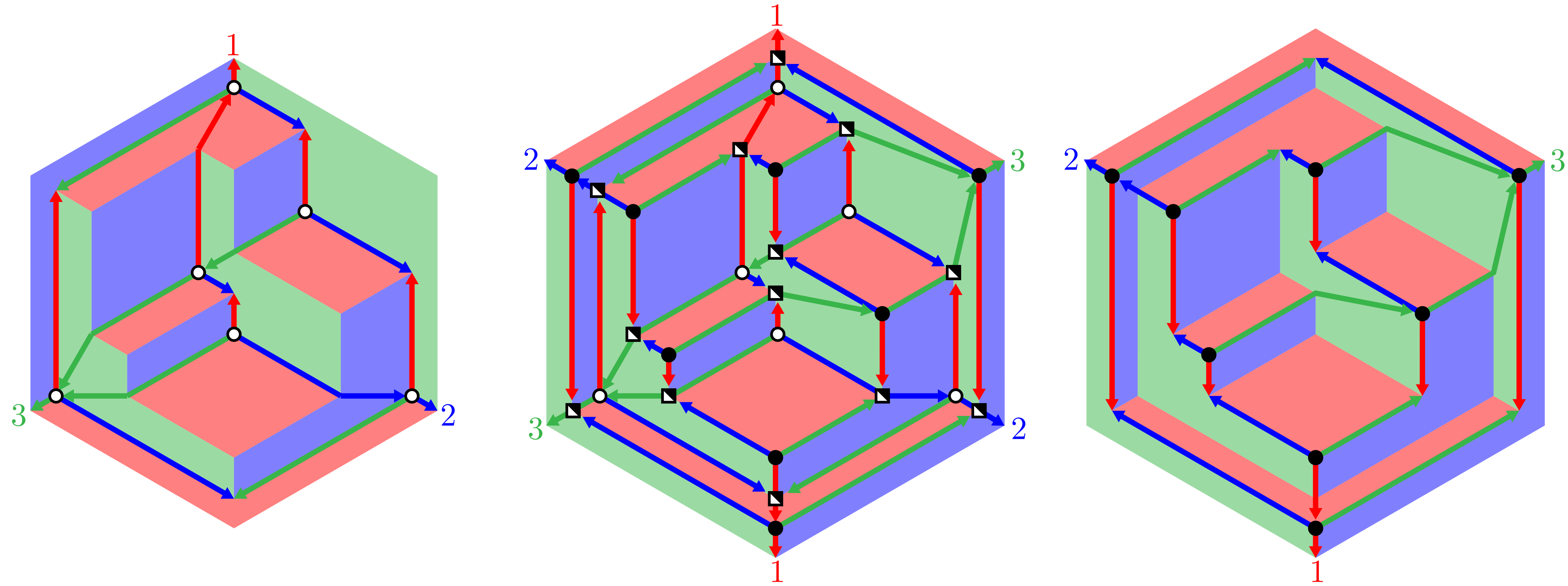
Schnyder woods and applications



V. PILAUD

MPRI 2-38-1. Algorithms and combinatorics for geometric graphs

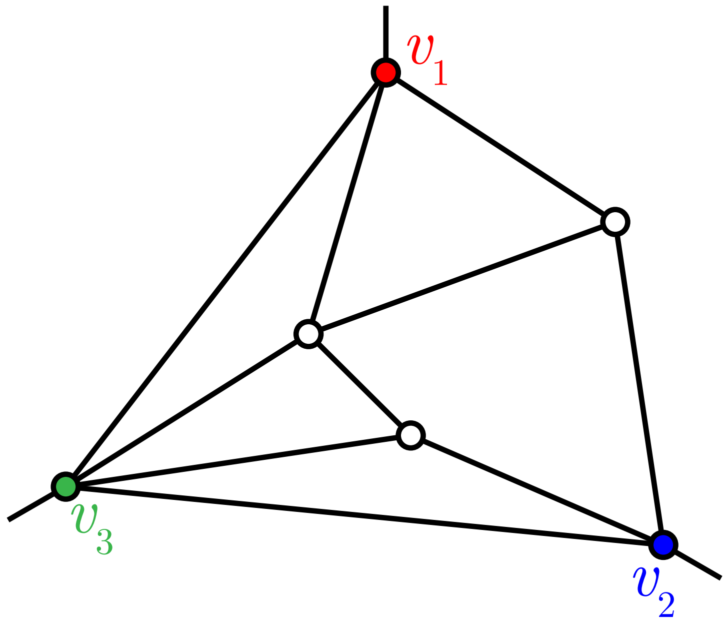
Course not given in 2022

slides available at: http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP-2.pdf

Course notes available at: https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI20.pdf

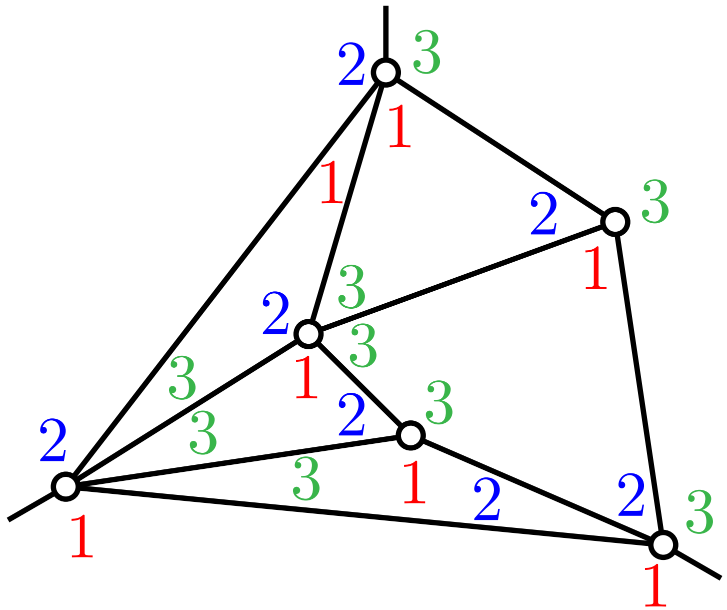
SCHNYDER LABELINGS AND WOODS

PLANAR MAP



$M =$ planar map with three distinguished vertices v_1, v_2, v_3 clockwise on the outer face where a half edge is pending in the outer face.

SCHNYDER LABELING

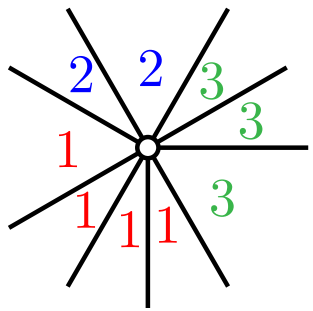


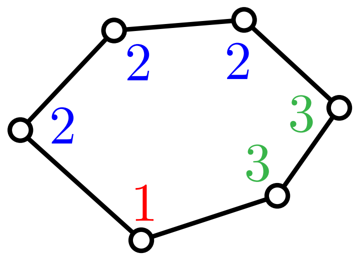
DEF. <u>Schnyder labeling</u> on M = labeling of the angles of M with labels $\{1, 2, 3\}$ st:

(L1) the angles at the half-edge of v_i are labeled $i+1$ and $i-1$ clockwise,

(L2) clockwise around each vertex, the labels form intervals of 1's, 2's and 3's,

(L3) clockwise around each face, the labels form intervals of 1's, 2's and 3's.





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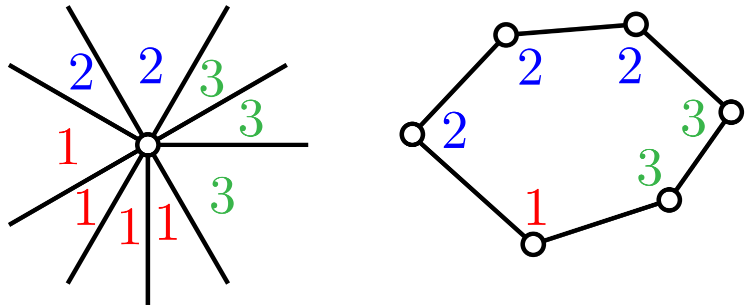


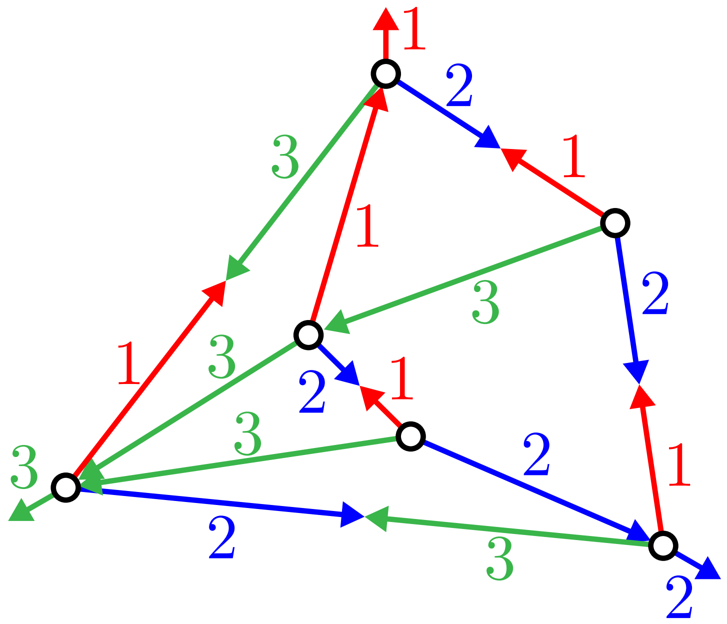
LEM. The three labels $\{1, 2, 3\}$ appear among the four angles surrounding any edge.

proof: Count the number of adjacent angles (same vertex and adjacent faces, or adjacent vertices and same face) with distinct labels. There are:

- 3 around each vertex,
- 3 around each face,
- 2 at each half-edge.

Since $3|V| + 3|F| = 3|E| + 6$ by Euler relation, there are also 3 for each edge.

SCHNYDER WOOD



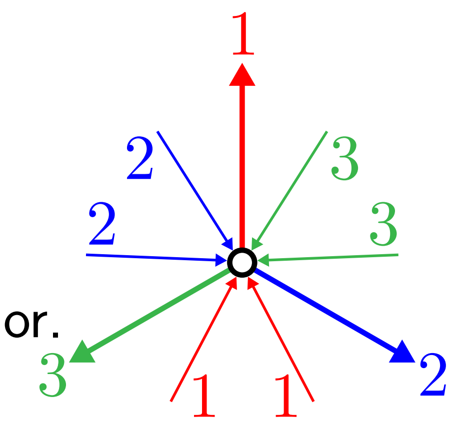
DEF. <u>Schnyder wood</u> on M = (bi-)orientation and (bi-)coloration of the edges of M with $\{1, 2, 3\}$ st:

(W0) bioriented edges get two distinct colors,

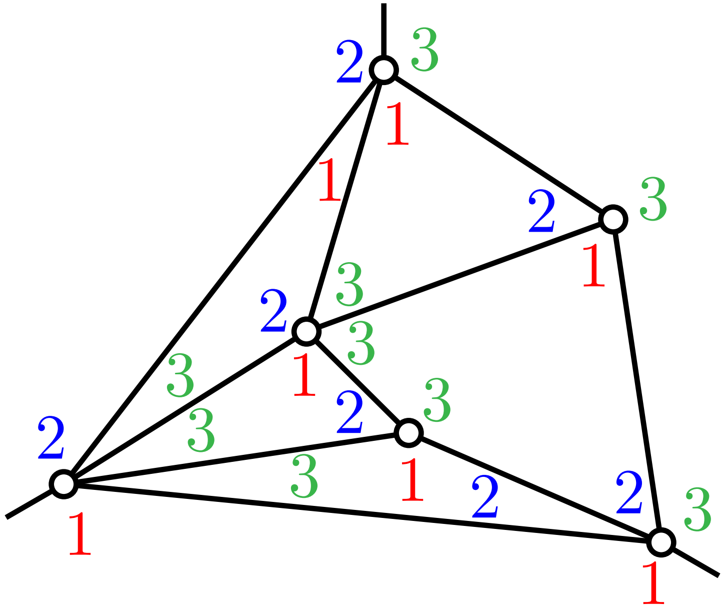
(W1) the half-edge at v_i is directed outwards and colored i,

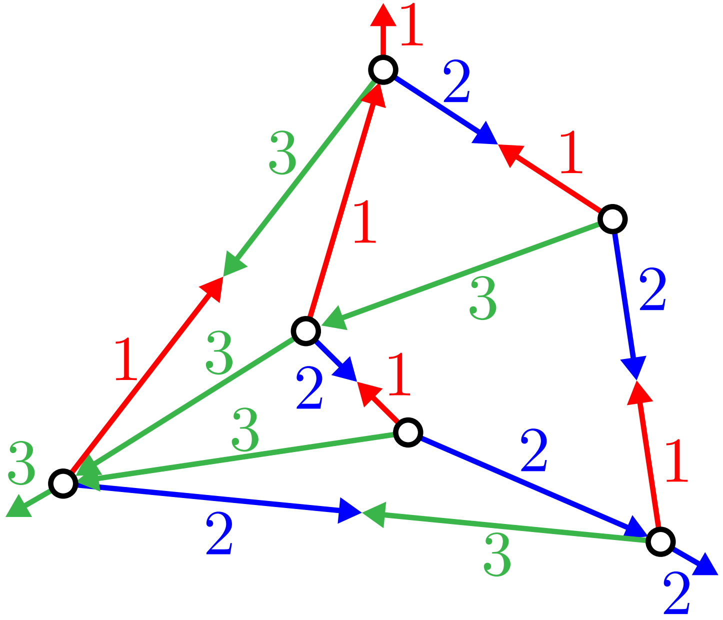
(W2) each vertex v has outdegree one in each label,

(W3) no interior face whose boundary is a directed cycle in one color.

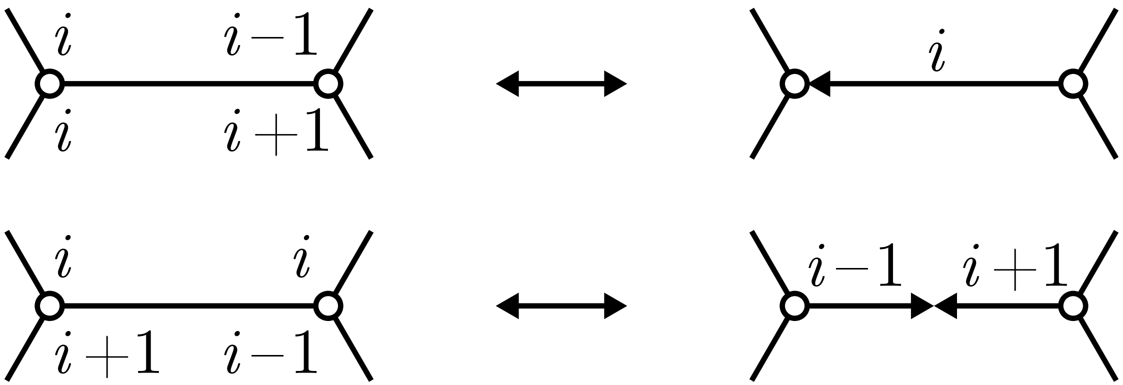


SCHNYDER LABELINGS VS SCHNYDER WOODS





THM. The transformation given by



is a bijection from Schnyder labelings to Schnyder woods.

SCHNYDER LABELINGS VS SCHNYDER WOODS

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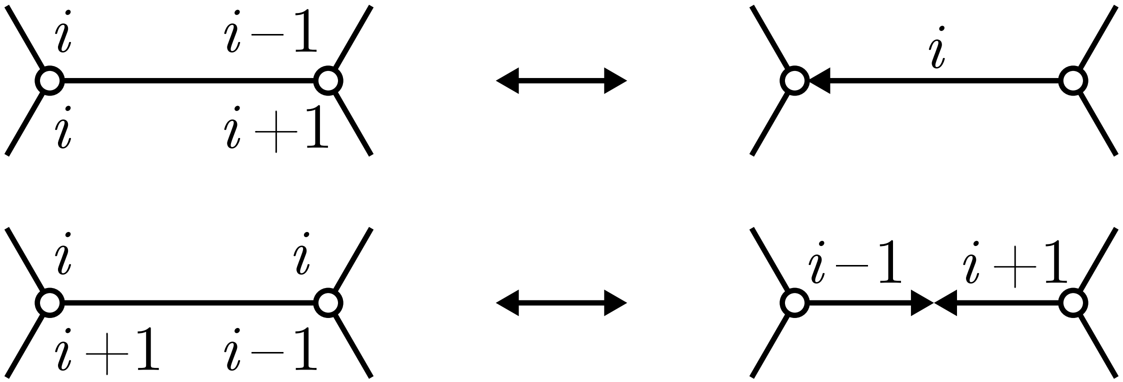


is a bijection from Schnyder labelings to Schnyder woods.

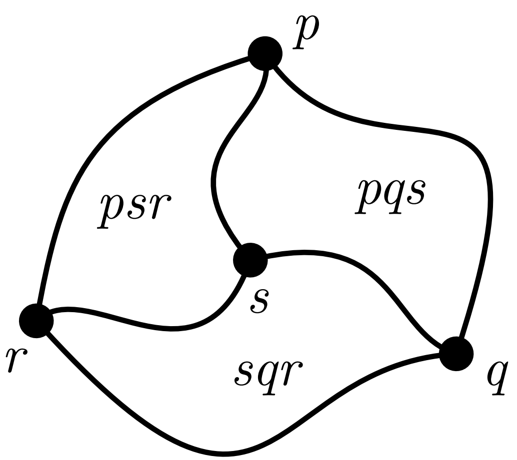
remarks:

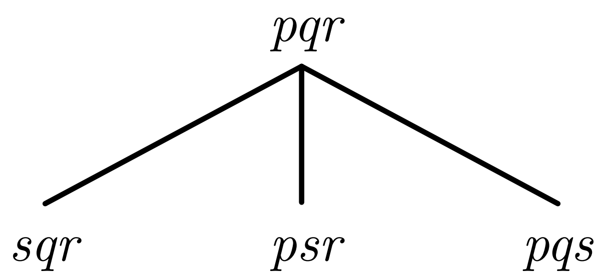
- Only two possible situations by the local rules around vertices, edges and faces.
- If M is triangulated, the second situation cannot occur except on the external face, so that there is no internal bioriented edge.

EXM: STACKED TRIANGULATIONS

DEF. stacked triangulation = triangulation obtained from an initial triangle abc by iteratively refining a triangle pqr into three triangles sqr, psr, and pqs.

construction tree = ternary tree where pqr is the parent of sqr, psr, and pqs.





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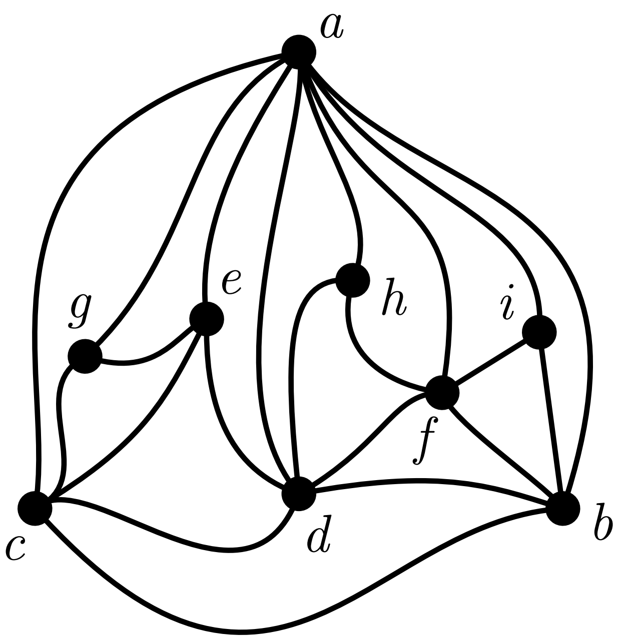

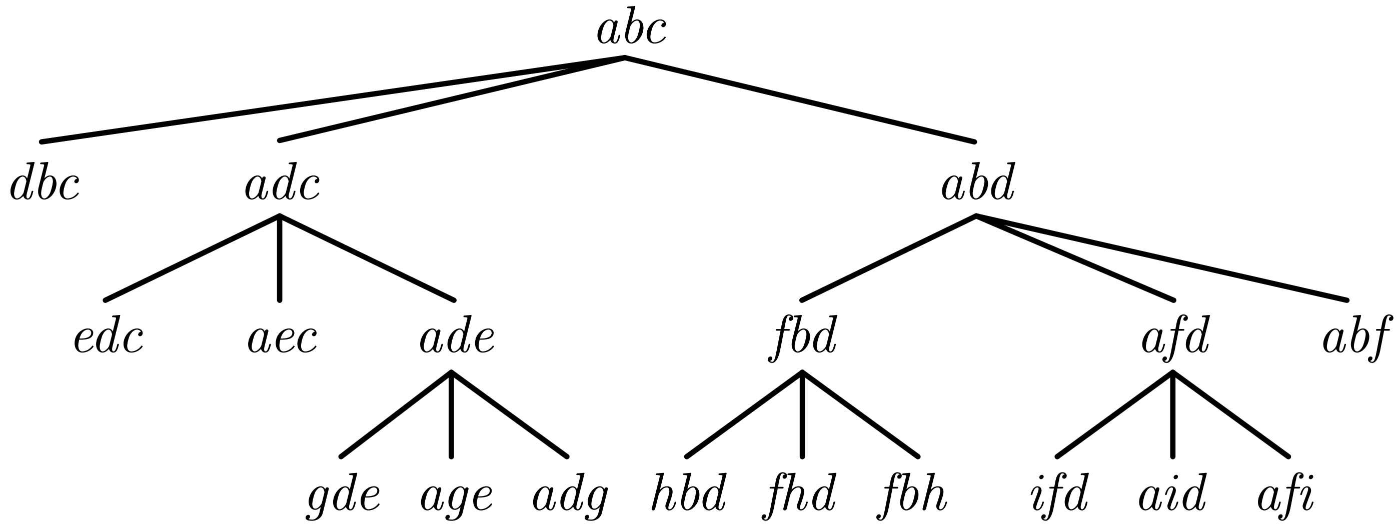

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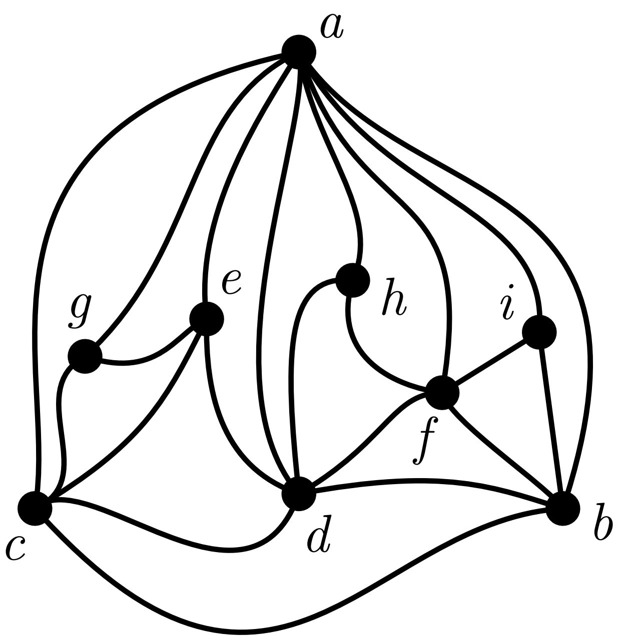

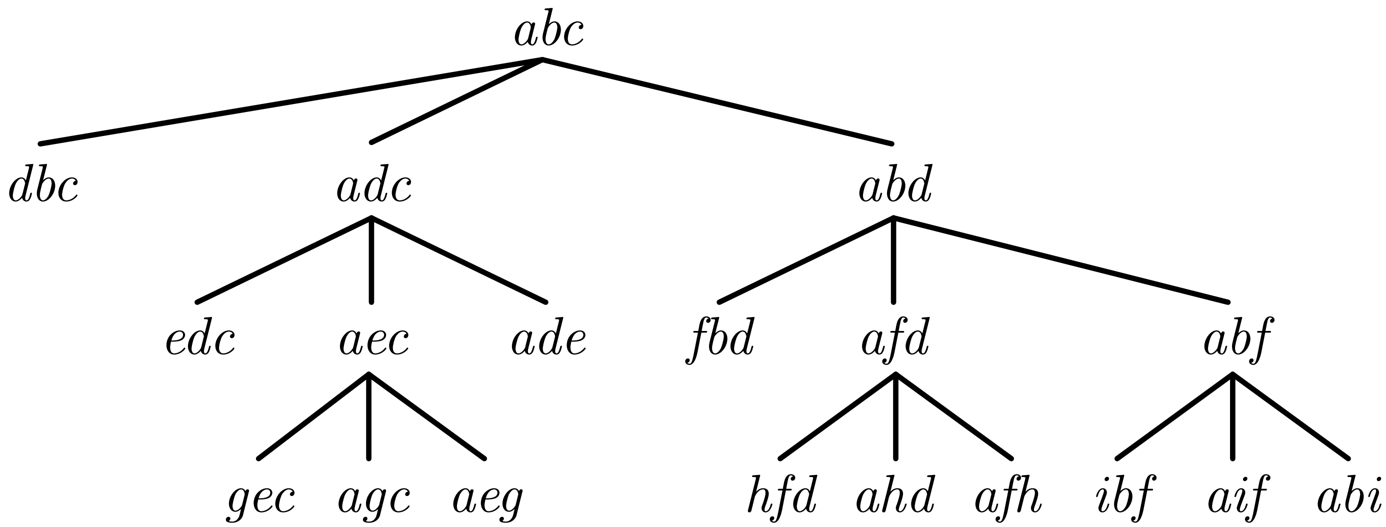

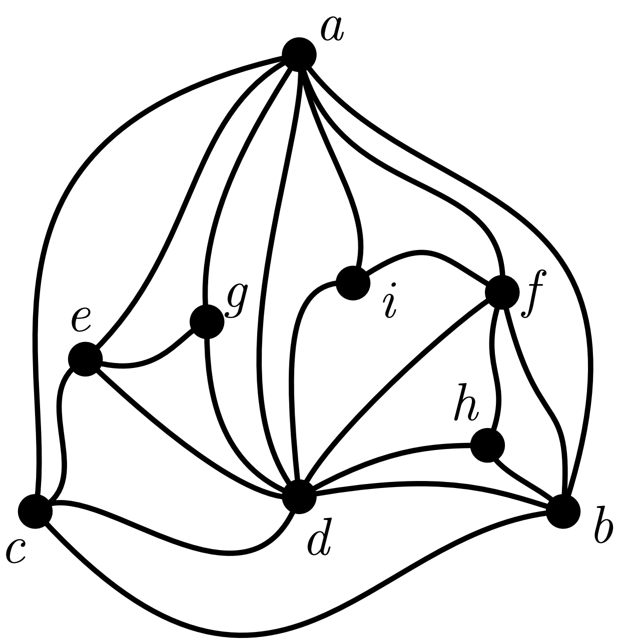

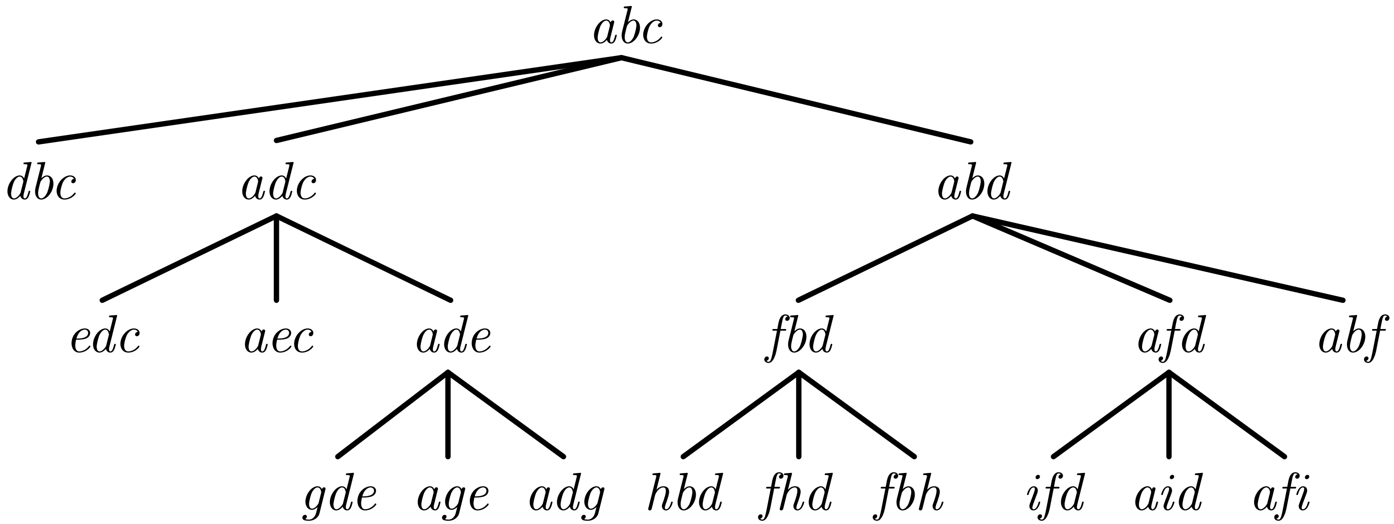

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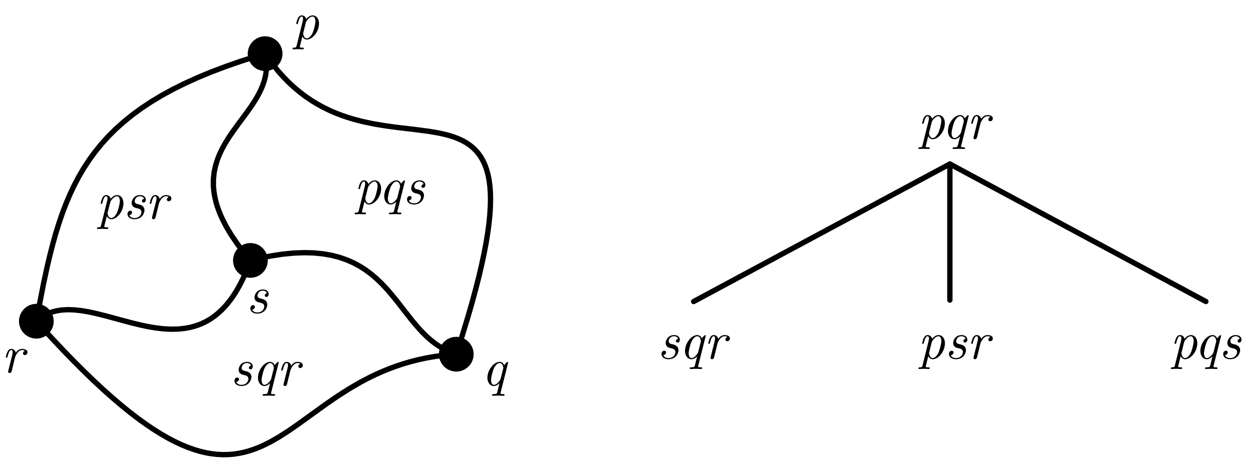

QU. Numbers of vertices, edges and faces of a stacked triangulation?

