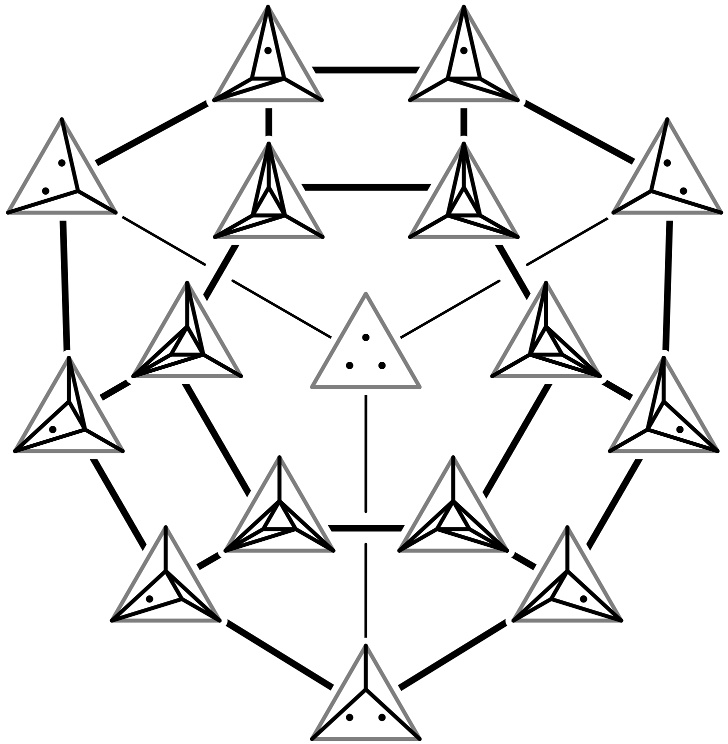
Triangulations



V. PILAUD

MPRI 2-38-1. Algorithms and combinatorics for geometric graphs

Fridays October 7th & October 14th, 2022

slides available at: http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP-4.pdf

Course notes available at: https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI21.pdf

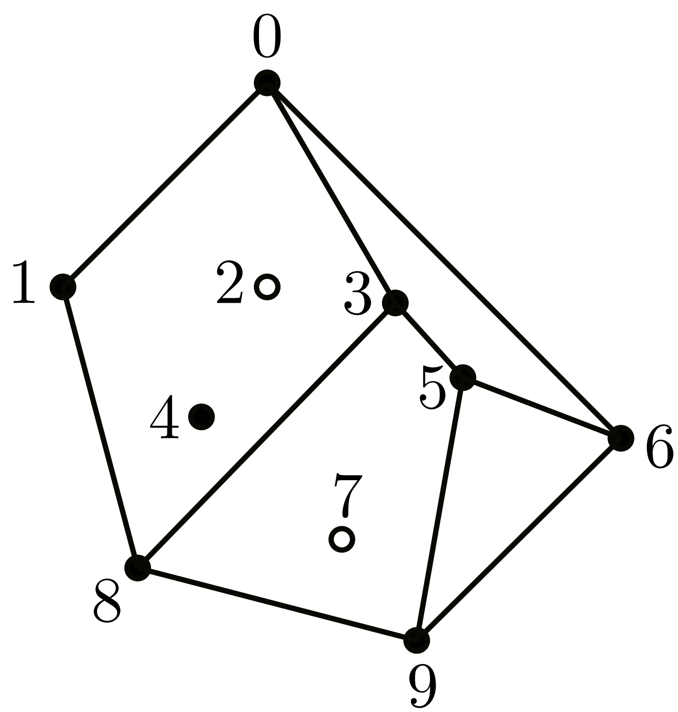
TRIANGULATIONS & SUBDIVISIONS

SUBDIVISIONS

DEF. $\boldsymbol{P}$ = point set in $\mathbb{R}^d$.

polyhedral subdivision of $\boldsymbol{P}$ = collection $\mathcal{S}$ of subsets of $\boldsymbol{P}$ st:

- closure property: if $\operatorname{conv}(\boldsymbol{X})$ is a face of $\operatorname{conv}(\boldsymbol{Y})$ and $\boldsymbol{Y} \in \mathcal{S}$, then $\boldsymbol{X} \in \mathcal{S}$,
- union property: $\operatorname{conv}(\boldsymbol{P}) = \bigcup_{\boldsymbol{X} \in \mathcal{S}} \operatorname{conv}(\boldsymbol{X})$,
- intersection property: $\operatorname{conv}(\boldsymbol{X})$ and $\operatorname{conv}(\boldsymbol{Y})$ have disjoint relative interiors and intersect along a face of both, for any $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{S}$.

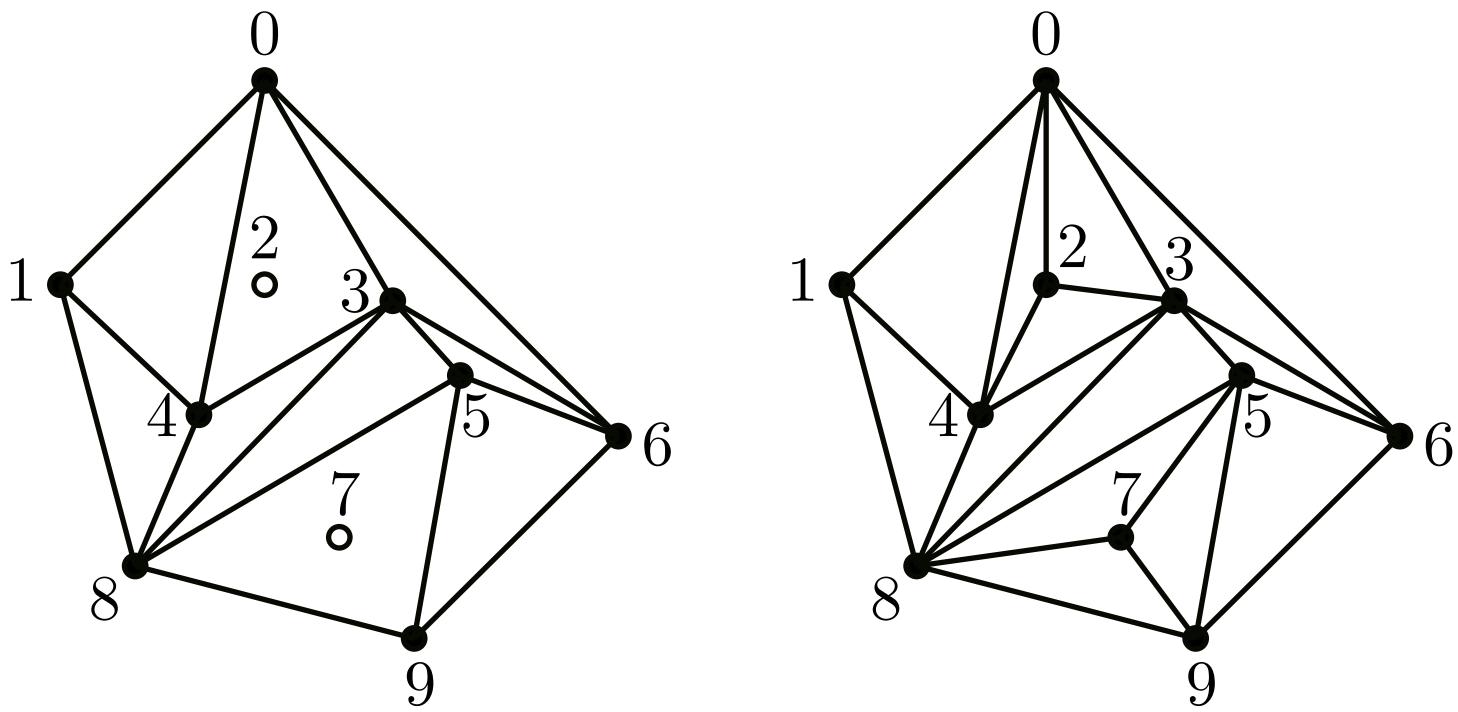


$\mathcal{S} = \{01348, 0356, 3589, 569\}$ + all faces...

TRIANGULATIONS

DEF. triangulation = subdivision $\mathcal{T}$ where all subsets are affinely independent.
(in particular, $\mathrm{conv}(\boldsymbol{X})$ is a simplex for all $\boldsymbol{X} \in \mathcal{T}$).

full triangulation = each point belongs to at least one simplex.



QU. Show that any full triangulation of a planar point set with i interior and b boundary points has $i + b$ vertices, $3i + 2b - 3$ edges, and $2i + b - 2$ triangles.

TRIANGULATIONS IN 3 DIMENSION

QU. What is the minimum / maximum number of simplices that triangulate the 3-cube?

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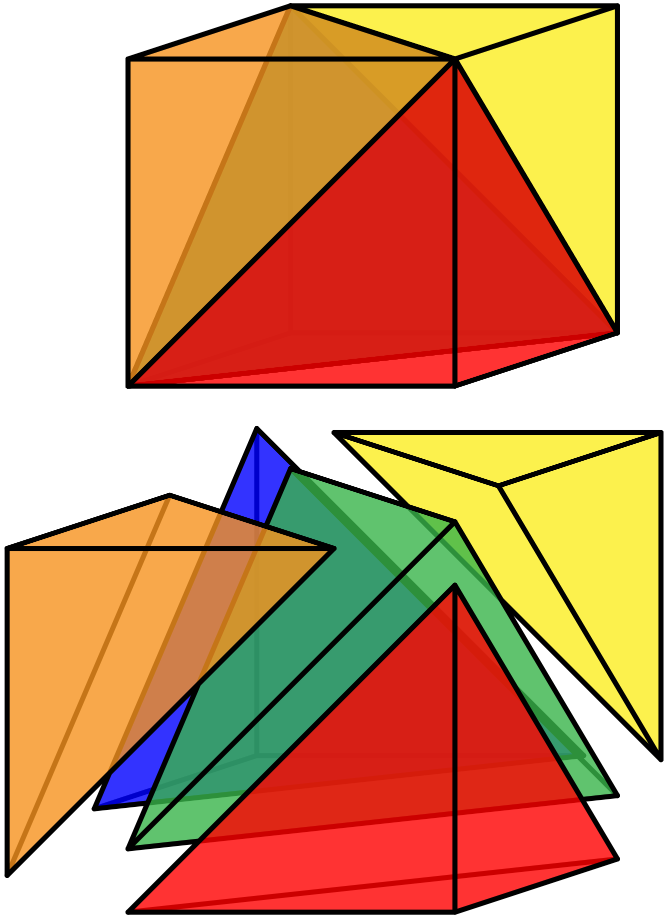
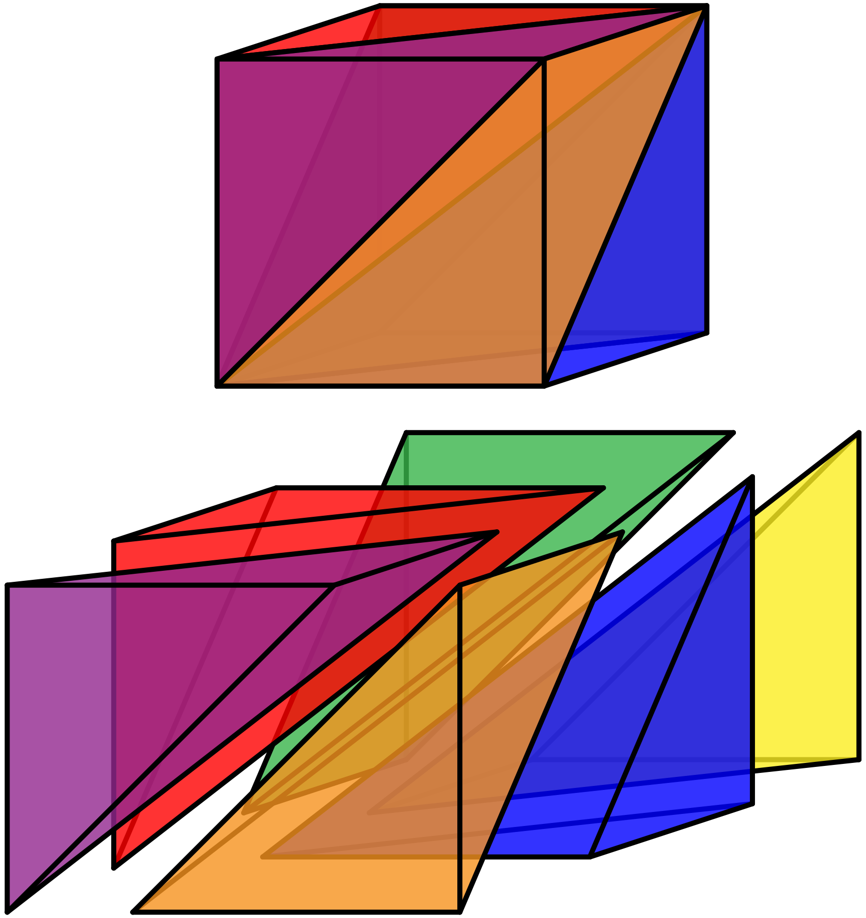


minimum $= 5$



maximum $= 6$

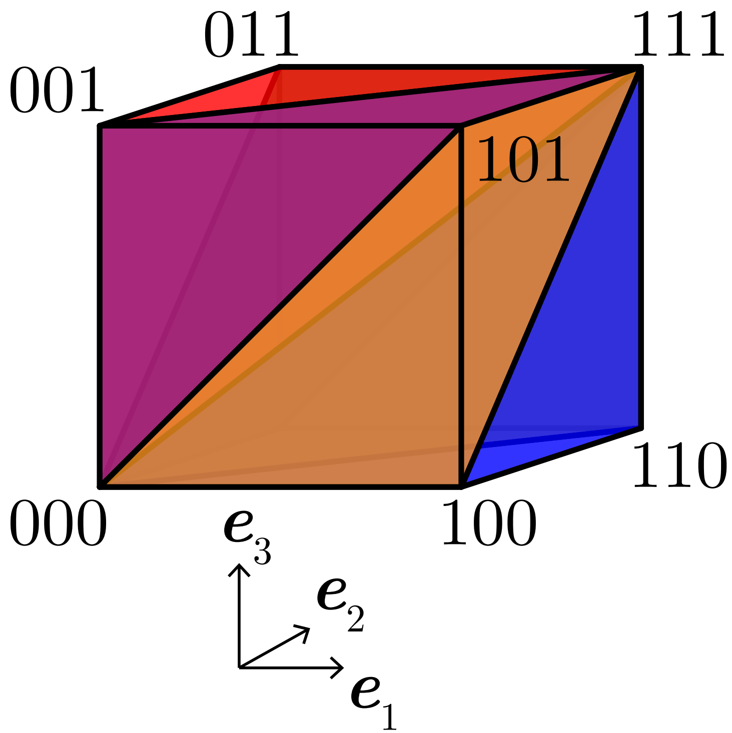
In dimension d, minimum is very difficult, maximum is $d!$

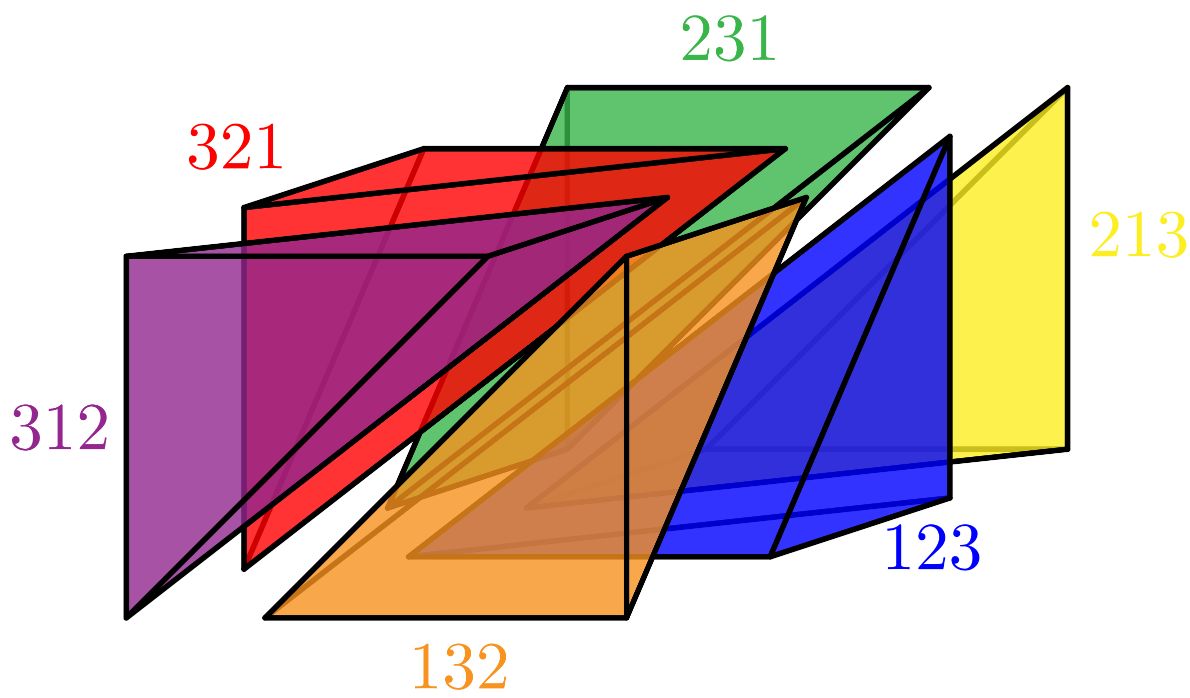
FREUDENTHAL TRIANGULATION

DEF. Freudenthal triangulation of the d-cube $\square_d$ = triangulation with a simplex

$$\triangle_\sigma = \Big\{ \sum_{i \le j} \boldsymbol{e}_{\sigma(i)} \;\Big|\; 0 \le j \le d \Big\} = \Big\{ \boldsymbol{x} \in \square_d \;\Big|\; x_{\sigma(1)} \ge x_{\sigma(2)} \ge \cdots \ge x_{\sigma(d)} \Big\}$$

for each permutation $\sigma \in \mathfrak{S}_d$.



