THE NUMBER OF Z-CONVEX POLYOMINOES

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ABSTRACT. In this paper we consider a restricted class of polyominoes that we call Z-convex polyominoes. Z-convex polyominoes are polyominoes such that any two pairs of cells can be connected by a monotone path making at most two turns (like the letter Z). In particular they are convex polyominoes, but they appear to resist standard decompositions. We propose a construction by "inflation" that allows to write a system of functional equations for their generating functions. The generating function $P(t)$ of Z-convex polyominoes with respect to the semi-perimeter turns out to be algebraic all the same and surprisingly, like the generating function of convex polyominoes, it can be expressed as a rational function of t and the generating function of Catalan numbers.

1. INTRODUCTION

1.1. **Convex polyominoes.** In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells lying on a vertical (horizontal) line. For the main definitions and results concerning polyominoes we refer to [S] and, for french aware readers, to [BM]. Invented by Golomb [G2] who coined the term *polyomino*, these combinatorial objects are related to many mathematical problems, such as tilings [BN, G1], or games [Ga] among many others. The enumeration problem for general polyominoes is difficult to solve and still open. The number a_n of polyominoes with n cells is known up to $n = 56$ [JG] and asymptotically, these numbers satisfy the relation $\lim_n (a_n)^{1/n} = \mu$, $3.96 < \mu < 4.64$, where the lower bound is a recent improvement of [BMRR].
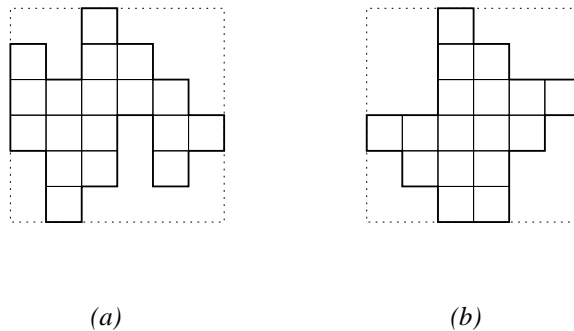

FIGURE 1. (*a*) a column-convex (but not convex) polyomino; (*b*) a convex polyomino.

In order to probe further, several subclasses of polyominoes have been introduced on which to hone enumeration techniques. One natural subclass is that of *convex* polyominoes. A polyomino is said to be *column-convex* [*row-convex*] when its intersection with any vertical [horizontal] line of cells in the square lattice is connected (see Fig. 1 (a)), and *convex* when it is both column and row-convex (see Fig. 1 (b)). The *area* of a polyomino is just the number of cells it contains, while its *semi-perimeter* is half the length of the boundary. Thus, in a convex polyomino the semi-perimeter is the sum of the numbers of its rows and columns. Moreover, any convex polyomino

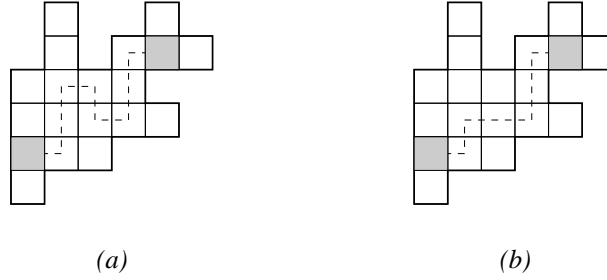<center>(a)   (b)</center>

FIGURE 2.   (a) a path between two highlighted cells in a polyomino; (b) a monotone path between the same cells, made only of north and east steps.

is contained in a rectangle in the square lattice which has the same semi-perimeter (called the *minimal bounding rectangle* of the polyomino).

The number f_n of convex polyominoes with semi-perimeter $n + 2$ was obtained by Delest and Viennot, in [DV]:

$$f_{n+2} = (2n + 11)4^n - 4(2n + 1)\binom{2n}{n}, \quad n \geq 0; \quad f_0 = 1, \quad f_1 = 2.$$

In particular the generating function of convex polyominoes with respect to the semi-perimeter

$$F(t) = \sum_{n \geq 0} f_n t^{n+2} = t^2 + 2t^3 + 7t^4 + 28t^5 + 120t^6 + 528t^7 + O(t^8)$$

is an algebraic series which is has a rational expression $R_0(t, d(t))$ in t and the Catalan generating function

$$c(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + O(t^7).$$

More precisely, the generating function of convex polyominoes with respect to the numbers of columns (variable x) and rows (variable y) is

$$F(x, y) = \frac{8x^2 y^2 d(x, y)}{\Delta^2} + \frac{xy(1 - x - xy - y)}{\Delta},$$

where $d(x, y)$ is the unique power series satisfying the relation $d = (x + d)(y + d)$,

$$d(x, y) = \frac{1}{2}(1 - x - y - \sqrt{\Delta}),$$

and,

$$\Delta = (1 - x - y)^2 - 4xy = (1 - x - y)^2(1 - \tfrac{4xy}{(1-x-y)^2}).$$

Observe that $d(t) := d(t, t)$ is just a shifted version of the Catalan generating function,

$$d(t) = t(c(t) - 1) = \frac{1}{2}(1 - 2t - \sqrt{1 - 4t}) = t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 + 132t^7 + O(t^8).$$

Incidentally, $d(x, y)$ is the generating function of parallel polyominoes with respect to the numbers of columns and rows.

1.2. **Monotone paths and k-convexity.** In [CR03] the authors observed that convex polyominoes have the property that every pair of cells is connected by a *monotone path*. More precisely, a *path* in a polyomino is a self-avoiding sequence of unitary steps of four types: north $N = (0, 1)$, south $S = (0, -1)$, east $E = (1, 0)$, and west $W = (-1, 0)$. A path is *monotone* if it is made with steps of only two types. Given a path $w = u_1 \ldots u_k$, with $u_i \in \{N, S, E, W\}$, each pair of steps $u_i u_{i+1}$ such that $u_i \neq u_{i+1}$, $0 < i < k$, is called a *change of direction*. These definitions are illustrated by Fig. 2, in which the non monotone path (a) has 6 changes of direction and the monotone path (b) has 4 changes of direction.

The authors of [CR03] further proposed a classification of convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino. More precisely,
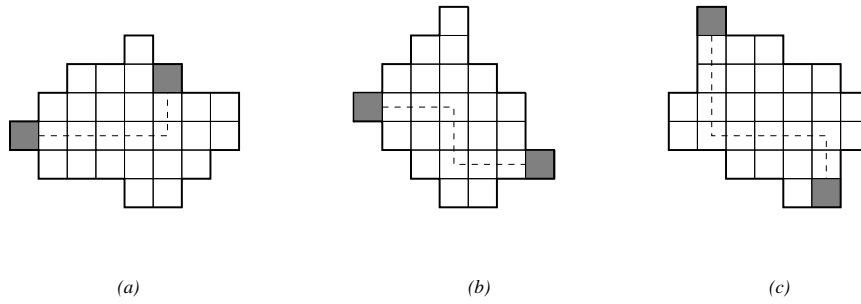
FIGURE 3. (a) a L-convex polyomino, and a monotone path with a single change of direction joining two of its cells; (b) a Z-convex but not L-convex polyomino: the two highlighted cells cannot be connected by a path with only one change of direction; (c) a centered polyomino (not L-convex).

a convex polyomino is k-convex if every pair of its cells can be connected by a monotone path with at most k changes of direction. In a convex polyomino of the first level of this classification, any two cells can be connected by a path with at most one change of direction: in view of the L-shape of these paths, 1-convex polyominoes are also called L-convex. The reader can easily check that in Fig. 3, the polyomino (a) is L-convex, while the polyominoes (b), (c) are not, but are 2-convex.