- in terms of the number of stacking operations,
- in terms of the construction tree.

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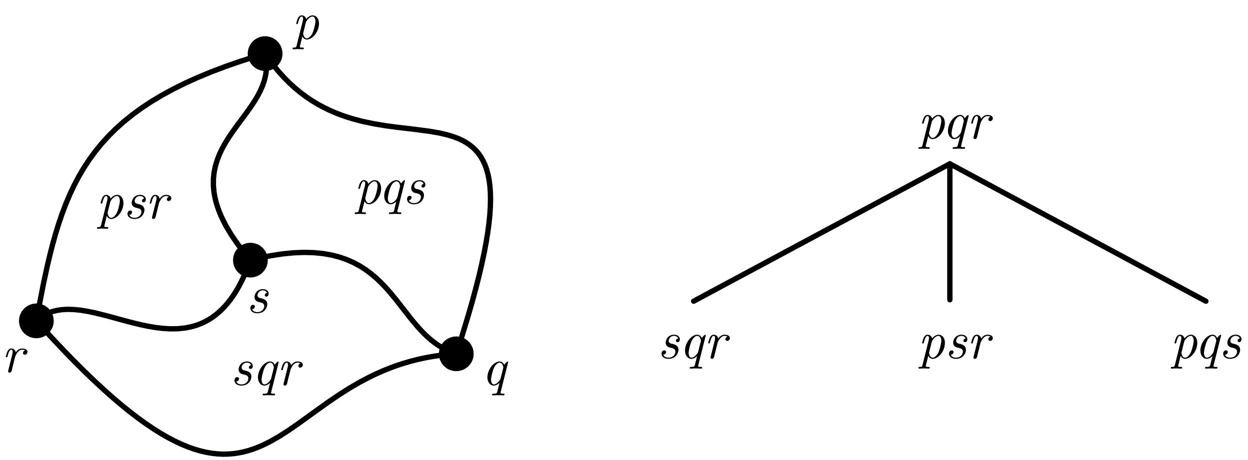

REM. In a stacked triangulation obtained after n stacking operations, and with construction tree C,

- number of vertices $= 3 + n = 3 +$ number interior nodes in C,
- number of edges $= 3(n+1) = 3 +$ number edges in C,
- number of faces $= 2n + 1 =$ number of leaves of C.

EXM: STACKED TRIANGULATIONS

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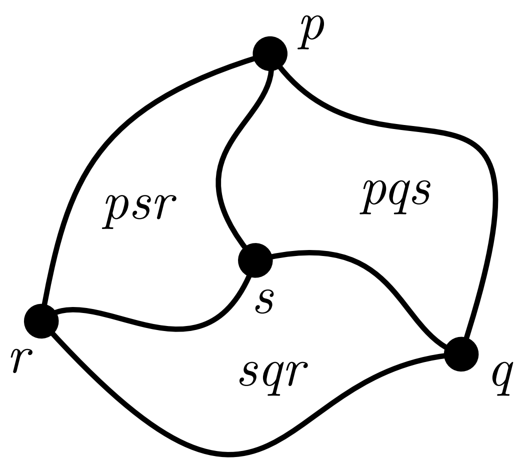

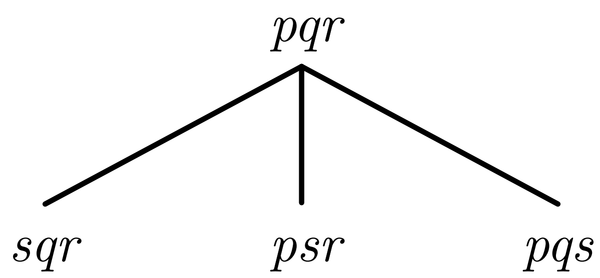

PROP. A stacked triangulation admits a unique Schnyder labeling and Schnyder wood.

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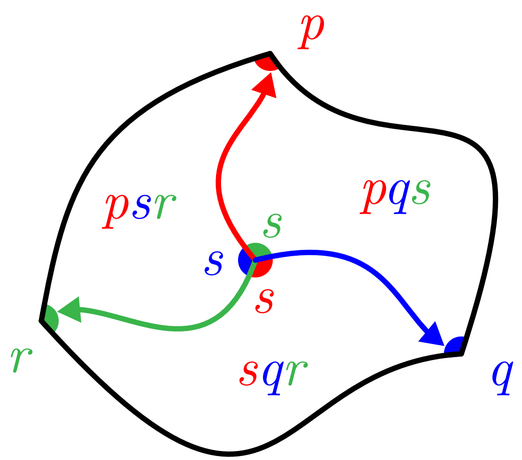
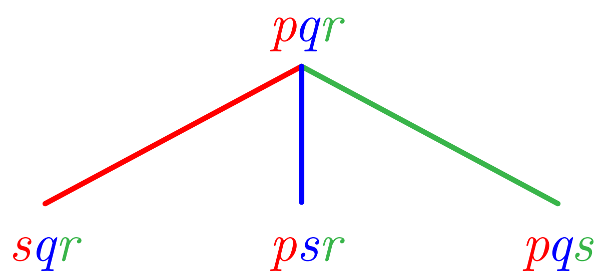
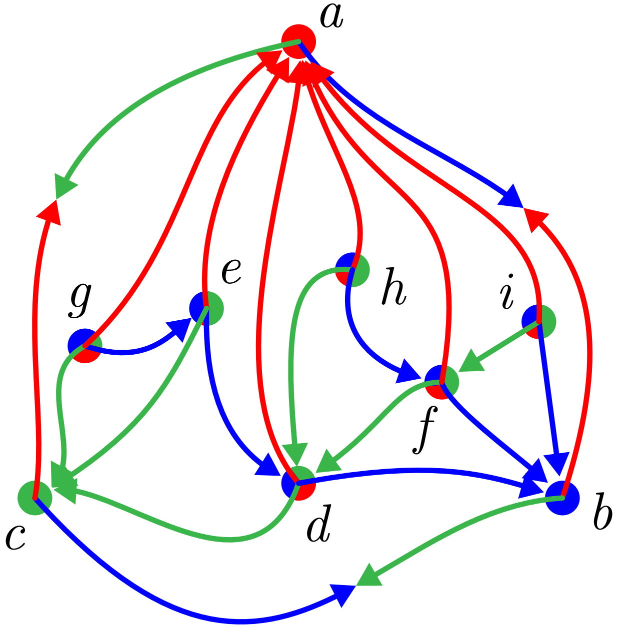


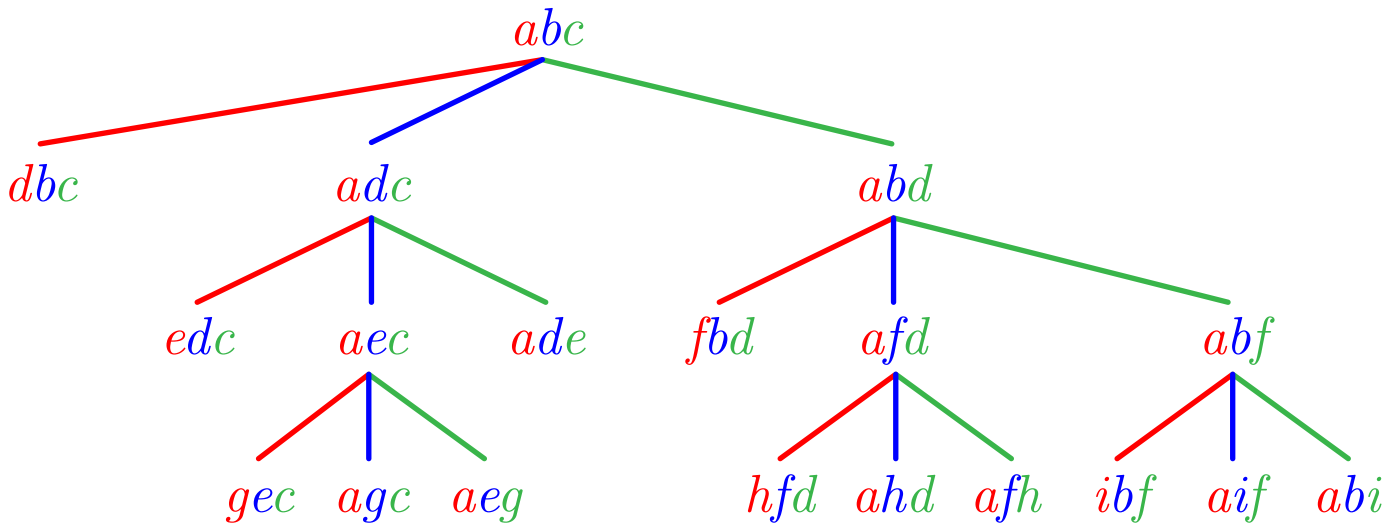


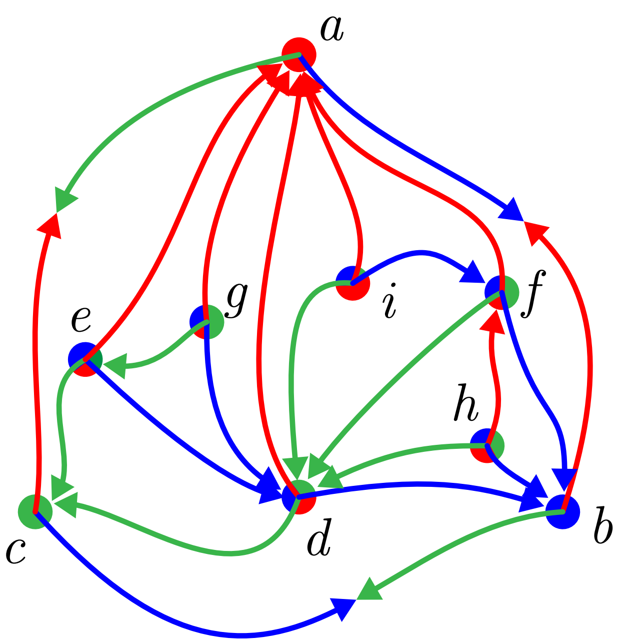
PROP. A stacked triangulation admits a unique Schnyder labeling and Schnyder wood.

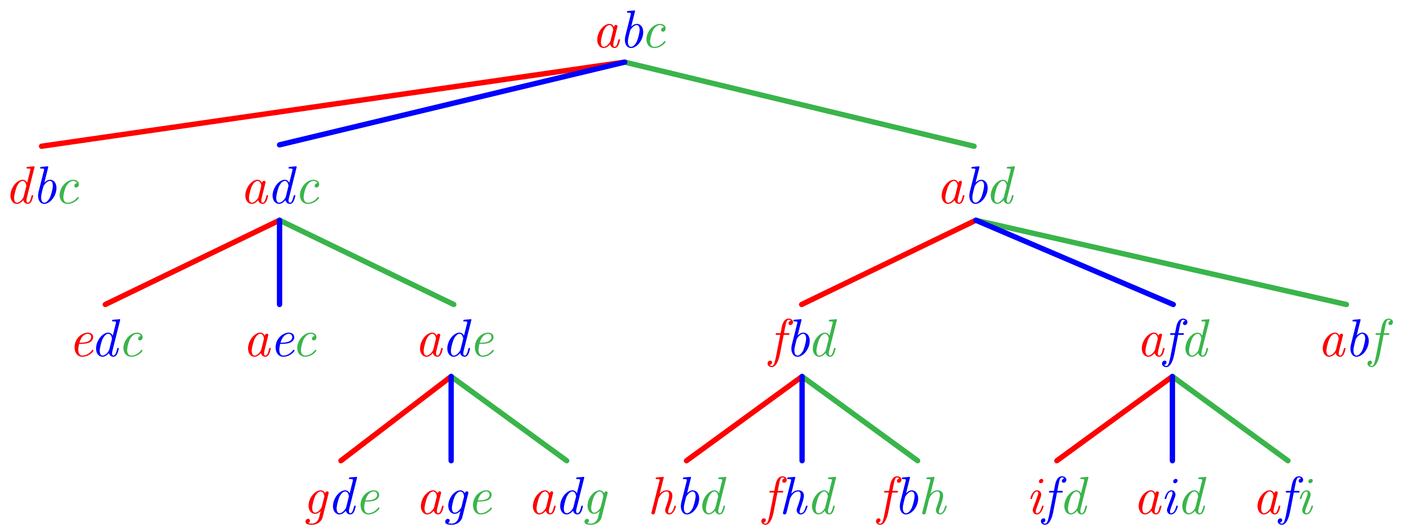
proof idea: induction.

EXM: STACKED TRIANGULATIONS



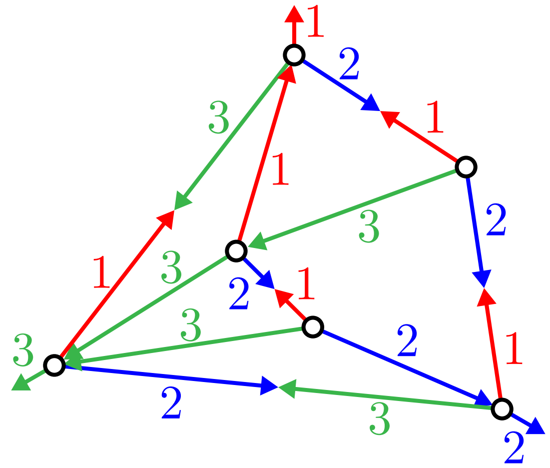






SCHNYDER EMBEDDING

DEF. M planar map with Schnyder wood.



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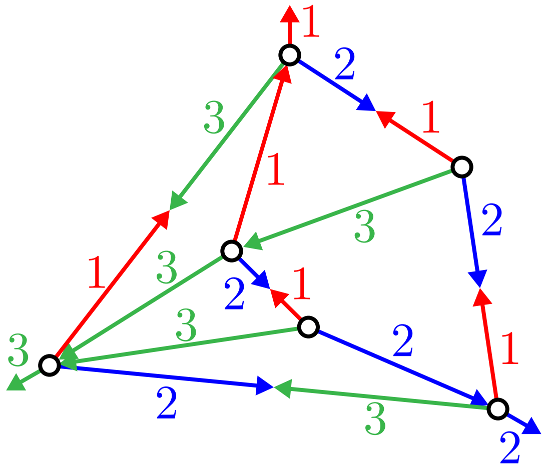
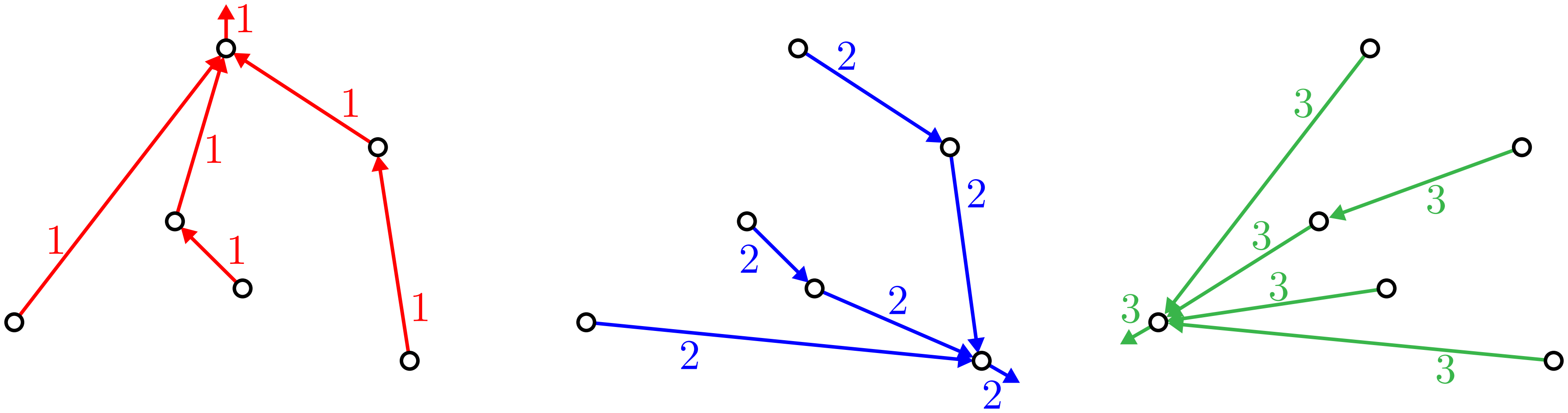


T_i = directed graph formed by edges colored i.



TREES

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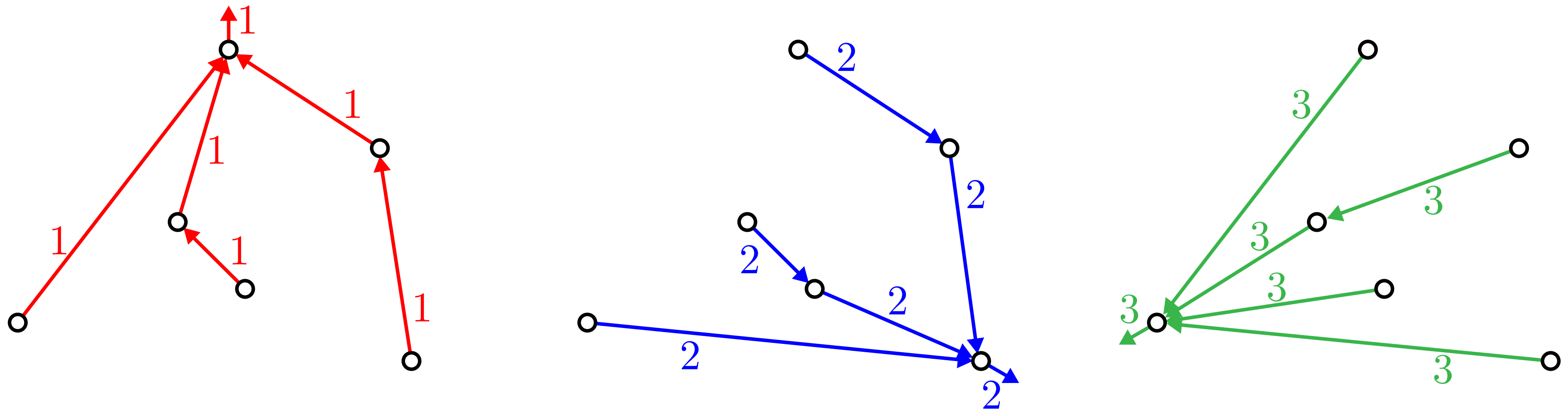


PROP. T_i is a directed tree rooted at v_i.

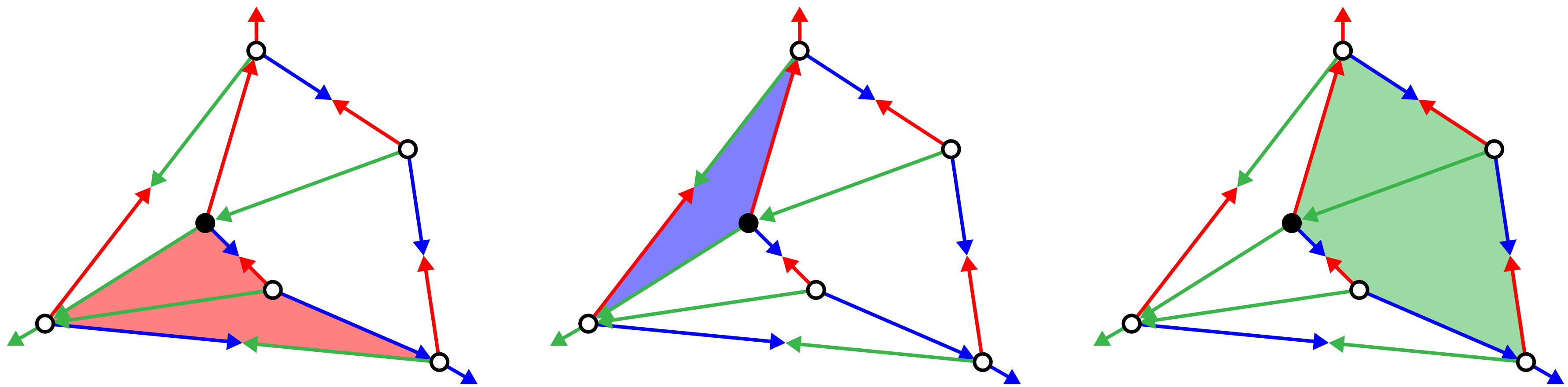
proof ideas:

- All vertices except v_i have outdegree 1, so enough to prove acyclicity.
- In fact, $D_i = T_i \cup T_{i-1}^{\text{rev}} \cup T_{i+1}^{\text{rev}}$ is already acyclic if we ignore bidirected edges or paths.
  If Z is an area minimal cycle in D_i, then:
  - Z bounds a single face F (otherwise, it has a chord or contains a vertex...),
  - if Z is clockwise, no angle of F has label $i + 1$.

REGIONS

DEF. For a vertex v of M, denote:

- $P_i(v)$ = directed path in T_i to the root v_i,
- $R_i(v)$ = region bounded by the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$,
- $r_i(v)$ = number of faces in region $R_i(v)$.



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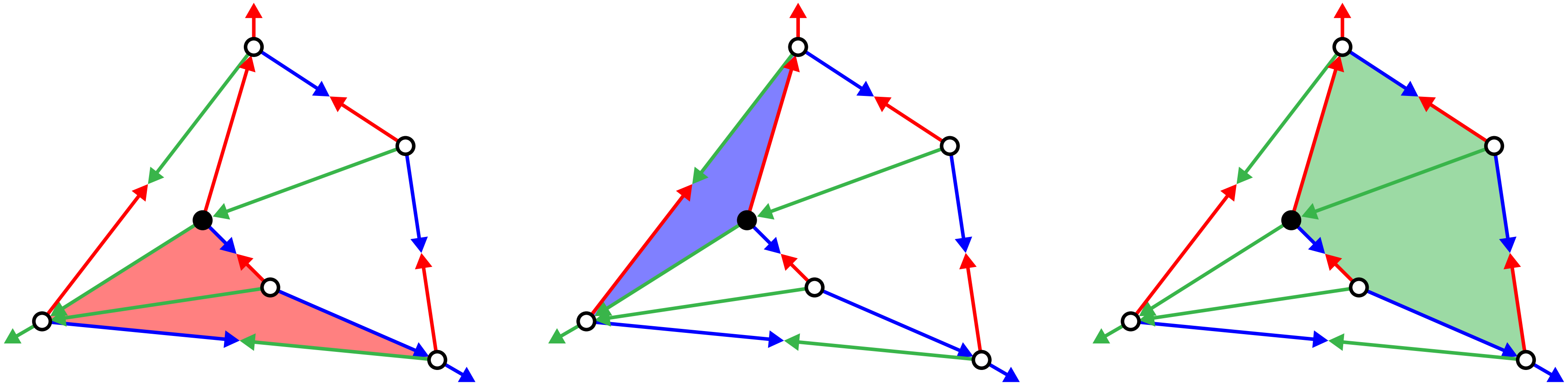
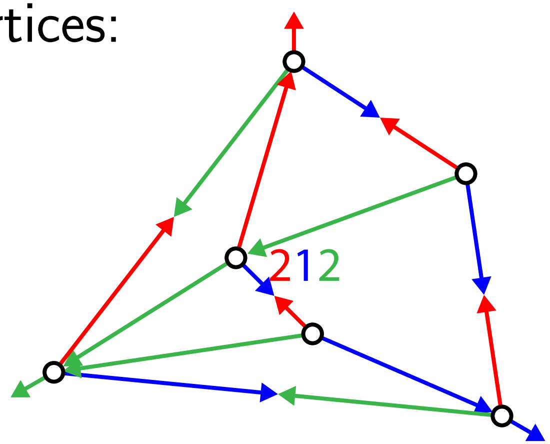


QU. Compute $r_1 r_2 r_3$ for all vertices:



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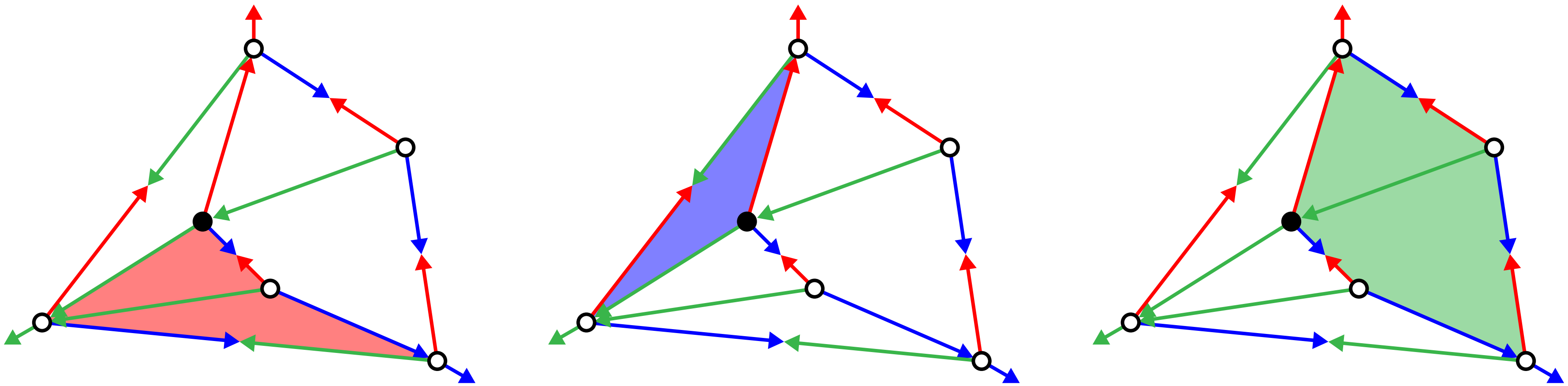
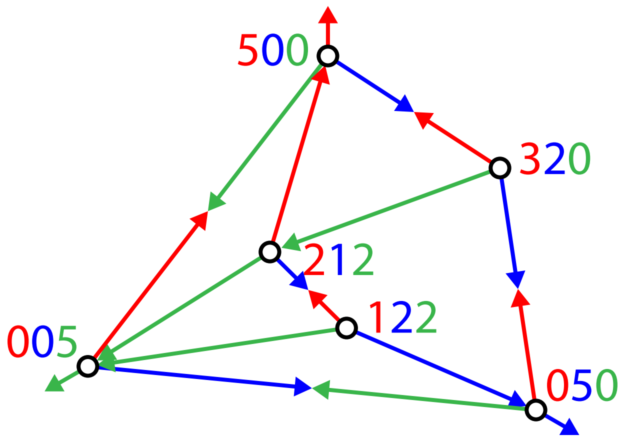


REM. $r_1 r_2 r_3$ are given by:



REGIONS

$R_i(v)$ = region bounded by the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$.