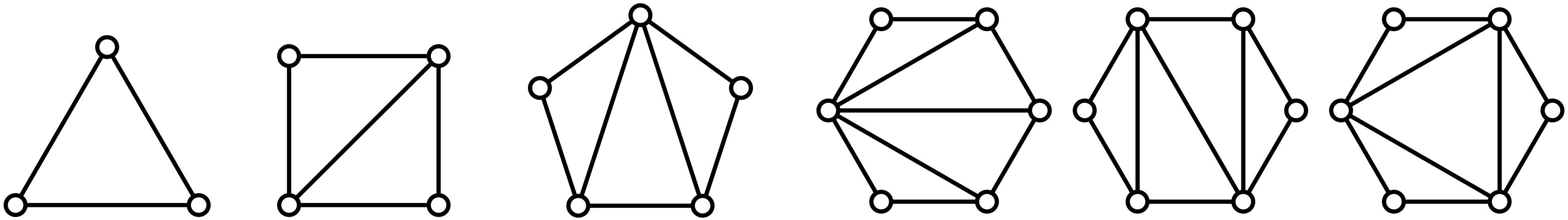


NUMBER OF TRIANGULATIONS

CONVEX POSITION & CATALAN NUMBERS

PROP. number triangulations convex n-gon = Catalan number $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
C_n	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	2674440



1 2 5 6 6 2

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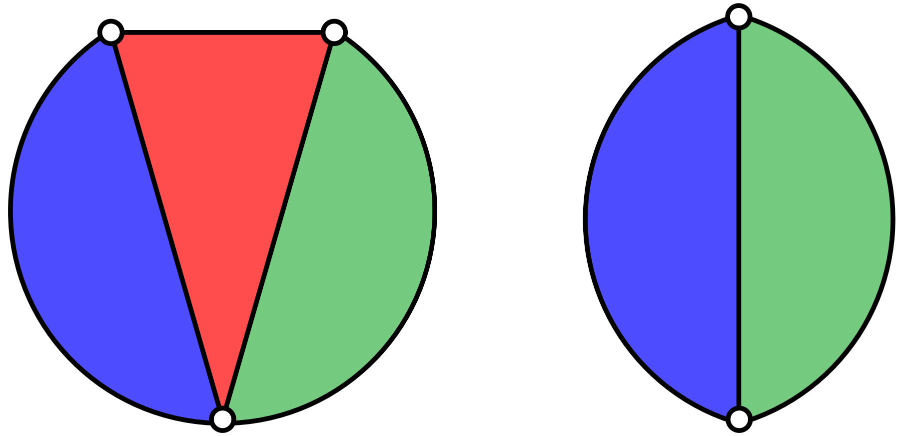
proof A: in the triangulations of the $(n-1)$-gon:

- number of edges $= 2n-5$
- average degree of a vertex $= 2(2n-5)/(n-1)$



Thus, contracting the triangle containing 1 and n, we get the induction formula

$$T_n = \frac{2(2n-5)}{n-1}T_{n-1} \qquad \text{thus} \qquad T_n = \frac{2^{n-3}(2n-5)(2n-7)\dots 3}{(n-1)(n-2)\dots 2}T_3 = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

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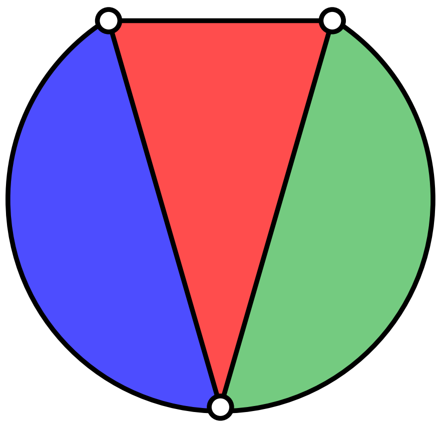
proof B: decomposing the triangulation by the triangle containing 1 and n, we have the summation formula



$$T_n = \sum_{2\le j\le n-1} T_j \cdot T_{n-j+1}$$

For the generating function $T(x) = \sum_{j\ge 2} T_j x^{j-2}$, this gives

$$T(x) = 1 + x \cdot T(x)^2 \qquad \text{thus} \qquad T(x) = \frac{1+\sqrt{1-4x}}{2x}.$$

We then get T_j developing the series.

CATALAND

QU. Show that the following are Catalan families (ie. counted by Catalan numbers):

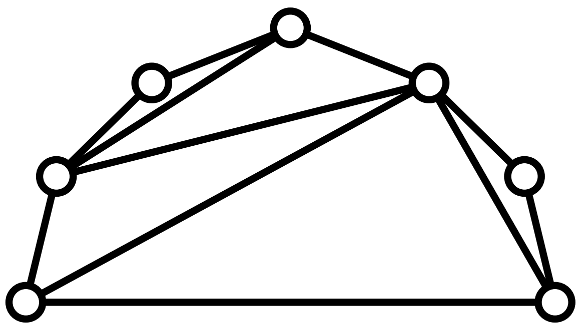
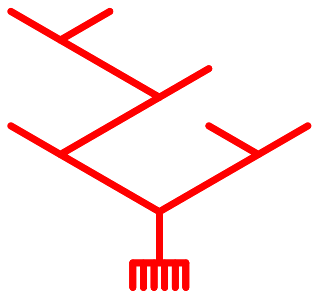
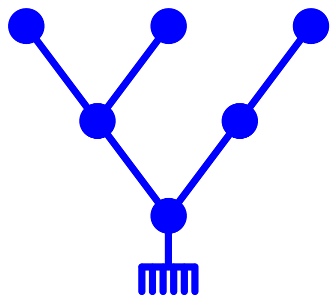
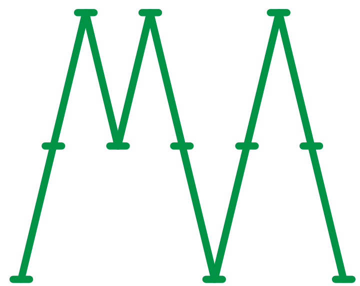
(i) triangulations of a convex n-gon,

(ii) binary trees with $n-2$ internal nodes,

(iii) rooted plane trees with $n-1$ nodes,

(iv) Dyck paths of length $2n-4$ (ie. paths with up steps $\nearrow$ and down steps $\searrow$ starting at $(0,0)$ finishing at $(2n-4,0)$ and which never go below the horizontal axis),

(v) valid bracketings of a non-associative product on $n-1$ elements.

PRO. The following are Catalan families (ie. counted by Catalan numbers):

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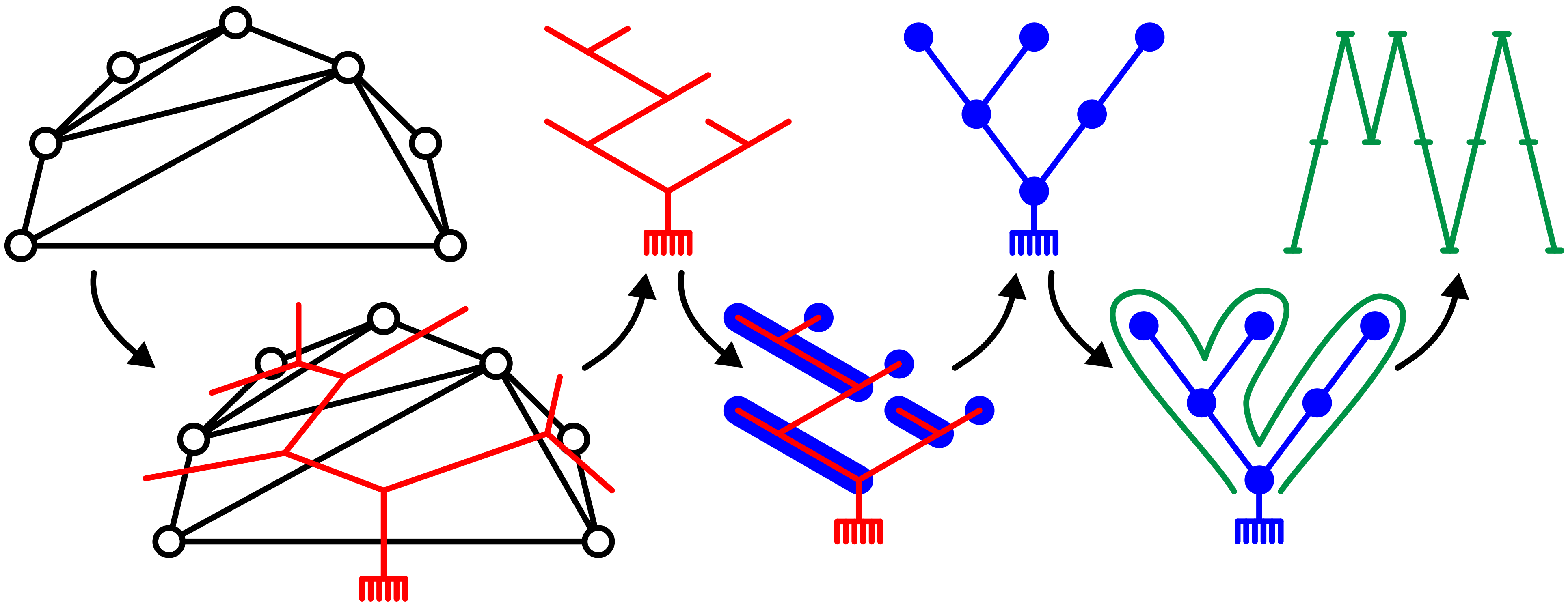
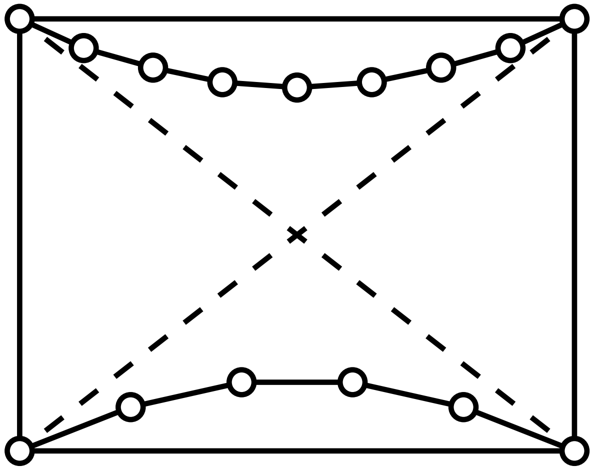
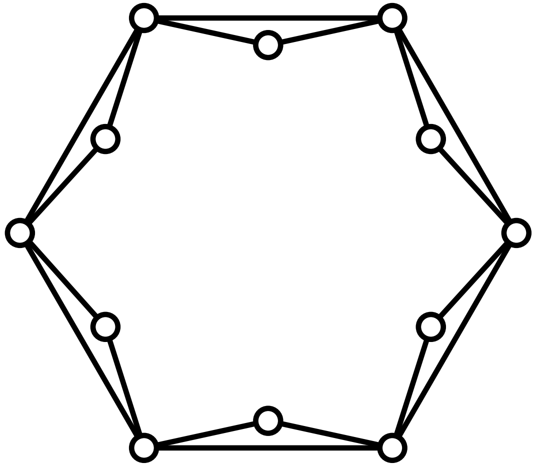


DOUBLE CHAIN AND DOUBLE CIRCLE



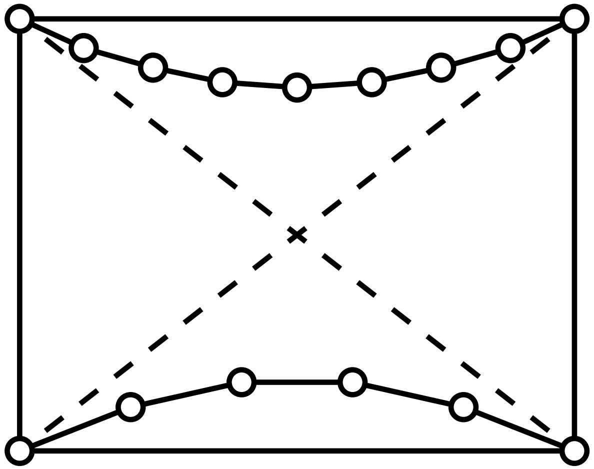
double chain



double circle

QU. Compute the numbers of full triangulations of the double chain and double circle.

DOUBLE CHAIN AND DOUBLE CIRCLE



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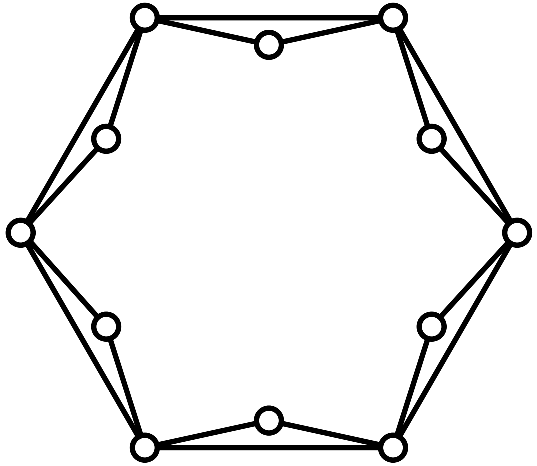


double circle

PROP. The numbers of full triangulations of the double chain and double circle are

$$C_m C_n \binom{m+n+2}{m+1} \qquad \text{and} \qquad \sum_{i \in [n]} (-1)^i \binom{n}{i} C_{n+i-2}.$$

proof:

- db chain: all edges of the chains belong to full triangulations...
- db circle: inclusion-exclusion for triangulations of convex polygon with no even ear.

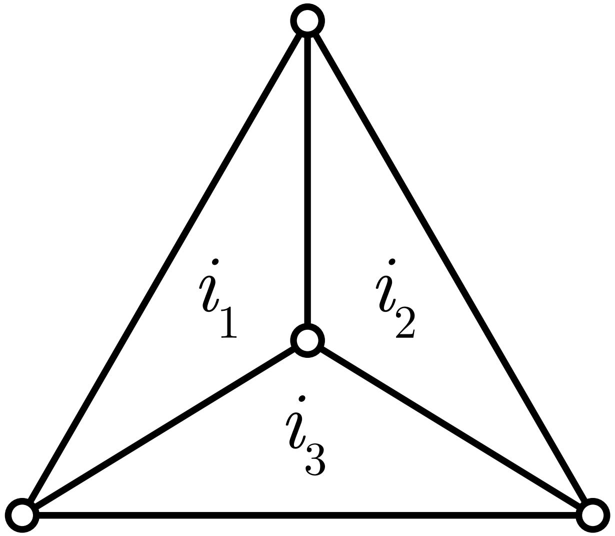
QU. What about all triangulations?

UPPER AND LOWER BOUNDS

THM. Any planar point set in general position with i interior and b boundary points has at least $C_{b-2}2^{i-b+2} = \Omega(2^n n^{-3/2})$ and at most $59^i\, 7^b / \binom{i+b+6}{6} \leq 59^n$ full triangulations.

proof: For the lower bound:

1. if $b = 3$:
   - check it for $i \leq 8$. This is a combinatorial problem!
   - use stacked triangulations:
     each point separates the triangle into three regions with $i = i_1 + i_2 + i_3 + 1$, thus defines at least $2^{i_1-1} \cdot 2^{i_2-1} \cdot 2^{i_3-1} = 2^{i-4}$ stacked triangulations
     thus in total, at least $i2^{i-4} \geq 2^{i-1}$ stacked triangulations.



2. if $b \geq 4$, choose a triangulation of the boundary, and stack in all triangles.

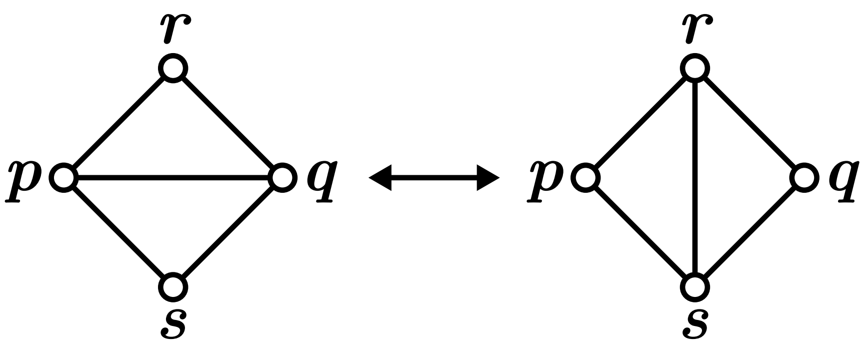
For the upper bound: see poly...

FLIPS

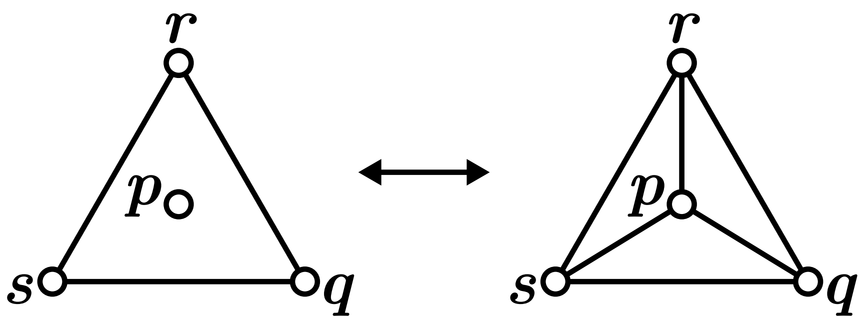
FLIPS

DEF. <u>flip</u> = local operation on triangulations of P defined as:

- <u>diagonal flip</u> = if pqr and prs form a convex quadrilateral $pqrs$, replace the diagonal pr by the other diagonal qs of $pqrs$.

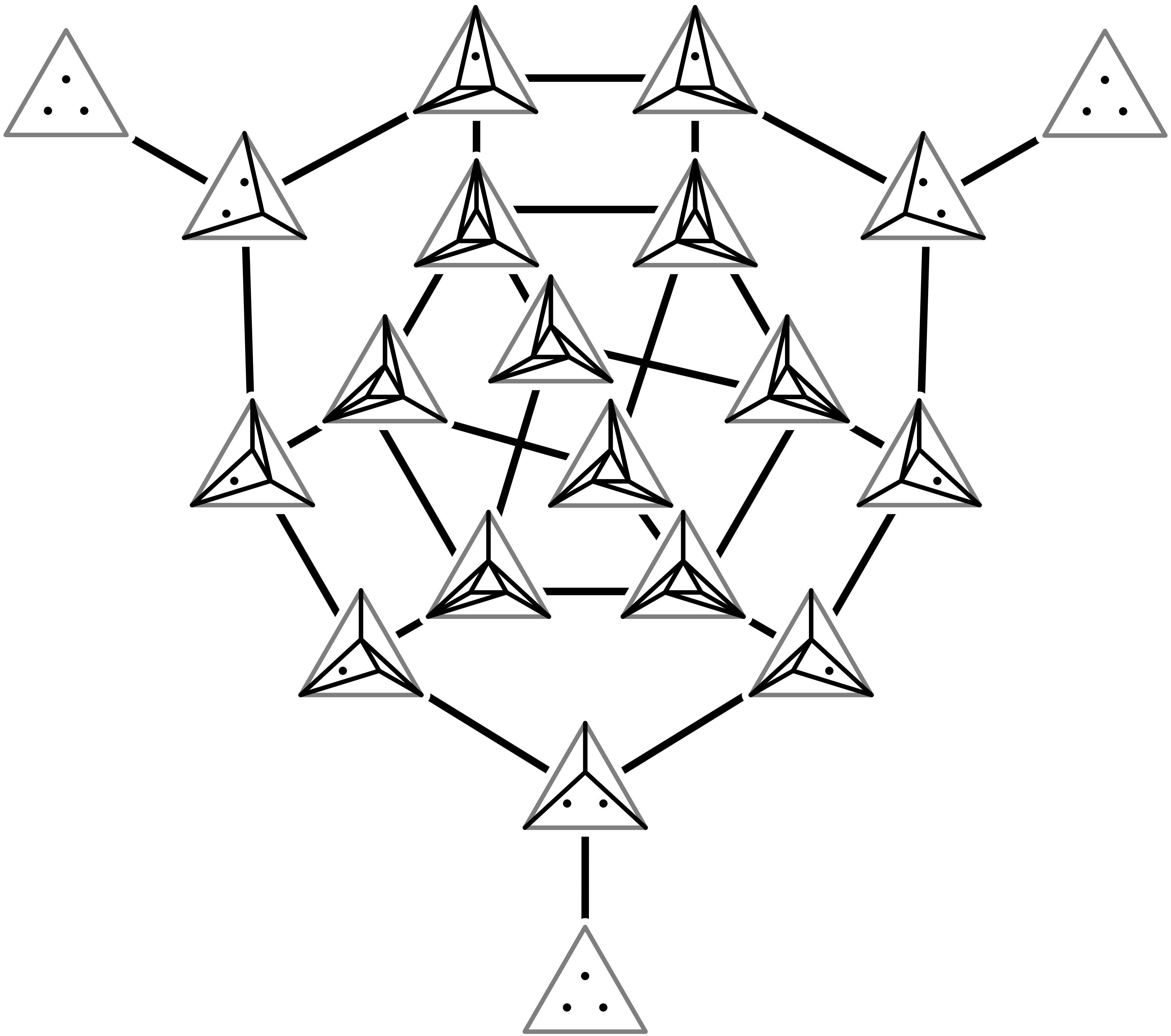


- <u>insertion/deletion flip</u> = if a point p is contained in the interior of a triangle uvw, then insert the edges pu, pv, and pw or vice-versa.