This class of polyominoes has been considered from several points of view: in [CR05] it is shown that the set of L-convex polyominoes is well-ordered with respect to the sub-picture order, in [CFRR1] the authors have investigated some tomographical aspects of this family, and have shown that L-convex polyominoes are uniquely determined by their horizontal and vertical projections. Finally, in [CFRR2] it is proved that the number g_n of L-convex polyominoes with semi-perimeter $n + 2$ satisfies the recurrence relation:

$$g_n = 4g_{n-1} - 2g_{n-2}, \qquad n \geq 3,$$

with $g_0 = 1$, $g_1 = 2$, $g_2 = 7$. In other terms the generating function of L-convex polyominoes is rational:

$$
\begin{aligned}
G(t) &= \sum_{n \geq 0} g_n t^{n+2} = t^2 + 2t^3 + 7t^4 + 24t^5 + 82t^6 + 280t^7 + O(t^8) \\
&= \frac{1 - 2t + t^2}{1 - 4t + 2t^2}.
\end{aligned}
$$

Indeed, in [CFMRR], the authors have provided an encoding of L-convex polyominoes by words of a regular language, and have furthermore studied the problem of enumerating L-convex polyominoes with respect to the area.

In view of the definition of L-convex polyominoes as 1-convex polyominoes, it is natural to investigate which of the previous properties remain true for some classes of k-convex polyominoes, with $k > 1$. Concerning enumeration in particular, one would like to know if the generating functions of k-convex polyominoes are rational, algebraic, or holonomic.

1.3. **Z-convex polyominoes.** In the present paper we deal with the family of *2-convex polyominoes*, which we rename Z-convex polyominoes in analogy with the L-convex notation. We shall prove the following results:

Theorem 1.1. *The generating function $P(t)$ of Z-convex polyominoes with respect to the semi-perimeter is*

$$
\begin{aligned}
P(t) &= \sum_{n \geq 0} p_n t^{n+2} = t^2 + 2t^3 + 7t^4 + 28t^5 + 116t^6 + 484t^8 + O(t^8), \\
&= \frac{2t^4(1 - 2t)^2 d(t)}{(1 - 4t)^2(1 - 3t)(1 - t)} + \frac{t^2(1 - 6t + 10t^2 - 2t^3 - t^4)}{(1 - 4t)(1 - 3t)(1 - t)},
\end{aligned}
$$

where

$$d(t) = \frac{1}{2}(1 - 2t - \sqrt{1 - 4t}).$$

More generally, the generating function $P(x, y)$ of Z-convex polyominoes with respect to the num-
bers of rows and columns is a rational power series $R(x, y, d(x, y))$ in x, y and the unique power
series $d(x, y)$ solution of the equation $d = (x + d)(y + d)$,

$$d(x, y) = \frac{1}{2}(1 - x - y - \sqrt{\Delta}),$$

where

$$\Delta = (1 - x - y)^2 - 4xy = (1 - x - y)^2 \left(1 - \frac{4xy}{(1 - x - y)^2}\right).$$

More precisely,

$$\begin{aligned}
P(x, y) &= \frac{2x^2 y^2 d(x, y)}{\Delta^2} \frac{(1 - x - y)^2}{((1 - x - y)^2 - xy)} \\
&+ \frac{xy(1 - x - y)^2(1 - x - y - xy) - x^2 y^2(1 - x - y - 3xy)}{\Delta\left((1 - x - y)^2 - xy\right)}.
\end{aligned}$$

As conjectured by Marc Noy [N], the asymptotic number of Z-convex polyominoes with semi-perimeter $n+2$ grows like $n \cdot 4^n$ (more precisely, $p_n \sim \frac{n}{24} \cdot 4^n$, so that $f_n/p_n \to 3$), while the number of L-convex polyominoes grows only like $(2 + \sqrt{2})^n$, and the number of centered polyominoes (see below) grows like 4^n.

The fact that the generating function ends up in the same algebraic extension as convex poly-ominoes looks surprising to us because we were unable to derive it using the standard approaches to convex polyomino enumeration (Temperley-like methods, wasp-waist decompositions, or in-clusion/exclusion on walks). Instead, one interesting feature of our paper is a construction of polyominoes by "inflating" smaller one along a hook. We believe that this approach could in principle allow for the enumeration of k-convex polyominoes in general.

The rest of the paper is organized as follows. The general strategy of decomposition is explained in Section 2. The different cases are listed and the corresponding relations for generating functions are derived in Section 3. Finally the resulting system of equations is solved in Section 4.

2. Classification and general strategy

In order to present our strategy for the decomposition, we need to distinguish between several types of Z-convex polyominoes.

2.1. **Centered polyominoes.** The first class we consider is the set $\mathcal{C}$ of *horizontally centered* (or simply *centered*) convex polyominoes. A convex polyomino is said to be centered if it contains at least one row touching both the left and the right side of its minimal bounding rectangle (see Fig. 3 (c)). Observe that centered polyominoes have a simple characterization in terms of monotone paths:

Lemma 2.1. *A convex polyomino is centered if and only if any pair of its cells can be connected by means of a path $S^{h_1} E^k S^{h_2}$ or $S^{h_1} W^k S^{h_2}$, with $h_1, h_2, k \geq 0$.*

In particular any L-convex polyomino is centered, and, more importantly for us, any centered polyomino is Z-convex, while the converse statements do not hold. Figure 3 (c) shows a centered polyomino which is not L-convex, and Figure 3 (b) a Z-convex polyomino which is not centered.

Centered convex polyominoes can also be described as made of two stack polyominoes glued together at their basis. As we shall see in Section 3.1, this decomposition allows to compute easily their generating function. In particular we shall not need here to deal with the monotone paths.
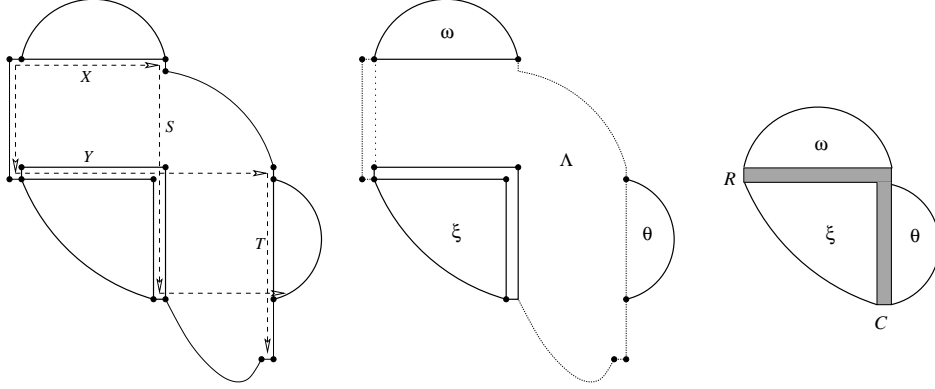
FIGURE 4. A non-centered polyomino of class $\mathcal{D}$ with its rows x, y, and columns s and t; its division into Regions ω, ξ, θ and Λ; its reduction.