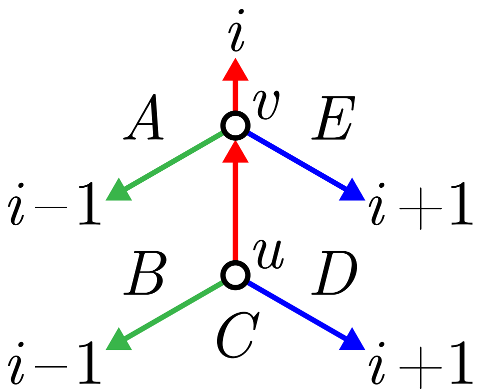
PROP. u, v = two adjacent vertices in the map M. Then:

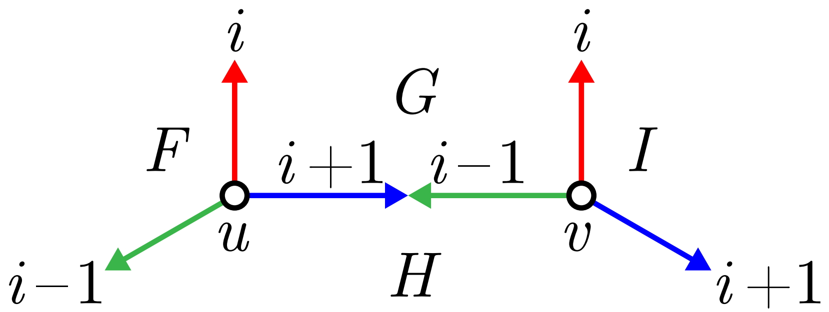
(R1) if there is a unidirected edge colored i from u to v, then

$$R_i(u) \subsetneq R_i(v), \qquad R_{i-1}(u) \supsetneq R_{i-1}(v), \qquad \text{and} \qquad R_{i+1}(u) \supsetneq R_{i+1}(v),$$

(R2) if there is a bidirected edge colored $i+1$ from u to v and $i-1$ from v to u, then

$$R_i(u) = R_i(v), \qquad R_{i-1}(u) \supsetneq R_{i-1}(v), \qquad \text{and} \qquad R_{i+1}(u) \subsetneq R_{i+1}(v).$$





$$R_i(u) = C \subsetneq B \cup C \cup D = R_i(v)$$
$$R_{i-1}(u) = D \cup E \supsetneq E = R_{i-1}(v)$$
$$R_{i+1}(u) = A \cup B \supsetneq A = R_{i+1}(v)$$

$$R_i(u) = H = R_i(v)$$
$$R_{i-1}(u) = G \cup I \supsetneq I = R_{i-1}(v)$$
$$R_{i+1}(u) = F \subsetneq F \cup G = R_{i+1}(v)$$

SCHNYDER EMBEDDING

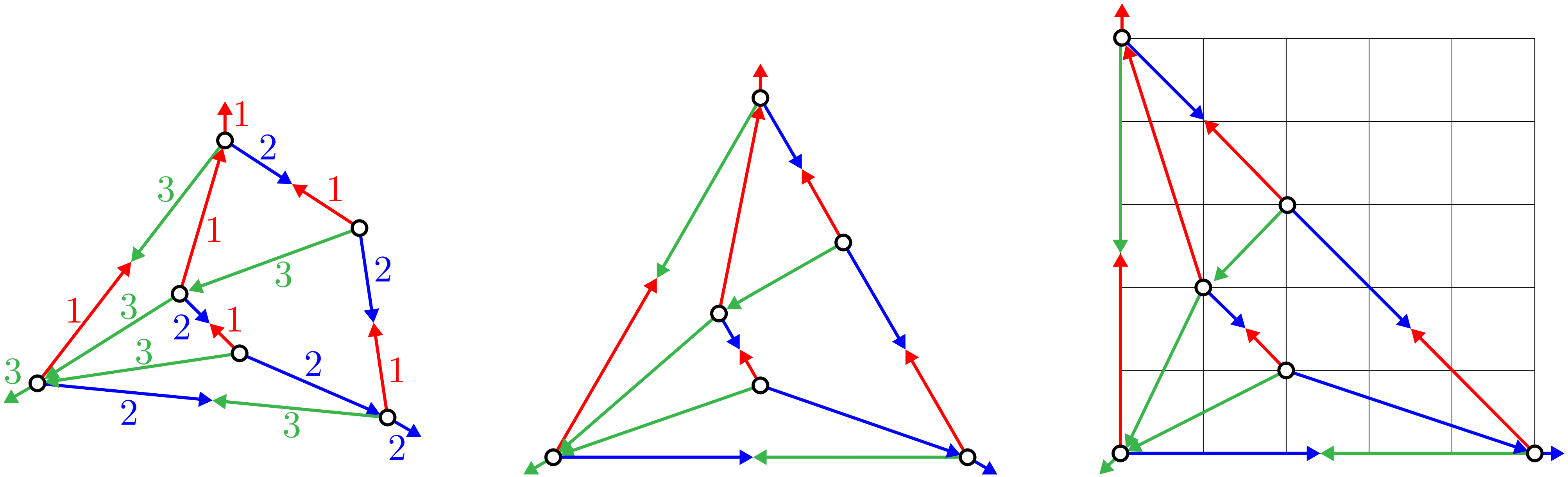
M = planar map with f faces (including the unbounded one),
endowed with a Schnyder wood.

$\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ = three arbitrary non-colinear points in the plane.

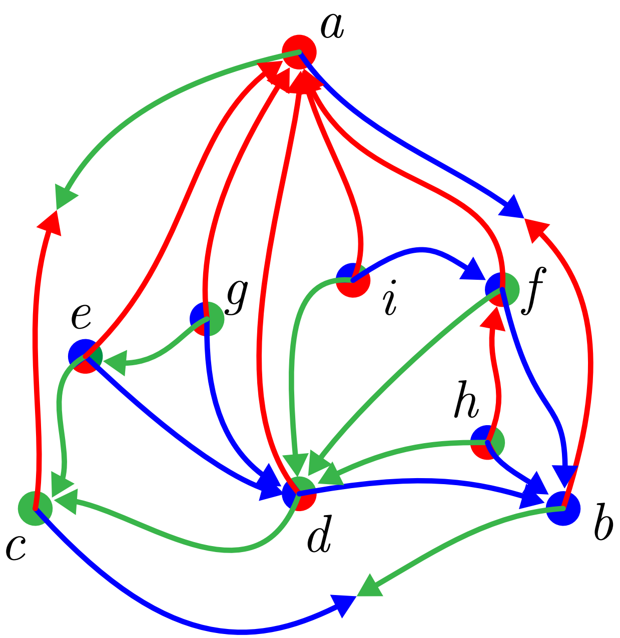
THM. The map

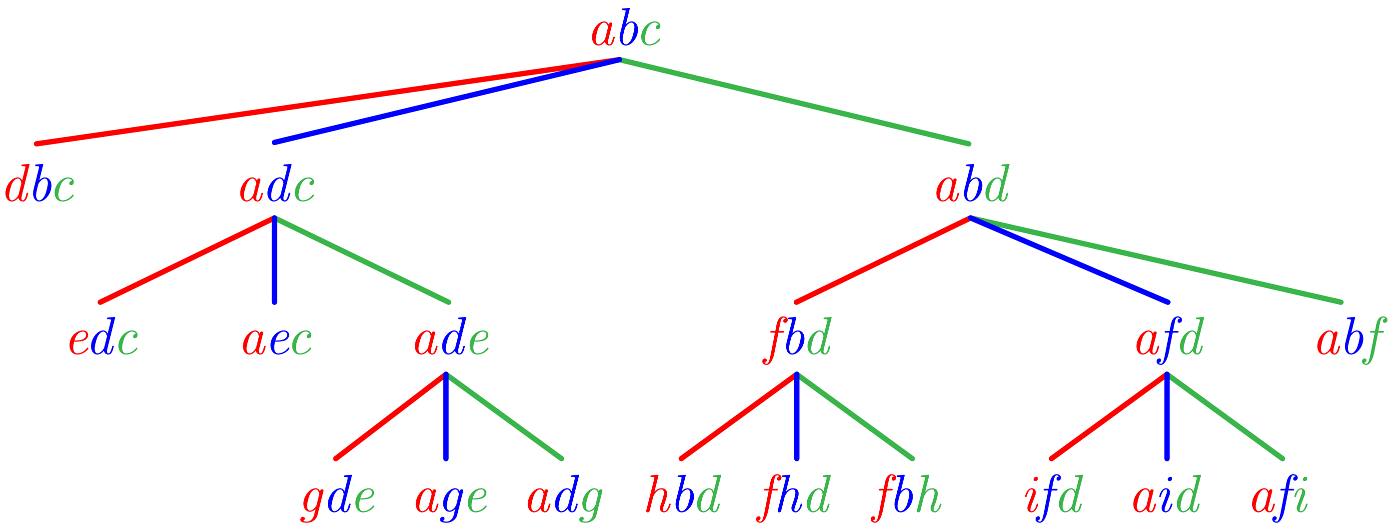
$$\mu : v \longmapsto \frac{1}{f-1}\big(r_1(v)\cdot \boldsymbol{p}_1 + r_2(v)\cdot \boldsymbol{p}_2 + r_3(v)\cdot \boldsymbol{p}_3\big)$$

defines a straightline embedding of M in the plane where all faces are convex.



EXM: STACKED TRIANGULATIONS





QU. Describe on the construction tree C of a stacked triangulation:

- the trees T_1, T_2 and T_3,
- the sizes $r_1(v)$, $r_2(v)$ and $r_3(v)$ of the regions of a vertex v.

Draw the Schnyder embedding for $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ being the vertices of an equilateral triangle.

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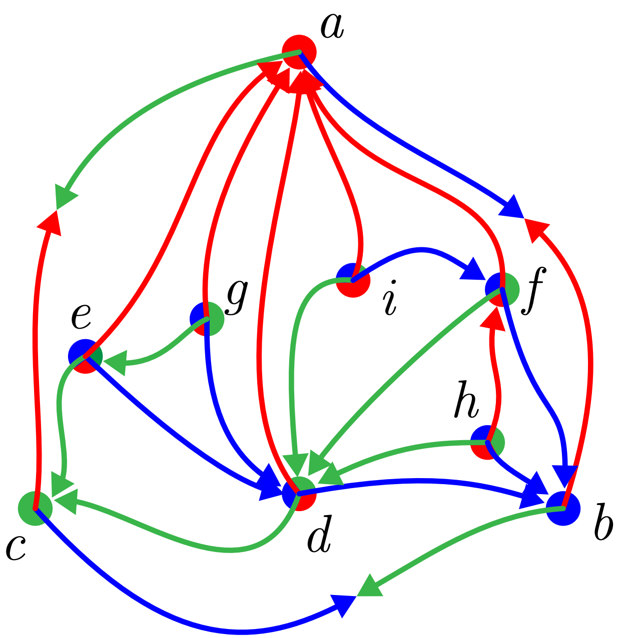

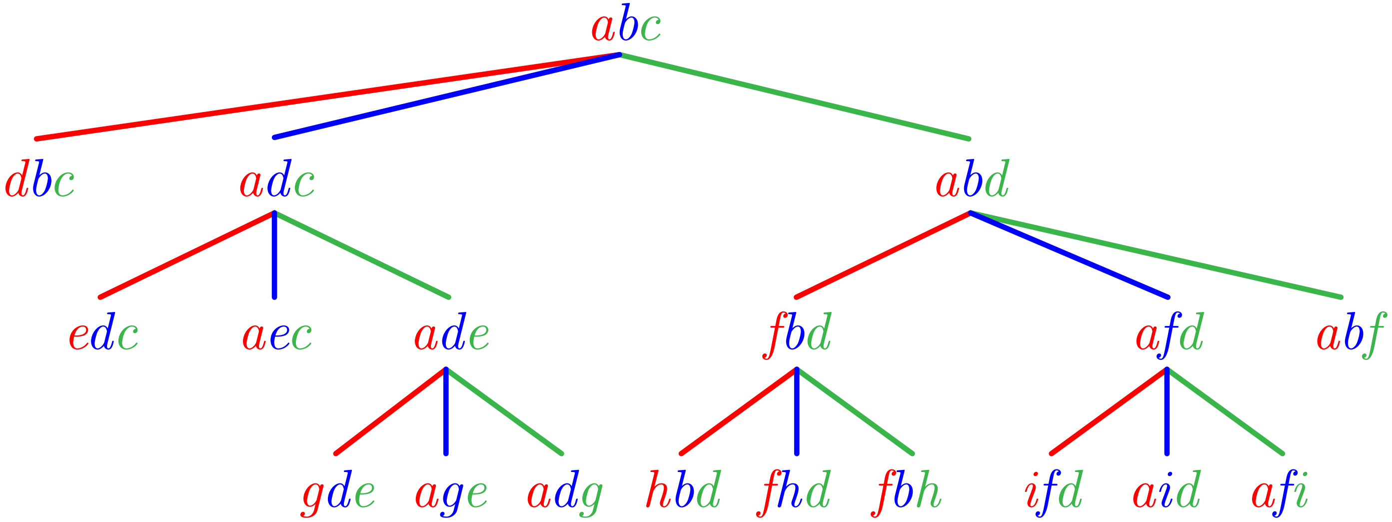

PROP. The tree T_i is obtained by contracting all edges colored $i-1$ and $i+1$ in C

PROP. Assume v is inserted in triangle t, and let γ be the path from t to the root in C. The size $r_i(v)$ is obtained by summing the number of leaves of the subtrees of the blue children of the nodes of γ that are not in γ.

<ins>proof idea:</ins> induction.

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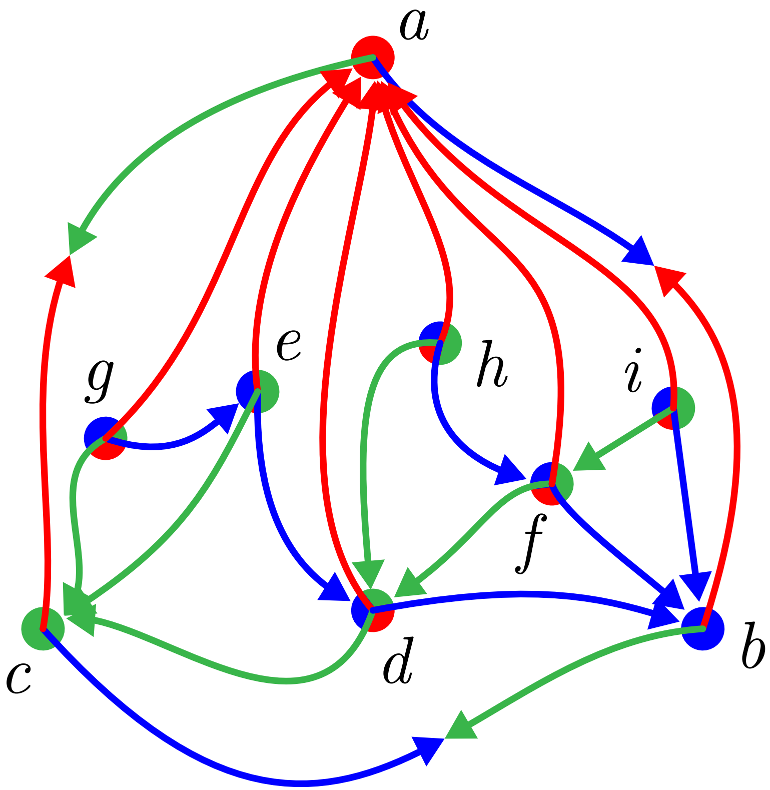

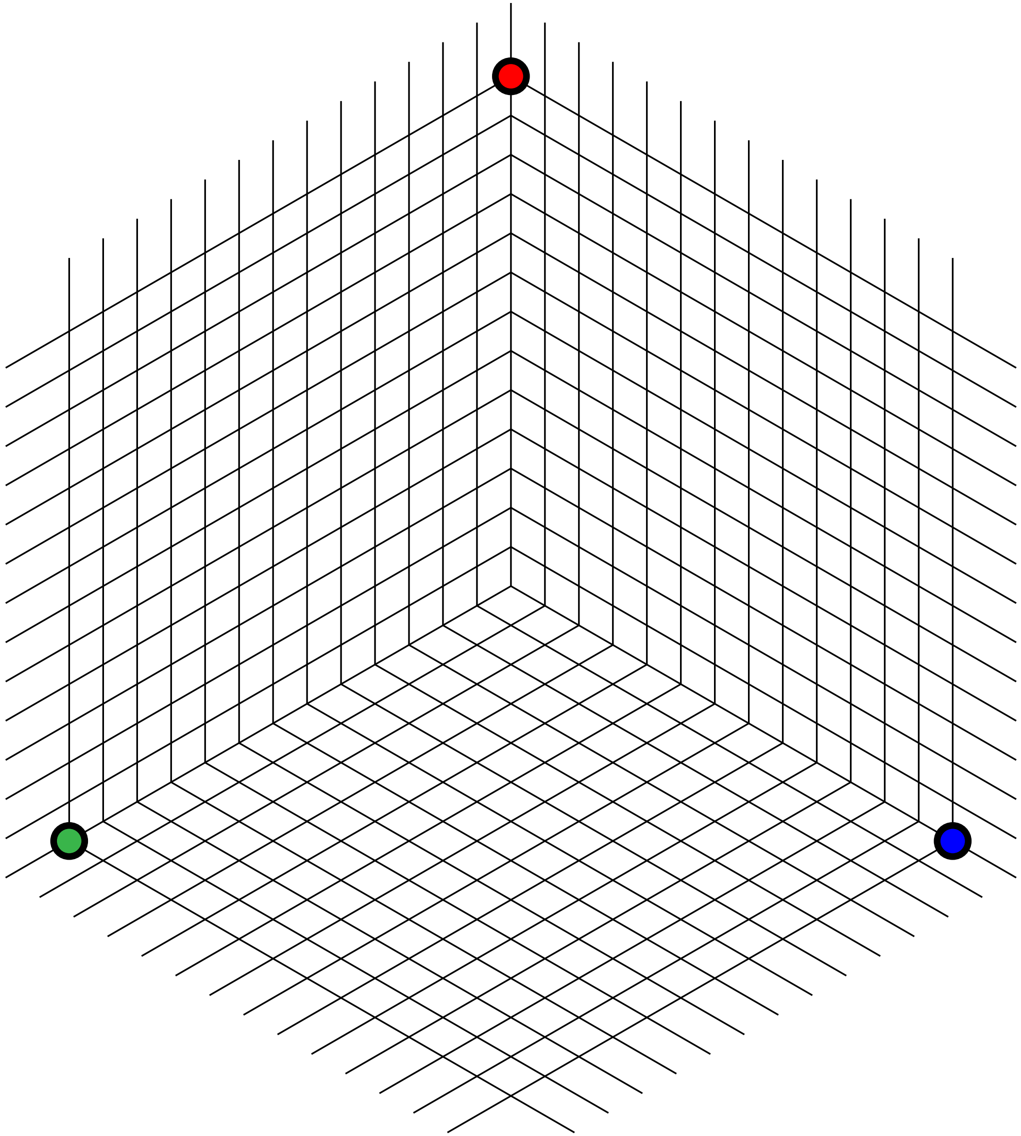

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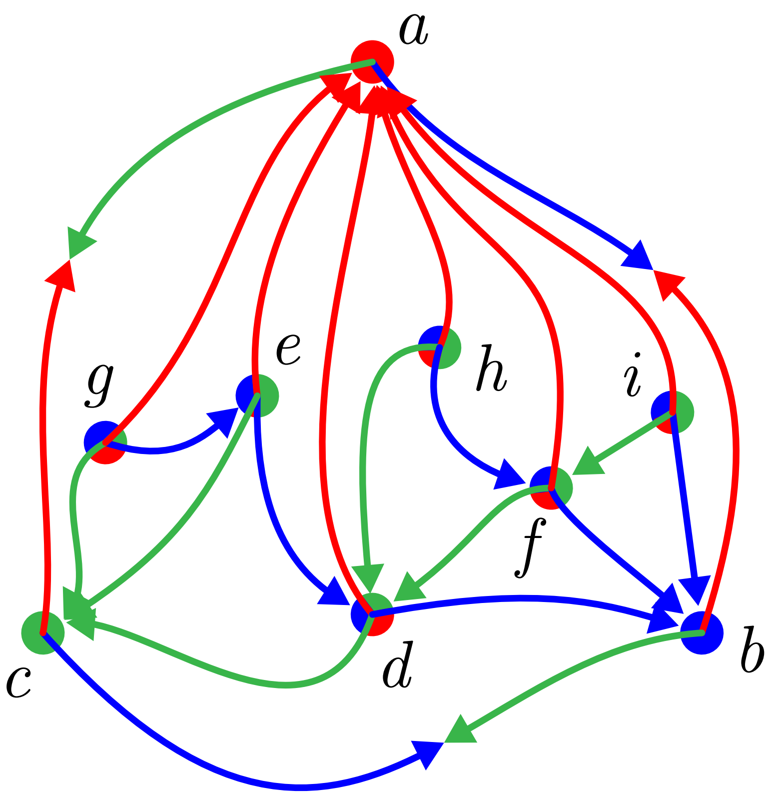
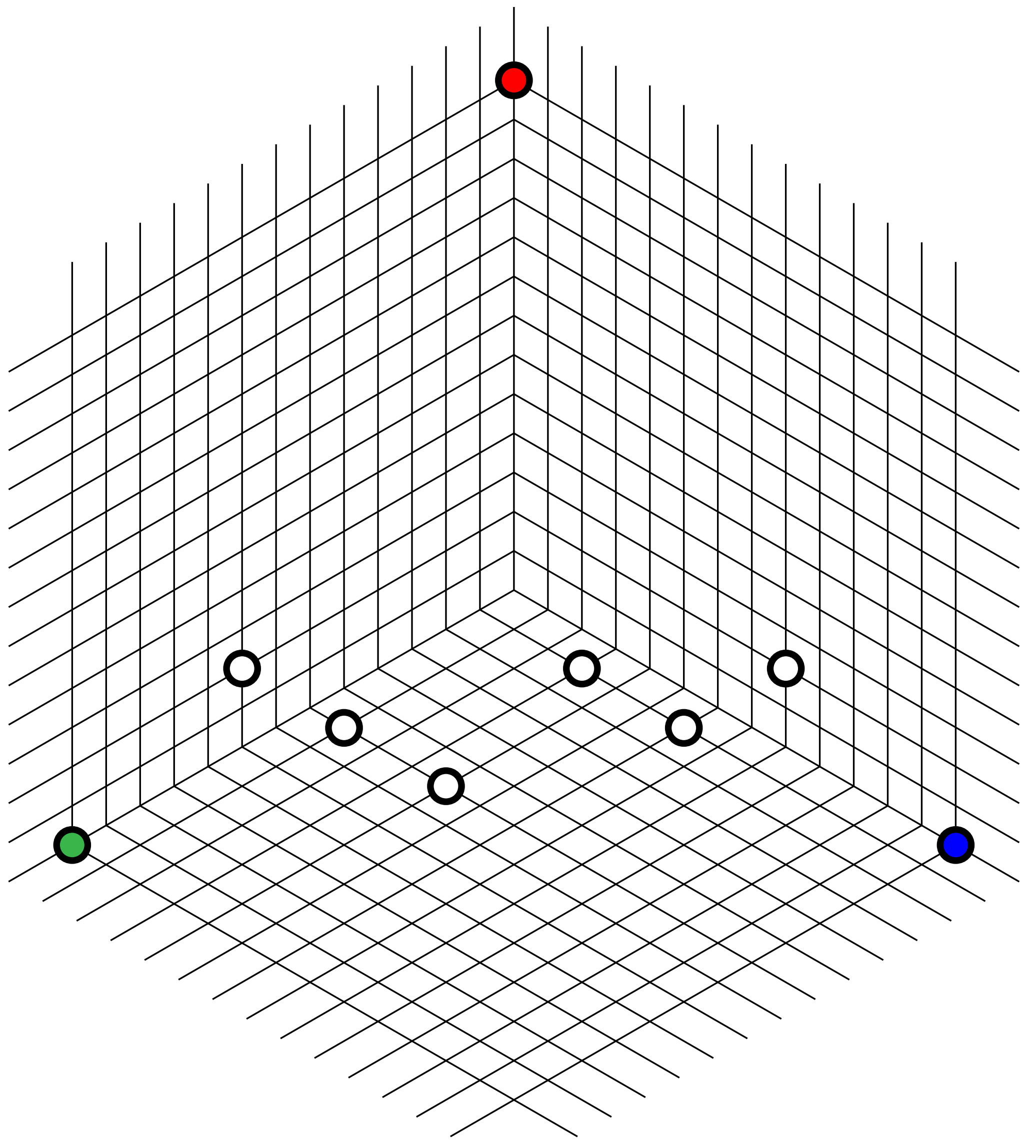
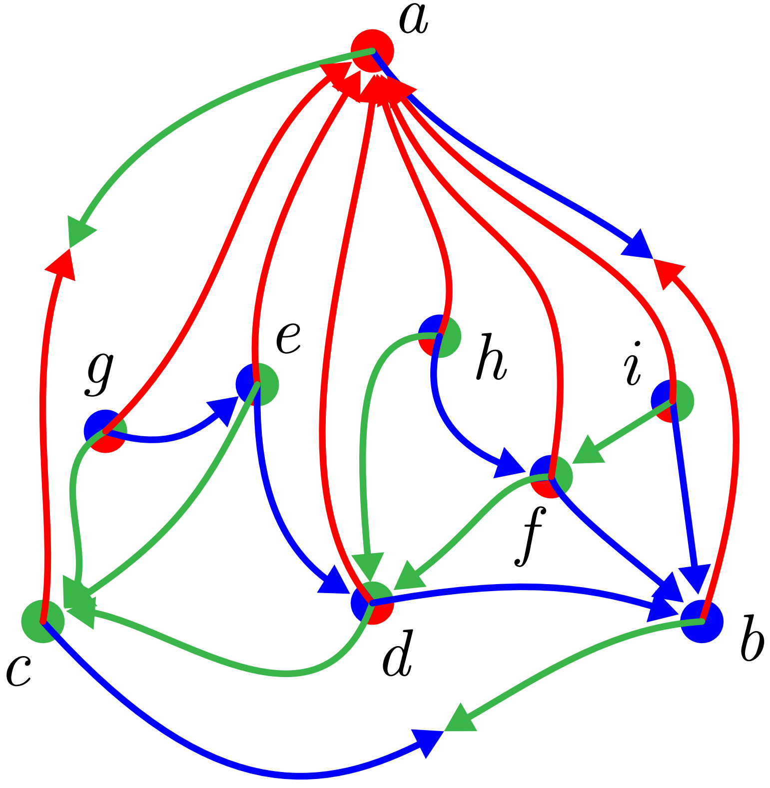
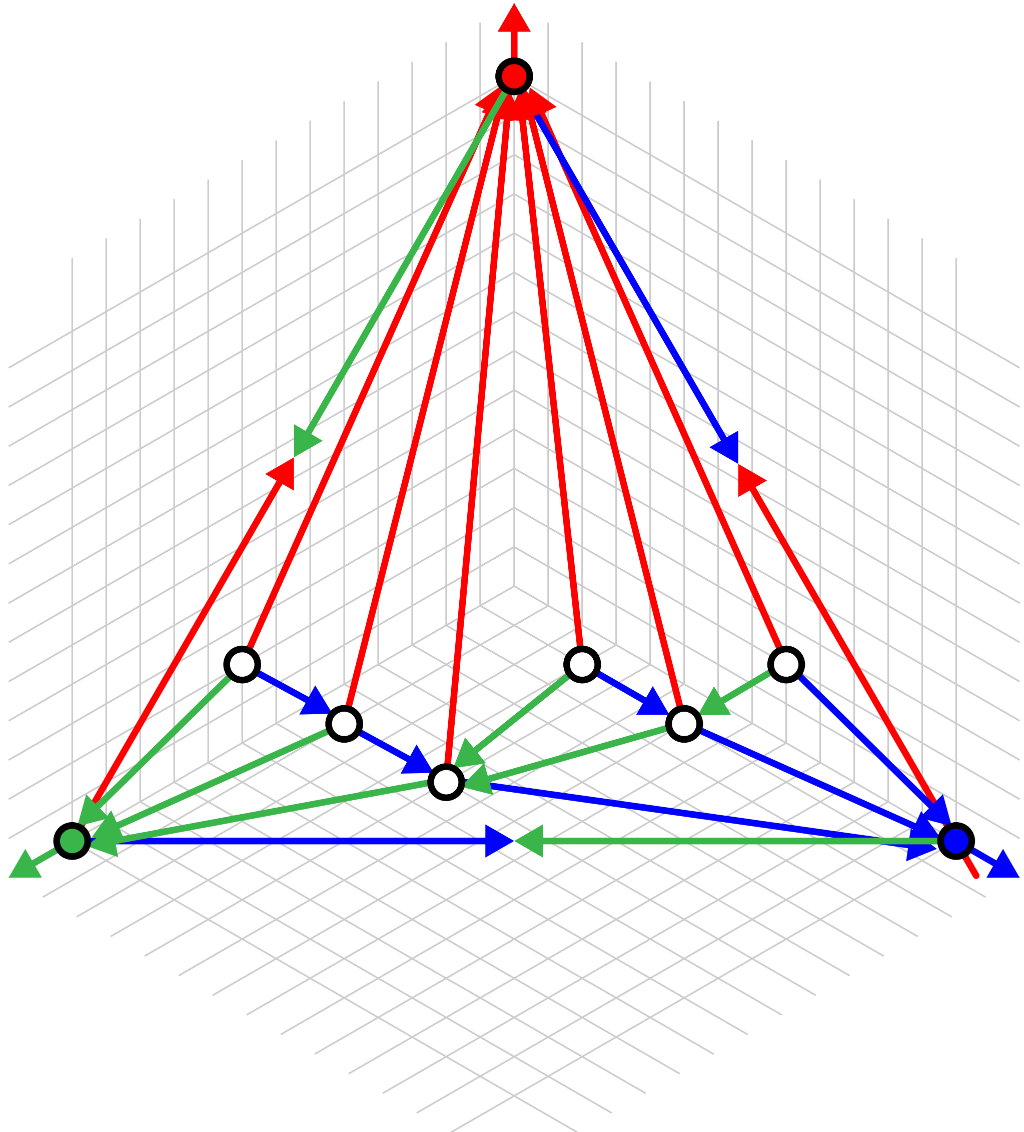




EXM: STACKED TRIANGULATIONS





GEODESIC MAPS ON ORTHOGONAL SURFACES

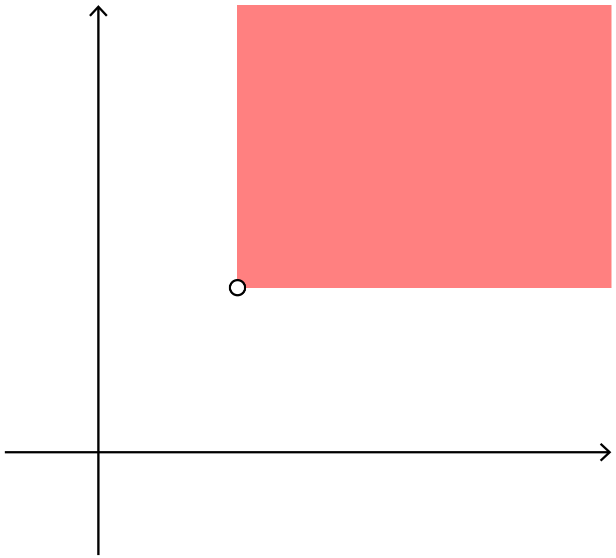
DOMINANCE ORDER

DEF. dominance order in $\mathbb{R}^3 = \boldsymbol{u} \leq \boldsymbol{v} \iff u_i \leq v_i$ for all $i \in [3]$ (componentwise).