DEF. <u>flip graph</u> = graph with vertices = triangulations and edges = flips.

FLIP GRAPH



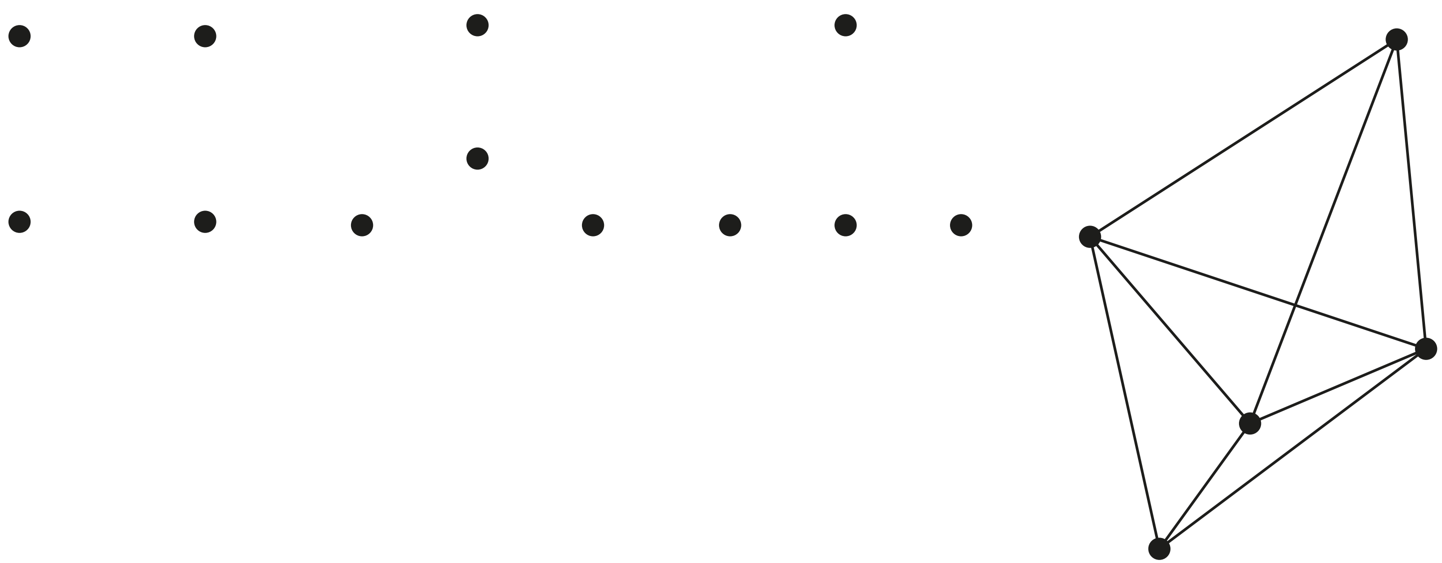
FLIPS IN HIGHER DIMENSION

THM. For any set $\boldsymbol{X}$ of $d+2$ points in $\mathbb{R}^d$, there exists a partition $\boldsymbol{X} = \boldsymbol{X}^+ \sqcup \boldsymbol{X}^- \sqcup \boldsymbol{X}^\circ$ such that $\operatorname{conv}(\boldsymbol{X}^+) \cap \operatorname{conv}(\boldsymbol{X}^-) \neq \varnothing$.

proof: There is an affine dependence $\sum_{\boldsymbol{x} \in \boldsymbol{X}} \lambda_{\boldsymbol{x}} \, \boldsymbol{x} = 0$ with $\sum_{\boldsymbol{x} \in \boldsymbol{X}} \lambda_{\boldsymbol{x}} = 0$ (to see it, linearize).

Let $\boldsymbol{X}^+ = \{\boldsymbol{x} \in \boldsymbol{X} \mid \lambda_{\boldsymbol{x}} > 0\}$ $\quad \boldsymbol{X}^- = \{\boldsymbol{x} \in \boldsymbol{X} \mid \lambda_{\boldsymbol{x}} < 0\}$ $\quad \boldsymbol{X}^\circ = \{\boldsymbol{x} \in \boldsymbol{X} \mid \lambda_{\boldsymbol{x}} = 0\}$.

Then $\Lambda = \sum_{\boldsymbol{x}^+ \in \boldsymbol{X}^+} \lambda_{\boldsymbol{x}^+} = \sum_{\boldsymbol{x}^- \in \boldsymbol{X}^-} (-\lambda_{\boldsymbol{x}^-})$ and $\frac{1}{\Lambda} \sum_{\boldsymbol{x}^+ \in \boldsymbol{X}^+} \lambda_{\boldsymbol{x}^+} \, \boldsymbol{x}^+ = \frac{1}{\Lambda} \sum_{\boldsymbol{x}^- \in \boldsymbol{X}^-} (-\lambda_{\boldsymbol{x}^-}) \, \boldsymbol{x}^-$.



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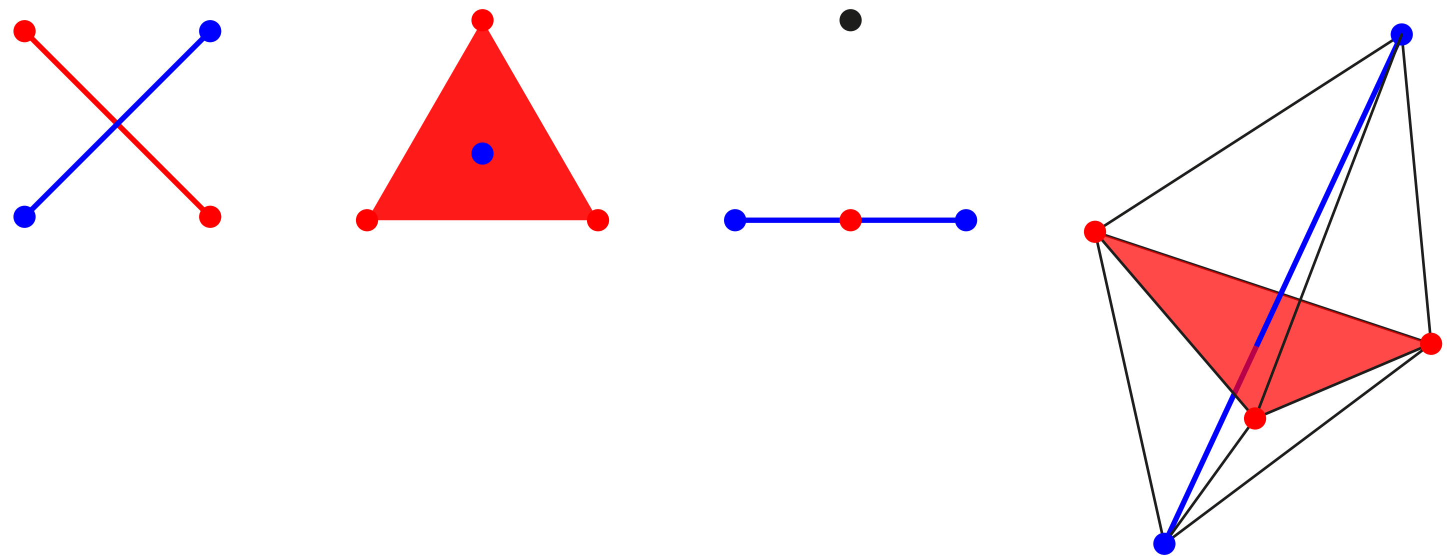

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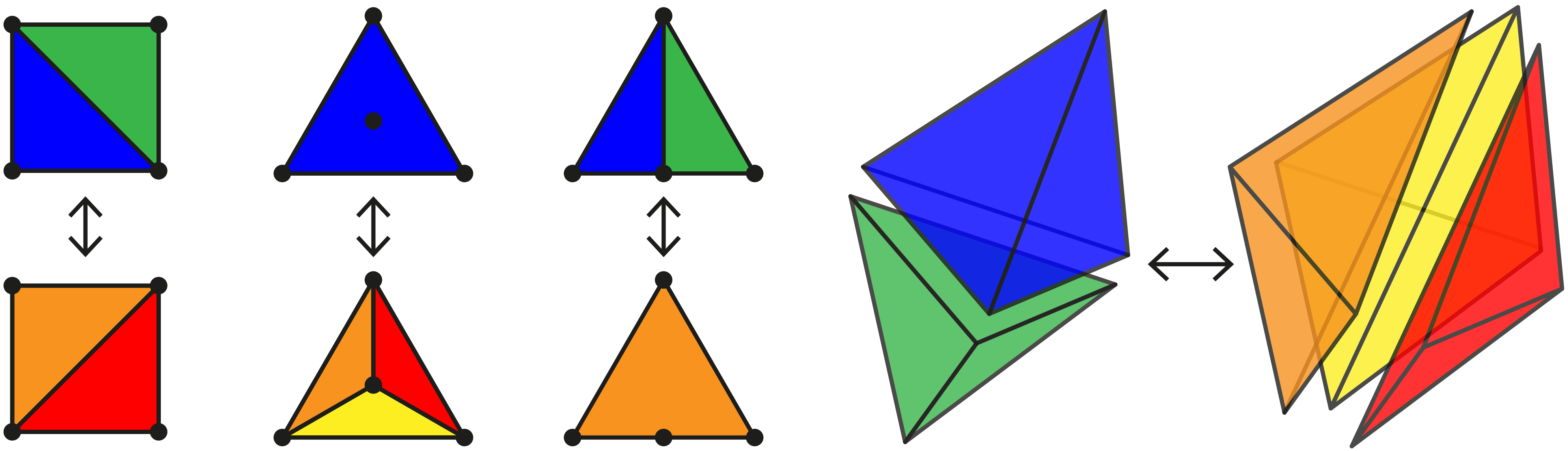
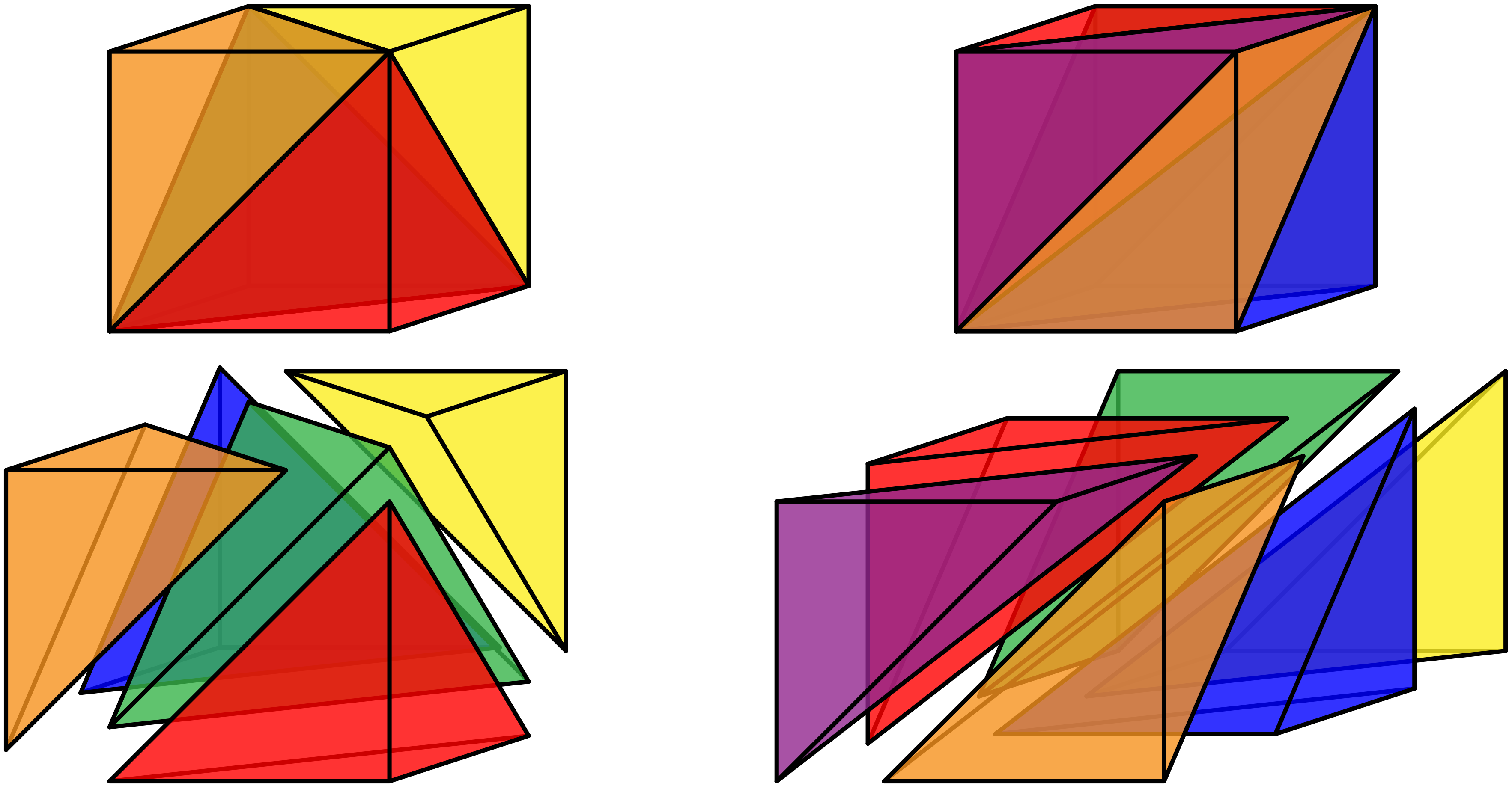
DEF. $\boldsymbol{X}$ set of $d+2$ points in $\mathbb{R}^d$.
$\boldsymbol{X} = \boldsymbol{X}^+ \sqcup \boldsymbol{X}^- \sqcup \boldsymbol{X}^\circ$ Radon partition of $\boldsymbol{X}$ with (inclusion) maximal $\boldsymbol{X}^\circ$.
<u>Bistellar flip</u> $= \left\{ \operatorname{conv}(\boldsymbol{X} \smallsetminus \{\boldsymbol{x}\}) \;\middle|\; \boldsymbol{x} \in \boldsymbol{X}^+ \right\} \longleftrightarrow \left\{ \operatorname{conv}(\boldsymbol{X} \smallsetminus \{\boldsymbol{x}\}) \;\middle|\; \boldsymbol{x} \in \boldsymbol{X}^+ \right\}$



QU. How many flips to connect these triangulations of the 3-cube?



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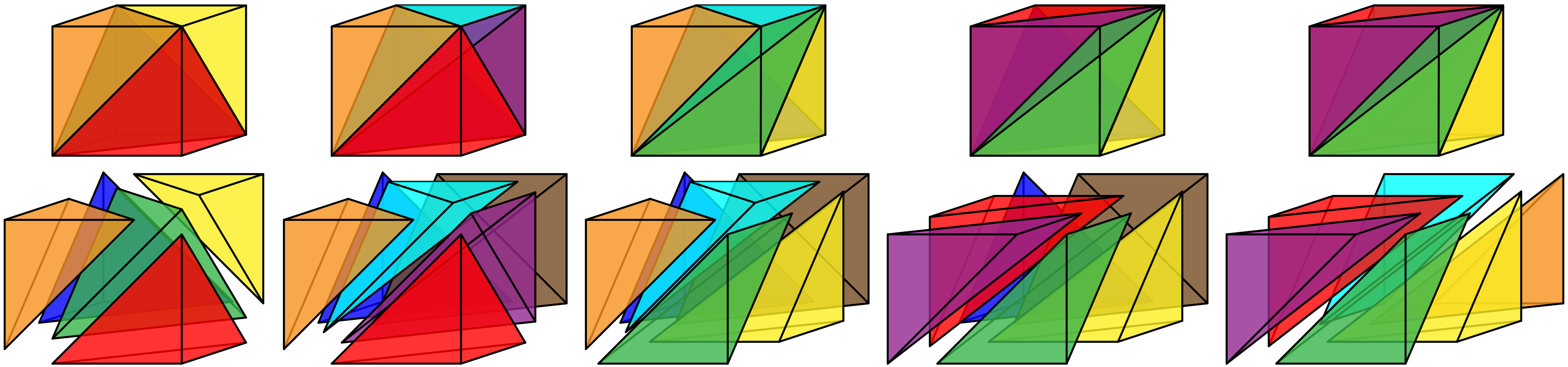


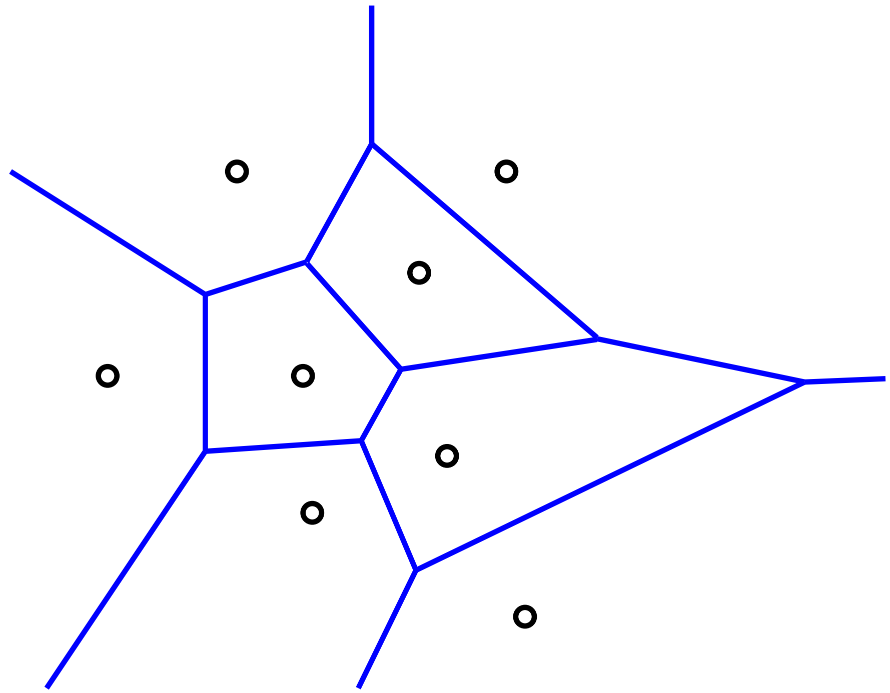
DELAUNAY TRIANGULATION (AGAIN)