2.2. Non centered polyominoes.
Let us thus turn to non centered polyominoes. The starting point of our decomposition is that we wish to remove the leftmost column. By definition of Z-convexity, any two cells must be connected by a path of type $S^{h_1}E^kS^{h_2}$, $S^{h_1}W^kS^{h_2}$, $E^{h_1}N^kE^{h_2}$ or $E^{h_1}S^kE^{h_2}$, with $h_1, h_2, k \geq 0$. In particular, we are interested in the set of cells that can be reached in this way from the cells of the leftmost column.

Let P be a non-centered convex polyomino, and let $c_1(P)$ (briefly, c_1) denote its leftmost column, and let us consider the following rows (as sketched in Fig. 4):

- The row X which contains the top cell of c_1.
- The row Y which contains the bottom cell of c_1.

Since the polyomino P is convex and non-centered, its rightmost column does not intersect any row between X and Y, hence it is placed entirely above X or below Y.

This remark leads to the following definitions:

- A non-centered convex polyomino is *ascending* if its rightmost column is above the row X. Let $\mathcal{U}$ denote the set of descending Z-convex polyominoes.
- A non-centered convex polyomino is *descending* if its rightmost column is below the row Y. Let $\mathcal{D}$ denote the set of ascending Z-convex polyominoes.

The whole set of Z-convex polyominoes is given by the union of the three disjoint sets $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{U}$. Moreover, by symmetry, for any fixed size, $\mathcal{D}$ and $\mathcal{U}$ have the same number of elements, thus, we will only consider non-centered polyominoes of the class $\mathcal{D}$, as the one represented in Fig. 4.

A first property of polyominoes of class $\mathcal{D}$ is the following consequence of their convexity: the boundary path from the end of row X to the end of row Y is made only of south and east steps.

2.3. The strategy.
Let us denote by S and T the columns starting from the rightmost cell of X and Y respectively, and running until they reach the bottom of the polyomino (see Fig. 4). The rows and columns X, Y, S and T allow us to individuate four connected sets of cells in a non-centered convex polyomino, as illustrated by Figure 4:

(1) the set of cells strictly above X, called ω;
(2) the set of cells strictly on the right of T, called θ;
(3) the set of cells that are at the same time below Y and on the left of S, called ξ;
(4) the remaining set of cells, called Λ: these cells are either between X and Y, or between S and T (or both).
- In the previous definitions, the hook H starting horizontally with the left hand part of Y and continuing down with the bottom part of S is included in ξ. The other cells of the row X, Y and columns S and T are included in Λ.

The cells of θ require at least two turns to be reached with a monotone path from the cells of c_1. The Z-convexity thus induces a restriction on the position of the lowest cells of θ.
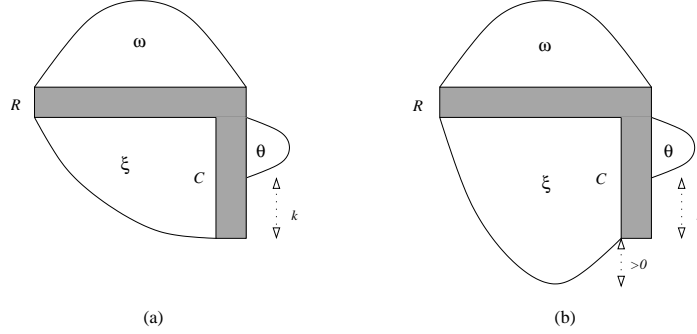
FIGURE 5. (a) a hooked polyomino with hook of type A and (b) one with hook of type B.

Property 2.1. *The region θ of a non-centered Z-convex polyomino contains no cell lower than the lowest cell of its column S.*

If a row between X and Y reaches the right side of the bounding box, the polyominio is centered:

Property 2.2. *The set θ of a non-centered convex polyomino is non empty.*

As already mentioned, we wish to decompose polyominoes of $\mathcal{D}$ by removing the leftmost column. For the decomposition to be bijective we then need to be able to replace a column to the left of a polyomino. But, as the reader can verify, if one takes a Z-convex polyomino and add a leftmost column, it is not so easy to grant *a priori* that Property 2.1 will be satisfied by the rows and columns X, Y, S, and T of the grown polyomino.

In order to circumvent this problem, our decomposition will consist into removing the whole region Λ together with the leftmost column. More precisely, given a descending polyomino P, let us define its *reduction* $\Phi(P)$ as the polyomino obtained as follows (see Figure 4):

- glue region ω to ξ, keeping the relative abscissa of cells between ω and ξ;
- glue region θ to $\omega \cup \xi$ by keeping the relative ordinates of cells between ξ and θ.

Since the hook H is kept in $\Phi(P)$, ω and ξ have at least one common column (as soon as ω is non empty) and ξ and θ have at least one common row, so that the reduction makes sense and it is a polyomino, in which we highlight the hook H. (The hook is highlighted in order to make easier the forthcoming description of the inverse construction.)

The following lemma explains our interest in this reduction.

Lemma 2.2. *A descending convex polyomino is Z-convex if and only if it satisfies Property 2.1 and its reduction $\Phi(P)$ is Z-convex.*

Proof. Assume first that P is Z-convex. Then Property 2.1 is satisfied and a monotone path connecting a cell x to a cell y of $\Phi(P)$ can easily be constructed from the monotone path connecting x and y in P: any section of the path in the deleted region Λ can be replaced by a simpler section in the hook.

Conversely assume that $\Phi(P)$ is Z-convex, that P satisfies Property 2.1 is satisfied, and let (x, y) be two cells of P. If x and y are not in Λ then there exists a monotone path in $\Phi(P)$ connecting these points, and there is no need to add a turn to extend this path into a monotone path in P. If x belongs to Λ, one easily construct the path in each case $y \in \omega$, $y \in \xi$ and, using Lemma 2.1, $u \in \theta$. $\qquad\square$

To characterize the set of polyominoes that can occur in the image of $\mathcal{D}$ by Φ, let us define a *hooked polyomino* as a polyomino P of $\mathcal{C} \cup \mathcal{D}$ in which a hook is highlighted, in such a way that

- the hook is made of a top row (the *arm* of hook) starting in the leftmost column of P and traversing the polyomino, and a partial column (the *leg* of the hook) starting in the right most cell of the top row and including all cells below in this column,
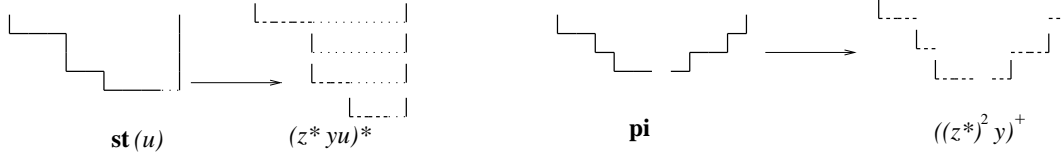- the region on the right hand side of the hook is non empty.