DEF.

cone dominating $\boldsymbol{y} \in \mathbb{R}^3$

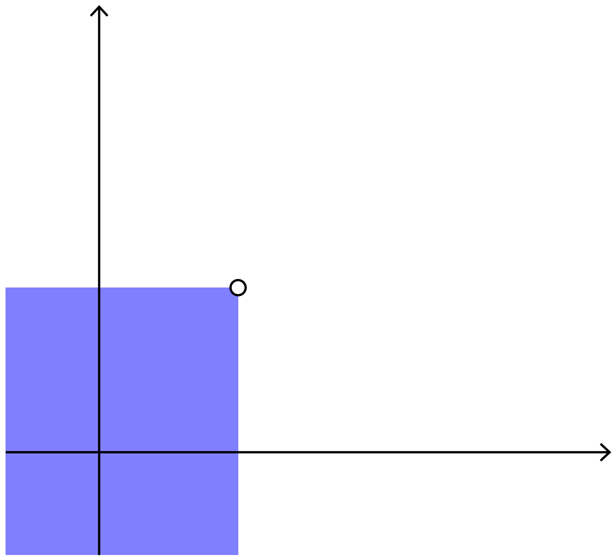
$$\Delta_{\boldsymbol{y}} = \left\{ \boldsymbol{z} \in \mathbb{R}^3 \;\middle|\; \boldsymbol{y} \leq \boldsymbol{z} \right\}$$



(= upper ideal of $\boldsymbol{y}$)

cone dominated by $\boldsymbol{y} \in \mathbb{R}^3$

$$\nabla_{\boldsymbol{y}} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \boldsymbol{x} \leq \boldsymbol{y} \right\}$$



(= lower ideal of $\boldsymbol{y}$)

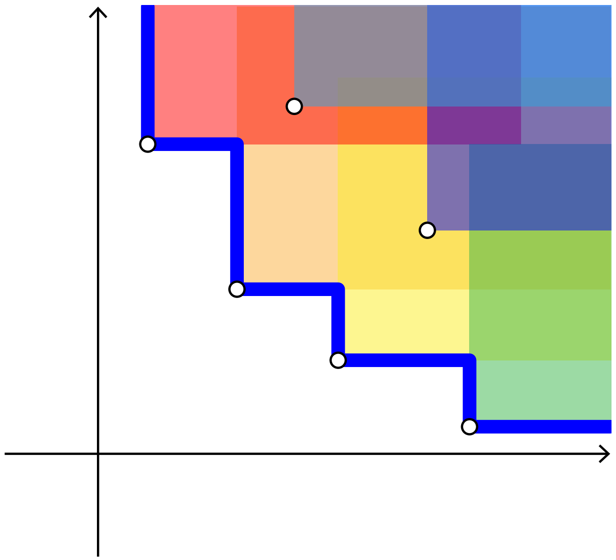
ORTHOGONAL SURFACE

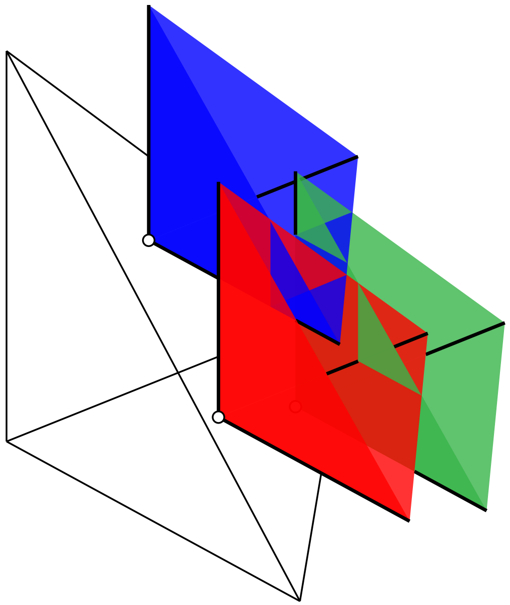
DEF. dominance order in $\mathbb{R}^3$ = $\boldsymbol{u} \le \boldsymbol{v} \iff u_i \le v_i$ for all $i \in [3]$ (componentwise).

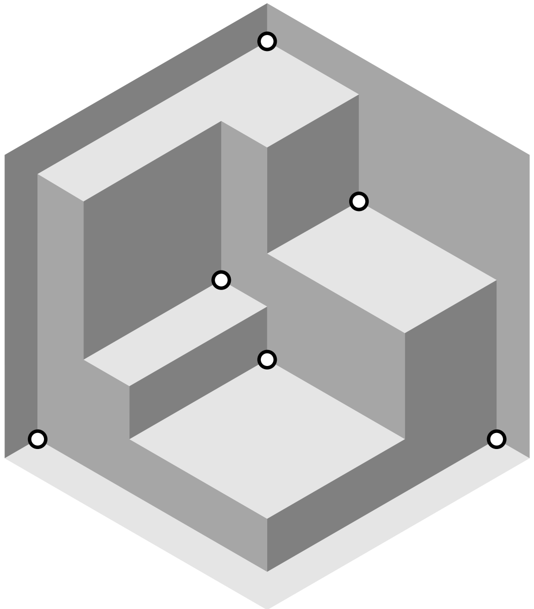
DEF. cone dominating $\boldsymbol{y} \in \mathbb{R}^3$ $\Delta_{\boldsymbol{y}} = \{\boldsymbol{z} \in \mathbb{R}^3 \mid \boldsymbol{y} \le \boldsymbol{z}\}$

cone dominated by $\boldsymbol{y} \in \mathbb{R}^3$ $\nabla_{\boldsymbol{y}} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} \le \boldsymbol{y}\}$

DEF. $\langle \boldsymbol{V} \rangle = \{\boldsymbol{z} \in \mathbb{R}^3 \mid \boldsymbol{v} \le \boldsymbol{z}$ for some $\boldsymbol{v} \in \boldsymbol{V}\} = \bigcup_{\boldsymbol{v} \in \boldsymbol{V}} \Delta_{\boldsymbol{v}}$.





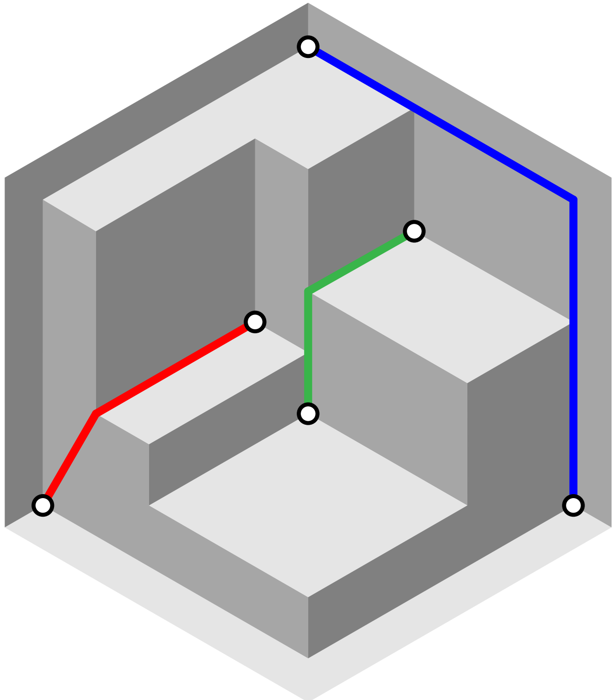


orthogonal surface $\mathcal{S}_{\boldsymbol{V}}$ = boundary of $\langle \boldsymbol{V} \rangle$ (assume now that $\boldsymbol{V}$ = antichain)

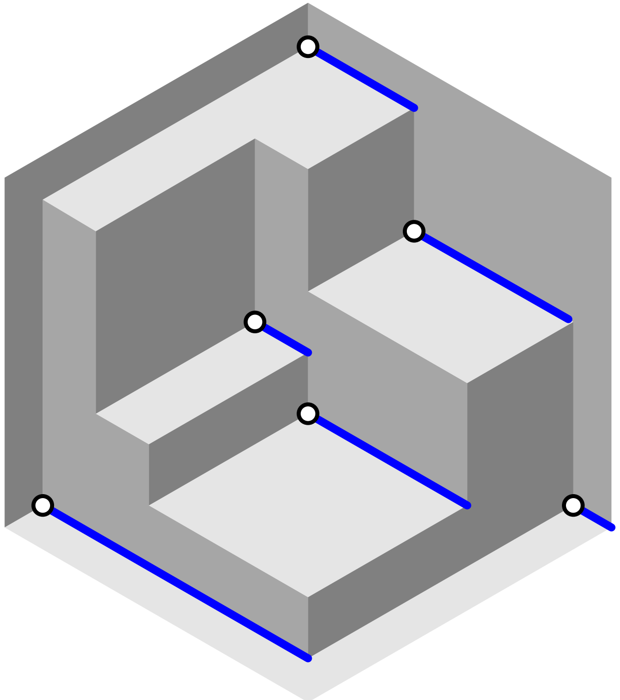
ELBOW GEODESICS AND COORDINATE ARCS

DEF. On an orthogonal surface $\mathcal{S}_{\boldsymbol{V}}$, define

- elbow geodesic $=$ union of the segments from $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ to $\boldsymbol{u} \vee \boldsymbol{v} = [\max(u_i, v_i)]_{i \in [n]}$,
- coordinate arcs $=$ (not always bounded) segments from $\boldsymbol{v} \in \boldsymbol{V}$ in an axis direction.



elbow geodesics



coordinate arcs

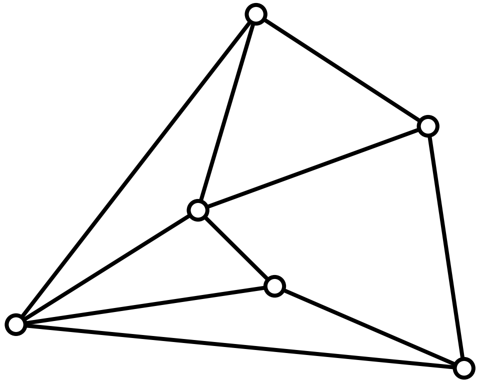
GEODESIC EMBEDDING

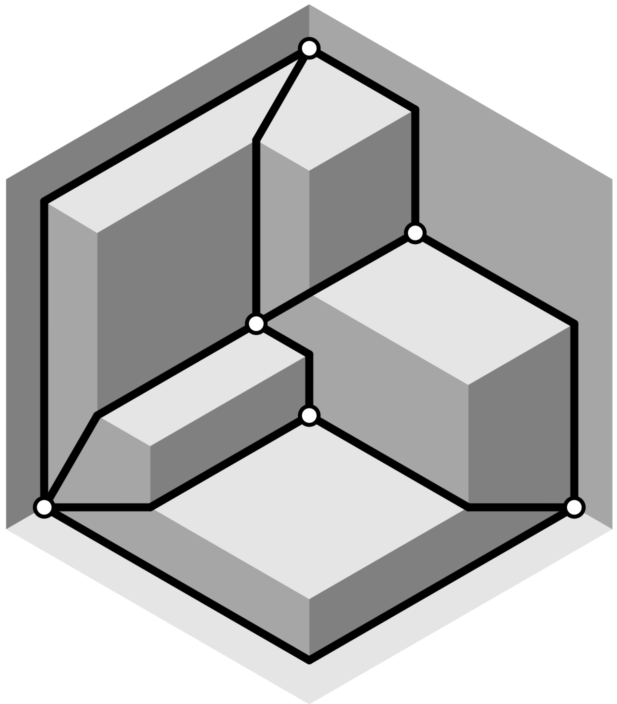
DEF. <u>geodesic embedding</u> of a map M on a surface $\mathcal{S}_{\boldsymbol{V}}$ = drawing of M on $\mathcal{S}_{\boldsymbol{V}}$ st:

(G1) there is a bijection between the points of $\boldsymbol{V}$ and the vertices of M,

(G2) every edge of M is an elbow geodesic in $\mathcal{S}_{\boldsymbol{V}}$ and every bounded coordinate arc is part of an edge of M,

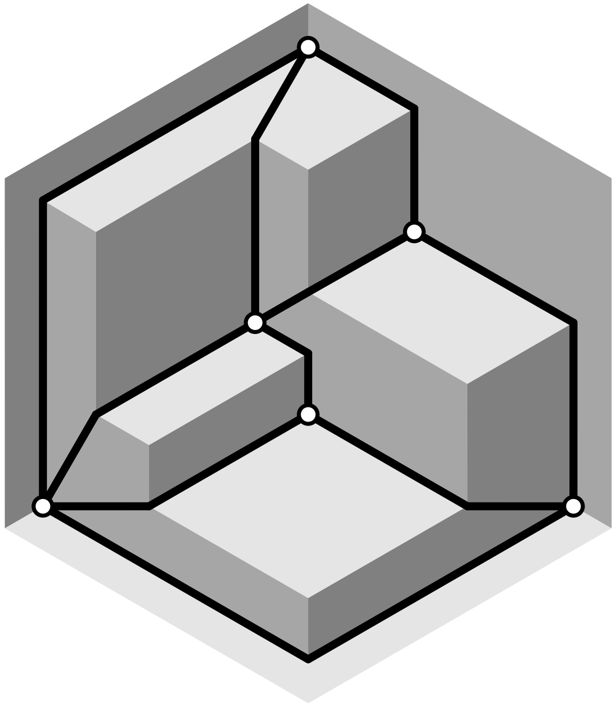
(G3) the drawing is crossing-free.





GEODESIC EMBEDDINGS VS SCHNYDER WOODS

THM. If $\boldsymbol{V}$ is an axial antichain, then a geodesic embedding of a map M on $\mathcal{S}_{\boldsymbol{V}}$ induces a Schnyder wood on M.



proof idea:

- label the angles according to the color of the flat region containing it,
- orient and color the edges according to the three axis. An elbow geodesic can get one or two colors depending on whether it contains one or two bounded coordinate arcs.

GEODESIC EMBEDDINGS VS SCHNYDER WOODS

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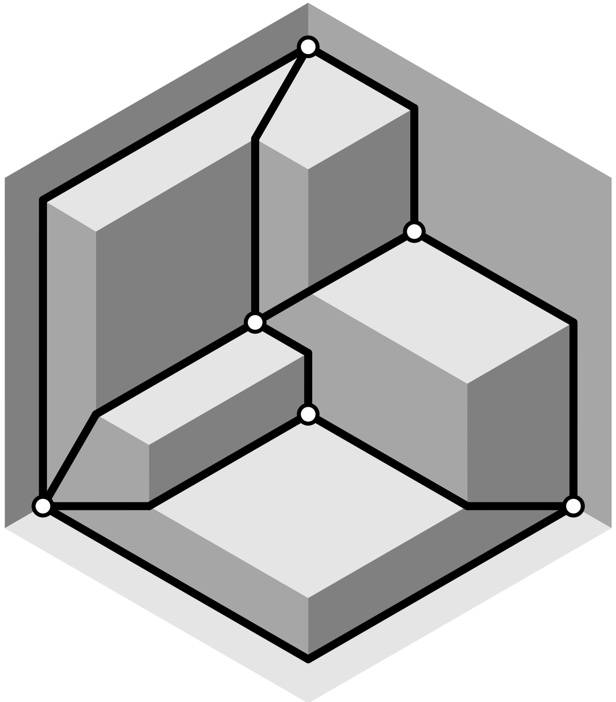

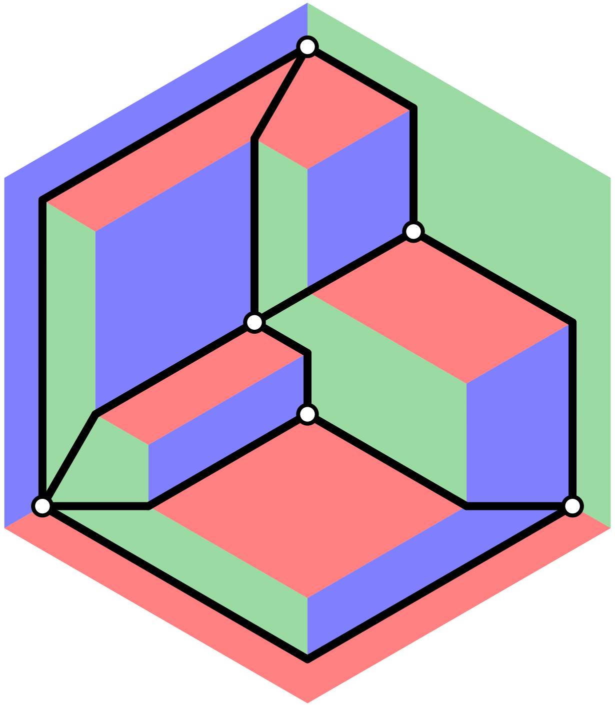

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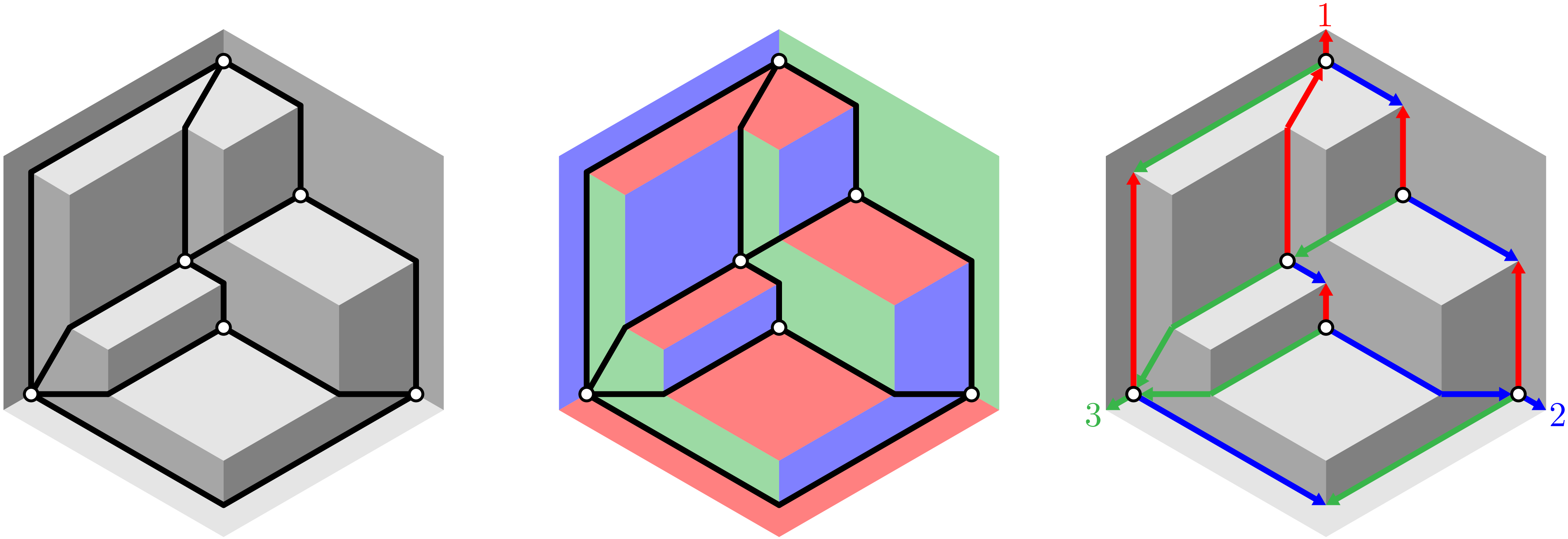

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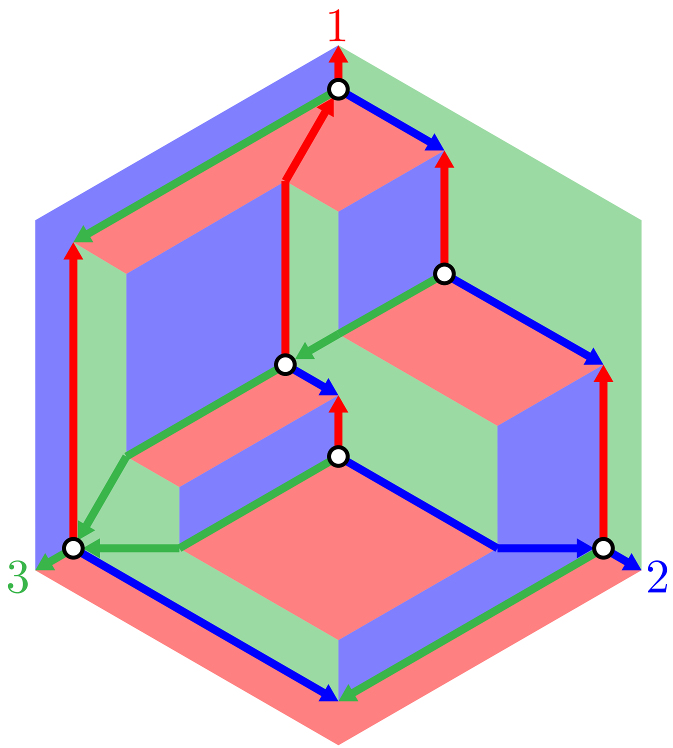

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GEODESIC EMBEDDINGS VS SCHNYDER WOODS

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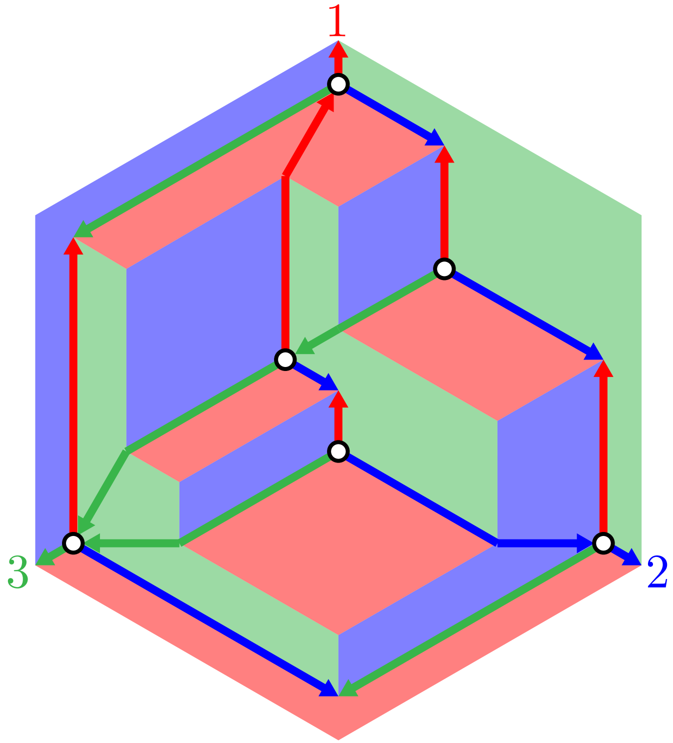
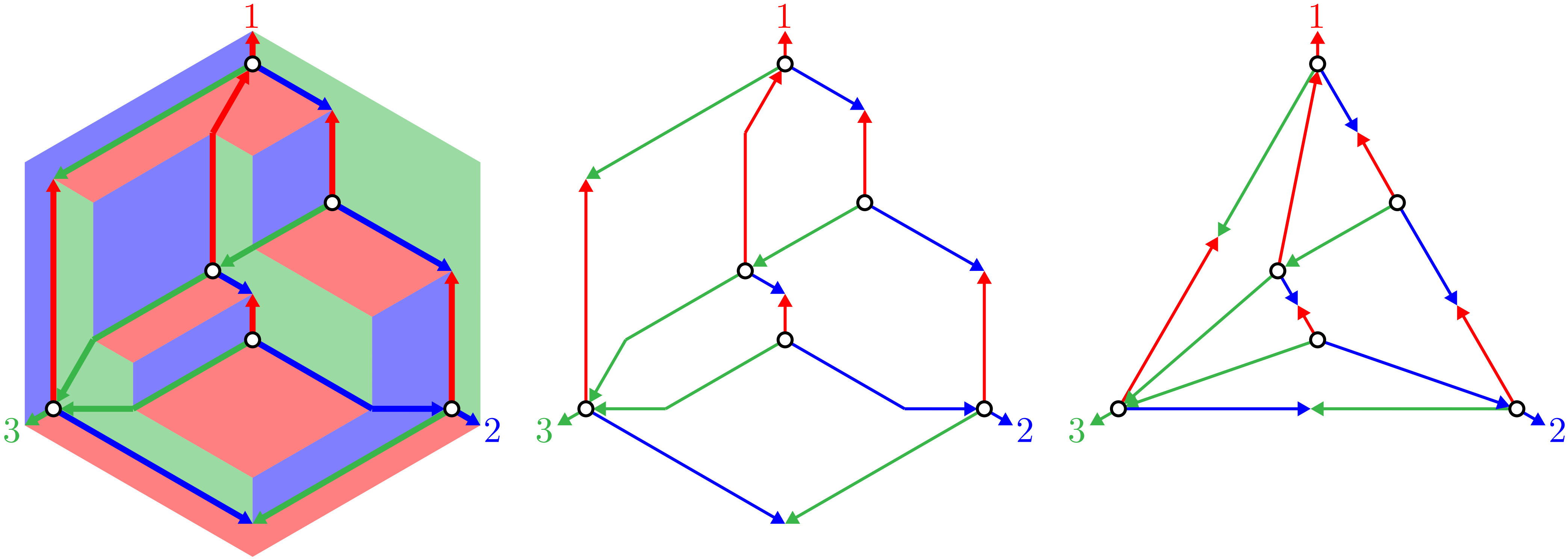


THM. Given a Schnyder wood W on a planar map M, the region vectors of the vertices of M with respect to W form an axial antichain $\boldsymbol{V}$ inducing a geodesic embedding of M on $\mathcal{S}_{\boldsymbol{V}}$.