VORONOI DIAGRAM

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x} - \boldsymbol{p}\| \leq \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P}\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



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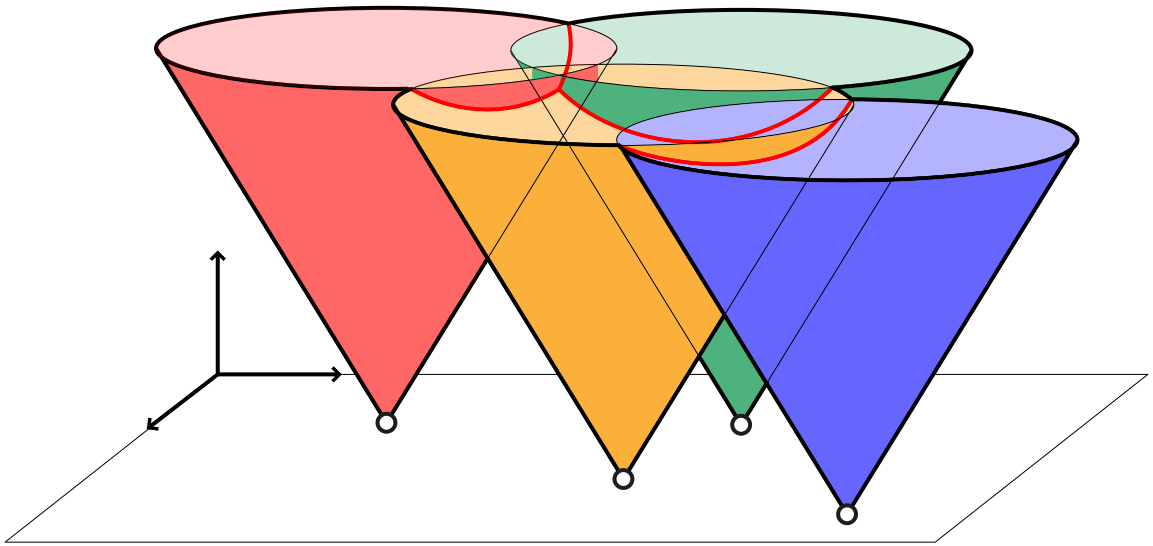
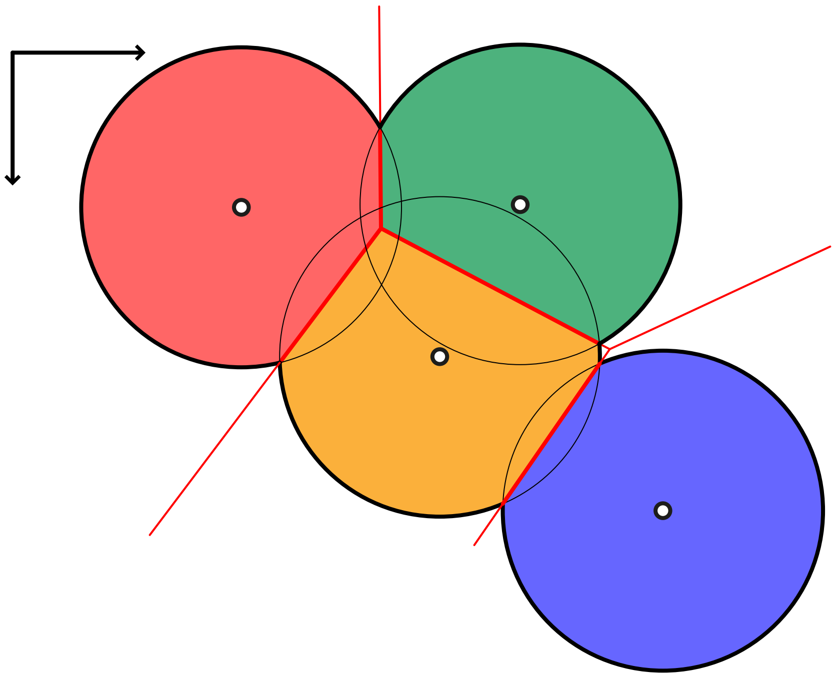
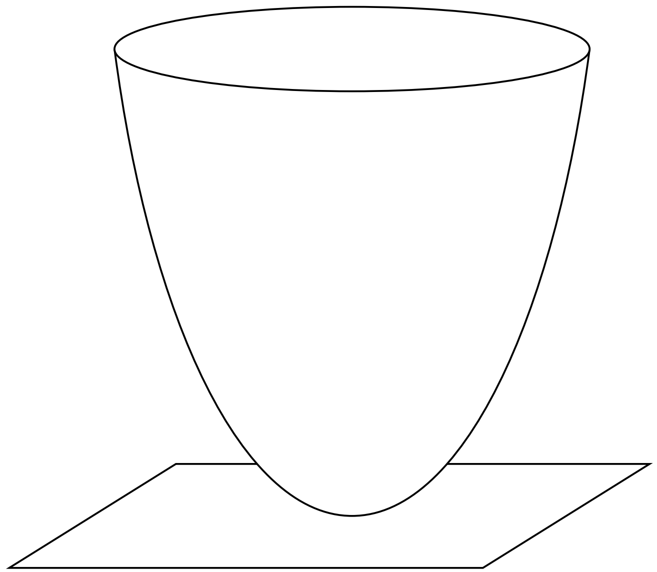




LIFTING POINTS ON THE PARABOLOID

paraboloïd $\mathcal{P}$ with equation $x_{d+1} = \sum_{i \in [d]} x_i^2$.



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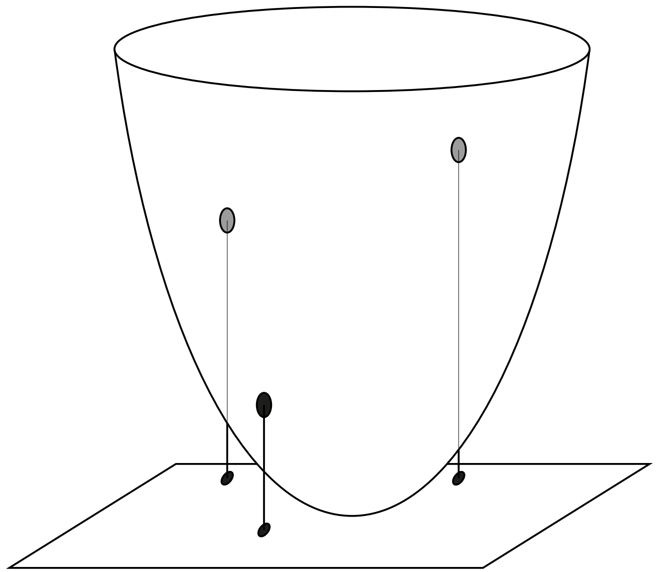

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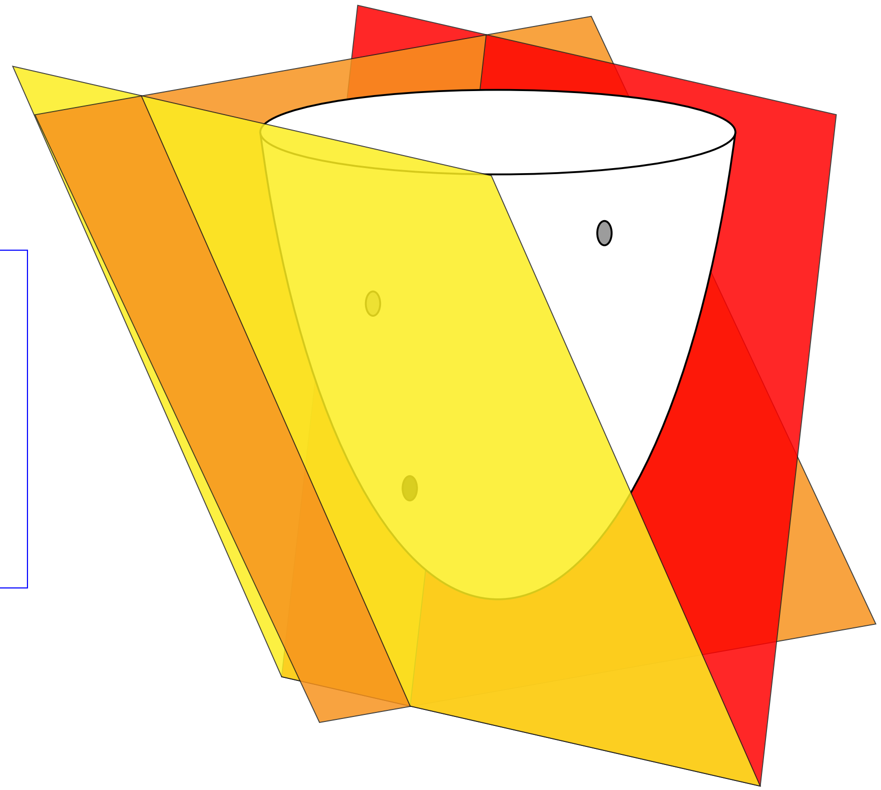

PROP. The Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ is the vertical projection of the upper enveloppe of the planes tangent to the paraboloïd $\mathcal{P}$ at the lifted points $\hat{\boldsymbol{p}}$ for $\boldsymbol{p} \in \boldsymbol{P}$.

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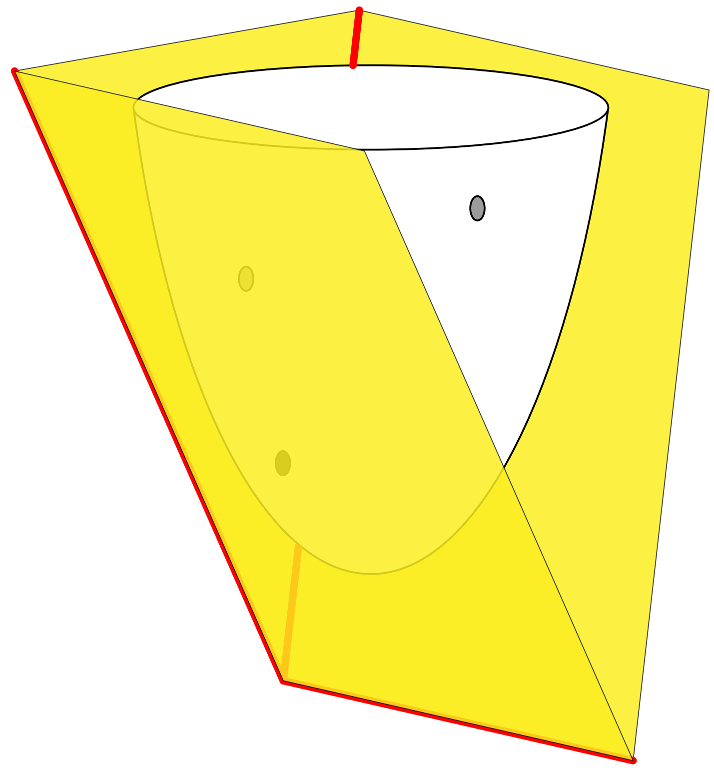

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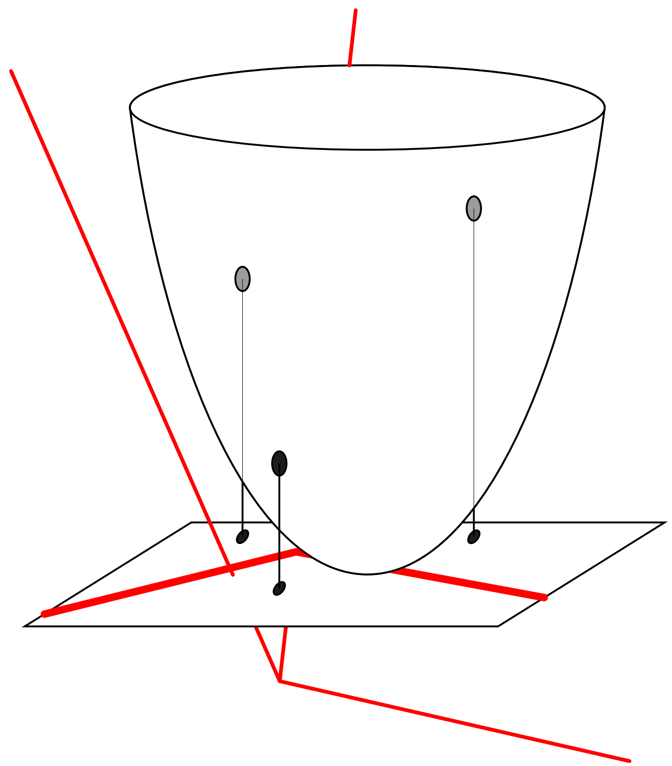
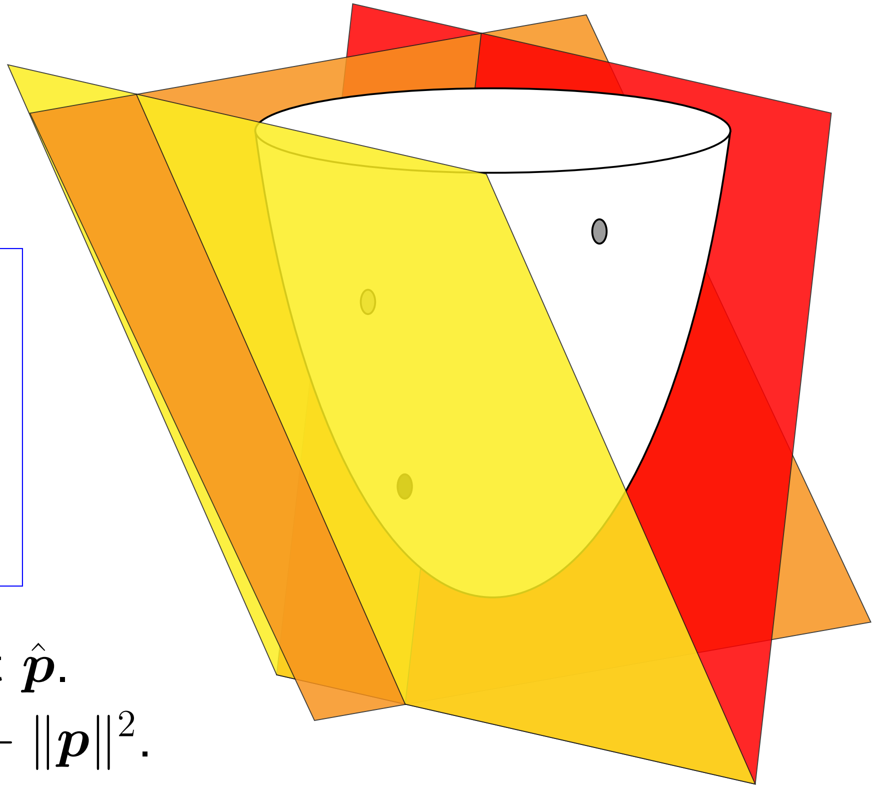


LIFTING POINTS ON THE PARABOLOID



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proof: $H(\boldsymbol{p})$ = tangent plane to the paraboloïd $\mathcal{P}$ at $\hat{\boldsymbol{p}}$.
= plane of equation $x_{d+1} = 2 \langle\, \boldsymbol{p} \mid \boldsymbol{x} \,\rangle - \|\boldsymbol{p}\|^2$.

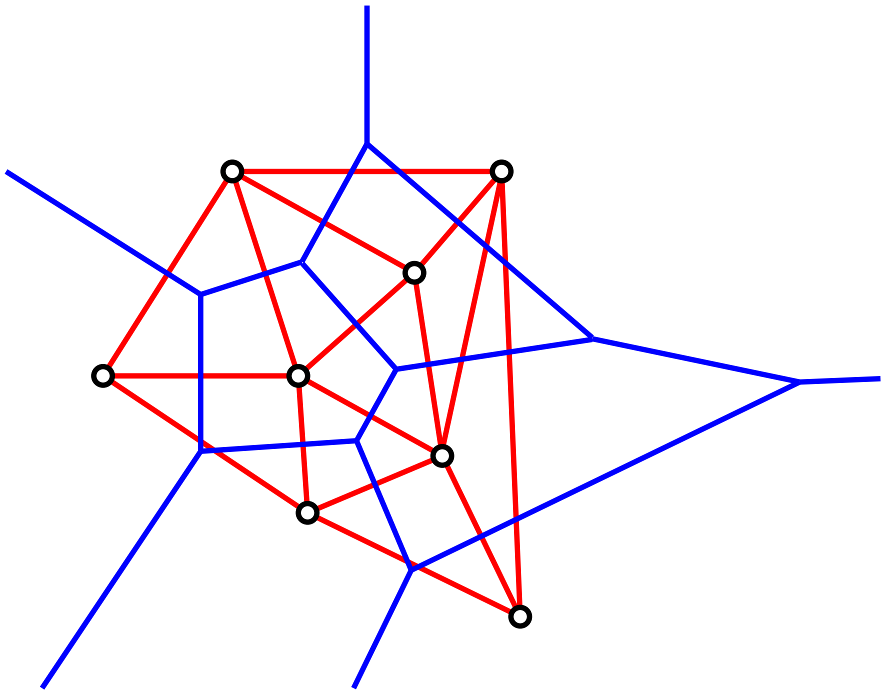
Therefore, $H(\boldsymbol{p})$ above $H(\boldsymbol{q})$ at point $\boldsymbol{x} \iff \|\boldsymbol{x} - \boldsymbol{p}\| \le \|\boldsymbol{x} - \boldsymbol{q}\|$.

DELAUNAY COMPLEX

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \|\boldsymbol{x} - \boldsymbol{p}\| \le \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P} \right\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



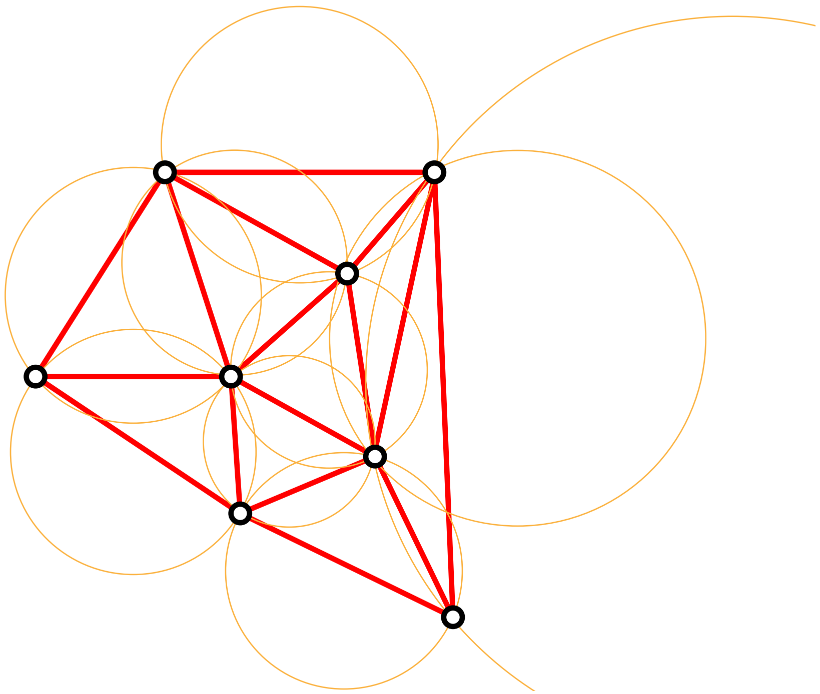
DEF. Delaunay complex $\mathrm{Del}(\boldsymbol{P})$ = intersection complex of $\mathrm{Vor}(\boldsymbol{P})$

$$\mathrm{Del}(\boldsymbol{P}) = \left\{ \mathrm{conv}(\boldsymbol{X}) \;\middle|\; \boldsymbol{X} \subseteq \boldsymbol{P} \text{ and } \bigcap_{\boldsymbol{p} \in \boldsymbol{X}} \mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) \neq \varnothing \right\}.$$

EMPTY CIRCLES

PROP. For any three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ of $\boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{pqr}$ is a triangle of $\mathrm{Del}(\boldsymbol{P}) \iff$ the circumcircle of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is empty.



proof idea: consider the circle centered at the intersection of the Voronoi regions and passing through the Voronoi sites.