FIGURE 6. The decomposition of a staircase $\mathbf{st}(u) = (z^*yu)^*$ and of a non empty pile $\mathbf{pi} = ((z^*)^2 y)^+$.

The hook is called a *hook of type A* if its bottom cell belongs to the lowest row of P, and a *hook of type B* otherwise (see Figure 5). The following lemma is an immediate consequence of the definition of hooked polyominoes.

Property 2.3. *The reduction of a polyomino of $\mathcal{D}$ is a hooked polyomino.*

In view of Lemma 2.2, our strategy will consist in the description of the types of region Λ that can be added to a hooked polyomino so that the "inflated" polyomino satisfies Property 2.1.

2.4. **Generating functions.** We shall compute the generating function $P(x, y)$ of Z-convex polyominoes with respect to the number of columns, or *width* (variable x) and to the number of rows, or *height* (variable y). In order to do that we shall need generating functions of hooked polyominoes with respect to the height and the width, but also with respect to an auxiliary parameter k which will be marked by a variable u: given a hooked polyomino, the parameter k is a non negative integer indicating the difference of ordinate between the lowest cell of the leg of the hook and the lowest cell of the next column to the right (see Figure 5). This definition makes sense since the region on the right hand side of the hook is assumed non empty.

We shall more precisely use the generating functions

- $C_A(x, y, u)$ of hooked centered polyominoes with hook of type A,
- $C_B(x, y, u)$ of hooked centered polyominoes with hook of type B,
- $A(x, y, u) = \sum_k a_k(x, y)u^k$ of hooked non-centered polyominoes with hook of type A,
- $B(x, y, u) = \sum_k b_k(x, y)u^k$ of hooked non-centered polyominoes with hook of type B.

Most of the time we drop the variables x, y and use the shorthand notation $A(u) = A(x, y, u)$, $a_k = a_k(x, y)$, etc.

3. DECOMPOSITIONS

We shall need the following elementary notations and results, illustrated by Figure 6:

- The sequence notation for formal power series is $w^* = \frac{1}{1-w}$. The non-empty sequence notation is $w^+ = w \cdot w^* = \frac{w}{1-w}$.
- The generating function of possibly empty staircases with width marked by z and height marked by yu is $\mathbf{st}(u) = (z^*yu)^*$.
- The generating function of non empty piles of lines with width marked by z and height marked by y is $\mathbf{pi} = ((z^*)^2 y)^+$.
- Given a generating function $F(u) = \sum_{n \geq 0} f_n u^n$ we define the series

$$F(u, v) = \sum_{n \geq 0} \sum_{i+j=n} f_n u^i v^j = \frac{uF(u)}{u - v} + \frac{vF(v)}{v - u},$$

where the inner summation is on non negative i and j with $i + j = n$, and

$$F(u, v, w) = \sum_{n \geq 0} \sum_{i+j+k=n} f_n u^i v^j w^k = \frac{u^2 F(u)}{(u - v)(u - w)} + \frac{v^2 F(v)}{(v - u)(w - u)} + \frac{w^2 F(w)}{(w - u)(w - v)},$$

where the inner summation is on non negative i, j and k with $i + j + k = n$. The values at $u = v$ of the previous series can be obtained by continuity:

$$A(u, u, w) = \left(\frac{u^2 A(u)}{u - w} \right)' + \frac{w^2 A(w)}{(w - u)^2} = \frac{u^2 A'(u)}{u - w} + \frac{u(u - 2w)A(u)}{(u - w)^2} + \frac{w^2 A(w)}{(w - u)^2}.$$
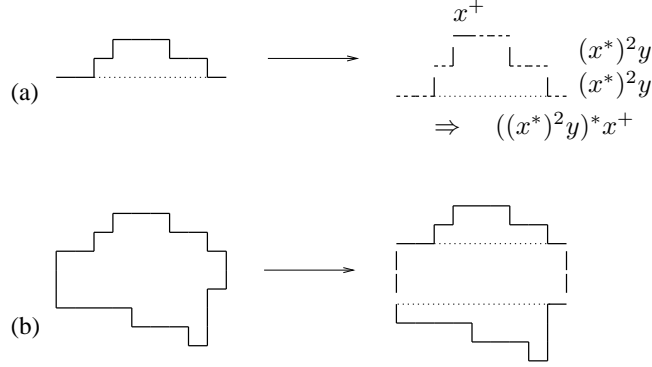
FIGURE 7. (a) The decomposition of a stack polyomino with baseline width marked by x and height marked by y. (b) The decomposition of a centered polyomino into a non-empty sequence of central rows and two stack polyominoes.

3.1. **Centered polyominoes.** Recall that a centered polyomino is a polyomino that contains at least one row touching both the left and the right hand side of its minimal bounding rectangle. We need to count polyominoes of the family $\mathcal{C}$ of centered polyominoes but also of the families $\mathcal{C}_A$ and $\mathcal{C}_B$ of hooked polyominoes with a hook of type A and B respectively.

Let $S(x, y)$ be the generating function of stack polyominoes with x marking the length of the baseline and y marking the height. In view of Figure 7(a),

$$S(x, y) = x^+ \cdot ((z^*)^2 y)^* = \frac{x(1 - x)}{(1 - x)^2 - y} = \left(\frac{1}{2} \frac{1}{1 - \frac{x}{1 - \sqrt{y}}} + \frac{1}{2} \frac{1}{1 - \frac{x}{1 + \sqrt{y}}} - 1 \right).$$

Observe then that for any power series $F(x, y)$ the Hadamard product $S(x, y) \odot_x F(x, y)$ is equal to:

$$\frac{1}{2} \left(F\left(\frac{x}{1 - \sqrt{y}}, y\right) + F\left(\frac{x}{1 + \sqrt{y}}, y\right) \right) - F(0),$$

which is a rational function of x and y if $F(x)$ is.

In view of Figure 7(b), centered polyominoes are formed of a centered rectangle supporting 2 strictly smaller stacks polyominoes:

$$C(x, y) = y^+ [S^>(x, y) \odot_x S^>(x, y)],$$

where $S^>(x, y)$ stands for the generating function of stack polyominoes with a first row strictly smaller than the baseline (so that the central rectangle is effectively given by the factor y^+). The series $S^>(x, y)$ is readily obtain by difference,