FROM GEODESIC EMBEDDINGS TO SCHNYDER EMBEDDINGS

THM. The projection of the geodesic embedding onto the plane $v_1 + v_2 + v_3 = f - 1$ gives a planar drawing of M whose edges are bended segments. Replacing them by straight segments preserves the non-crossing-freeness.



proof idea: when straightening the geodesic embedding, the elbow geodesic joining $\boldsymbol{u}$ and $\boldsymbol{v}$ is controled by $\nabla_{\boldsymbol{u} \vee \boldsymbol{v}}$.

SCHNYDER EMBEDDING

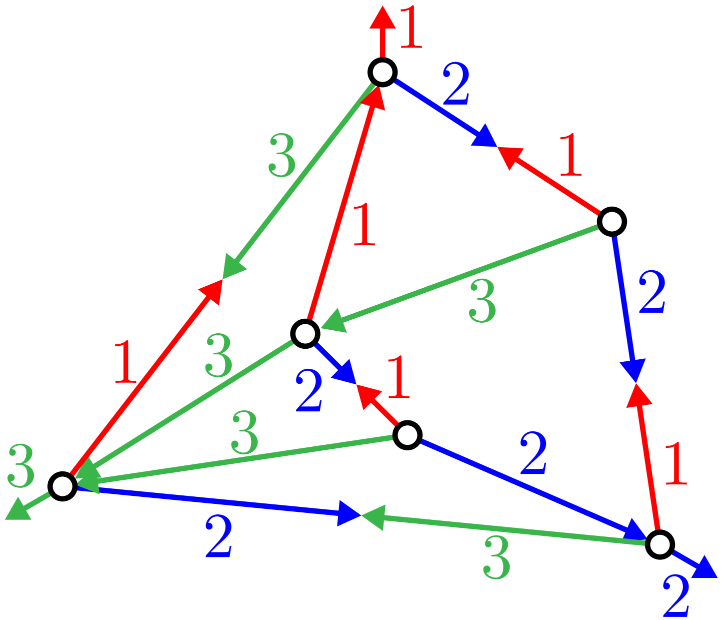
M = planar map with f faces (including the unbounded one),
endowed with a Schnyder wood.

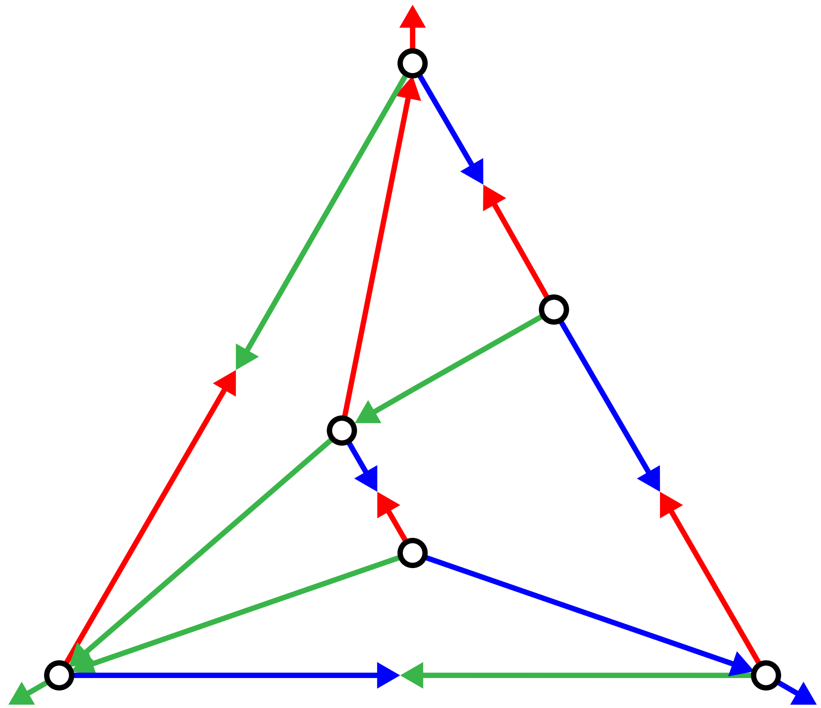
$\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ = three arbitrary non-colinear points in the plane.

THM. The map

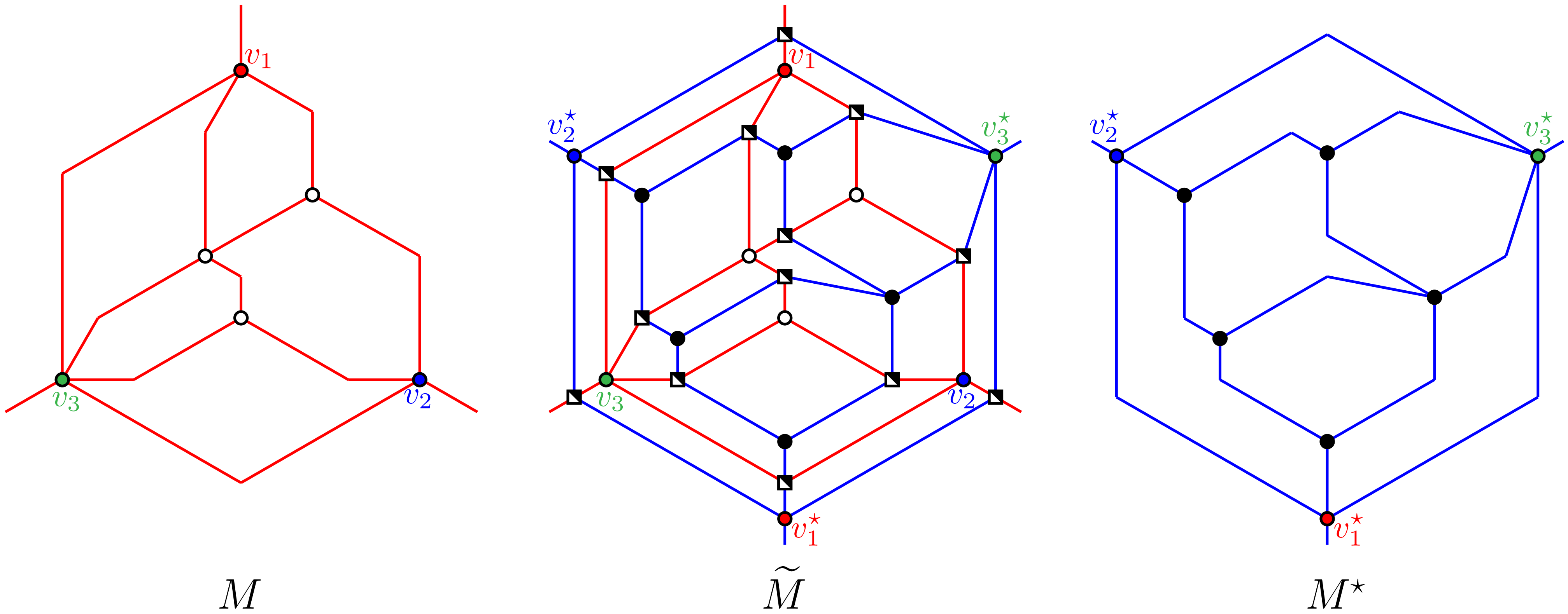
$$\mu : v \longmapsto \frac{1}{f-1}\left(r_1(v) \cdot \boldsymbol{p}_1 + r_2(v) \cdot \boldsymbol{p}_2 + r_3(v) \cdot \boldsymbol{p}_3\right)$$

defines a straightline embedding of M in the plane where all faces are convex.





PRIMAL-DUAL MAP

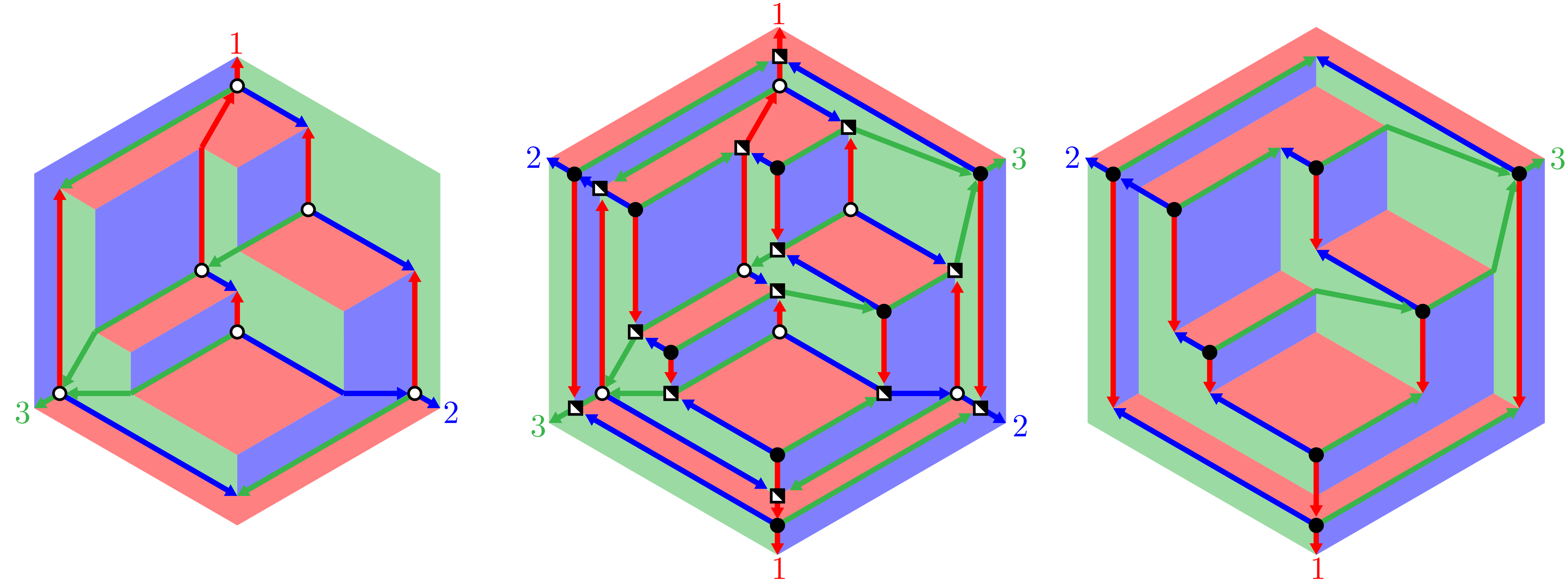


DEF. dual map of M = exchange vertices $\longleftrightarrow$ faces.

suspended dual map $M^\star$ = dual map of M where the vertex corresponding to the external face is split into three vertices.

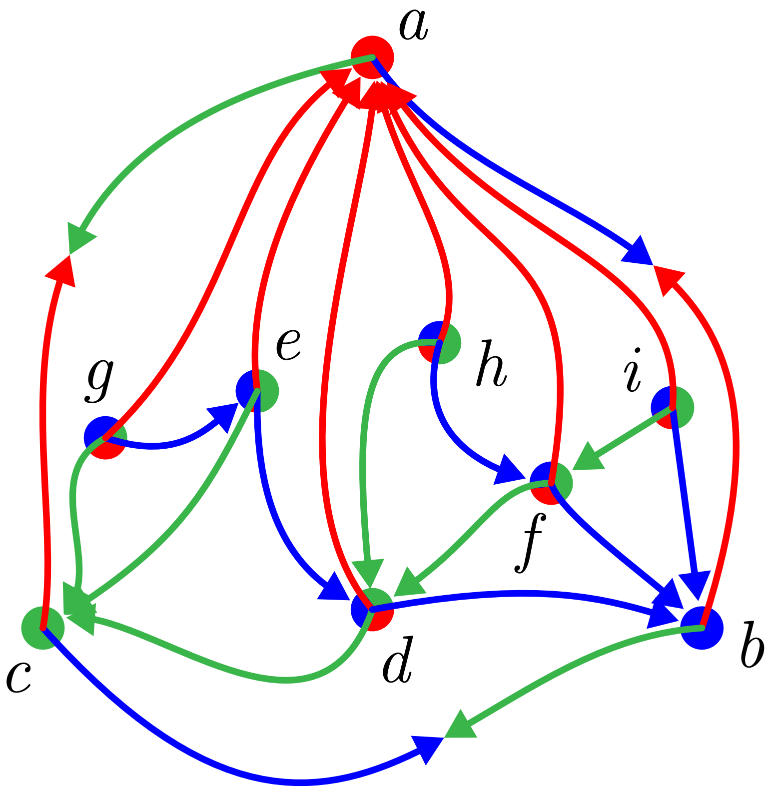
primal-dual map $\widetilde{M}$ = superimposition of the map M and its suspended dual map $M^\star$ with additional vertices at the edge intersections.

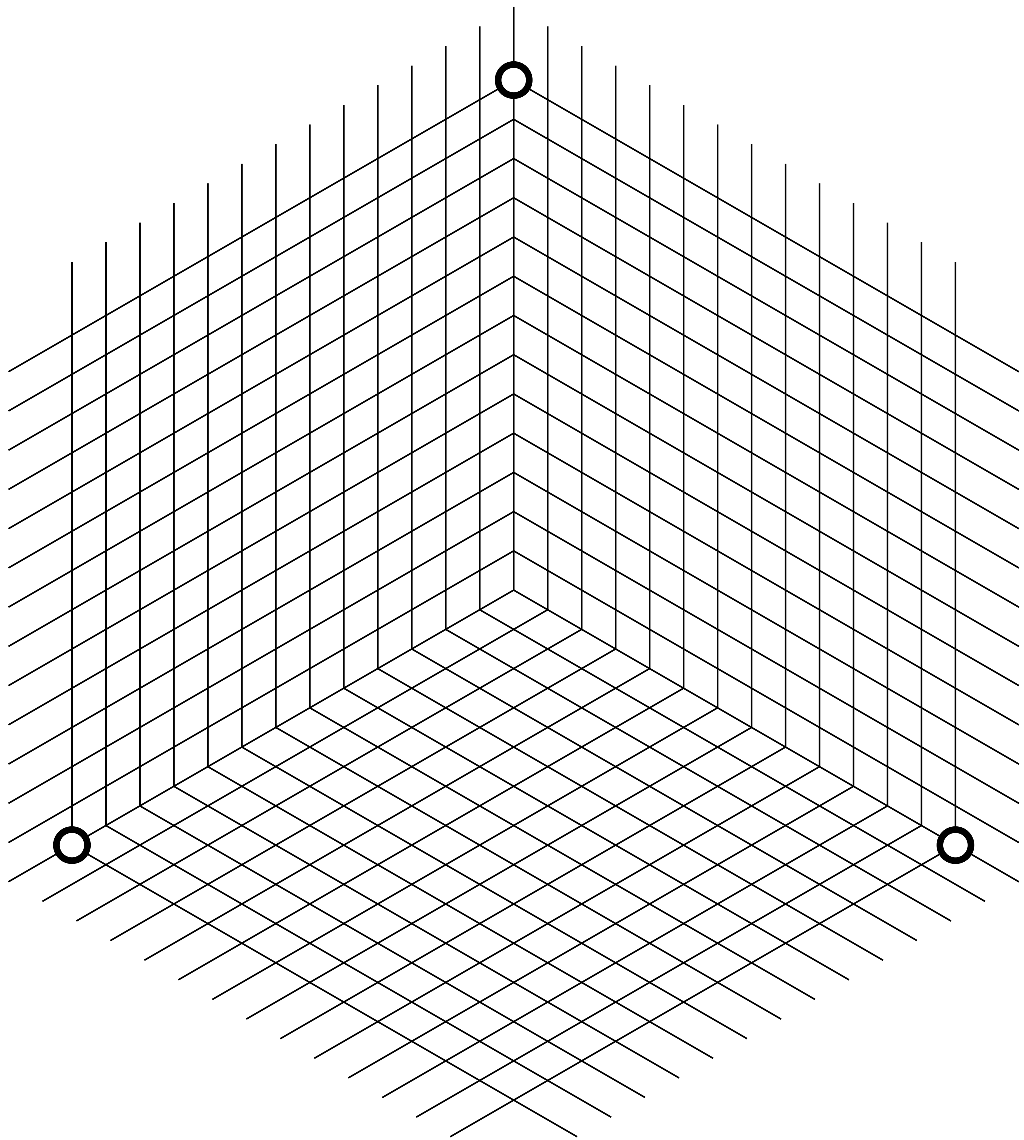
PRIMAL-DUAL GEODESIC EMBEDDING



THM. Reversing the orientation, the same orthogonal surface admits a geodesic embedding of the map M, of its suspended dual map $M^\star$, and of its primal-dual map $\widetilde{M}$.

EXM: STACKED TRIANGULATIONS





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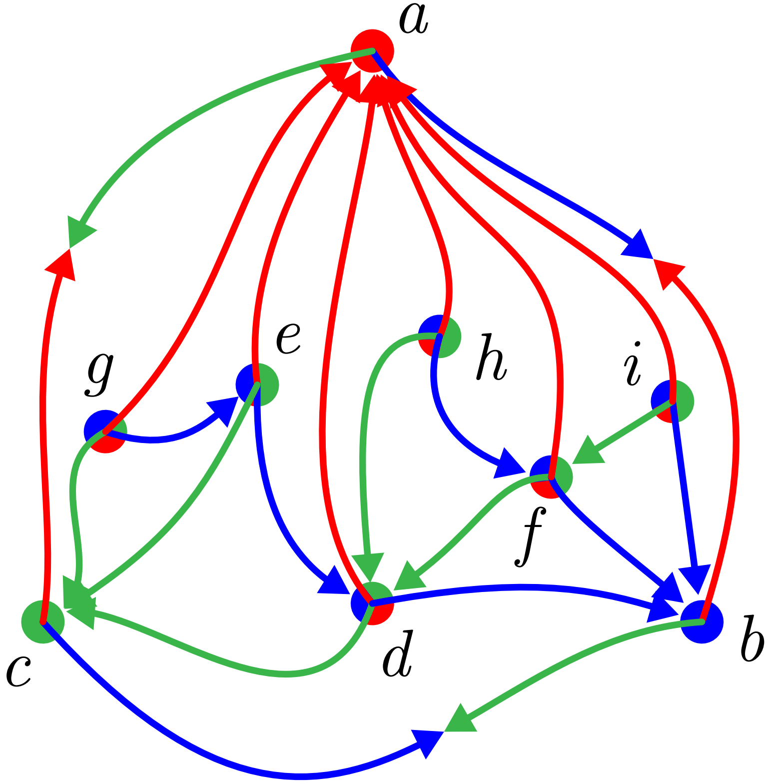

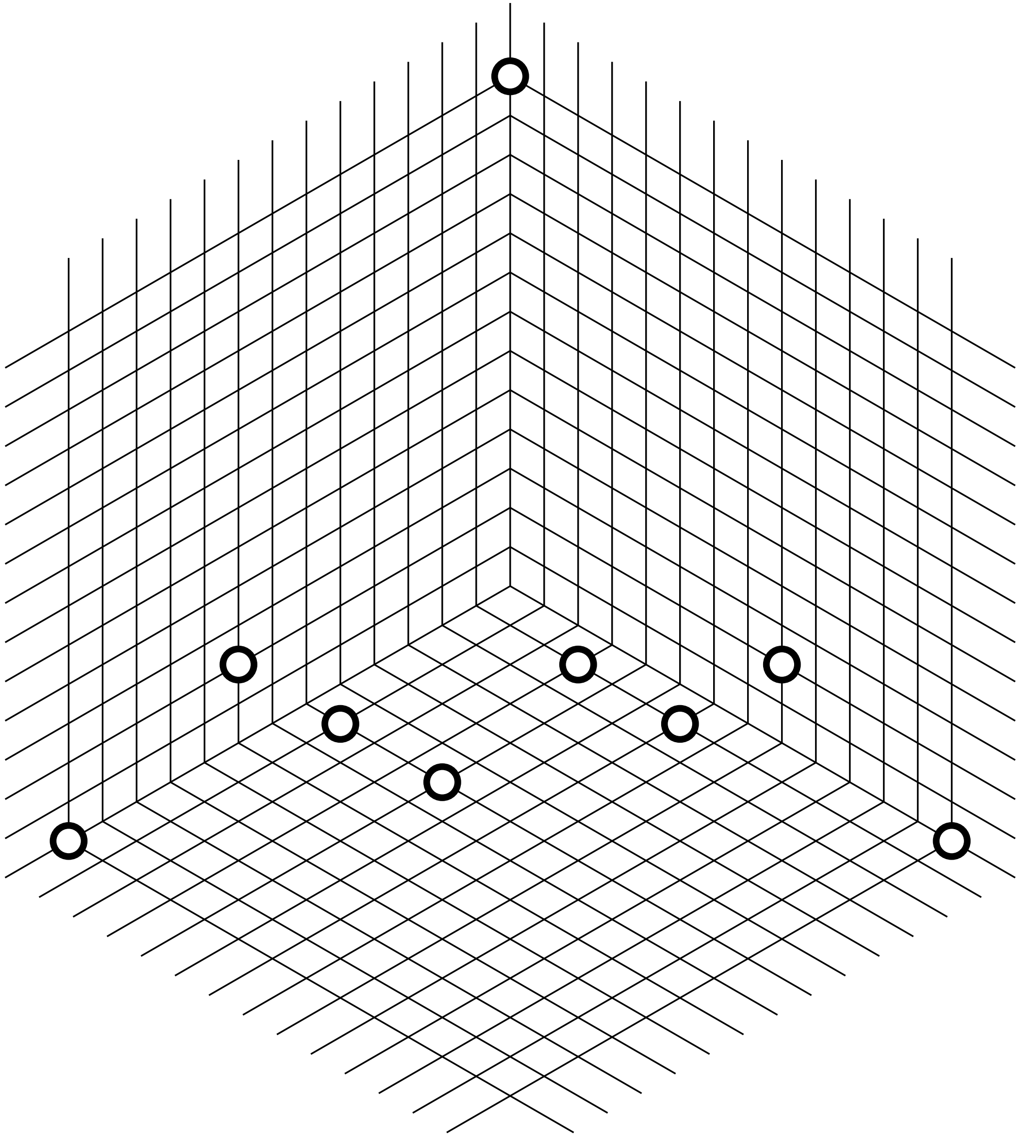

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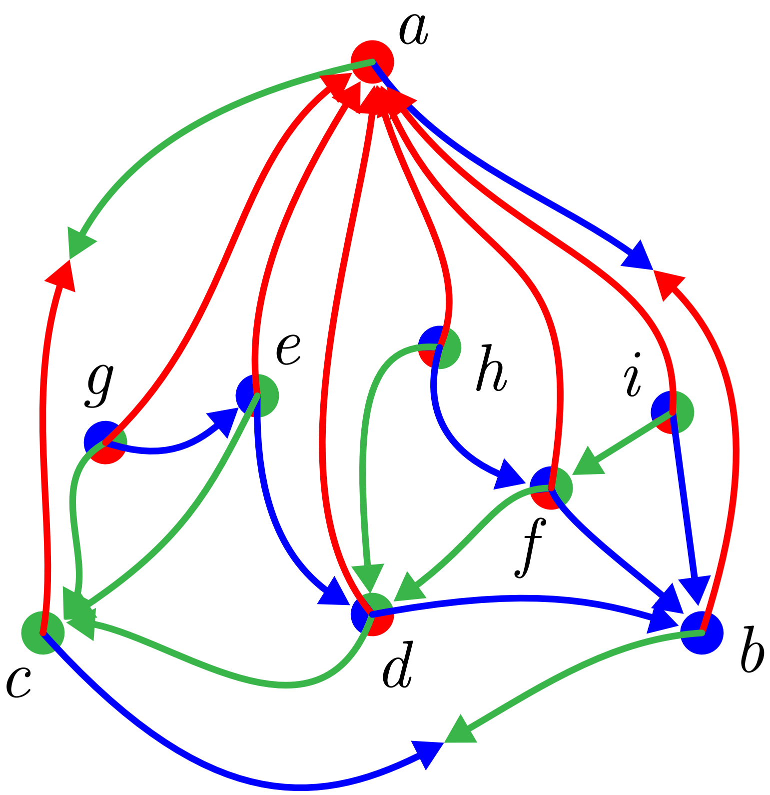

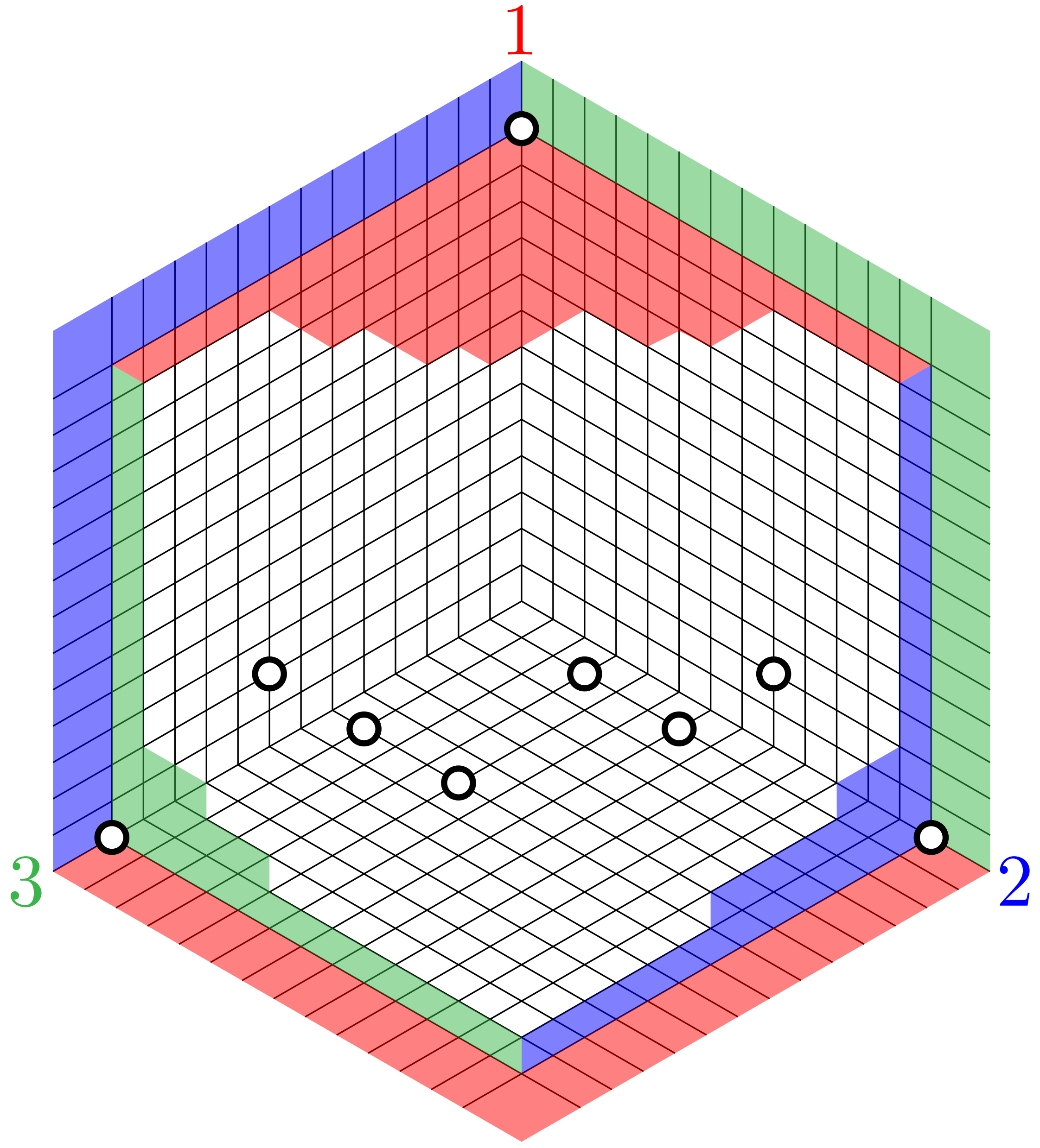

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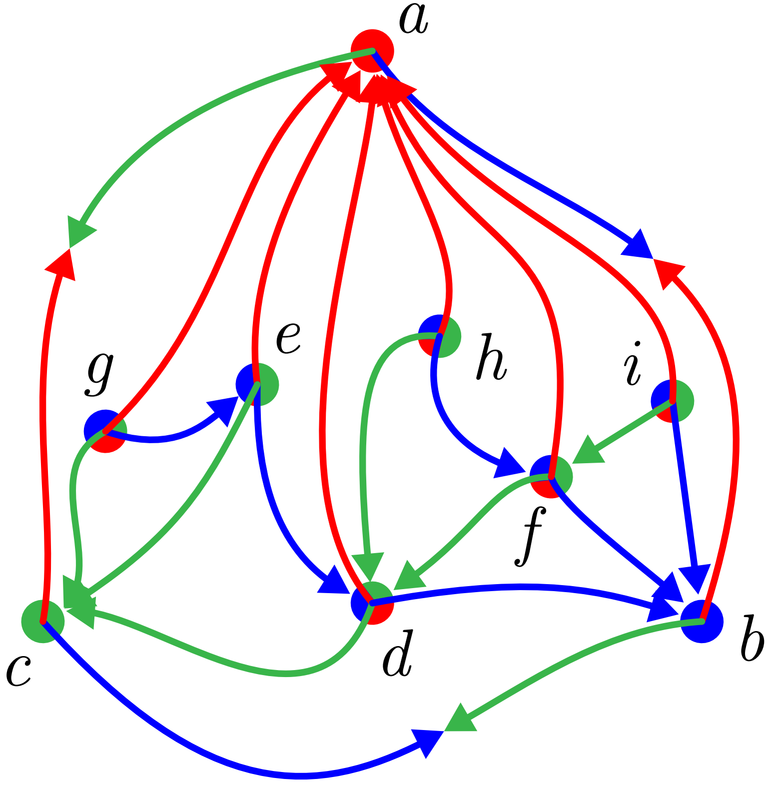