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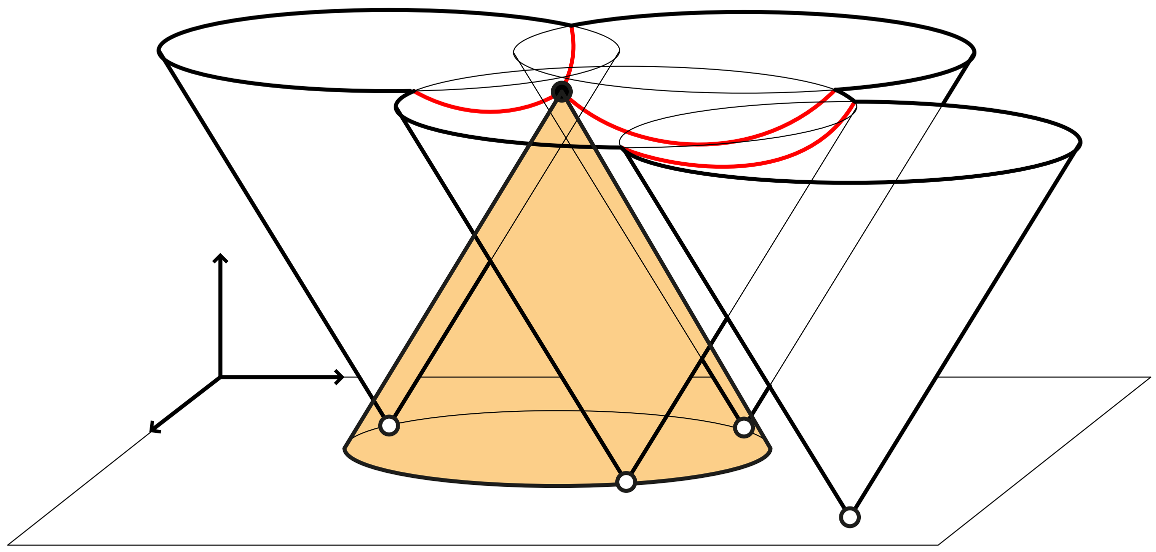

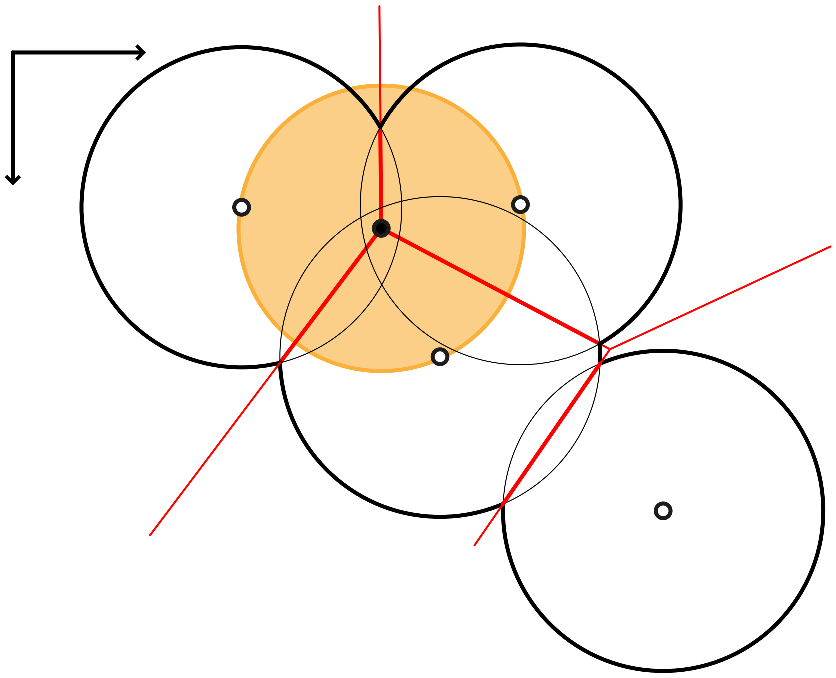

LIFTING POINTS ON THE PARABOLOID

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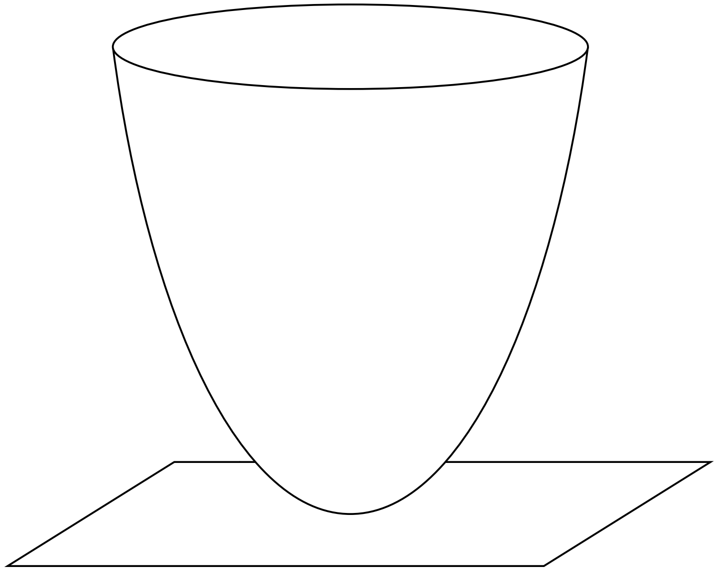

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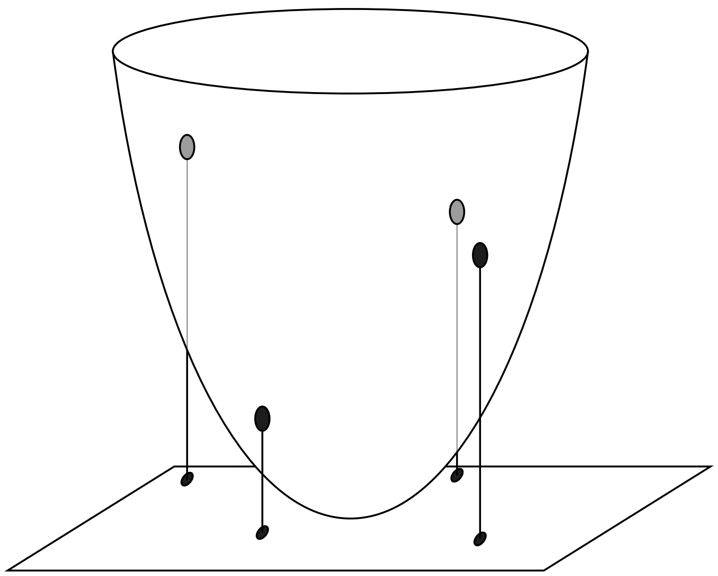

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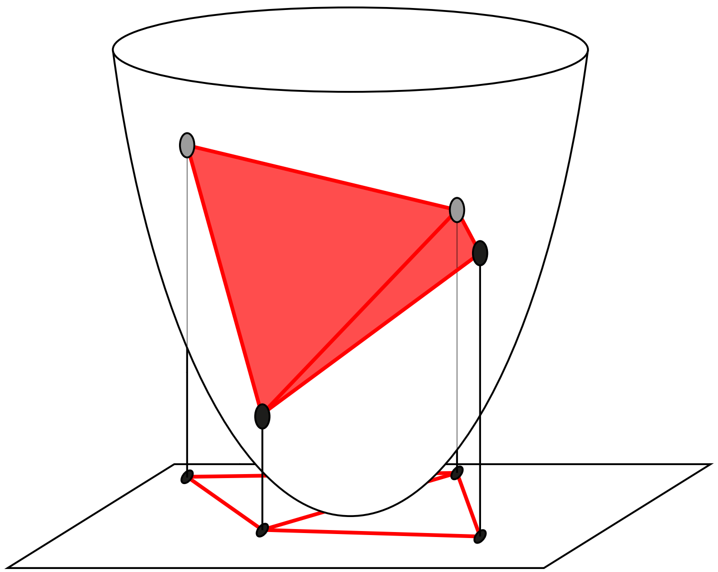

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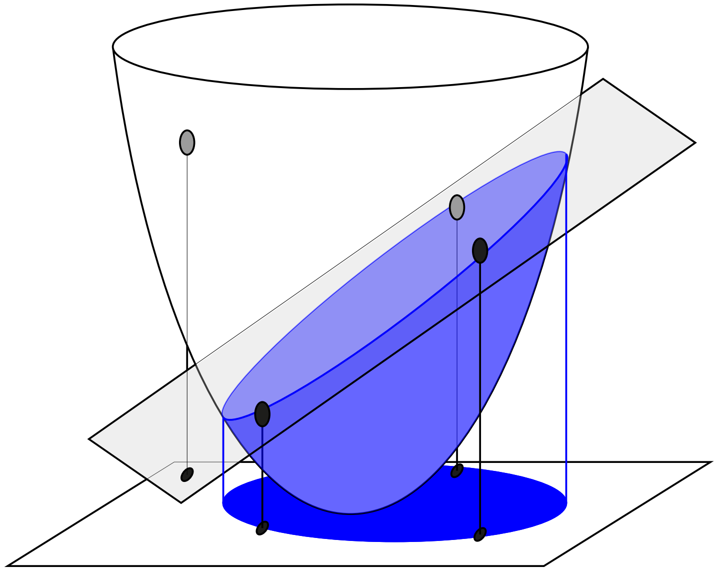

PROP. The Delaunay complex $\mathrm{Del}(\boldsymbol{P})$ is the vertical projection of the lower convex hull of the lifted points $\hat{\boldsymbol{p}}$ for $\boldsymbol{p} \in \boldsymbol{P}$.

proof: Paraboloïd cap below a hyperplane:

$$x_{d+1} = \sum_{i\in[d]} x_i^2 \quad \text{and} \quad x_{d+1} \le \sum_{i\in[d]} \lambda_i \, x_i.$$

Projection of this cap:

$$\sum_{i\in[d]} (x_i - \lambda_i/2)^2 \le \sum_{i\in[d]} \lambda_i^2/4.$$

LIFTING POINTS ON THE PARABOLOID

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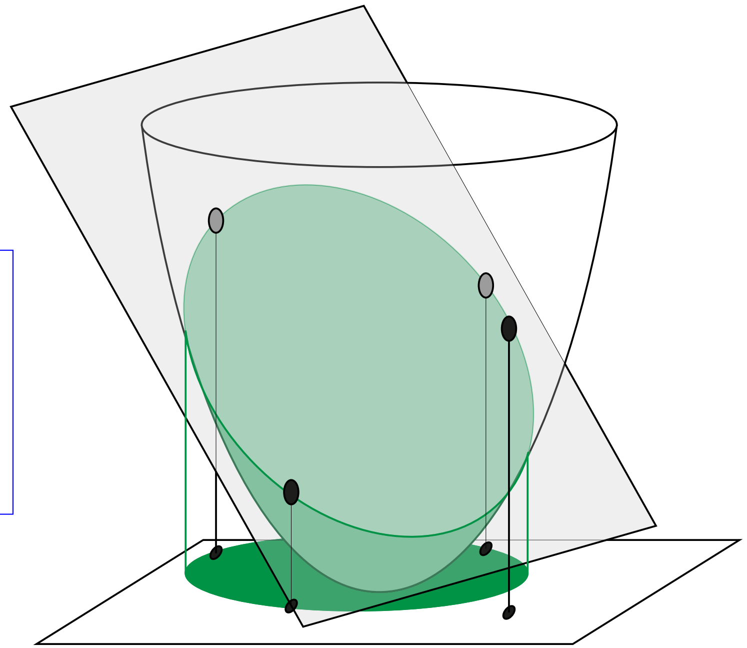

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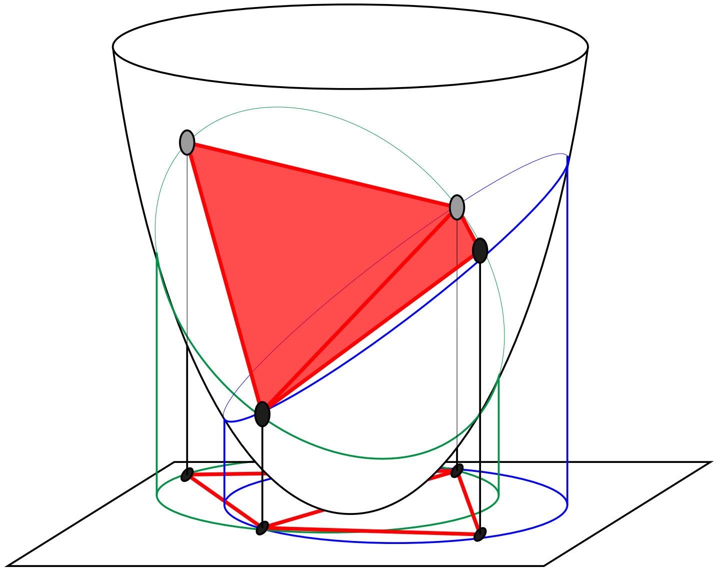

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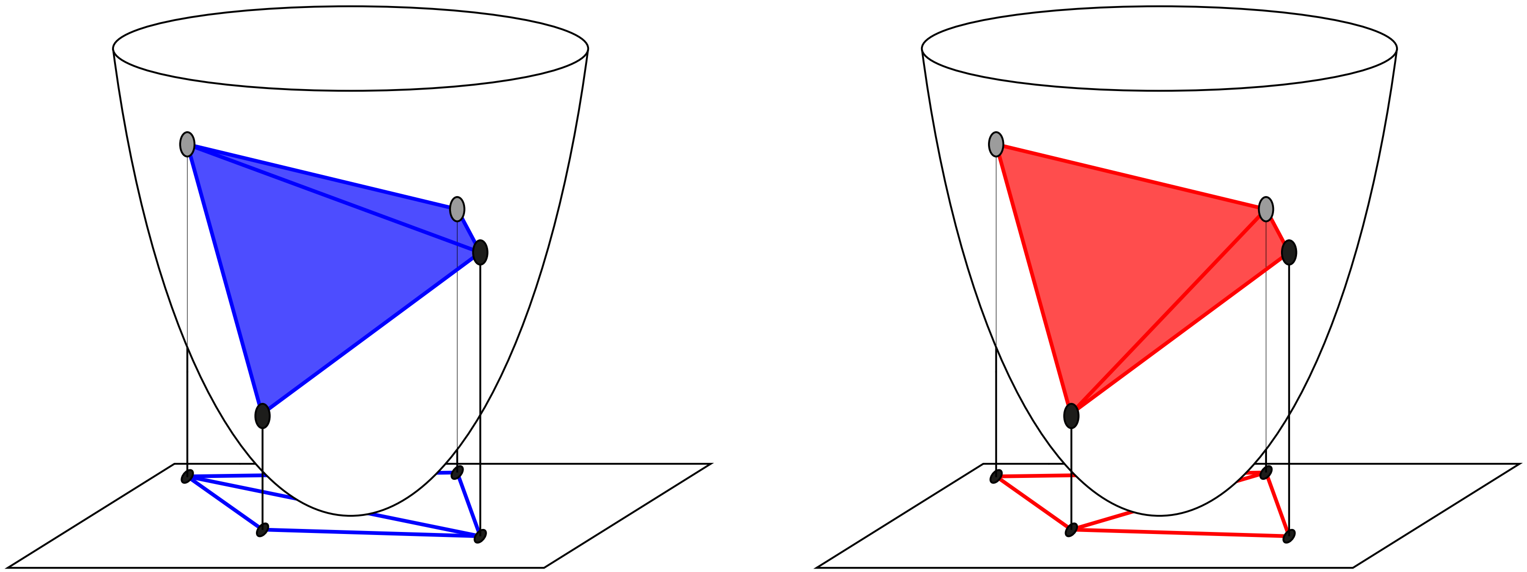
LAWSON FLIPS IN DIMENSION 2

DEF. Lawson flip = flip of an edge pq contained in two triangles pqr and pqs such that s is inside the circumcircle of pqr and r is inside the circumcircle of pqs.

PROP. Lawson flips are always possible, and lead to the Delaunay triangulation.



CORO. For any 2-dimensional point configuration, the flip graph is connected.

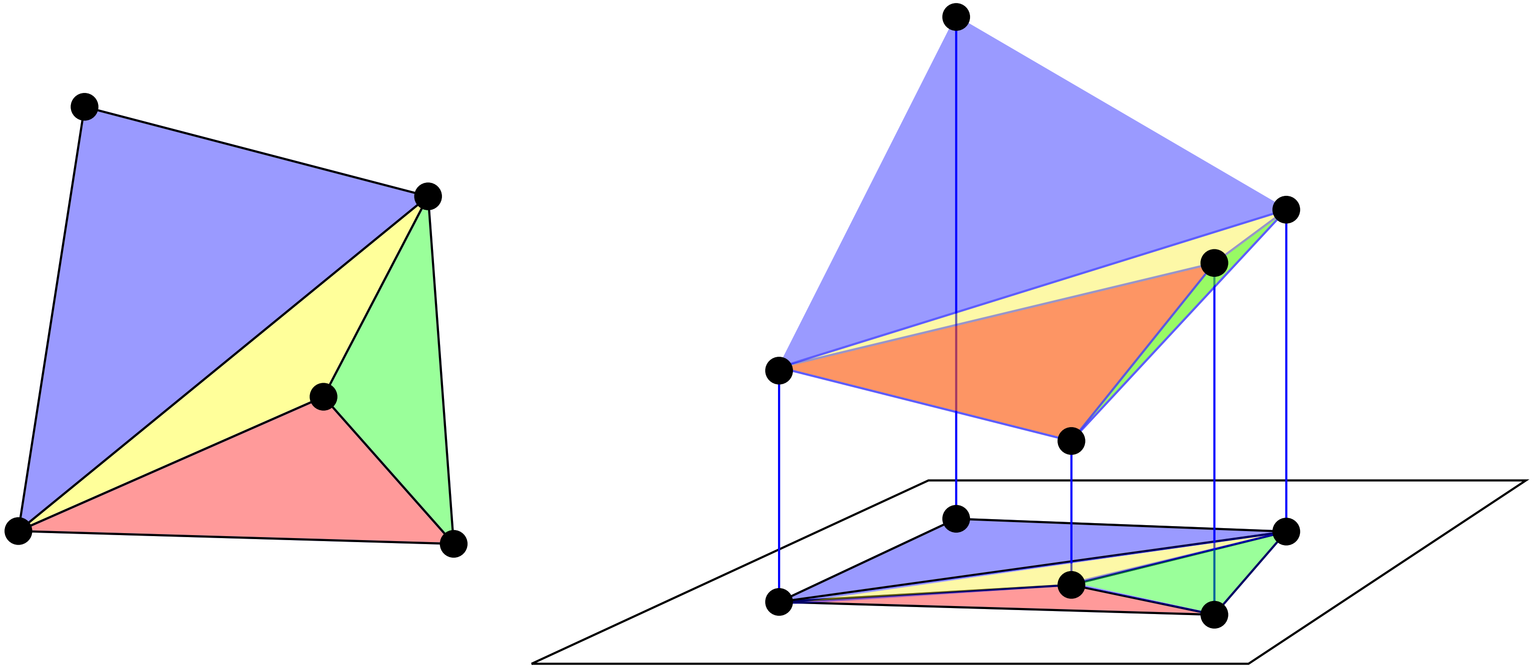
THM. (Santos) In dimension ≥ 5, some point sets have disconnected flip graphs.