$$S^>(x, y) = S(x, y) - yS(x, y) = \frac{x(1 - x)(1 - y)}{(1 - x)^2 - y},$$
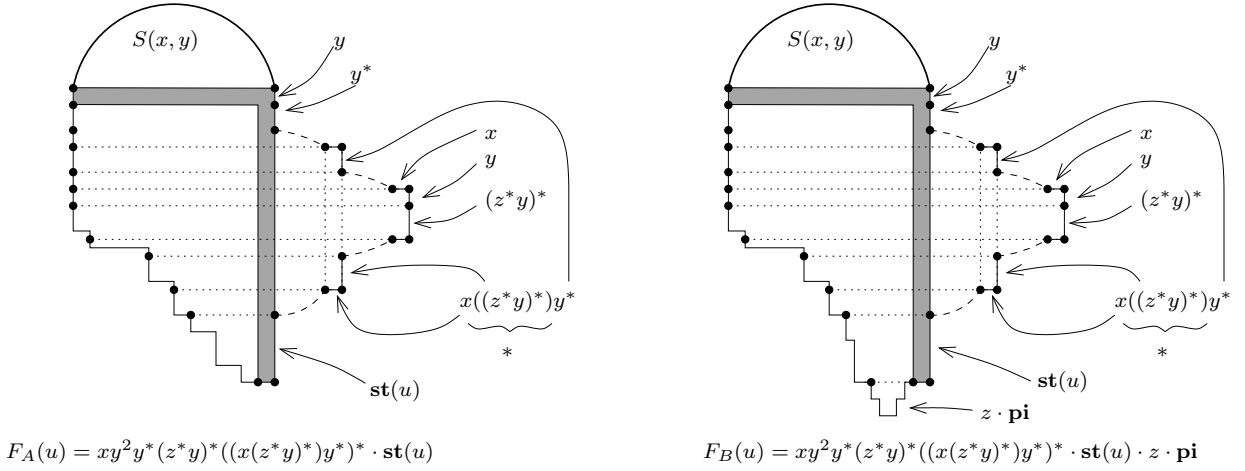
and computing the Hadamard product with the previous formula yields:

$$C(x, y) = \frac{xy(-y - xy + 1 - 2x + x^2)(1 - y)}{(1 - x - y)(x^2 - 2xy - 2x + y^2 - 2y + 1)}.$$

We shall also need centered polyominoes with a marked hook, of type A and B. As illustrated by Figure 8, the series for the first type is $C_A(u) = S(z, y) \odot_z F_A(zx, x, y, u)$ where

$$F_A(z, x, y, u) = y^2 z^+ \cdot \mathbf{st}(u) \cdot (xy^*(yz^*)^*)^+.$$

Indeed, with z marking columns on the left hand side of the leg of the hook, the Hadamard product accounts for gluing, along the arm of the hook, a staircase, with generating series $S(z, y)$, to the rest of the polyomino, with generating series $F_A(z, x, y, u)$: in this later series, a factor $xy(z^*y)^*$ corresponds to the central rectangle; each factor $x((z^*y)^*)y^*$ corresponds to a column on the right of the hook and to the lines having their rightmost cell in that column; the factor $\mathbf{st}(u)$ corresponds to the bottom staircase made of lines having their rightmost cell in the hook.

$$F_A(u) = xy^2y^*(z^*y)^*((x(z^*y)^*)y^*)^* \cdot \mathbf{st}(u) \qquad F_B(u) = xy^2y^*(z^*y)^*((x(z^*y)^*)y^*)^* \cdot \mathbf{st}(u) \cdot z \cdot \mathbf{pi}$$

FIGURE 8. Construction of elements of the classes $C_A(u)$ and $C_B(u)$.

Similarly, the series for the second type is $C_B(u) = S(z, y) \odot_z F_B(zx, x, y, u)$ where

$$F_B(z, x, y, u) = F_A(z, x, y, u) \cdot z \cdot \mathbf{pi},$$

with the extra factor corresponding to cells lower than the leg of the hook.

3.2. **Hooked polyominoes with hooks of type A.** A hooked polyomino with hook ok type A can be a hooked centered polyomino (with gf $C_A(u)$ already computed) or can be obtained from its reduction which must be a hooked polyomino with a hook of type A (recall that type A means that the leg of the hook reaches the lowest row of the polyomino). Let us describe the different cases, with respect to the properties of the resulting inflated polyomino:

- The leg of the hook and the two columns S and T have same abscissa (Figure 9, left): let

$$A_1(u) = x(y^*)^2 \cdot A(u).$$

- The leg of the hook has same abscissa as the column T but not as S (Figure 9, middle): by definition of type A, the column S cannot be longer than the leg of the hook, and

$$A_2(u) = xy^*y^+ \cdot \mathbf{st}(u) \cdot z^+A(u).$$

The series $A_2(u)$ apparently does not takes into account the construction of the staircase starting on the righthand side of the column S and connecting it to the top-right angle of the hook. Instead each column between column S (excluded) and the leg of the hook (included) is marked by a factor z. However upon setting $z = xy^*$, each column marked by z gets a factor x and a factor y^* that accounts for the rows ending in that column. The generating function of polyominoes of this case is thus $A_2(u)\big|_{z=xy^*}$.

In all forthcoming cases, we describe similarly generating functions of polyominoes without the staircase connecting X to the top-right corner of the hook. In other terms, in the following pages z is to be understood as a shorthand notation for xy^*.

- The leg of the hook has same abscissa as the column S but not as T (Figure 9, right): by definition of type A, the leg of the hook is at least as long as the column T, and

$$A_3(u) = xy^*y^+ \cdot z^+A(u, z^*).$$

As suggested by Figure 9, the factor $A(u, z^*)$ accounts for the fact that from a polyomino of type A with parameter k, one constructs a new polyomino with parameter i with $0 \le i \le k$.

- The abscissa of the leg of the hook is strictly between S and T and the cells marked by a factor u are strictly below the lowest cells of columns S and T (Figure 10, left):

$$A_4(u) = x(y^+)^2 \cdot \mathbf{st}(u) \cdot \mathbf{pi} \cdot (z^+)^2A(z^*).$$
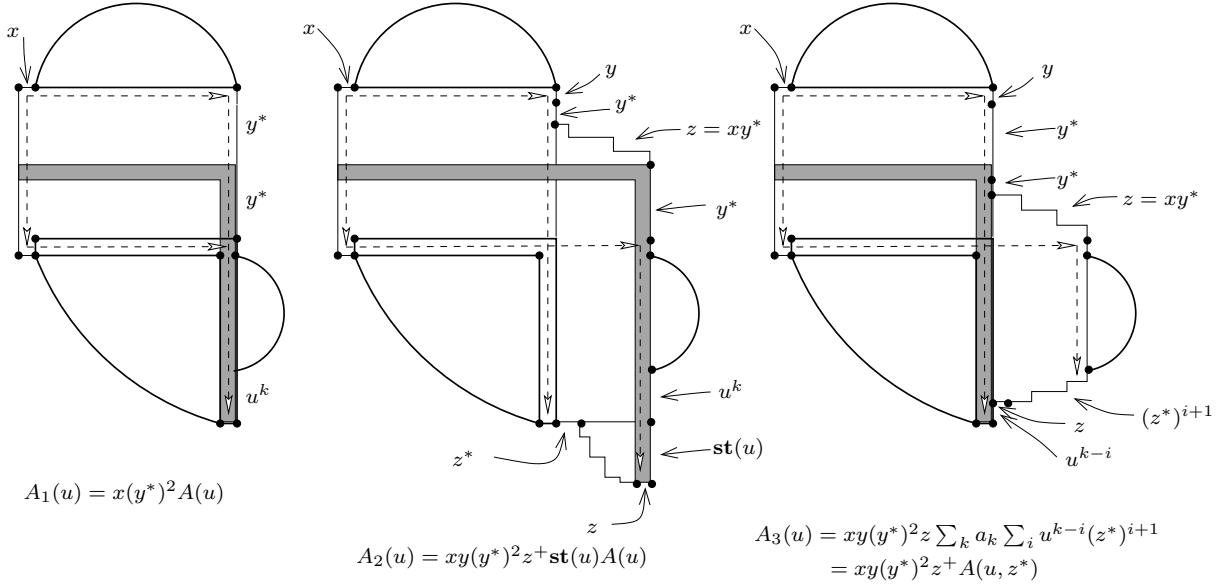
$$A_1(u) = x(y^*)^2 A(u)$$

$$A_2(u) = xy(y^*)^2 z^+ \mathbf{st}(u) A(u)$$

$$A_3(u) = xy(y^*)^2 z \sum_k a_k \sum_i u^{k-i} (z^*)^{i+1}$$
$$= xy(y^*)^2 z^+ A(u, z^*)$$

