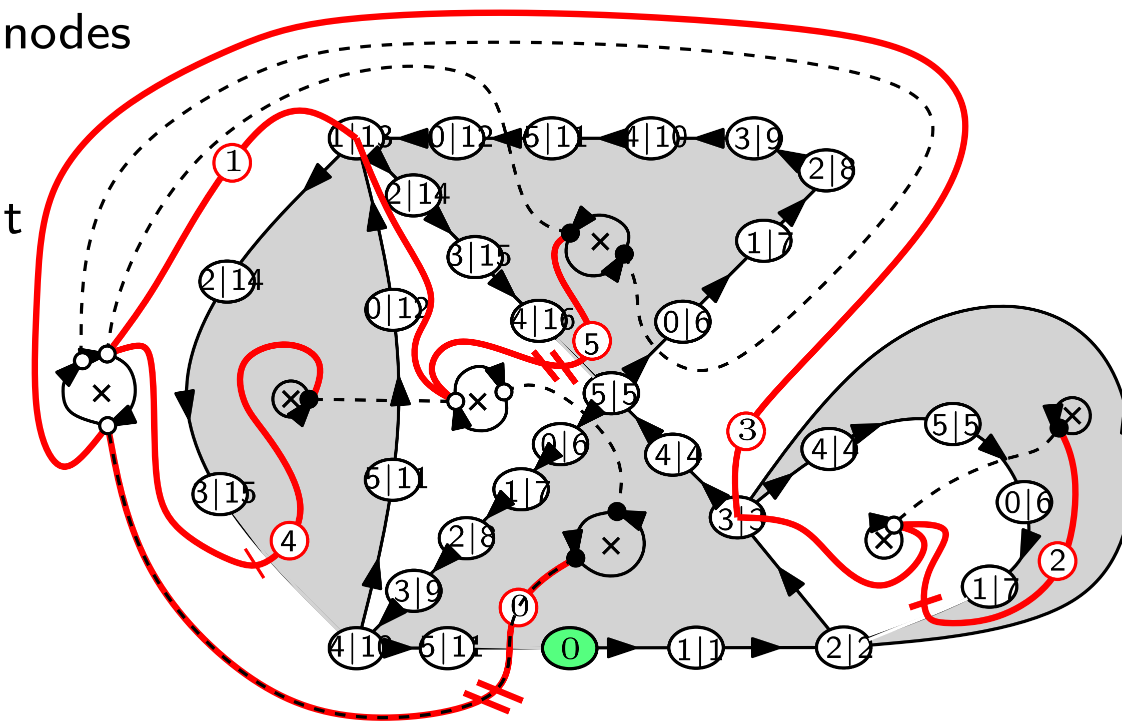




In the special case of simple critical points

In the case $\lambda = \mu = 1^n$, the Hurwitz mobiles simplify:

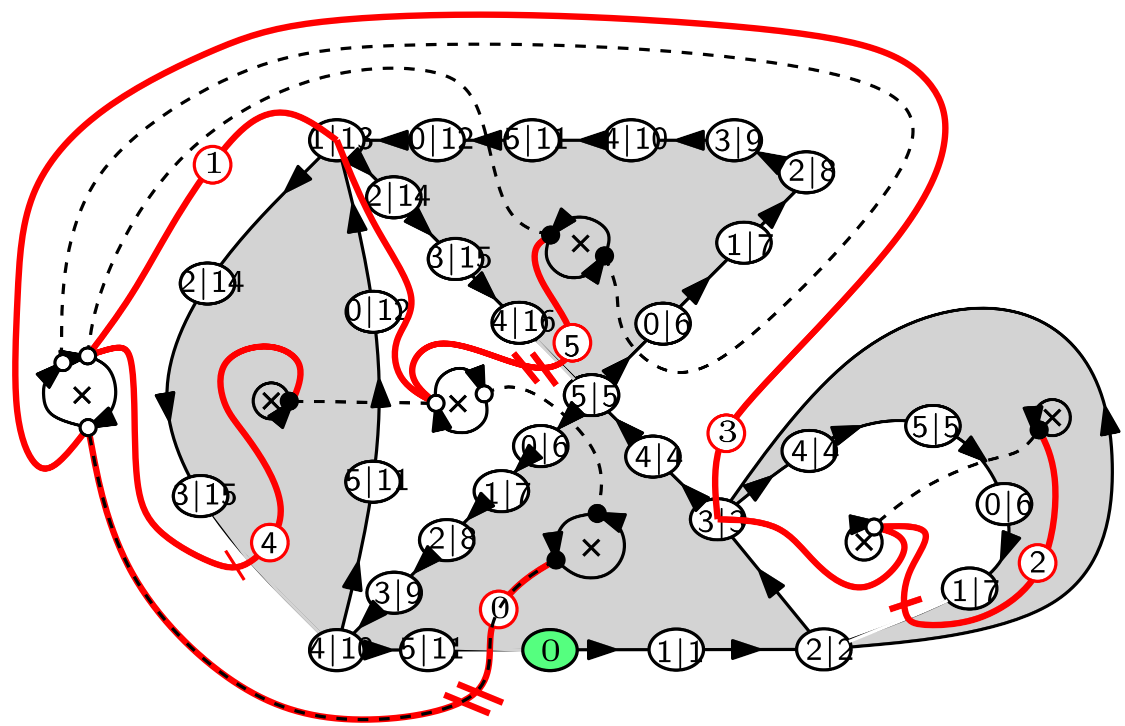
Black faces of galaxy have degree $m+1 \Rightarrow$ Black polygons are 1-gons.

White faces of galaxy have degree $m+1 \Rightarrow$ white polygons are 1-gons

Sum of weight of edges at each vertex is 1

$\Rightarrow$ black 1-gons are leaves.

$\Rightarrow$ each white 1-gons is incident to one leaf.



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$\Rightarrow$ black 1-gons are leaves.

$\Rightarrow$ each white 1-gons is incident to one leaf.

So in the end we get:

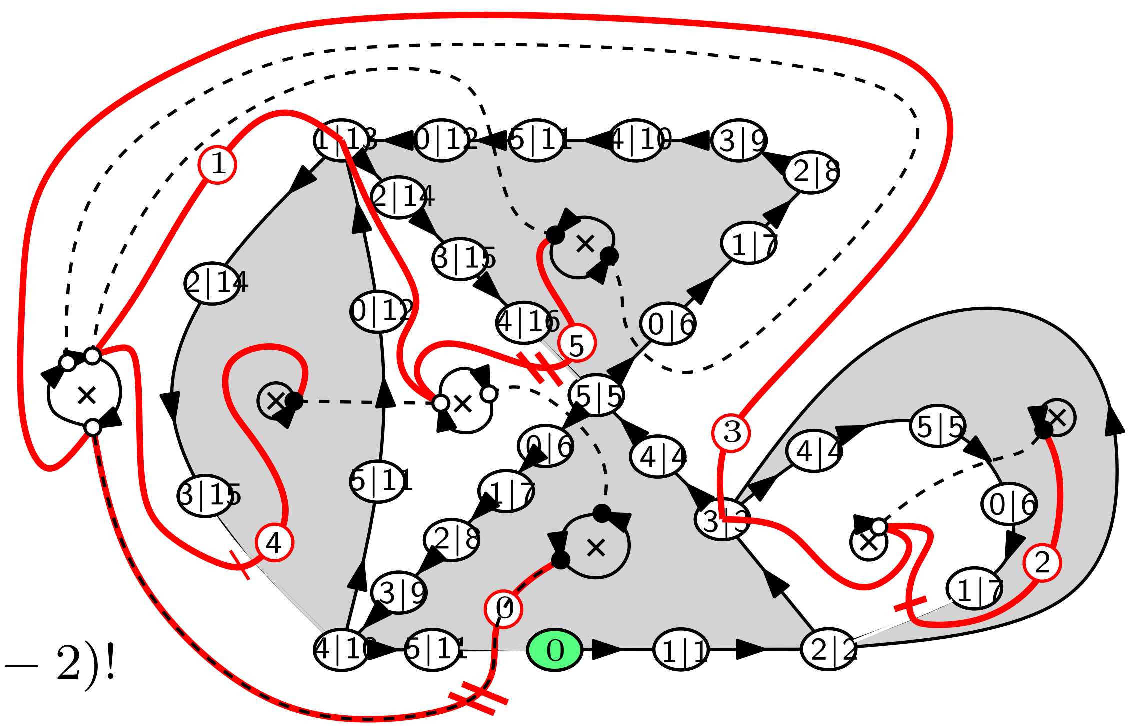
Trees with $2n-1$ labeled edges

such that each inner vertex is incident to one leaf.

Their EGF with respect to edges is

$$G(z) = z\exp(zG(z))$$

from which we recover $h_n = n^{n-3}(2n-2)!$



Distances in galaxies and branched covers

It is possible to keep track on the Hurwitz mobile of distances in the initial galaxy.

But planar galaxies of type $(1^n, 1^n)$ are made of $2n$ labelled $(2n-1)$-gons, glued by 1 vertex of degree 2 for each color.

$\Rightarrow$ distance have to be at least of order n.

In fact a rougth analysis of the labels of trees show that $n^{5/4}$ should be right order.

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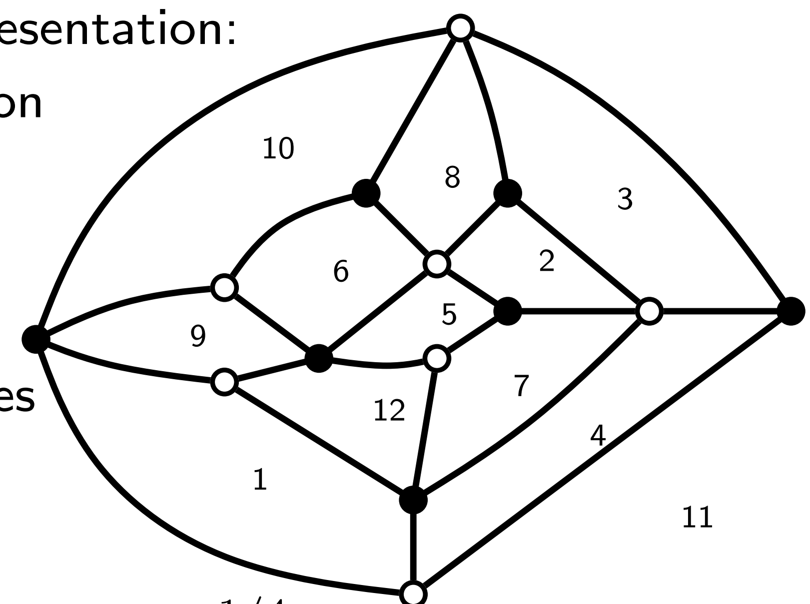
However these galaxies also have a nice dual representation:

An increasing quadrangulation is a quadrangulation with n black and n white vertices and faces with labels $1, \ldots, n$ that are

– "clockwise increasing" around white vertices

– "counterclockwise "increasing" around black ones



We expect distances in this representation to be of order $n^{1/4}$.

Distances in galaxies and branched covers

Recall that the number of quadrangulations with n faces is $\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$

Theorem [Le Gall, Miermont] (informal statement): Let $\mathbb{Q}_n$ be a uniform random rooted planar quadrangulation:

$$\Pr(\mathbb{Q}_n = q) = \frac{1}{\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}} \qquad \text{for all } q \in \mathcal{Q}_n,$$

then $\mathbb{Q}_n$ with distances rescaled by $n^{-1/4}$ converges to the Brownian map.