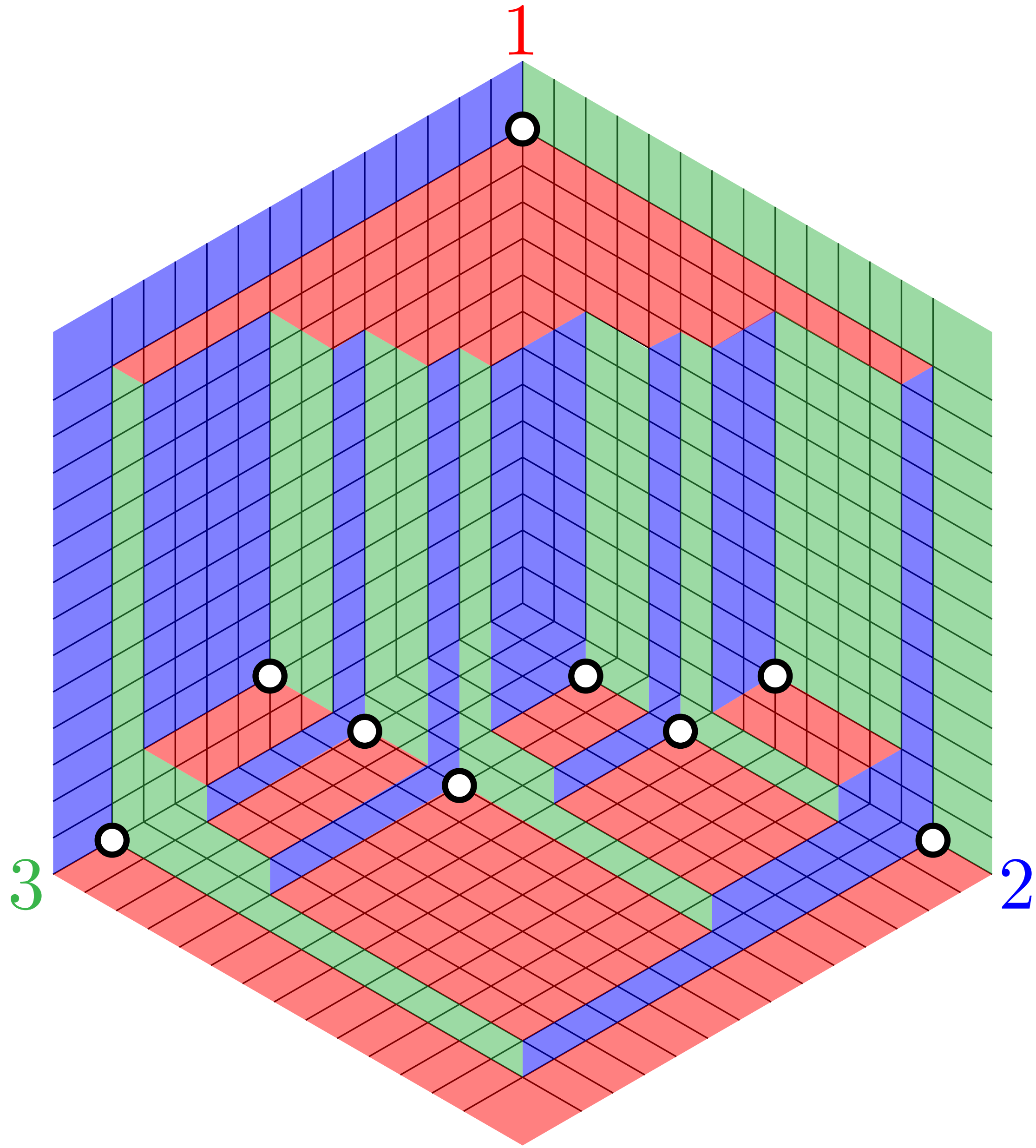

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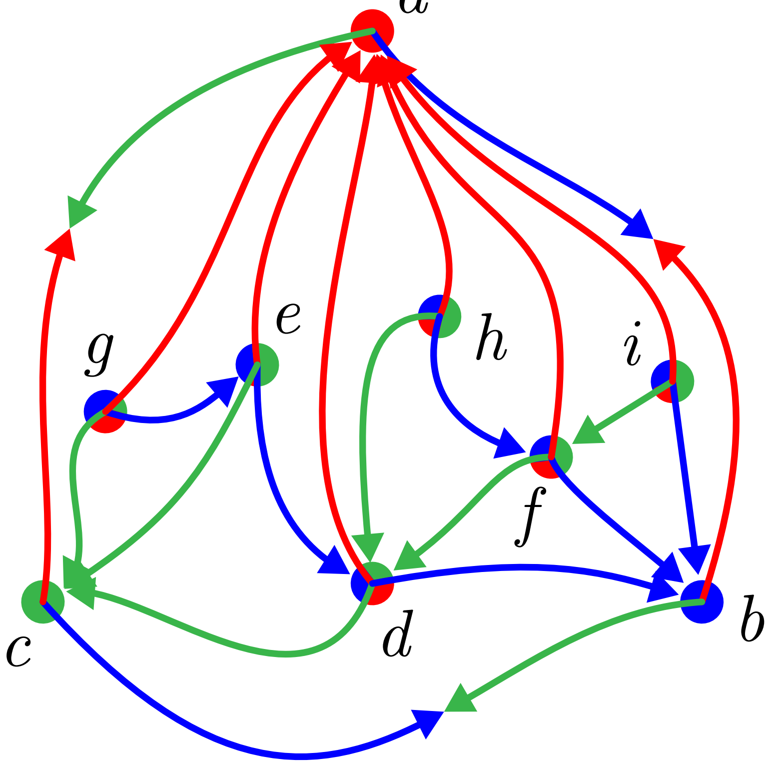

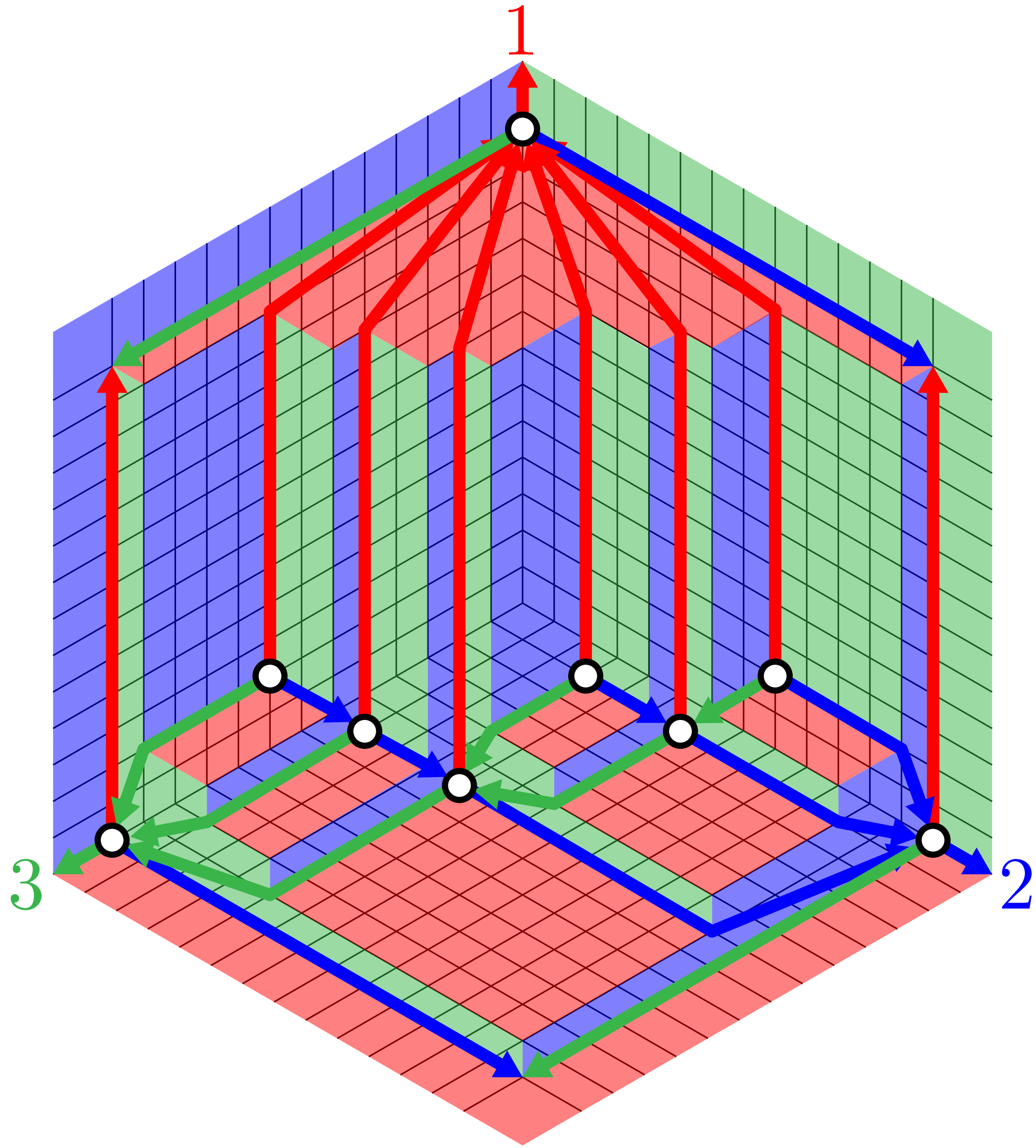

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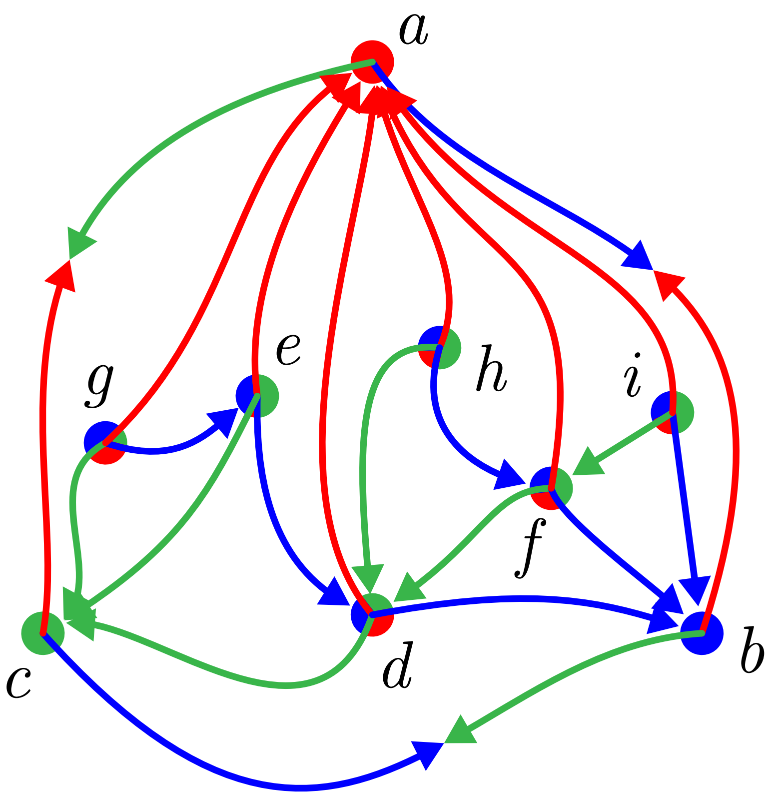
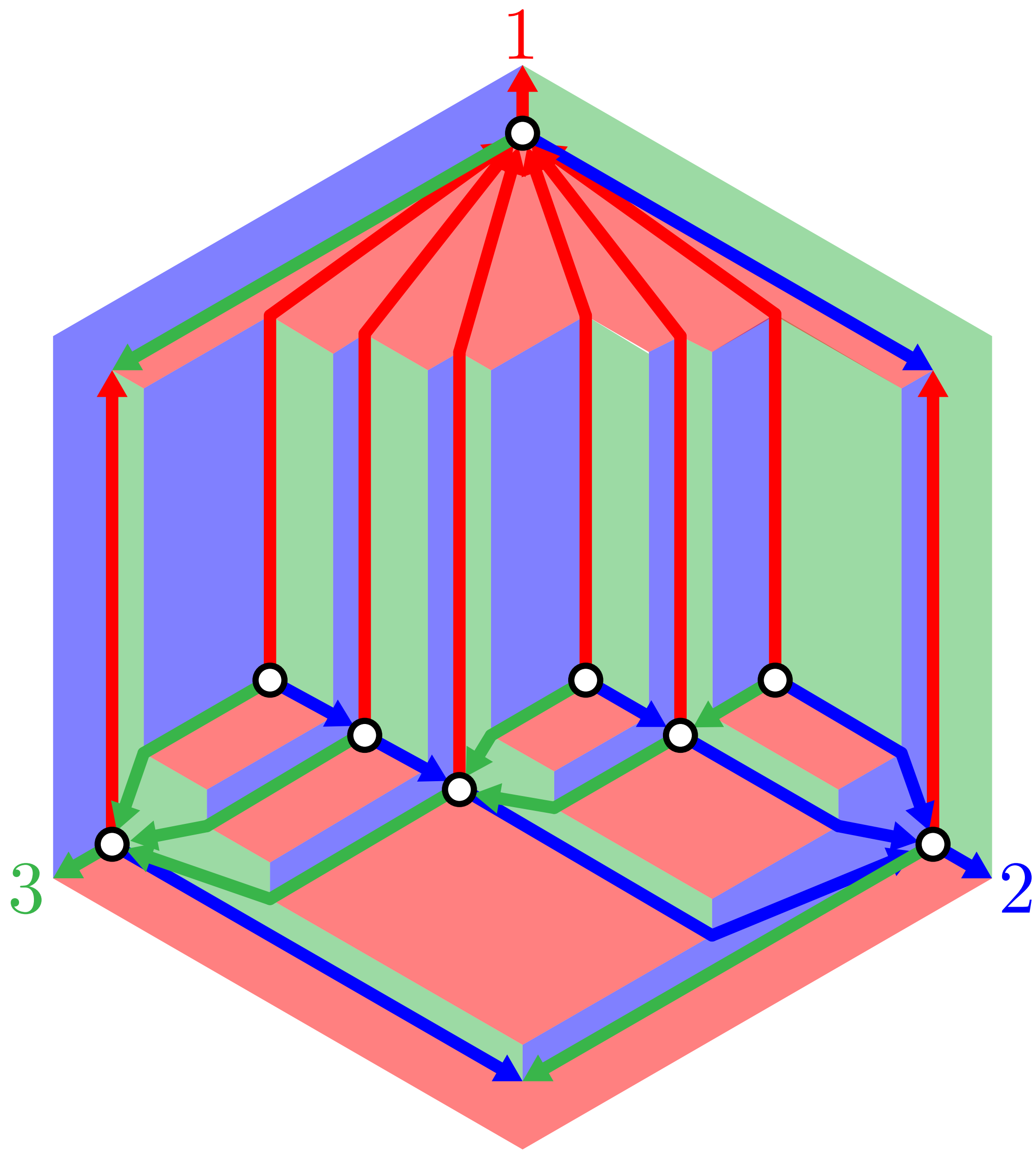




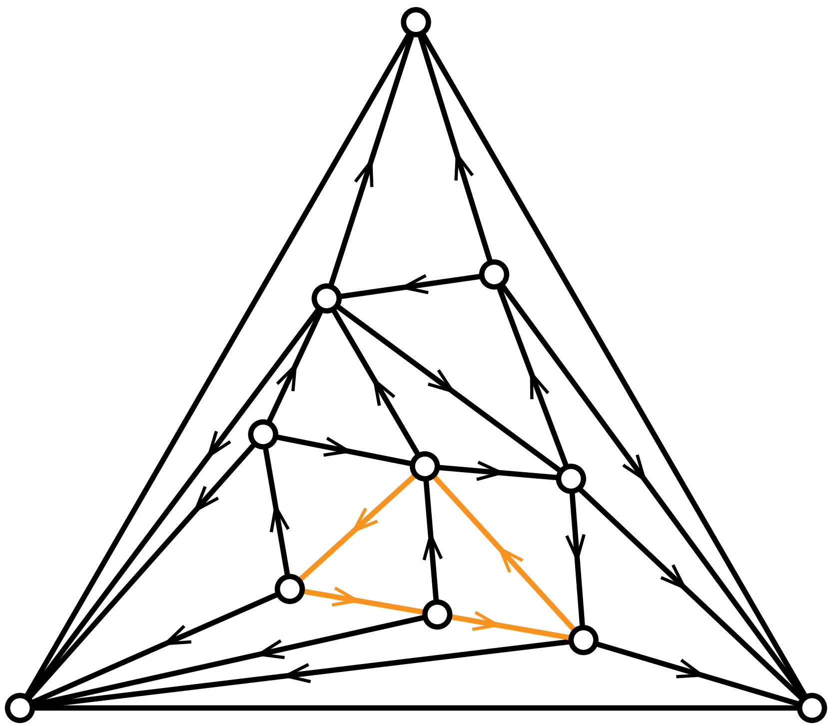
ALPHA-ORIENTATIONS

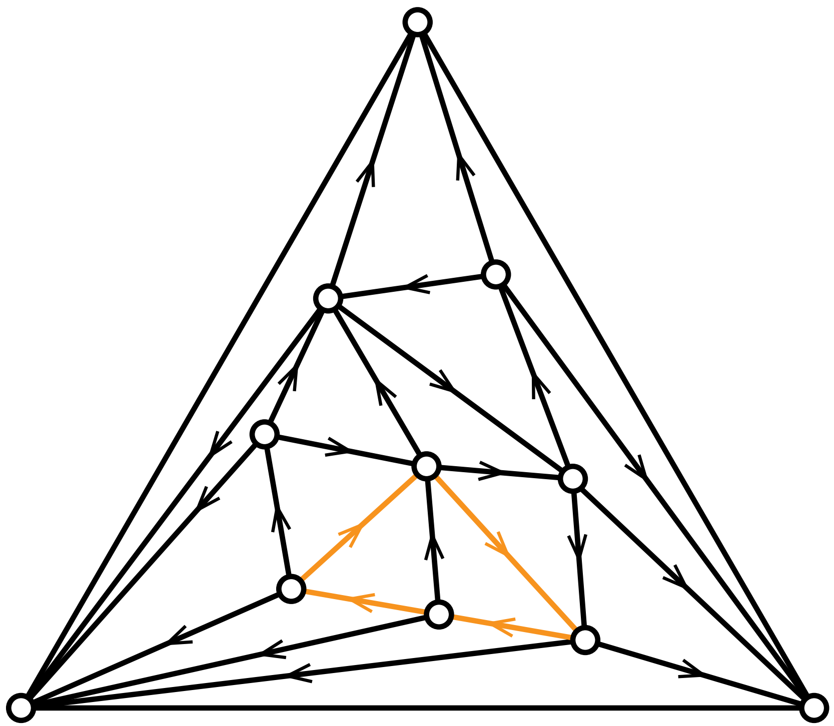
α-ORIENTATION

DEF. $G = (V, E)$ a graph, $\alpha : V \to \mathbb{N}$.
α-orientation = edge orientation of G such that any vertex v has $\alpha(v)$ outgoing edges.

remark: α-orientation do not always exists,
even when $\sum_{v \in V} \alpha(v) = |E|$ and $\alpha(v) \leq \deg(v)$ for all $v \in V$.

PROP. Reversing an oriented cycle in an α-orientation yields another α-orientation.

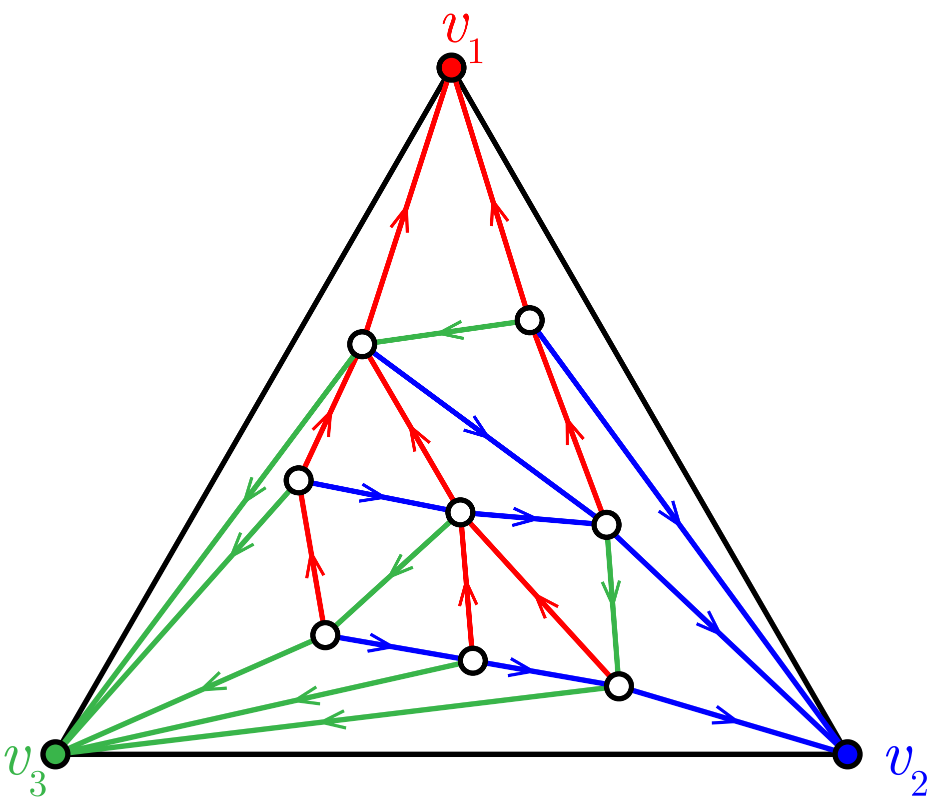


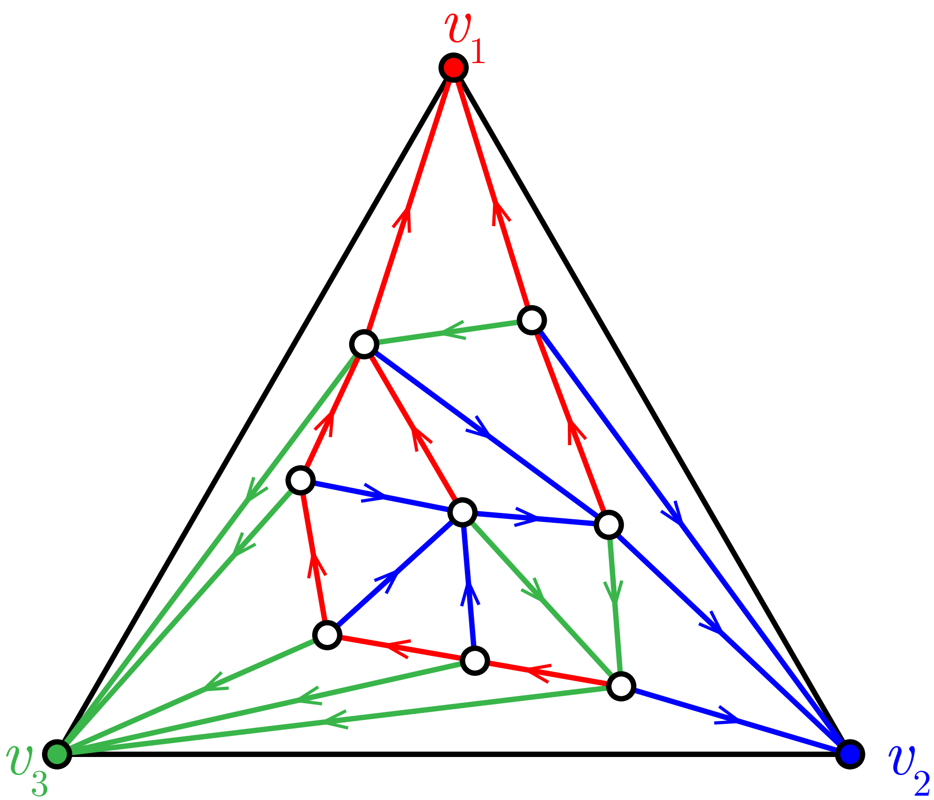


3-ORIENTATIONS IN TRIANGULATIONS

DEF. M = triangulated planar map with external vertices v_1, v_2, v_3, and edges e_1, e_2, e_3
3-orientation = α-orientation of $M \setminus \{e_1, e_2, e_3\}$,
where $\alpha(v) = 3$ except $\alpha(v_1) = \alpha(v_2) = \alpha(v_3) = 0$.

THM. For a triangulated triangulated map M, there is a bijection
3-orientations of $M \longleftrightarrow$ Schnyder woods of M.