FIGURE 9. Construction of elements of classes A_1, A_2 and A_3, from the hooked polyomino of Figure 5(a).

$$A_4(u) = xy^2(y^*)^2(z^+)^2 \cdot \mathbf{st}(u) \cdot \mathbf{pi} \cdot A(z^*) \qquad A_5(u) = xy^2(y^*)^2(z^+)^2 \cdot \mathbf{st}(u) \cdot A(u, z^*)$$
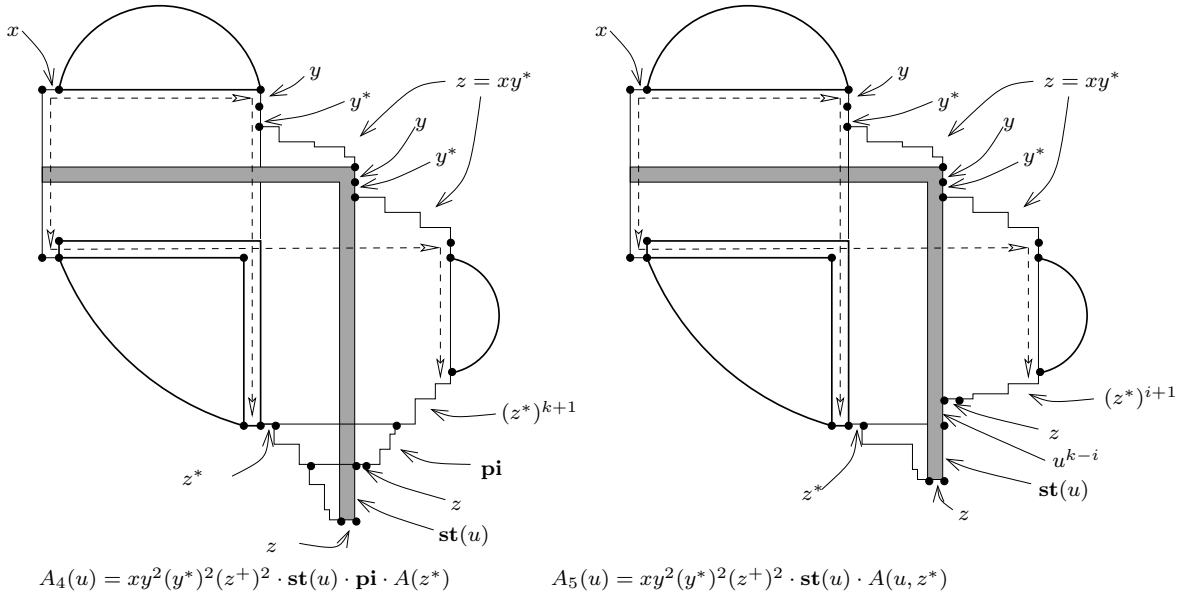
FIGURE 10. Construction of elements of classes A_4, and A_5, from the hooked polyomino of Figure 5(a).

- The abscissa of the leg of the hook is strictly between S and T and the cells marked by a factor u intersects the baseline of column S (Figure 10, right):
$$A_5(u) = x(y^+)^2 \cdot \mathbf{st}(u) \cdot (z^+)^2 A(u, z^*).$$

The generating series of hooked polyominoes with a hook of type A is then
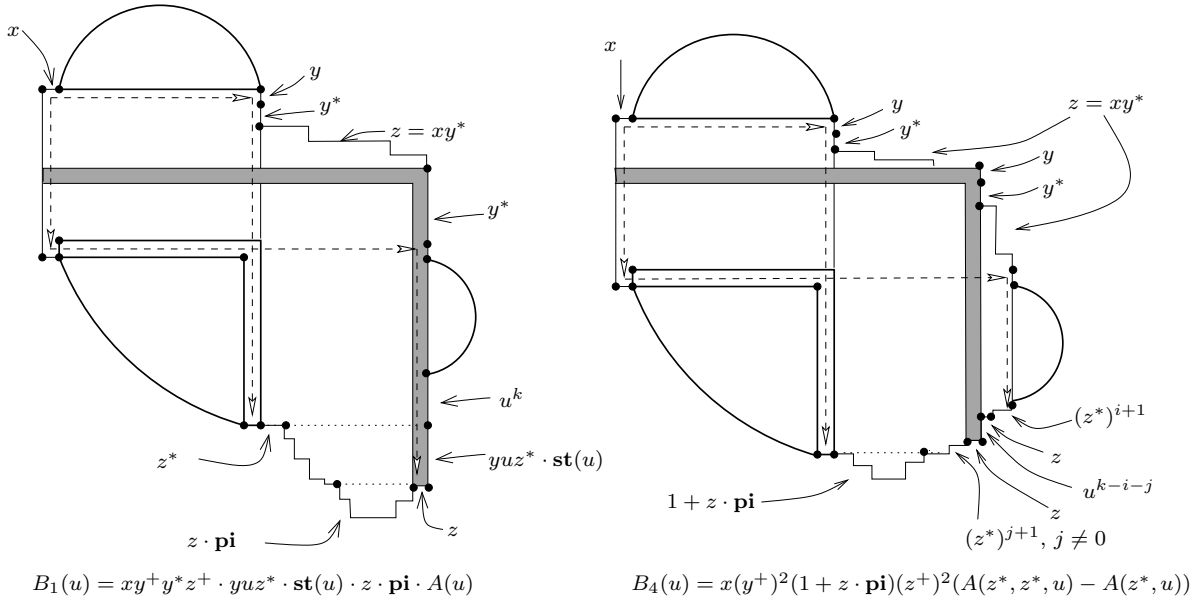$$A(u) = C_A(u) + \sum_{i=1}^{5} A_i(u)\big|_{z=xy^*}.$$

$$B_1(u) = xy^+ y^* z^+ \cdot yuz^* \cdot \mathbf{st}(u) \cdot z \cdot \mathbf{pi} \cdot A(u)$$

$$B_4(u) = x(y^+)^2(1 + z \cdot \mathbf{pi})(z^+)^2(A(z^*, z^*, u) - A(z^*, u))$$

FIGURE 11. Construction of elements of classes B_1, and B_4, from the hooked polyomino of Figure 5(a).