REGULAR TRIANGULATIONS & SUBDIVISIONS

LIFTINGS AND REGULAR SUBDIVISIONS

DEF. $\boldsymbol{P}$ = point configuration. $h : \boldsymbol{P} \to \mathbb{R}$ height function.

$\mathcal{S}(\boldsymbol{P}, h)$ = subdivision of $\boldsymbol{P}$ obtained as the projection of the lower convex hull of the lifted point set $\{(\boldsymbol{p}, h(\boldsymbol{p})) \mid \boldsymbol{p} \in \boldsymbol{P}\}$.



A subdivision $\mathcal{S}$ is regular if there is a height function $h : \boldsymbol{P} \to \mathbb{R}$ st $\mathcal{S} = \mathcal{S}(\boldsymbol{P}, h)$.

PROP. If $g : \mathbb{R}^n \to \mathbb{R}$ is affine, then $\mathcal{S}(\boldsymbol{P}, g + h) = \mathcal{S}(\boldsymbol{P}, h)$ for any $h : \boldsymbol{P} \to \mathbb{R}$.

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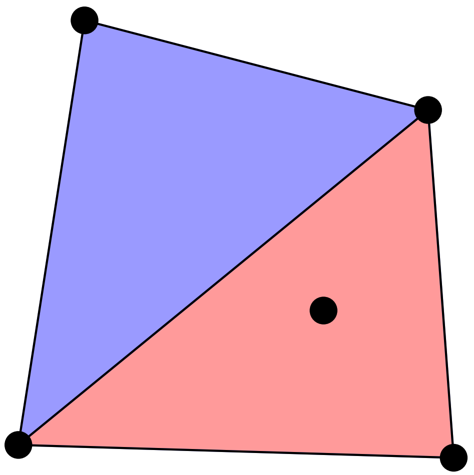

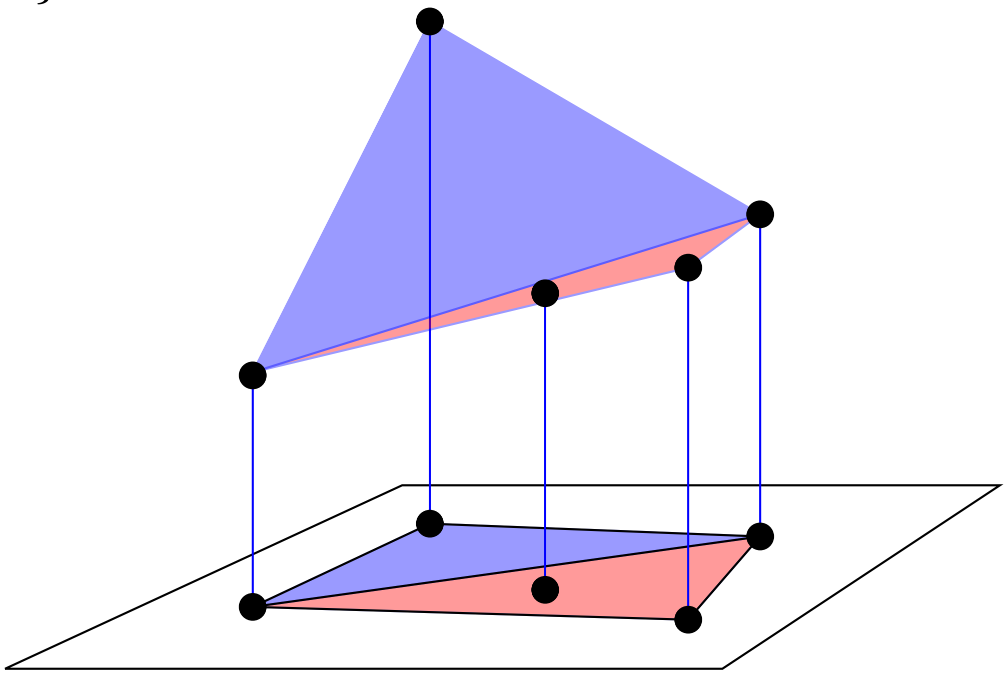

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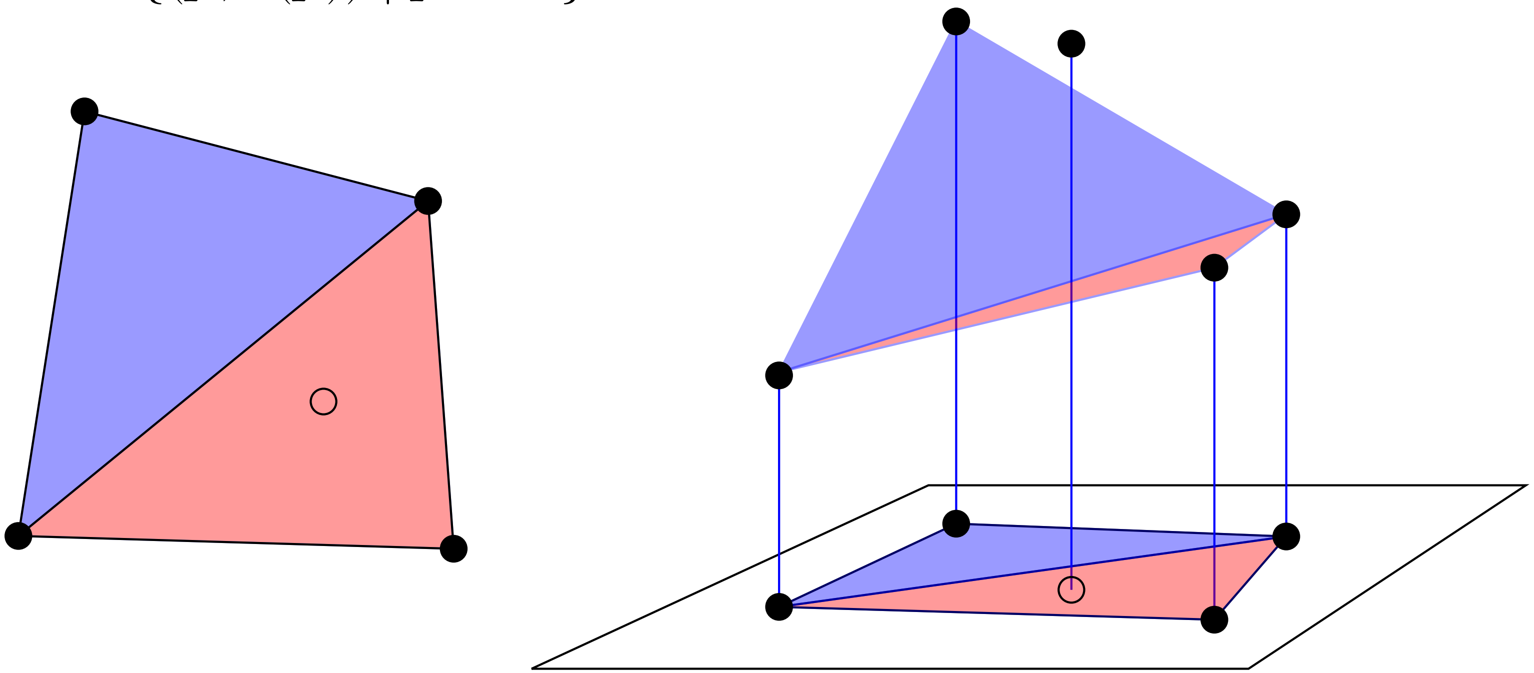

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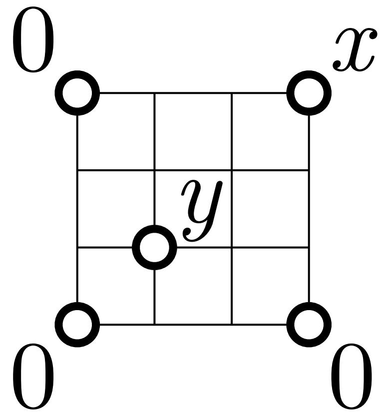
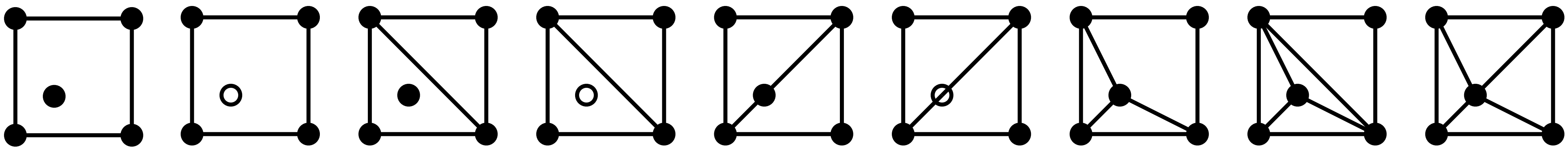
EXM OF REGULAR SUBDIVISIONS

Point configuration $\boldsymbol{P} = \{(0,0),(3,0),(0,3),(3,3),(1,1)\}$.
Restrict to height functions h with $h((0,0)) = h((3,0)) = h((0,3)) = 0$.
Let $x = h((3,3))$ and $y = h((1,1))$.



QU. Give conditions on x and y to obtain the following regular subdivisions:



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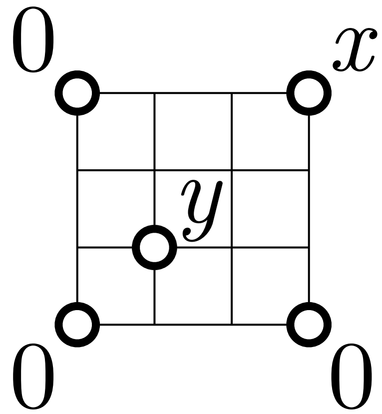

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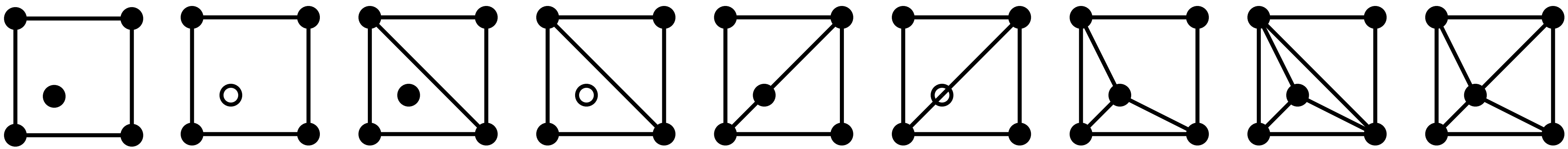

$x = 0$	$x = 0$
$y = 0$	$y > 0$

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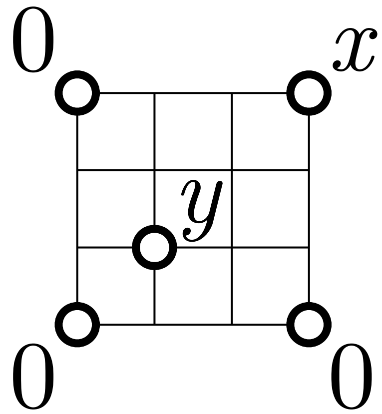

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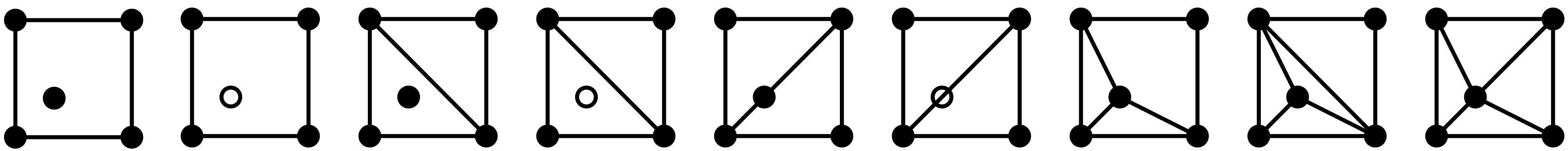

$x = 0$	$x = 0$	$x > 0$	$x > 0$					
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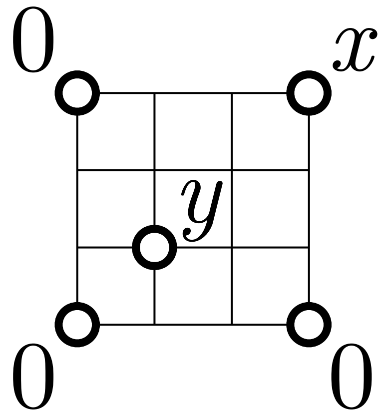

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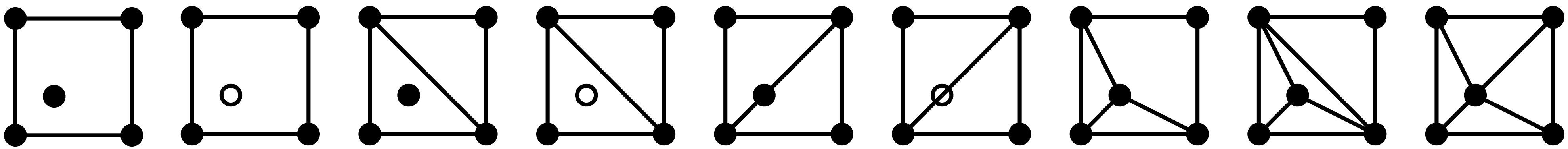

$x = 0$	$x = 0$	$x > 0$	$x > 0$	$x < 0$	$x < 0$			
$y = 0$	$y > 0$	$y = 0$	$y > 0$	$x - 3y = 0$	$x - 3y < 0$			

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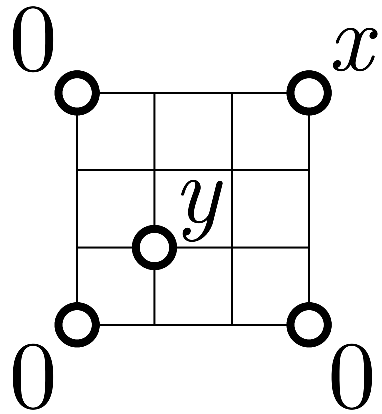

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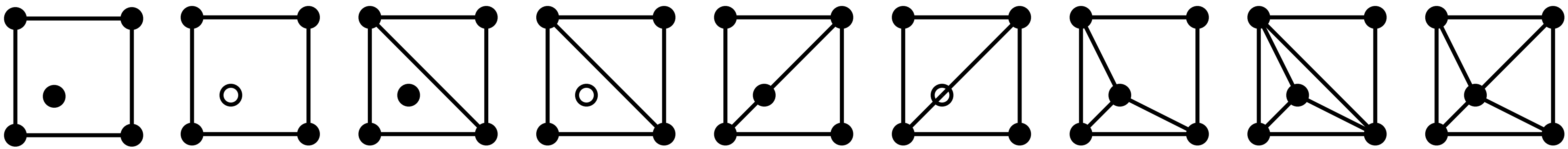
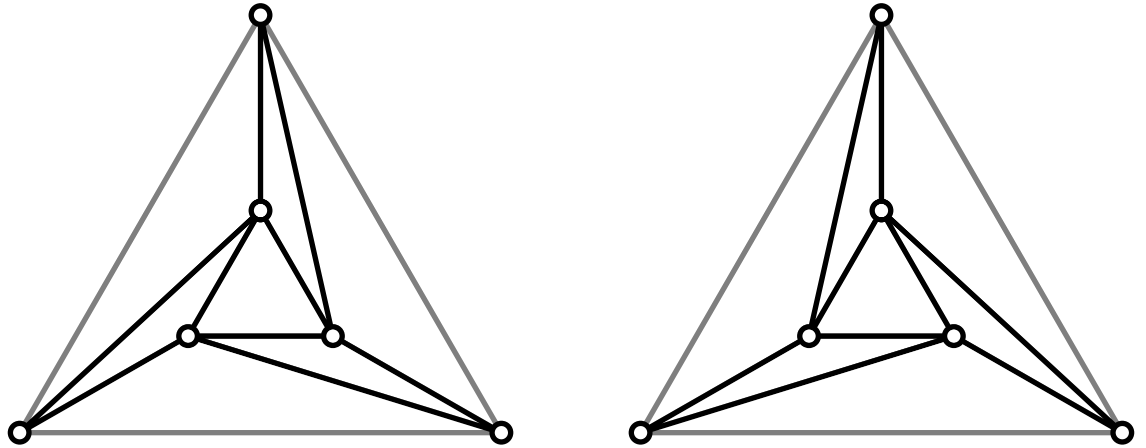


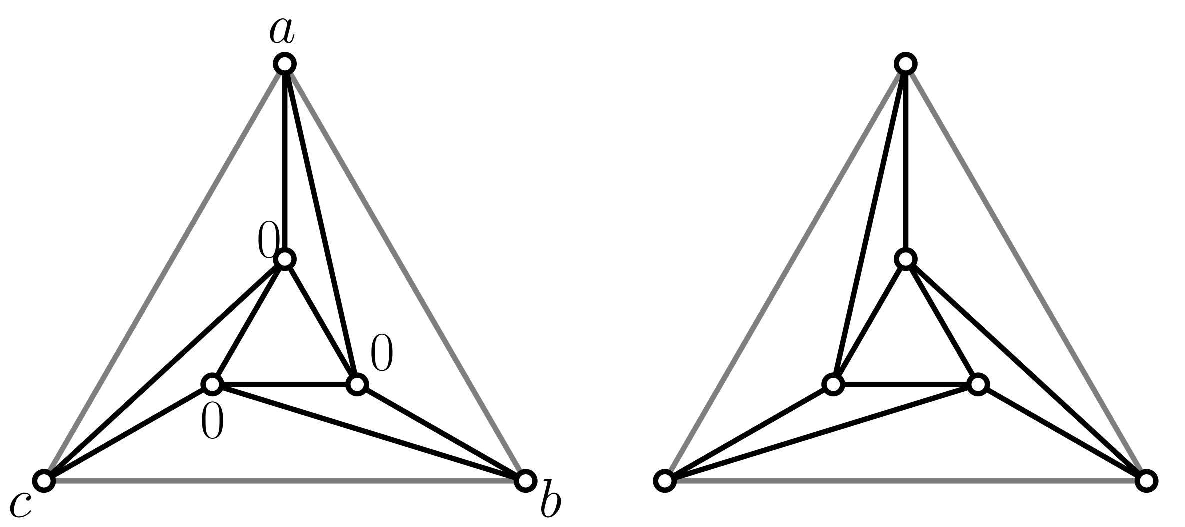
$x = 0$	$x = 0$	$x > 0$	$x > 0$	$x < 0$	$x < 0$	$x+3y=0$	$x+3y>0$	$x+3y<0$
$y = 0$	$y > 0$	$y = 0$	$y > 0$	$x-3y=0$	$x-3y<0$	$y < 0$	$y < 0$	$x-3y>0$

QU. Show that the following two triangulations are not regular:



NON REGULAR TRIANGULATIONS

PROP. The following two triangulations are not regular:



proof: assume the left one regular, and pick a height function. Up to an affine function, height 3 for the 3 internal vertices. The heights of the 3 external vertices satisfy: $a < b < c < a$. Contradiction.