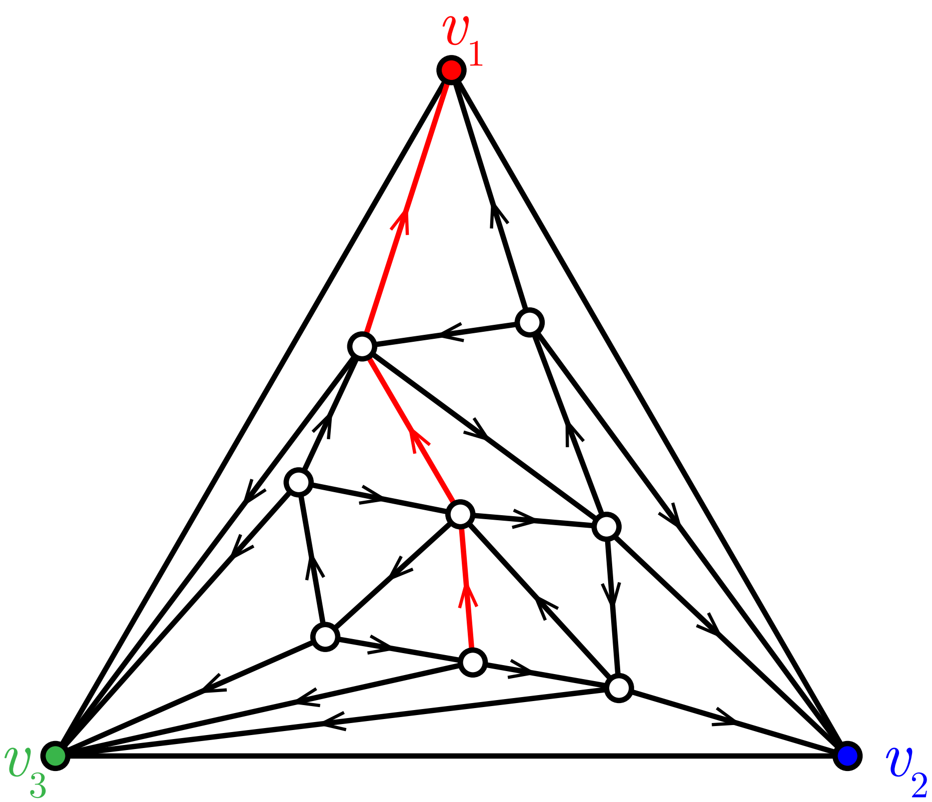
3-ORIENTATIONS IN TRIANGULATIONS

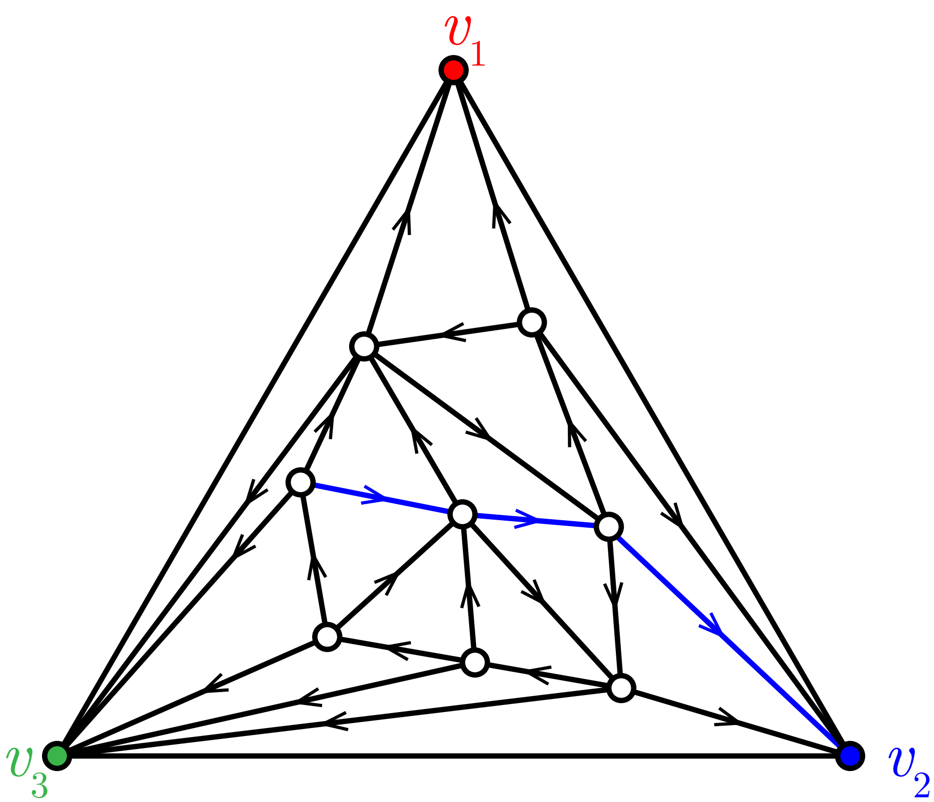
THM. For a triangulated triangulated map M, there is a bijection

$$3\text{-orientations of } M \longleftrightarrow \text{Schnyder woods of } M.$$

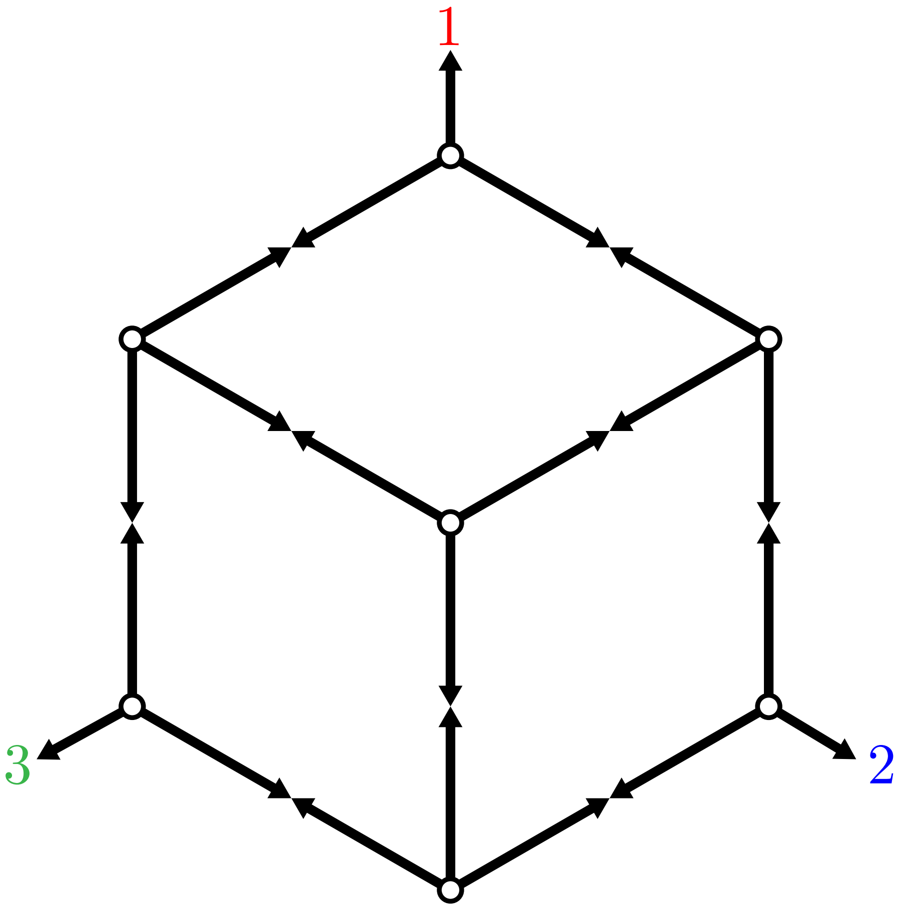
proof idea:

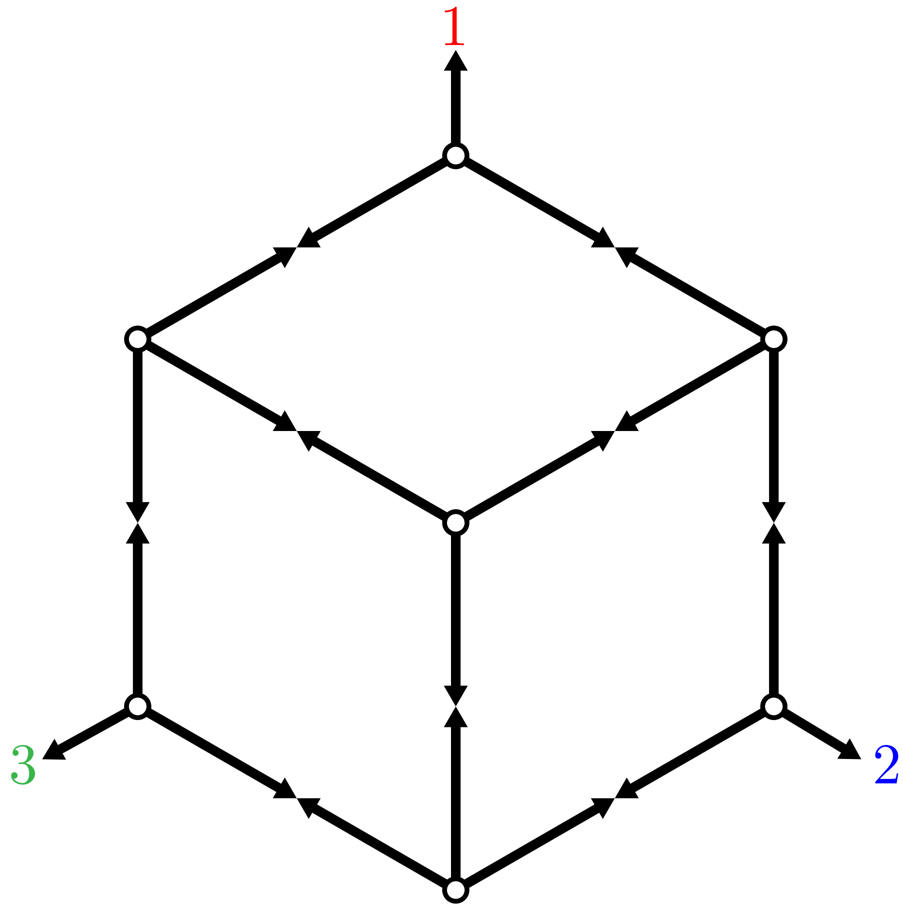
- A Schnyder woods clearly gives a 3-orientation.
- Conversely, consider the central paths in a 3-orientation and prove that they never self-intersect, nor intersect twice.



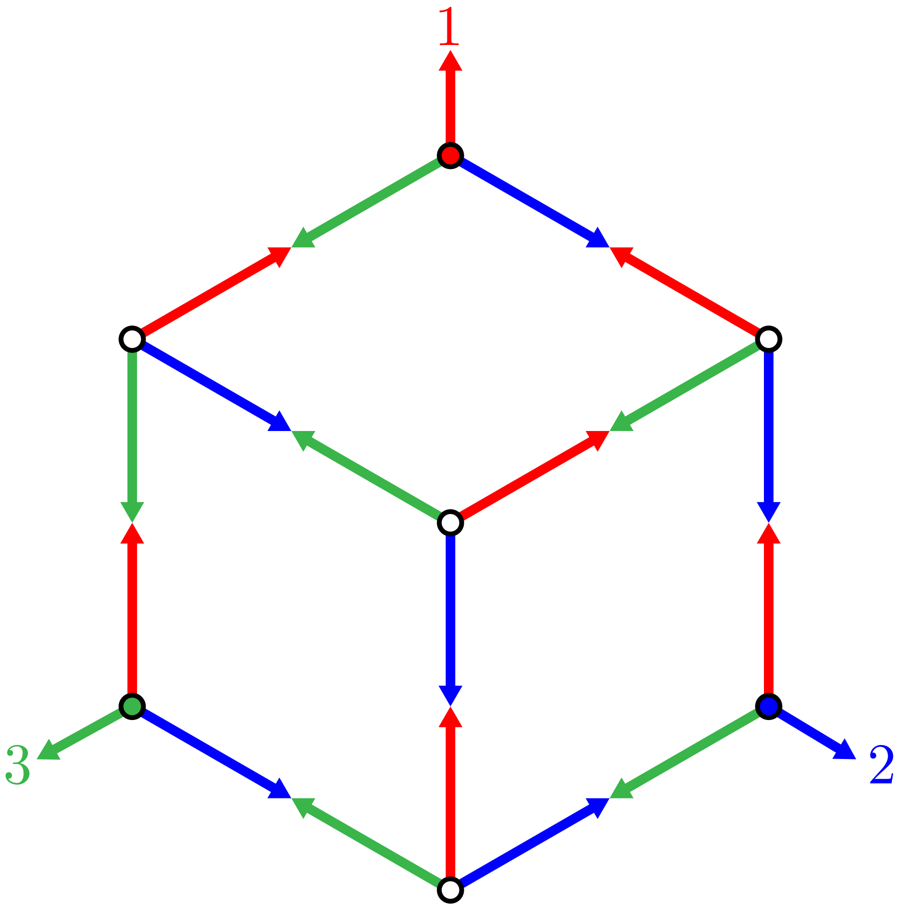


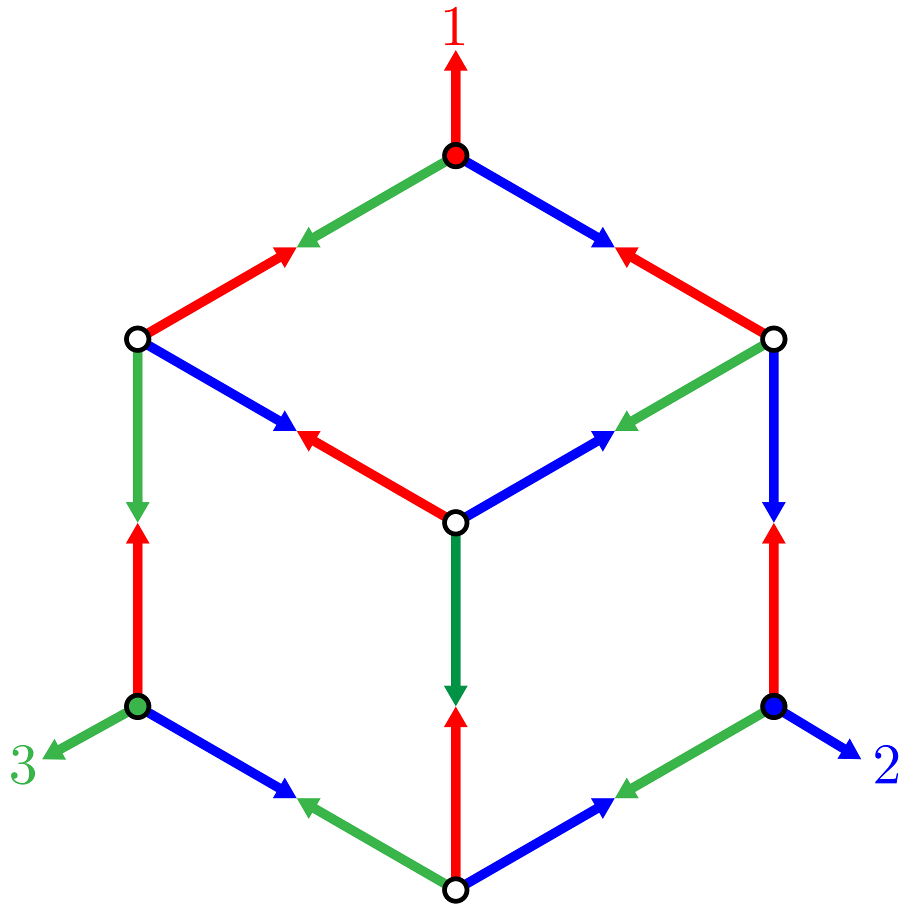
BEYOND TRIANGULATIONS





remark: for an arbitrary planar map, there are more Schnyder woods than 3-orientations...





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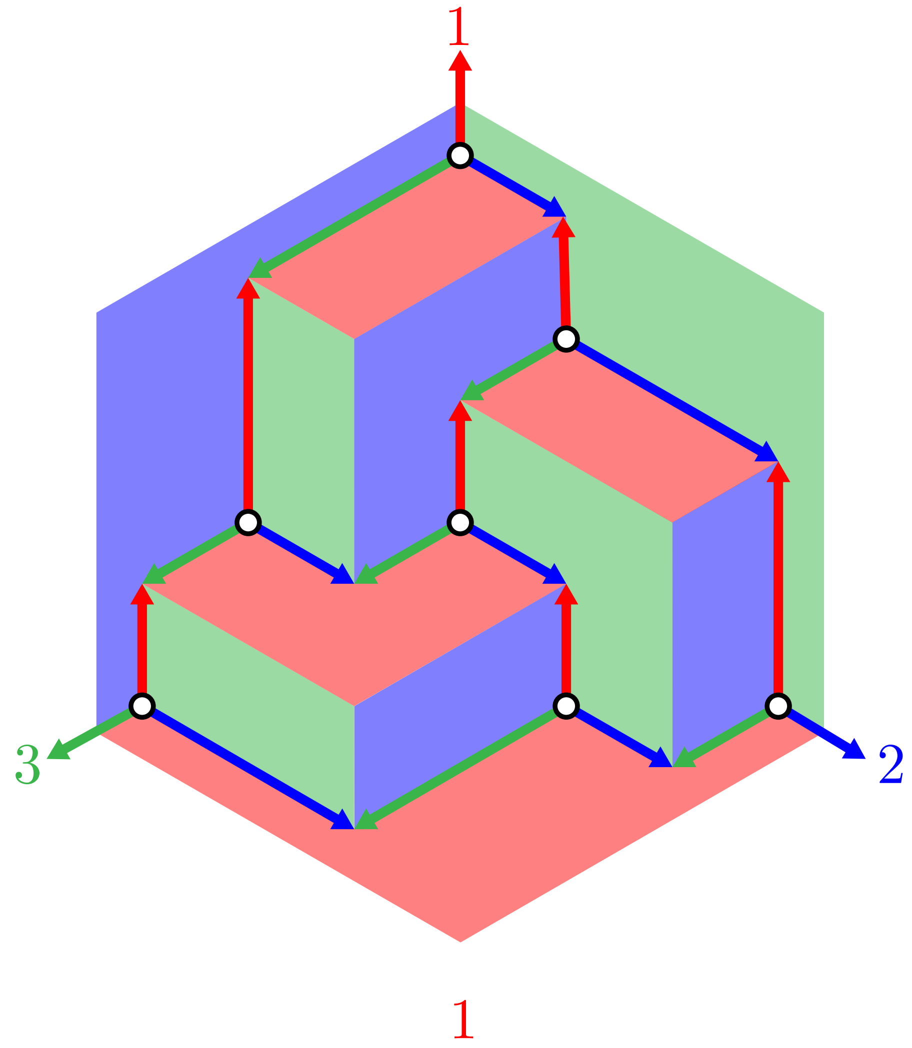
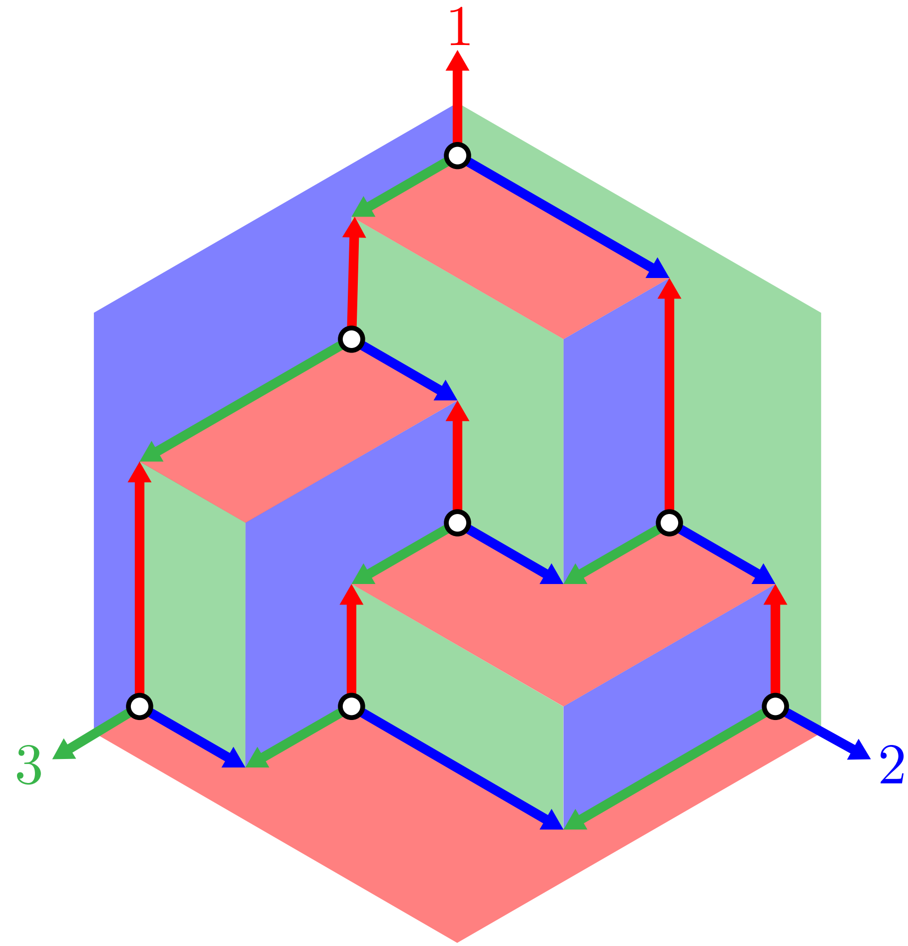
BEYOND TRIANGULATIONS

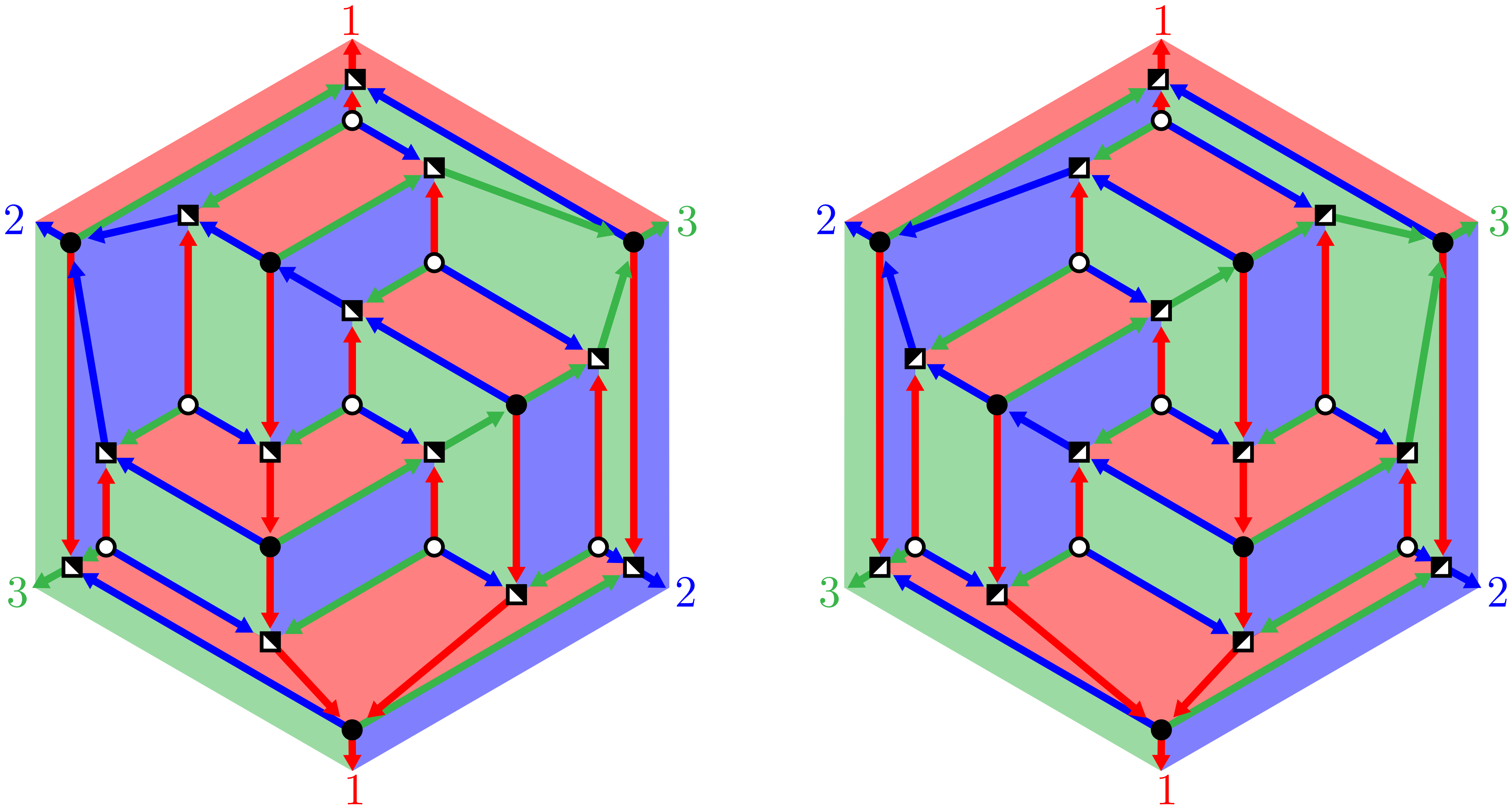




remark: for an arbitrary planar map, there are more Schnyder woods than 3-orientations...

... but distinct Schnyder woods yield geodesic embeddings on distinct orthogonal surfaces...

BEYOND TRIANGULATIONS

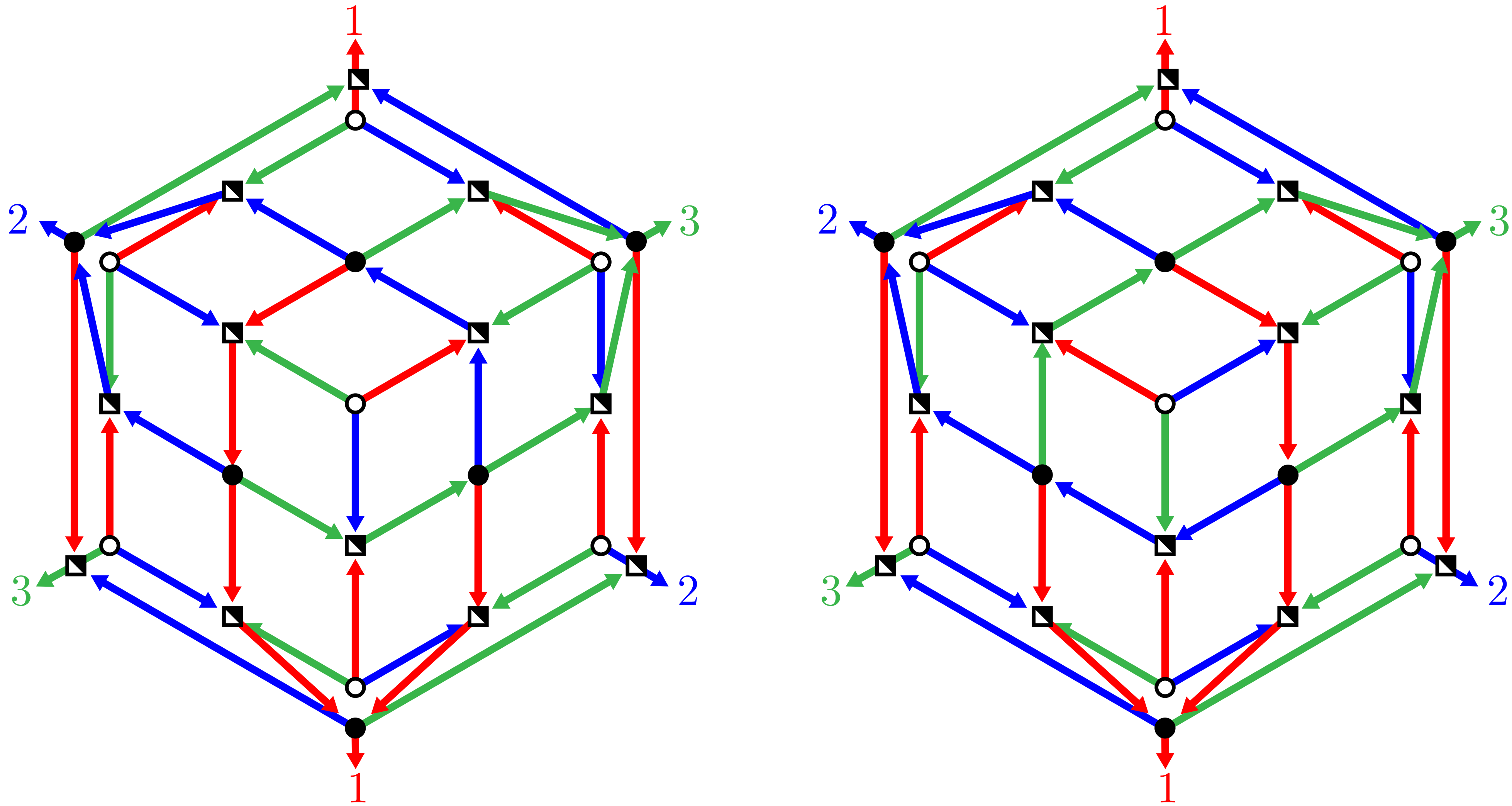


remark: for an arbitrary planar map, there are more Schnyder woods than 3-orientations...

... but distinct Schnyder woods yield geodesic embeddings on distinct orthogonal surfaces...

... with distinct orientations for the suspended duals...