$$B_5(u) = x(y^*)^2 B(u)$$

$$B_6(u) = 2 \cdot xy^* y^+ z^* B(z^*, u)$$
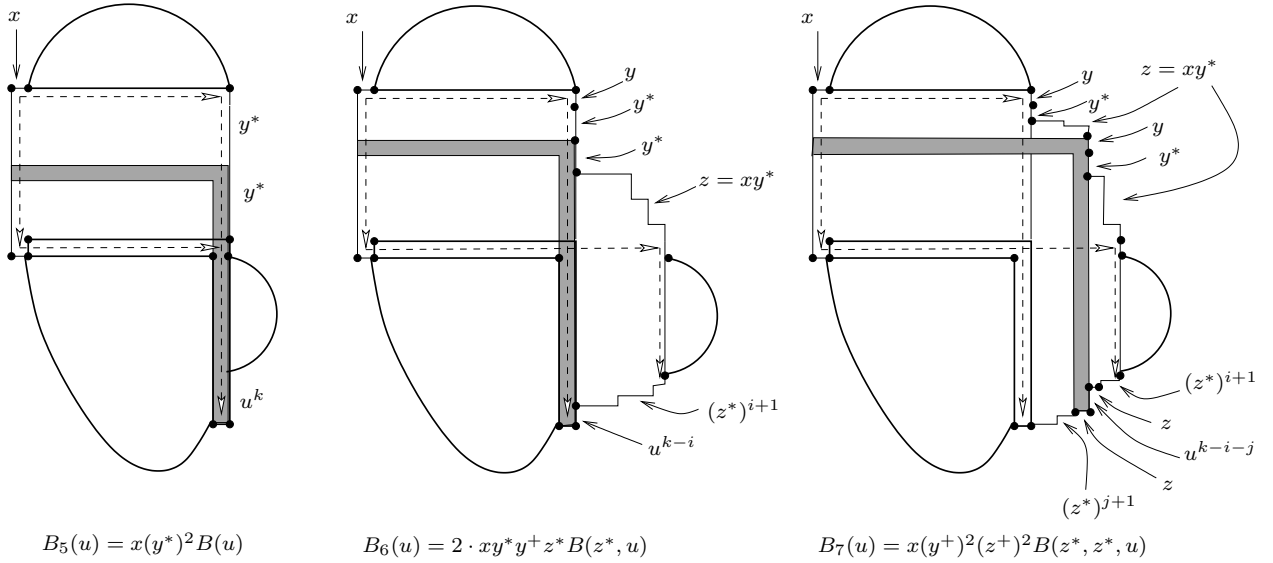
$$B_7(u) = x(y^+)^2(z^+)^2 B(z^*, z^*, u)$$

FIGURE 12. Construction of elements of classes B_5, B_6 and B_7, from the hooked polyomino of Figure 5(b).

3.3. **Hooked polyominoes with hooks of type** B. A hooked polyomino with hook of type B can be a hooked centered polyomino (with gf $C_B(u)$ already computed) or can be obtained by inflating its reduced polyomino. We start with those that are produced from a hooked polyomino with hook of type A, and we give again the different cases with respect to the properties of the obtained inflated polyomino:

- The leg of the hook has same abscissa as column T and it is strictly longer than column S: in order to produce a hook of type B, some other column between S and T must be

even longer, and

$$B_1(u) = A_2(u) \cdot z \cdot \mathbf{pi} = xy^*y^+ \cdot \mathbf{pi} \cdot z \cdot \mathbf{st}(u) \cdot z^*yu \cdot z^+A(u).$$

In agreement with the relation $B_1(u) = A_2(u) \cdot z \cdot \mathbf{pi}$, polyominoes of B_1 (Figure 11, left) can be obtained from polyominoes of A_2 (Figure 9, middle) by adding a non-empty pile of rows just before the leg of the hook.

- The abscissa of the leg of the hook is strictly between S and T, and the cells marked by a factor u are strictly below the lowest cells of columns S and T:

$$B_2(u) = A_4(u) \cdot z \cdot \mathbf{pi} = x(y^+)^2 \cdot \mathbf{pi} \cdot z \cdot \mathbf{st}(u) \cdot \mathbf{pi} \cdot (z^+)^2A(z^*).$$

These polyominoes are obtained from the polyominoes of A_4 upon adding a non-empty pile of rows just before the leg of the hook.

- The abscissa of the leg of the hook is strictly between S and T, and the cells marked by a factor u intersects the baseline of column S.

$$B_3(u) = A_5(u) \cdot z \cdot \mathbf{pi} = x(y^+)^2 \cdot \mathbf{pi} \cdot z \cdot \mathbf{st}(u) \cdot (z^+)^2A(u, z^*).$$

These polyominoes are obtained from the polyominoes of A_5 upon adding a non-empty pile of rows just before the leg of the hook.

- The abscissa of the leg of the hook is strictly between S and T, and the cells marked by a factor u are strictly above the lowest cell of column S.

$$B_4(u) = x(y^+)^2 \cdot (1 + \mathbf{pi} \cdot z)(z^+)^2(A(z^*, z^*, u) - A(z^*, u)).$$

Observe that difference is due to the restriction $j \neq 0$, as illustrated by the Figure 11: the leg of the hook must end strictly above the lowest cell of columns S, so that one must have $j \geq 1$.

Now we present the cases produced from a hooked polyomino with hook of type B, again arranged according to the properties of the resulting polyomino. Observe that in these cases the column S is at least as long as the column T:

- The leg of the hook and the columns S and T have the same abscissa:

$$B_5(u) = x(y^*)^2 \cdot B(u)$$

- The leg of the hook has the same abscissa as the column S or the same abscissa as the column T (but not both):

$$B_6(u) = 2 \cdot xy^*y^+ \cdot z^*B(z^*, u).$$

- The abscissa of the leg of the hook is strictly between column S and T:

$$B_7(u) = x(y^+)^2 \cdot (z^+)^2B(z^*, z^*, u).$$

The generating series of hooked polyominoes with a hook of type B is then

$$B(u) = C_B(u) + \sum_{i=1}^{7} B_i(u)\big|_{z=xy^*}.$$

3.4. **Z-convex polyominoes.** Again we start with polyominoes that are produced from a hooked polyomino with hook of type A:

- The columns S and T have the same abscissa:

$$P_1 = xy^*A(1).$$

- The columns S and T have distinct abscissa and the column S is strictly shorter than T:

$$P_2 = xy^+ \cdot \mathbf{pi} \cdot zA(1).$$

- The columns S and T have distinct abscissa and the column S is at least as long as T:

$$P_3 = xy^+ \cdot (1 + z \cdot \mathbf{pi})(z^*A(z^*) - A(1)).$$

The difference is due to the fact that at least one horizontal column must be inserted at the level of the rows that were marked by the factor u to ensure that the column T is not longer that S.
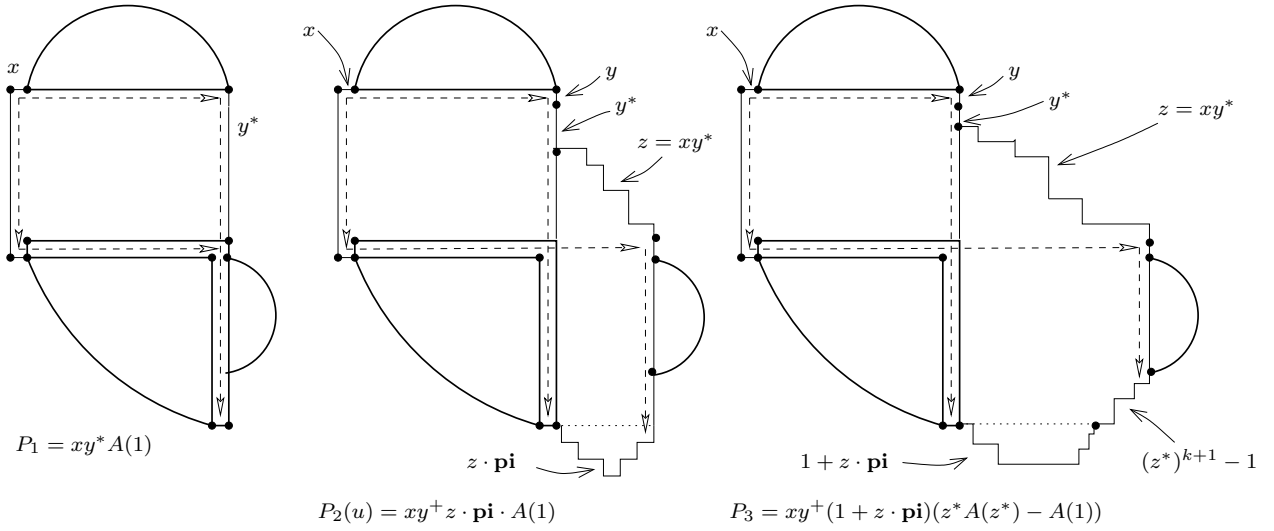
$P_1 = xy^*A(1)$

$P_2(u) = xy^+z \cdot \mathbf{pi} \cdot A(1)$

$P_3 = xy^+(1 + z \cdot \mathbf{pi})(z^*A(z^*) - A(1))$

FIGURE 13. Construction of elements of classes P_1, P_2 and P_3, from the hooked polyomino of Figure 5(a).
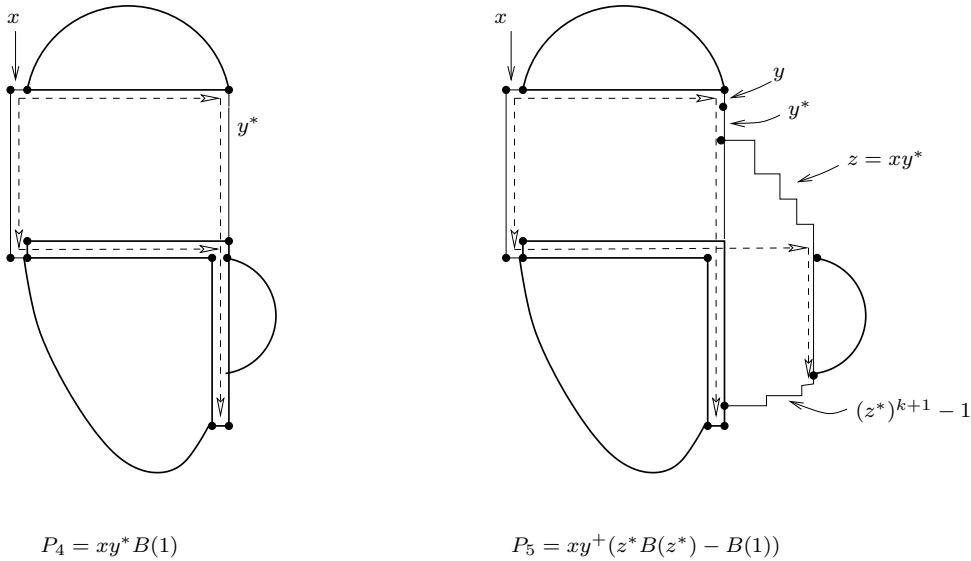

$P_4 = xy^*B(1)$

$P_5 = xy^+(z^*B(z^*) - B(1))$

FIGURE 14. Construction of elements of classes P_4, and P_5, from the hooked polyomino of Figure 5(b).

Next we present the polyominoes obtained from a hooked polyomino with hook of type B:

- The columns S and T have the same abscissa:

$$P_4 = xy^*B(1).$$

- The columns S and T have distinct abscissa and the lowest cell of S is below or at the same level as the lowest cell of T:

$$P_5 = xy^+ \cdot (z^*B(z^*) - B(1)).$$

Finally the generating function of Z-convex polyominoes is

$$P = C + \sum_{i=1}^{5} P_i\big|_{z=xy^*}.$$

4. Resolution

In view of the previous section, upon setting as announced $z = xy^*$, the system of equations defining the series P has the following form:

$$
\begin{aligned}
A(u) &= C_A(u) + a_1(u)A(u) + a_2(u)A(z^*) \\
B(u) &= C_B(u) + b_1(u)A(u) + b_2(u)A(z^*) + b_3(u)A'_u(z^*) + b_4(u)B(u) + b_5(u)B(z^*) + b_6(u)B'_u(z^*) \\
P &= C_0 + p_1A(z^*) + p_2A(1) + p_3B(z^*) + p_4B(1),
\end{aligned}
$$

where the $C_A(u)$, $C_B(u)$ and C_0 are the rational generating series of centered polyominoes computed in Section 3.1, the $a_i(u)$ and $b_i(u)$ are explicit rational functions of x, y and u, and the p_i are explicit rational functions of x and y.

The first step of the resolution is to apply the kernel method to the first equation, which involves only $A(u)$ and $A(z^*)$ as unknown. The kernel equation $1 - a_1(u) = 0$ contains a factor that can be written

$$
u = 1 + (x - y)u + yu^2,
$$

so that it clearly admits a power series root $c(x, y)$, which is a refinement of the Catalan generating function

$$
c(t, t) = \frac{1 - \sqrt{1 - 4t}}{2t}.
$$

Setting $u = c(x, y)$ in the first equation, the kernel is canceled and $A(z^*)$ is obtained as

$$
A(z^*) = \frac{C_A(c)}{a_2(c)}.
$$

Then, using again the first equation of the system we derive $A(u)$. Once $A(u)$ is known, $A'_u(z^*)$ can also be computed.

The second step consists in applying now the kernel method to the second equation of the system, which now has three unknowns $B(u)$, $B(z^*)$ and $B'_u(z^*)$. The kernel $1 - b_4(u)$ admits two roots R_1 and R_2 that are rational power series in $x^{1/2}$ and y:

$$
R_1 = \frac{1 - y + x^{1/2}}{1 - y - x + (1 - x)x^{1/2}}, \quad \text{and} \quad R_2 = \frac{1 - y - x^{1/2}}{1 - y - x - (1 - x)x^{1/2}}.
$$

Using these two roots we write two linear equations for $B(z^*)$ and $B'_u(z^*)$ and solve the system. The resulting series are rational series in x, y and $C(x, y)$ (in particular fractional powers of x cancel, as one could expect from the symmetry with respect to $\pm x^{1/2}$). Returning to the second equation of the system, we obtain $B(u)$ and finally, turning to the third equation, the generating function P of Z-convex polyominoes.

It should be remarked that our method leads to heavy computations in the intermediary steps, involving big rational expressions. The fact that things dramatically simplify when all pieces are put together in P calls for a simpler, more combinatorial, derivation. In particular, the expression are nicer in terms of the more symmetric parametrization $d(x, y) = y(c(x, y) - 1)$ satisfying $d = (x + d)(y + d)$.

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