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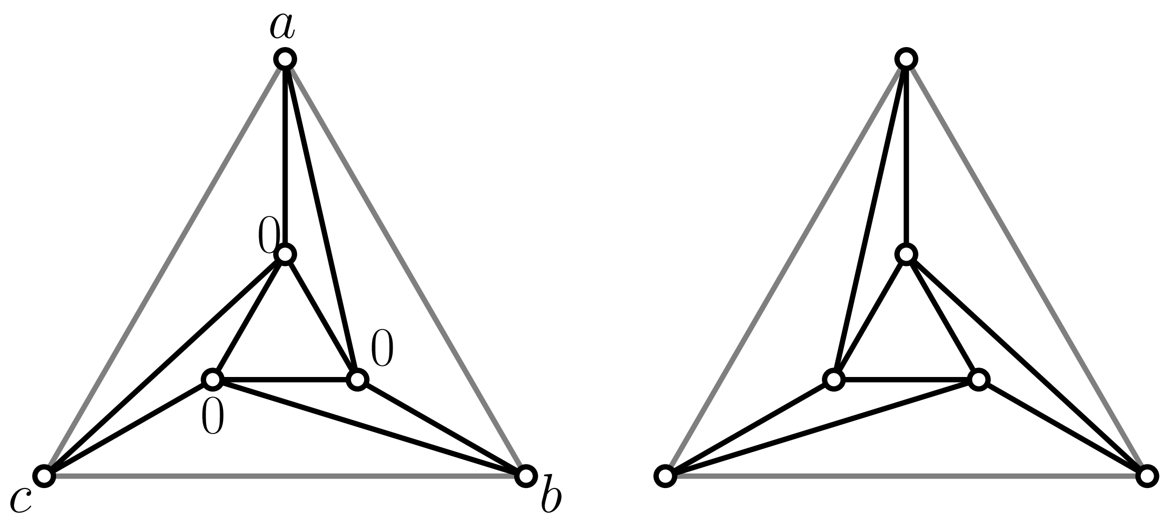

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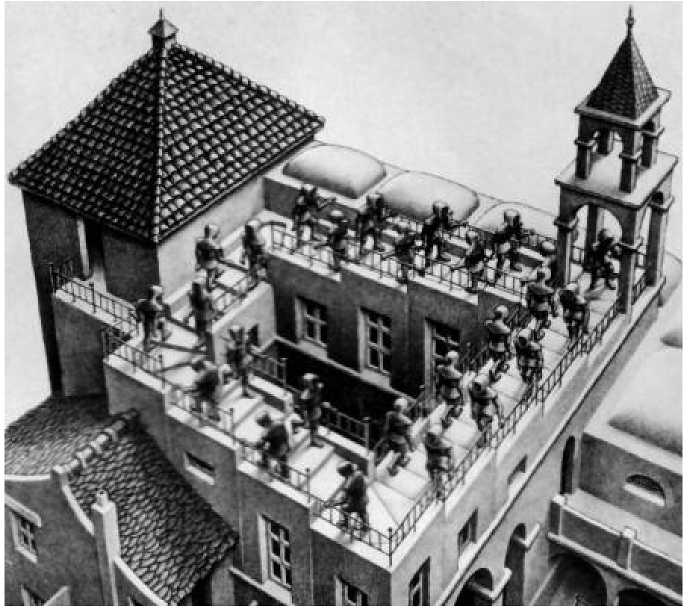
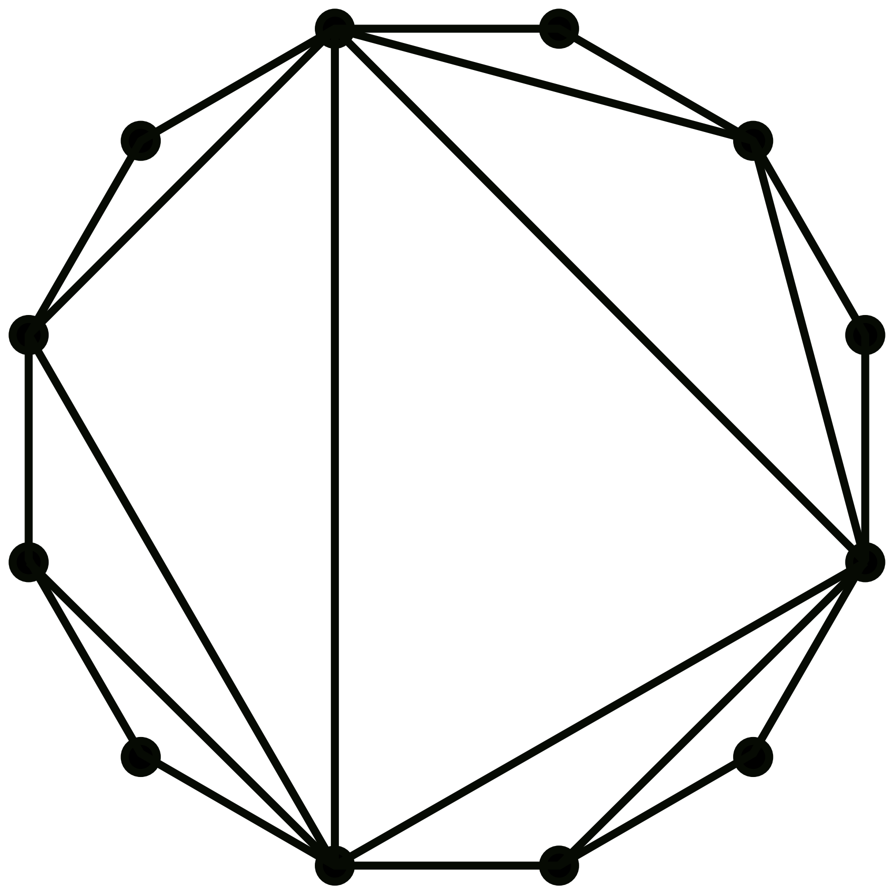


QU. Show that all subdivisions of a planar point set in convex position are regular.



CONVEX POSITION

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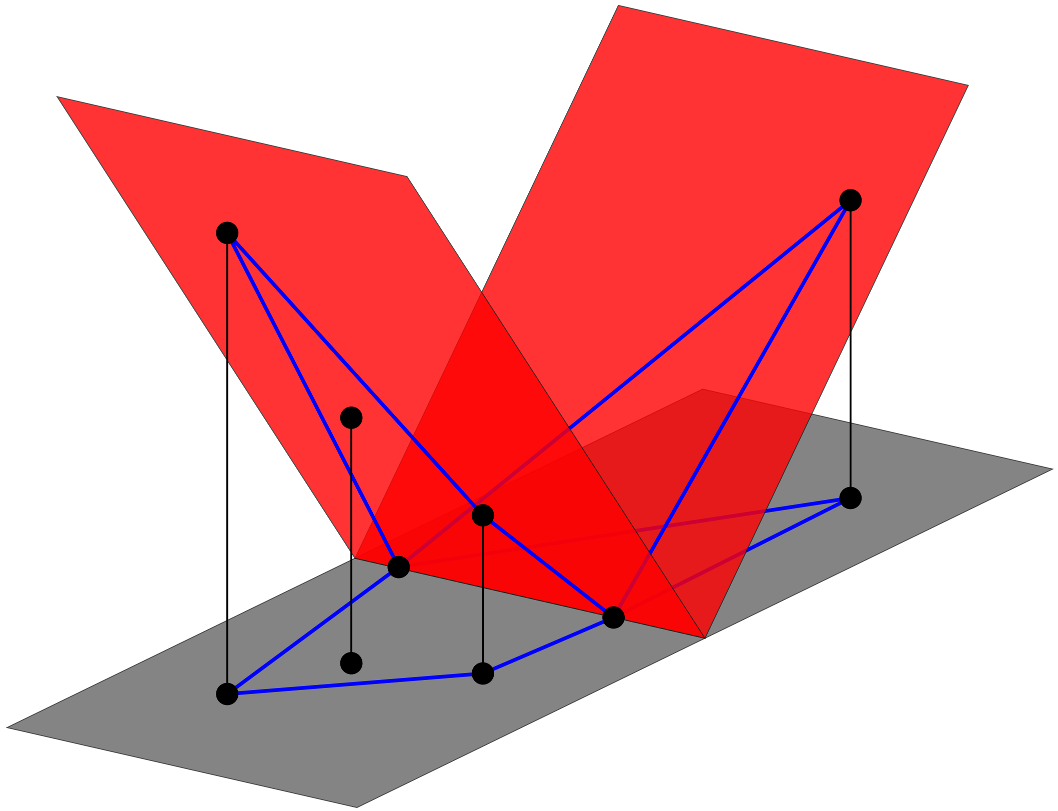
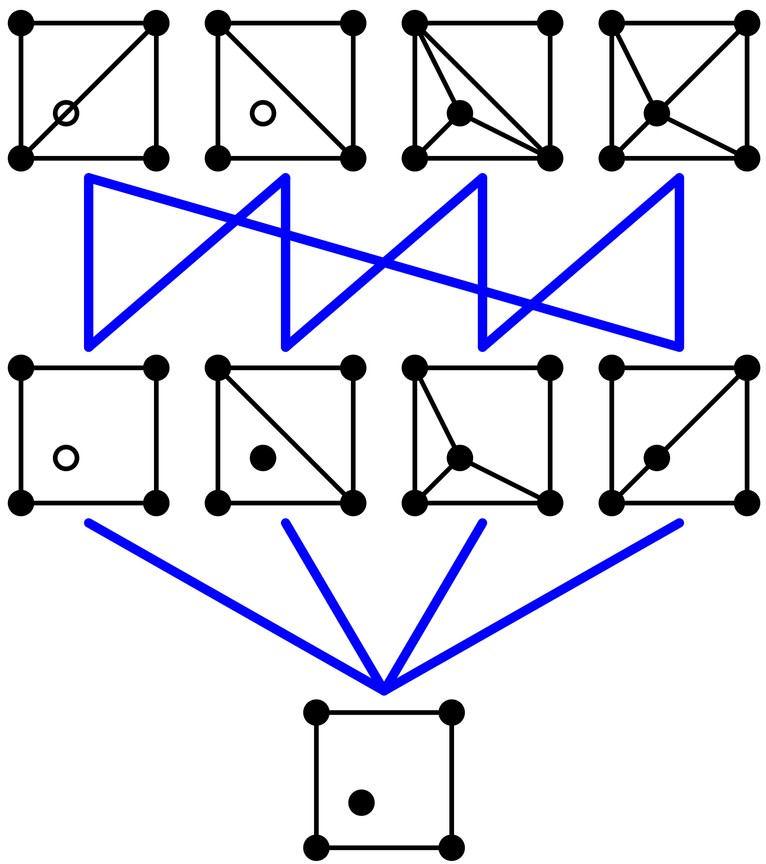


Use $h(\boldsymbol{p}) = \sum_{\delta \in \mathcal{S}} d(\delta, \boldsymbol{p})$ where $d(\delta, \boldsymbol{p})$ is the distance of $\boldsymbol{p}$ to the line spanned by δ.

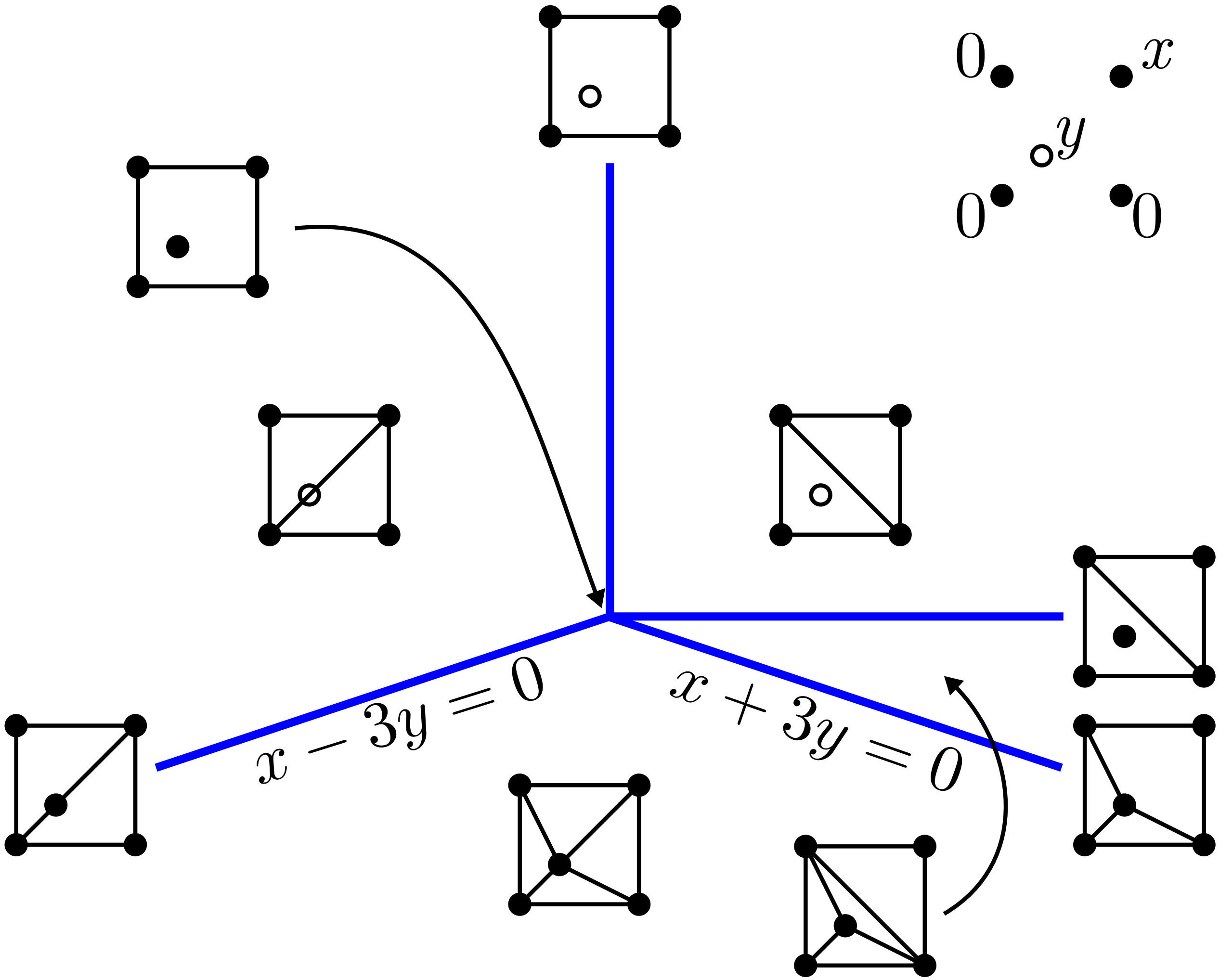
DEF. $\mathcal{S}$ refines $\mathcal{S}'$ when for any $\boldsymbol{X} \in \mathcal{S}$, there is $\boldsymbol{X}' \in \mathcal{S}'$ st $\boldsymbol{X} \subseteq \boldsymbol{X}'$.
regular subdivision lattice $=$ regular subdivisions of $\boldsymbol{P}$ ordered by refinement.



SECONDARY FAN AND POLYTOPE

SECONDARY FAN

DEF. secondary cone of subdivision $\mathcal{S}$ of $\boldsymbol{P} = \Sigma\mathbb{C}(\mathcal{S}) = \left\{ \boldsymbol{h} \in \mathbb{R}^{\boldsymbol{P}} \;\middle|\; \mathcal{S}(\boldsymbol{P}, \boldsymbol{h}) = \mathcal{S} \right\}$.
secondary fan of $\boldsymbol{P}$ = fan formed by the secondary cones of all (regular) subdivisions.



SECONDARY POLYTOPE

DEF. $\mathcal{T}$ triangulation of a point set $\boldsymbol{P} \subseteq \mathbb{R}^d$.