Conjecture (informal statement): Let $\mathbb{Q}_n^\ell$ be a uniform random increasing planar quadrangulation:

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A pattern in enumeration and random structures

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Plane trees	$P = 1 + zP^2$	$\frac{4^n}{n^{3/2}}$	Continuum Random Trees (CRT)
Simple trees	$\mathbb{N}$-algebraic OGF	$\frac{\rho^n}{n^{3/2}}$	Universality (proved)
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Planar maps	(OGF)$' = T$	$\frac{12^n}{n^{5/2}}$	Brownian map
Planar maps with degree constraints	$\mathbb{N}$-algebraic (OGF)'	$\frac{\rho^n}{n^{5/2}}$	Universality (many examples)
Simple branched covers	EGF' = H	$\frac{n!\rho^n}{n^{5/2}}$	Universality (to be proved)

Conclusion

Conclusion and open problems

Bijections for planar maps have labelled analogs in the context of simple branched covers.

The bijection allow to obtain new (complicated) formulas about double Hurwitz numbers

Some open problems:

We claim that random increasing quadrangulations are like random map: what about standard map parameters? Start with the degree of a random vertex?

Prove convergence to the Brownian map of galaxies or increasing quadrangulations..

Compute exactly the GF of a galaxy with marked point at distance k.

Equivalently, compute the GF of embedded Hurwitz mobiles with a vertex at position k.

Thank you

Plane trees and Catalan numbers

Some like Catalan trees and OGF (chapter 1 of The Book of Analytic Combinatorics)

Some prefer Cayley trees and EGF (chapter 2 of The Book of Analytic Combinatorics)

But everyone here knows the symbolic method (I hope...):

Exercice (Chapuy). Consider the example of m-ary trees:

Spec of m-ary trees: $\quad \Delta = \underbrace{(\cdot + \Delta)\cdots(\cdot + \Delta)}_{m}$

Let $m \to \infty$ with n fixed and conclude that the EGF $C(y)$ of rooted Cayley trees satisfies $C(y) = y\exp(C(y))$

Plane trees and Catalan numbers

Recall that Cayley tree with n vertices are:

Acyclic connected graph with n labeled vertices.

or equivalently

Acyclic connected graphs with $n-1$ labeled edges and a marked vertex

Proof (Chapuy): We have $T = z(1+T)^M$.

When $m \to \infty$ with n fixed, a random m-ary tree almost surely does not have two edges with same index:

$$T_{m,n} \underset{n\to\infty}{\sim} \binom{m}{n-1} \mathrm{Cay}(n) \underset{n\to\infty}{\sim} \frac{m^{n-1}}{(n-1)!} \mathrm{Cay}(n)$$

Hence $C(y) = mT(y/m) + O(\frac{1}{m})$

And $C(y) = z(1+\frac{1}{m}C(y))^m = z\exp(C(y)) + O(\frac{1}{m})$

Plane trees and Catalan numbers

We will use a variant of the previous derivation.

Now return to m-ary trees:

$$\Delta = \underbrace{(\cdot + \Delta) \cdots (\cdot + \Delta)}_{m}$$

Translate to OGF: $\quad T = (1 + x_1 T) \cdots (1 + x_m T)$

where x_i keeps track of the number of edges of type i

Then the number $\mathrm{Cay}(n)$ of Cayley trees with n vertices is $[x_1 \cdots x_{n-1}]T$.

Now $T = z\Phi(T)$ so $[z^n]T = \frac{1}{n}[y^{n-1}]\Phi(z)^n$, by Lagrange Inversion (Chapitre 1)

Hence we want the coefficient of $x_1 \cdots x_{n-1}$ in $\frac{1}{n}((1 + x_1) \cdots (1 + x_{n-1}))^n$

That is: $\mathrm{Cay}(n) = \frac{1}{n} n^{n-1} = n^{n-2}$