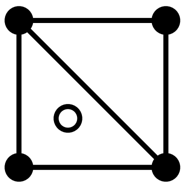
volume vector of $\mathcal{T}$:

$$\Phi(\mathcal{T}) = \left(\sum_{\boldsymbol{p} \in \triangle \in \mathcal{T}} \mathrm{vol}(\triangle) \right)_{\boldsymbol{p} \in \boldsymbol{P}}$$

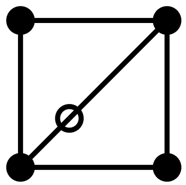
secondary polytope of $\boldsymbol{P}$:

$$\Sigma\mathbb{P}(\boldsymbol{P}) := \mathrm{conv}\left\{ \Phi(\mathcal{T}) \mid \mathcal{T} \text{ triangulation of } \boldsymbol{P} \right\}.$$

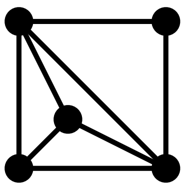
exm:



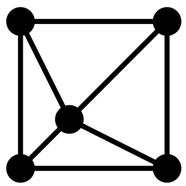
(9, 18, 18, 9, 0)



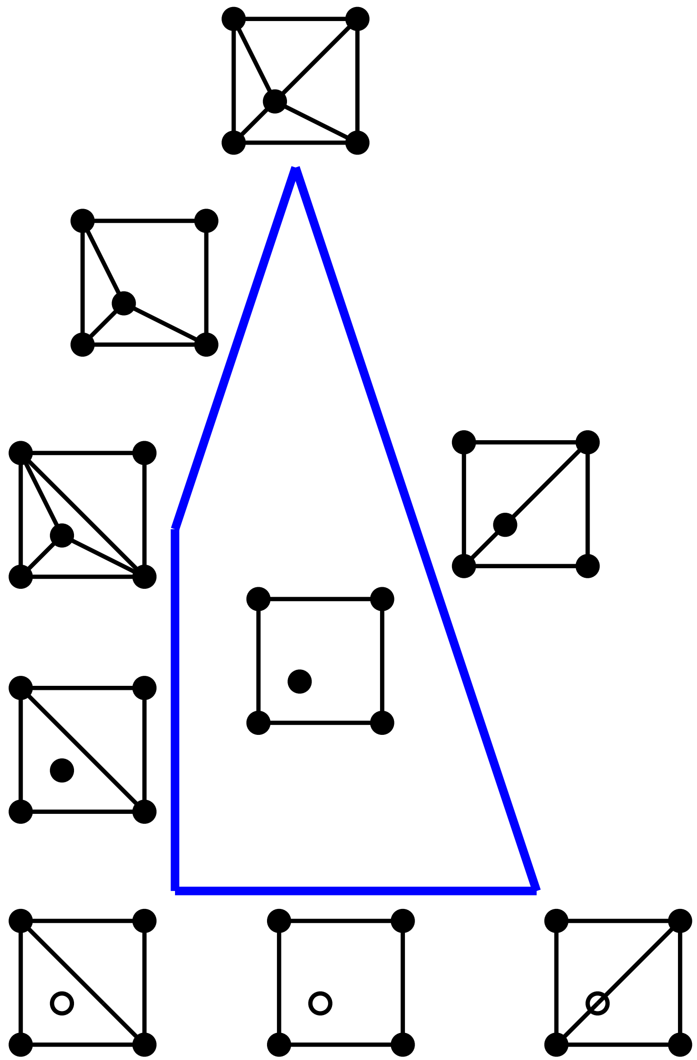
(18, 9, 9, 18, 0)



(6, 15, 15, 9, 9)



(6, 9, 9, 12, 18)



THM. (Gelfand, Kapranov, and Zelevinsky) For $\boldsymbol{P}$ in general position in $\mathbb{R}^d$,

- $\Sigma\mathbb{P}(\boldsymbol{P})$ has dimension $|\boldsymbol{P}| - d - 1$,
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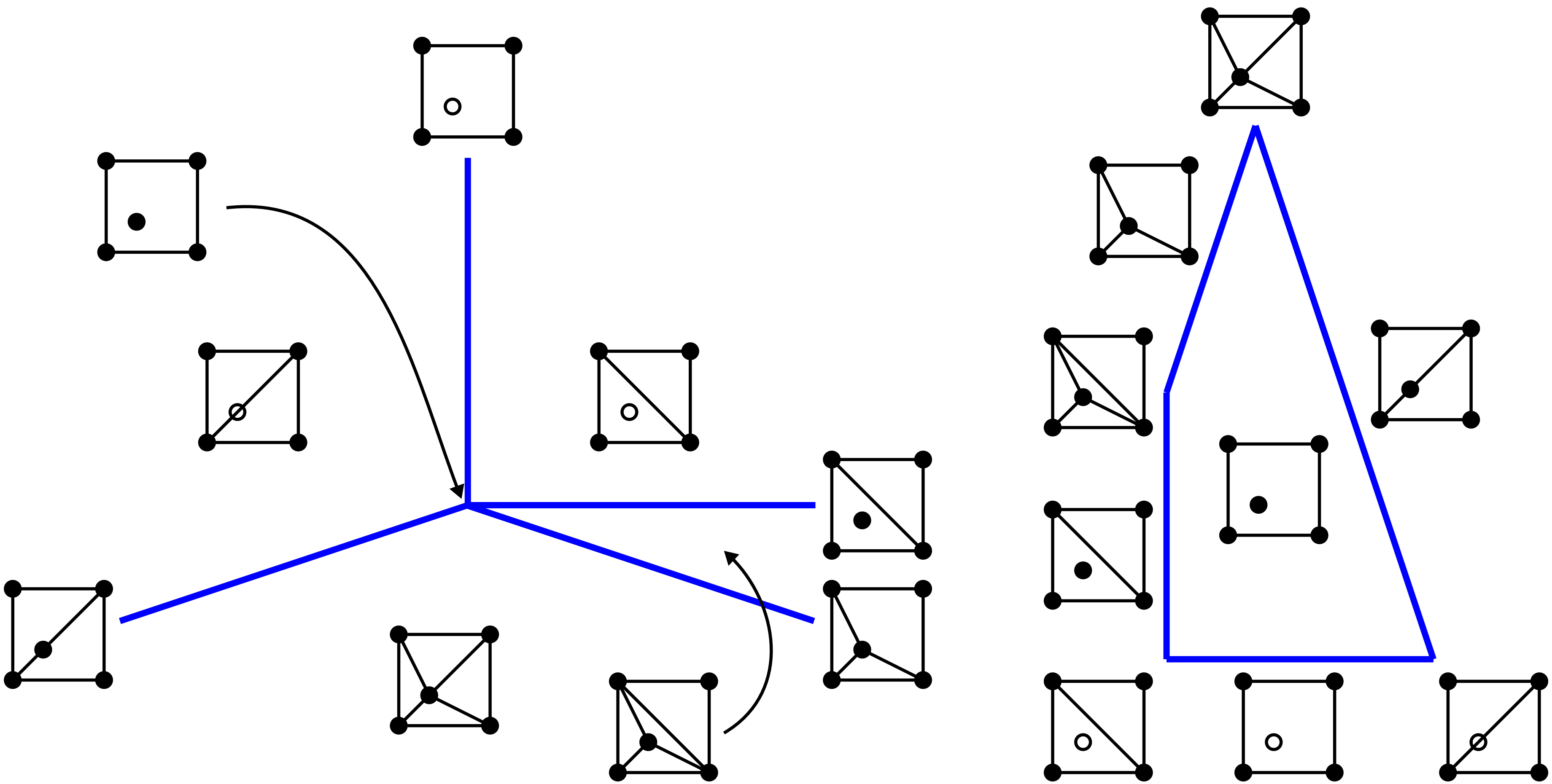

SECONDARY FAN AND POLYTOPE

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proof: lower bound on $\dim(\Sigma\mathbb{P}(\boldsymbol{P}))$ by induction on $|\boldsymbol{P}|$:

- when $|\boldsymbol{P}| = 3$, $\Sigma\mathbb{P}(\boldsymbol{P})$ is a single point,
- for $|\boldsymbol{P}| \geq 4$ and any $\boldsymbol{p} \in \boldsymbol{P}$, $\Sigma\mathbb{P}(\boldsymbol{P} \smallsetminus \boldsymbol{p}) = \Sigma\mathbb{P}(\boldsymbol{P}) \cap \left\{\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{P}} \;\middle|\; x_{\boldsymbol{p}} = \alpha\right\}$ where

$$\alpha = \begin{cases} 0 & \text{if } \boldsymbol{p} \text{ inside } \operatorname{conv}(\boldsymbol{P}), \\ \operatorname{vol}(\operatorname{conv}(\boldsymbol{P})) - \operatorname{vol}(\operatorname{conv}(\boldsymbol{P} \smallsetminus \boldsymbol{p})) & \text{if } \boldsymbol{p} \text{ on the boundary of } \operatorname{conv}(\boldsymbol{P}). \end{cases}$$

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upper bound on $\dim(\Sigma\mathbb{P}(\boldsymbol{P}))$ from the volume and center of mass of $\operatorname{conv}(\boldsymbol{P})$:

$$\operatorname{vol}(\boldsymbol{P}) = \sum_{\triangle \in \mathcal{T}} \operatorname{vol}(\triangle) = \sum_{\triangle \in \mathcal{T}} \sum_{\boldsymbol{p} \in \triangle} \frac{\operatorname{vol}(\triangle)}{d+1} = \frac{1}{d+1} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \sum_{\boldsymbol{p} \in \triangle \in \mathcal{T}} \operatorname{vol}(\triangle) = \frac{1}{d+1} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \Phi(\mathcal{T})_{\boldsymbol{p}}.$$

$$\operatorname{vol}(\boldsymbol{P}) \cdot \operatorname{cm}(\boldsymbol{P}) = \sum_{\triangle \in \mathcal{T}} \operatorname{vol}(\triangle) \cdot \operatorname{cm}(\triangle) = \sum_{\triangle \in \mathcal{T}} \operatorname{vol}(\triangle) \cdot \left(\frac{1}{d+1} \sum_{\boldsymbol{p} \in \triangle} \boldsymbol{p} \right) = \frac{1}{d+1} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \Phi(\mathcal{T})_{\boldsymbol{p}} \cdot \boldsymbol{p}.$$

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proof: $\mathcal{T}$ triangulation of $\boldsymbol{P}$ and a height vector $\boldsymbol{h} \in \mathbb{R}^{\boldsymbol{P}}$.
$f_{\mathcal{T},\boldsymbol{h}} : \mathrm{conv}(\boldsymbol{P}) \to \mathbb{R} =$ piecewise linear map on the simplices of $\mathcal{T}$ such that $f_{\mathcal{T},\boldsymbol{h}}(\boldsymbol{p}) = \boldsymbol{h}_{\boldsymbol{p}}$.
Then the volume below the hypersurface defined by $f_{\mathcal{T},\boldsymbol{h}}$ is

$$\int_{\mathrm{conv}(\boldsymbol{P})} f_{\mathcal{T},\omega}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\triangle\in\mathcal{T}} \int_{\triangle} f_{\mathcal{T},\omega}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\triangle\in\mathcal{T}} \frac{\mathrm{vol}(\triangle)}{d+1} \sum_{\boldsymbol{p}\in\triangle} \boldsymbol{h}_{\boldsymbol{p}}$$

$$= \frac{1}{d+1} \sum_{\boldsymbol{p}\in\boldsymbol{P}} \boldsymbol{h}_{\boldsymbol{p}} \cdot \sum_{\boldsymbol{p}\in\triangle\in\mathcal{T}} \mathrm{vol}(\triangle) = \frac{\langle\, \Phi(\mathcal{T}) \mid \boldsymbol{h}\, \rangle}{3}.$$

SECONDARY FAN AND POLYTOPE

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Then the volume below the hypersurface defined by $f_{\mathcal{T},\boldsymbol{h}}$ is

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$$= \frac{1}{d+1} \sum_{\boldsymbol{p}\in\boldsymbol{P}} \boldsymbol{h}_{\boldsymbol{p}} \cdot \sum_{\boldsymbol{p}\in\triangle\in\mathcal{T}} \operatorname{vol}(\triangle) = \frac{\langle\, \Phi(\mathcal{T}) \mid \boldsymbol{h} \,\rangle}{3}.$$

Therefore, if $\mathcal{T} = \mathcal{S}(\boldsymbol{P}, \boldsymbol{h}) \neq \mathcal{T}'$ then

$$\langle\, \Phi(\mathcal{T}) \mid \boldsymbol{h} \,\rangle < \langle\, \Phi(\mathcal{T}') \mid \boldsymbol{h} \,\rangle\,.$$

In other words, the normal cone of $\Phi(\mathcal{T})$ in $\Sigma\mathbb{P}(\boldsymbol{P})$ is the secondary cone of $\mathcal{T}$.

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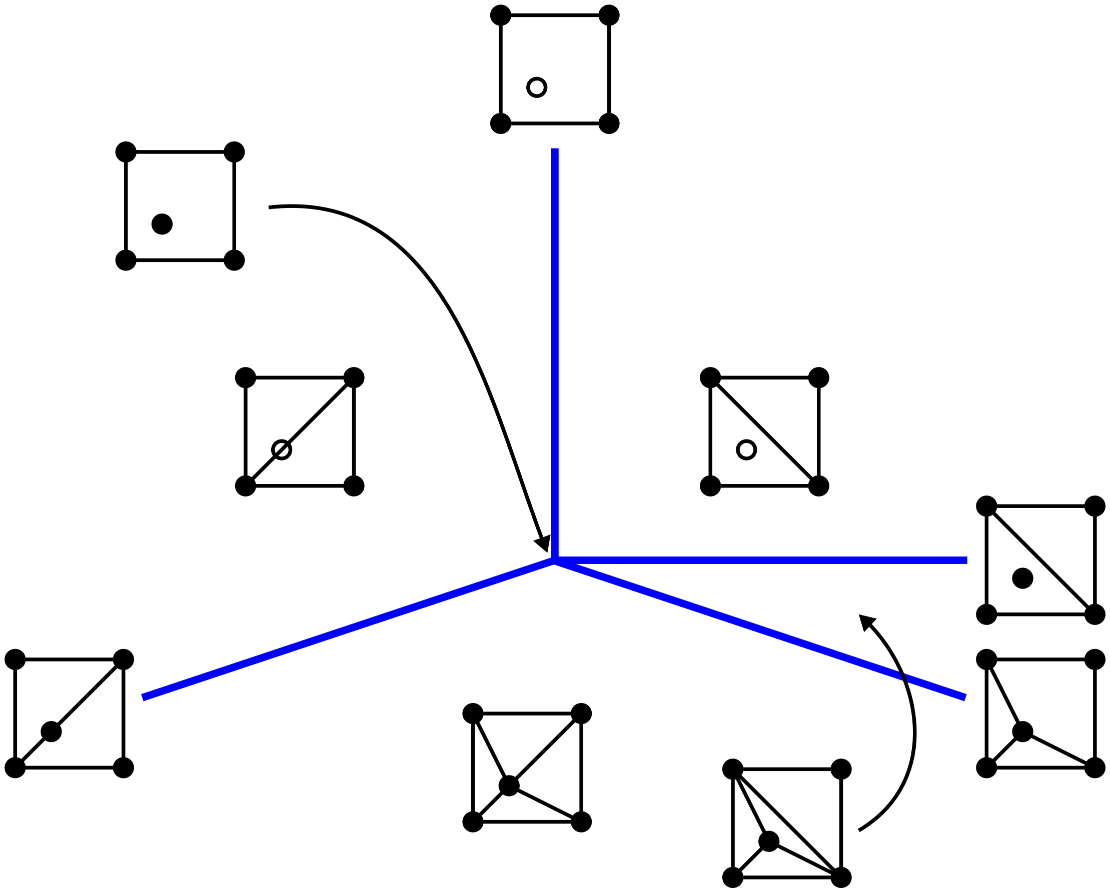
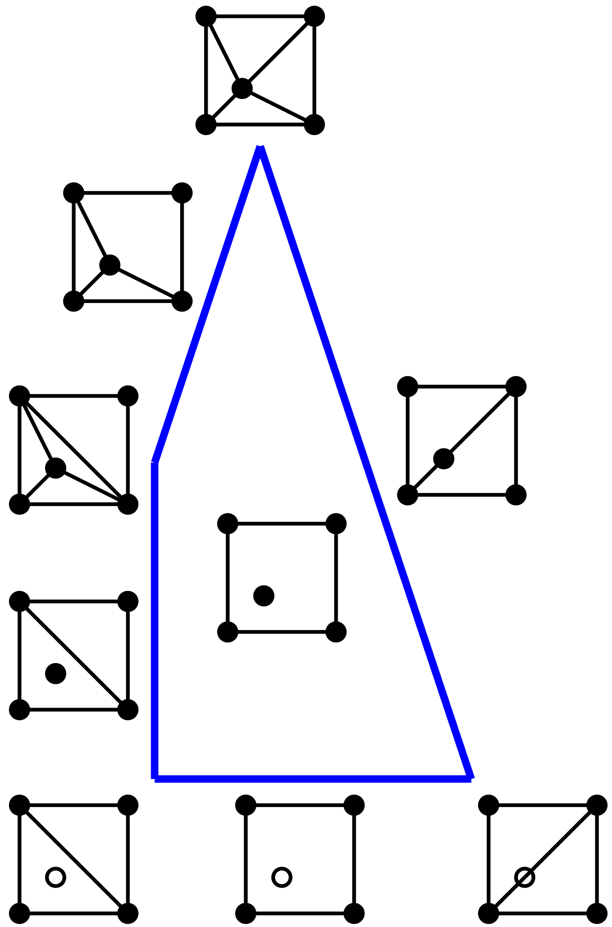
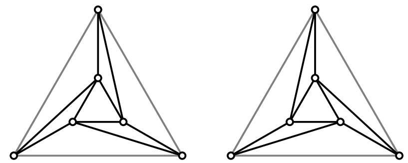
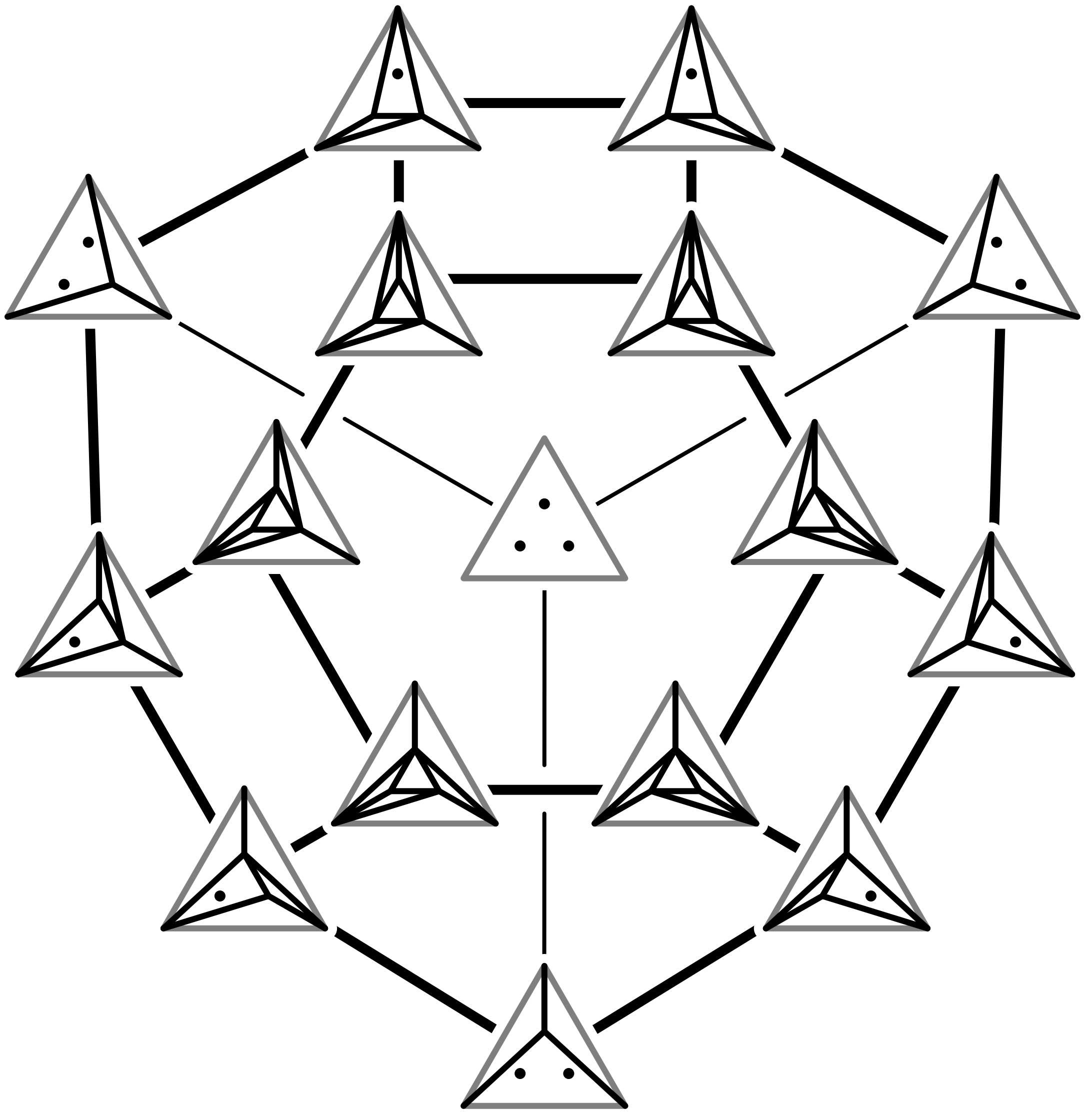




QU. Locate the volume vectors of the non-regular triangulations  in



SOME REFERENCES

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