BEYOND TRIANGULATIONS

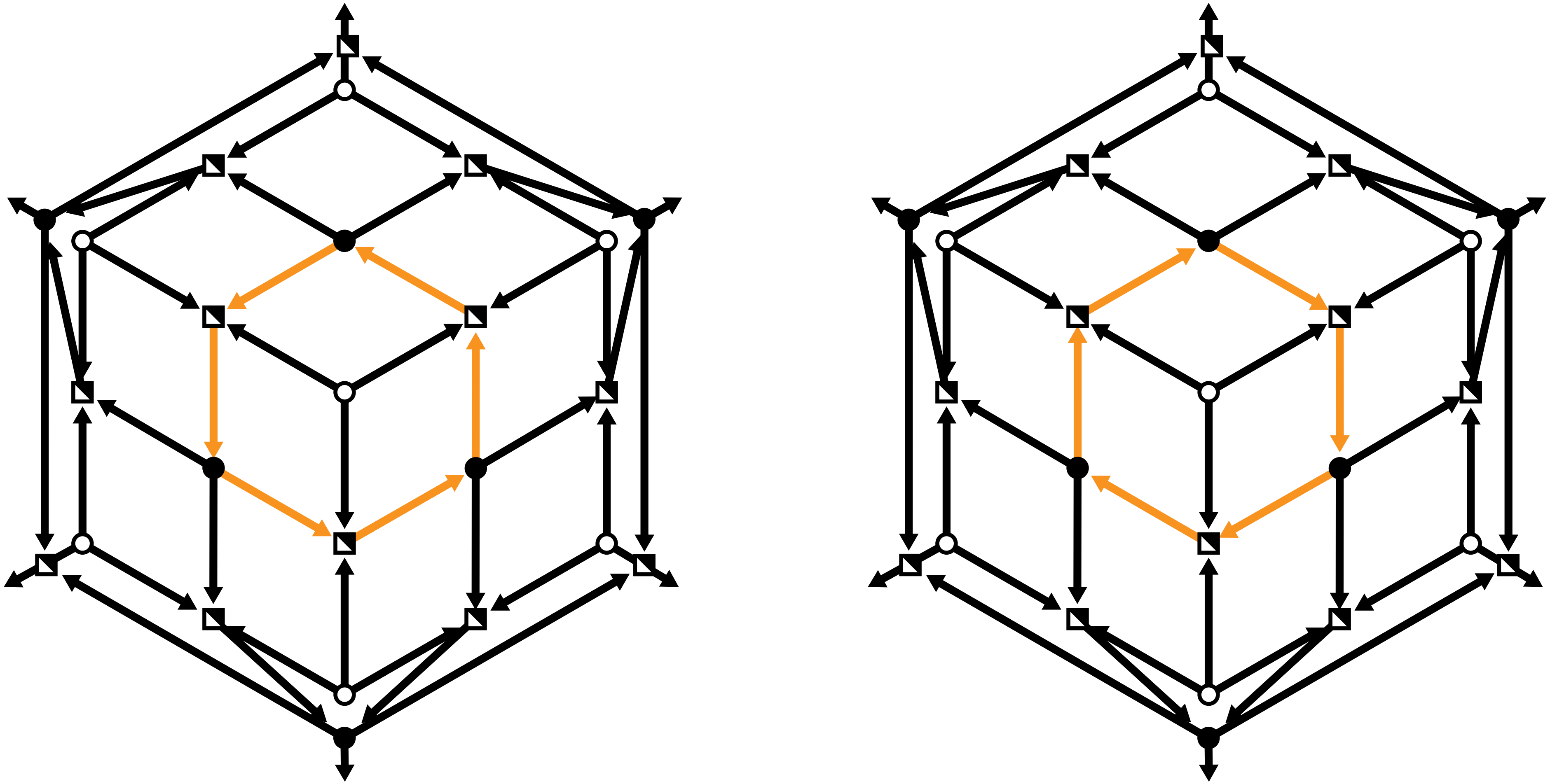


THM. For a 3-connected planar map M, there is a bijection

α-orientations of the primal-dual $\widetilde{M} \longleftrightarrow$ Schnyder woods of M

where $\alpha(\circ) = \alpha(\bullet) = 3$ while $\alpha(\square) = 1$.

BEYOND TRIANGULATIONS



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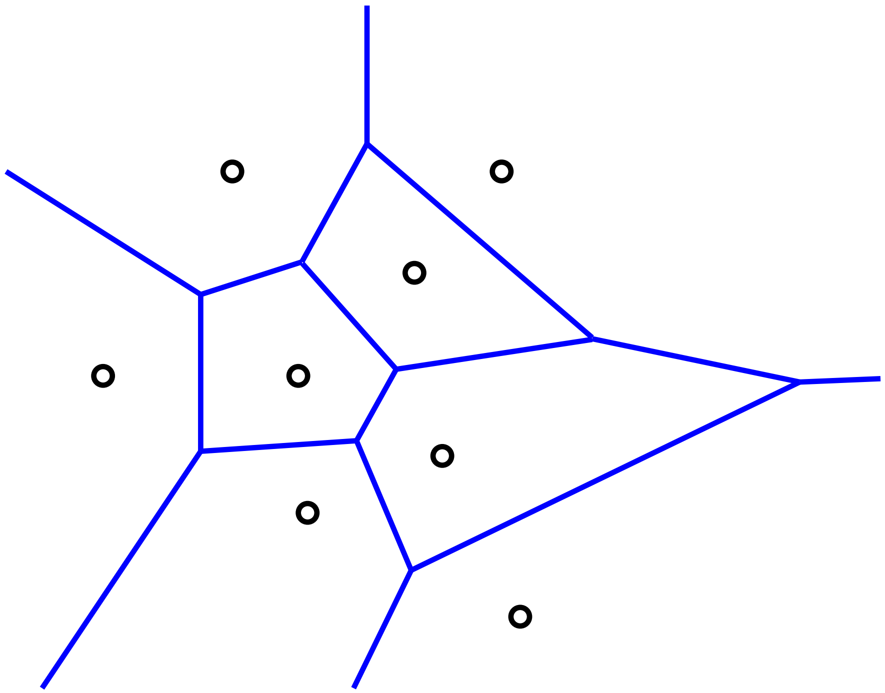
TD-DELAUNAY TRIANGULATIONS

VORONOI DIAGRAM

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p},\boldsymbol{P}) = \left\{\boldsymbol{x}\in\mathbb{R}^2 \;\middle|\; \|\boldsymbol{x}-\boldsymbol{p}\| \le \|\boldsymbol{x}-\boldsymbol{q}\| \text{ for all } \boldsymbol{q}\in\boldsymbol{P}\right\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p},\boldsymbol{P})$ for $\boldsymbol{p}\in\boldsymbol{P}$.



VORONOI DIAGRAM

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \left\{\boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \|\boldsymbol{x} - \boldsymbol{p}\| \le \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P}\right\}$.

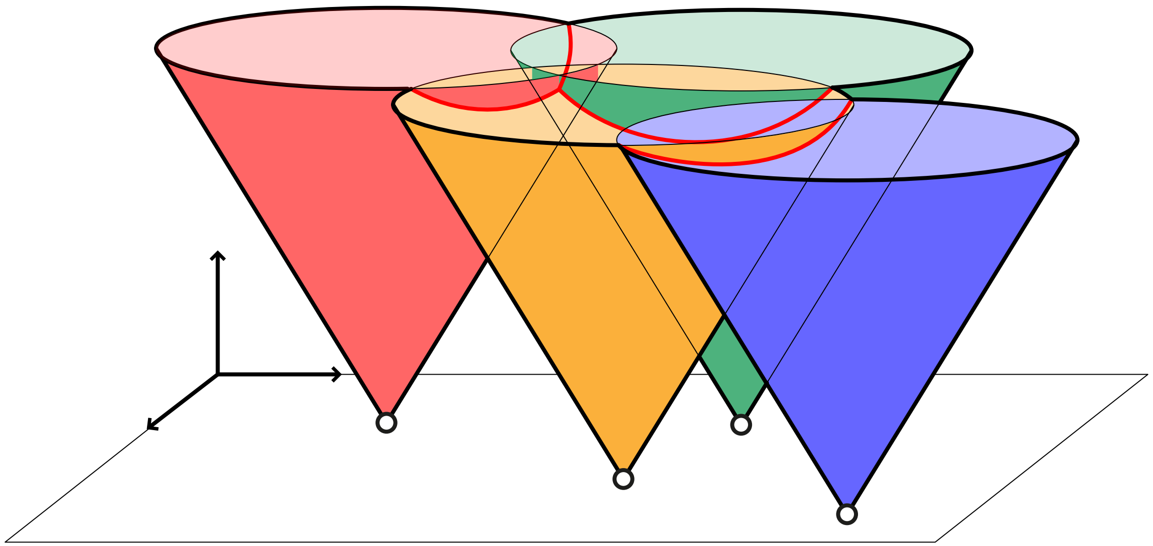
Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.

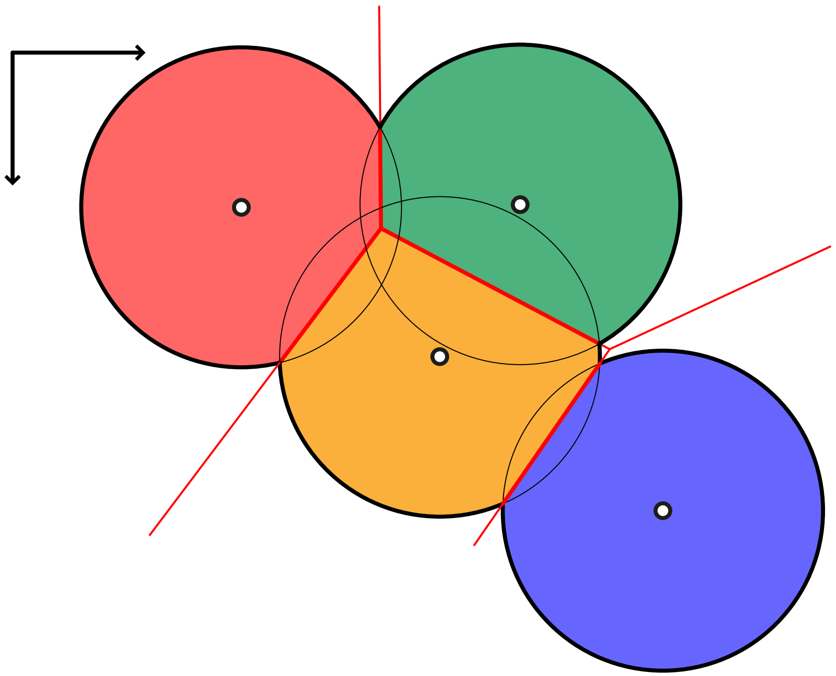
VORONOI DIAGRAM

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x} - \boldsymbol{p}\| \le \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P}\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



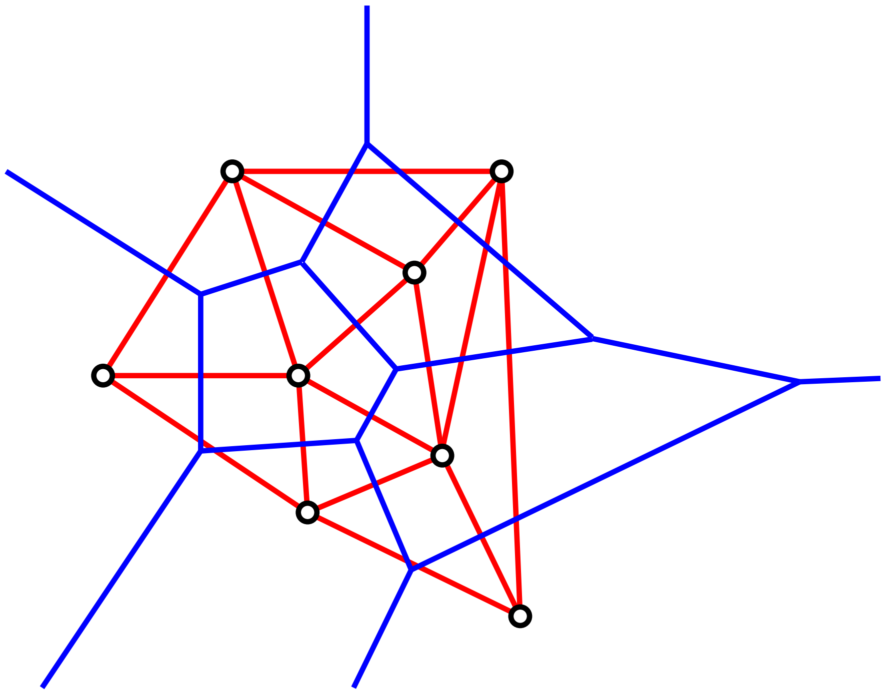


DELAUNAY COMPLEX

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x} - \boldsymbol{p}\| \leq \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P}\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



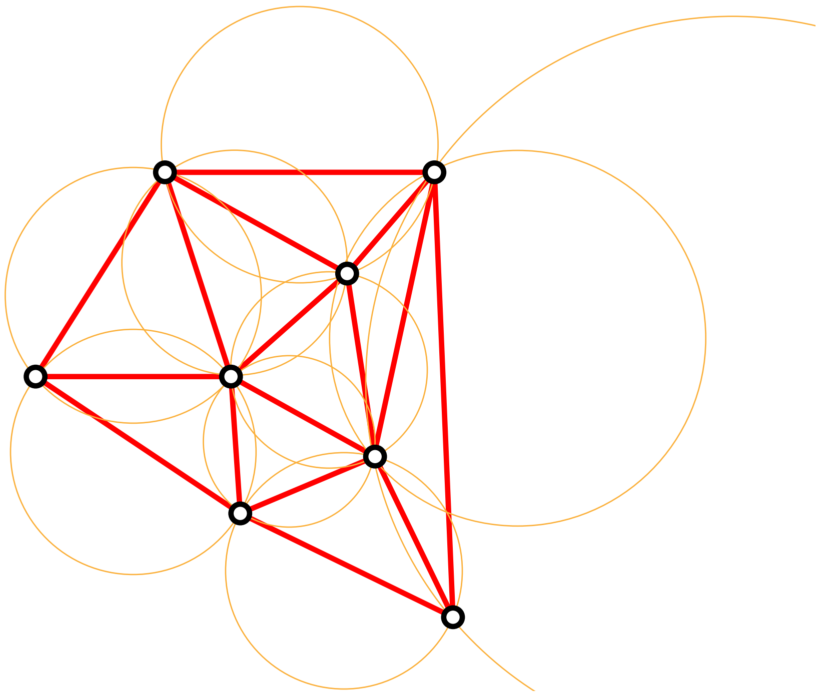
DEF. Delaunay complex $\mathrm{Del}(\boldsymbol{P})$ = intersection complex of $\mathrm{Vor}(\boldsymbol{P})$

$$\mathrm{Del}(\boldsymbol{P}) = \Big\{ \mathrm{conv}(\boldsymbol{X}) \;\Big|\; \boldsymbol{X} \subseteq \boldsymbol{P} \text{ and } \bigcap_{\boldsymbol{p} \in \boldsymbol{X}} \mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) \neq \varnothing \Big\}.$$

EMPTY CIRCLES

PROP. For any three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ of $\boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{pqr}$ is a triangle of $\mathrm{Del}(\boldsymbol{P}) \iff$ the circumcircle of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is empty.

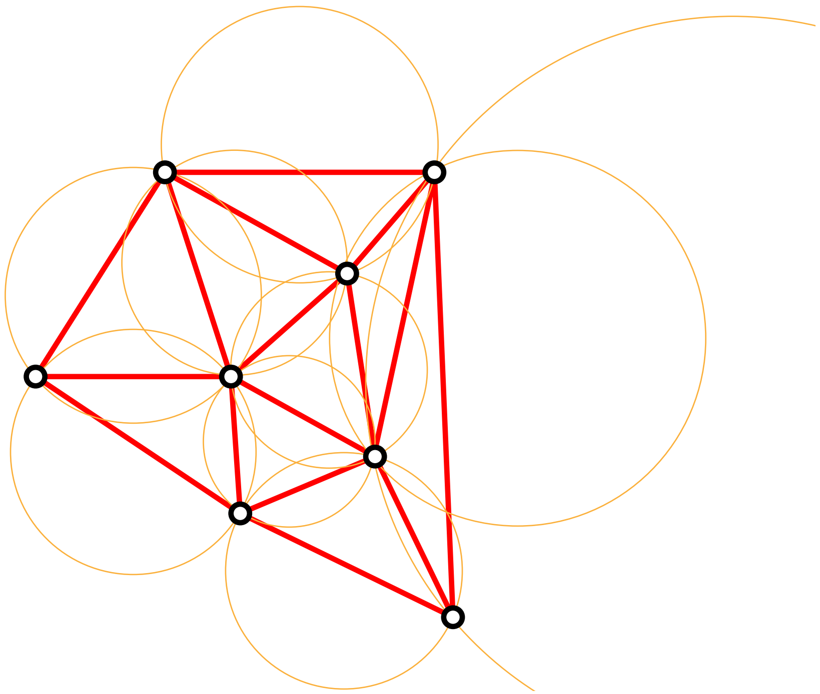


proof idea: consider the circle centered at the intersection of the Voronoi regions and passing through the Voronoi sites.

EMPTY CIRCLES

PROP. For any three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ of $\boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{pqr}$ is a triangle of $\mathrm{Del}(\boldsymbol{P}) \iff$ the circumcircle of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is empty.

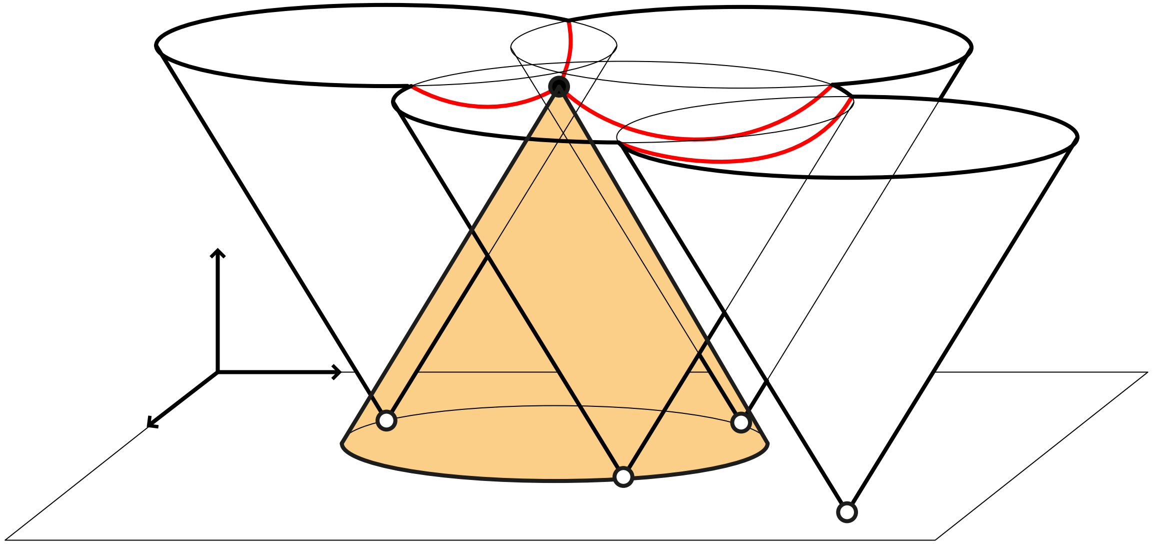


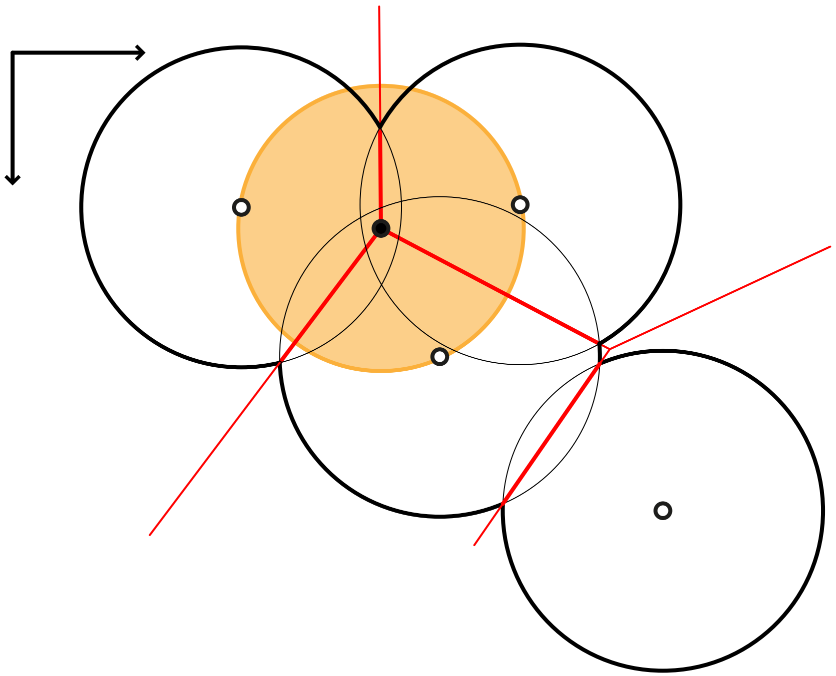
CORO. In two adjacent triangles of a Delaunay triangulation, the sum of the two opposite angles is at most π.

EMPTY CIRCLES

PROP. For any three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ of $\boldsymbol{P}$,

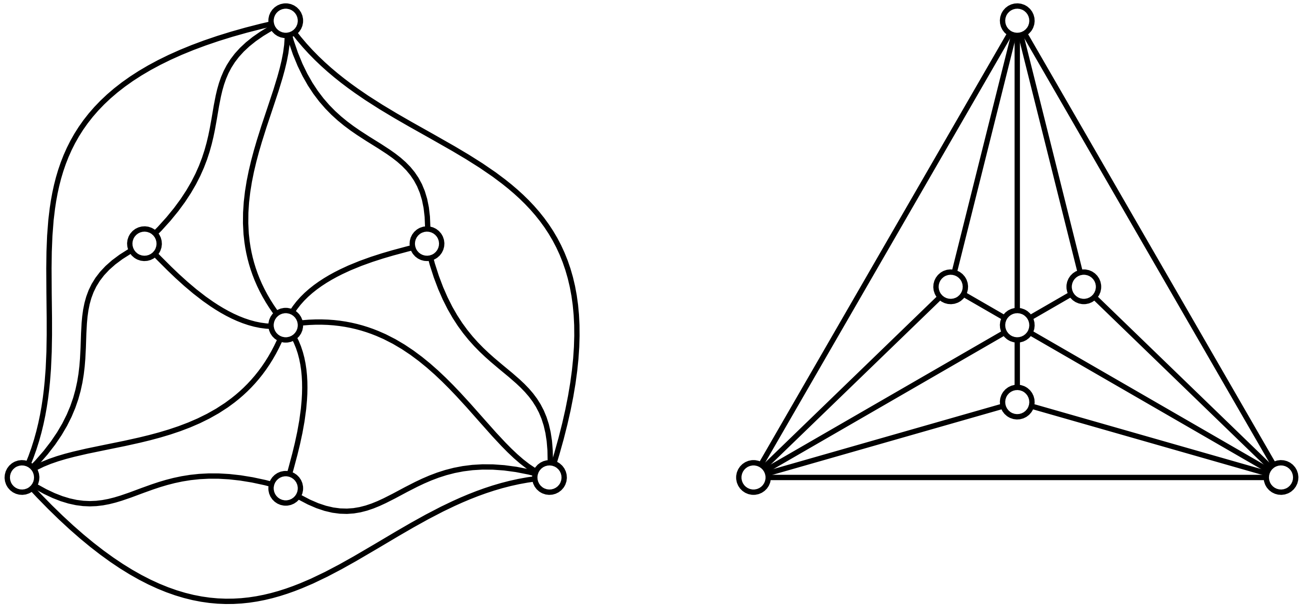
- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{pqr}$ is a triangle of $\mathrm{Del}(\boldsymbol{P}) \iff$ the circumcircle of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is an empty circle.





EXM: STACKED TRIANGULATIONS

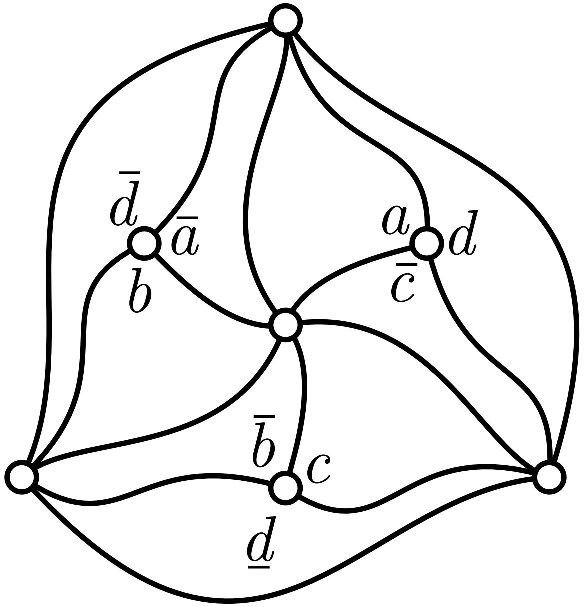
QU. Consider the stacked triangulation

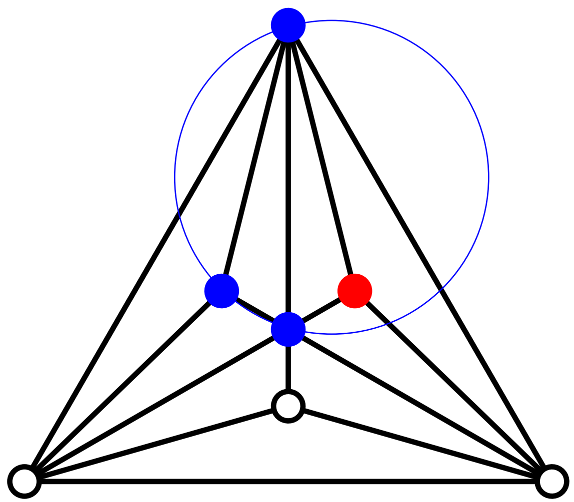


Does this realization look Delaunay? Can you provide a Delaunay realization?

EXM: STACKED TRIANGULATIONS

REM. The stacked triangulation





has no Delaunay realization.

proof: In a Delaunay realization of this stacked triangulation, we would have

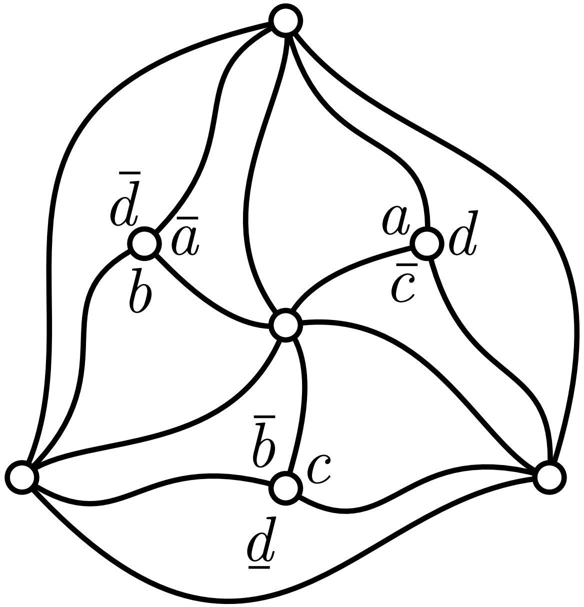
$$a + \overline{a} < \pi, \qquad b + \overline{b} < \pi, \qquad c + \overline{c} < \pi,$$

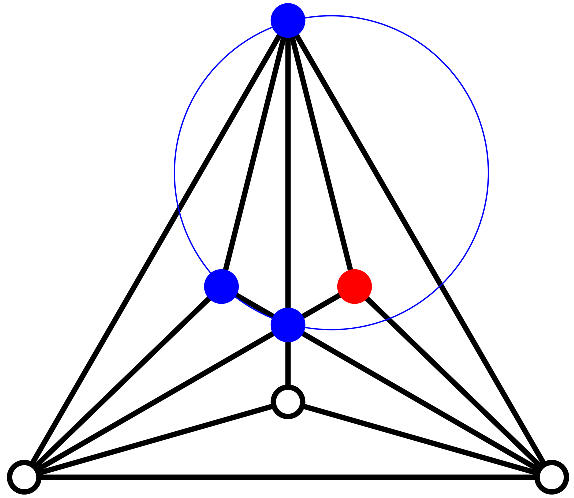
$$\text{and} \qquad a + \overline{c} + d = \overline{a} + b + \overline{d} = \overline{b} + c + \underline{d} = 2\pi.$$

Thus $d + \overline{d} + \underline{d} > 3\pi$ and at least one of d, $\overline{d}$ and $\underline{d}$ is larger than π, a contradiction.

EXM: STACKED TRIANGULATIONS

REM. The stacked triangulation





has no Delaunay realization.

THM. A stacked triangulation admits a Delaunay realization if and only if its construction tree has no ternary node after deletion of all its leaves.

proof ideas:

- one direction follows from the example above,
- for the opposite direction, find an explicit construction (see Exercise 113 course notes).

QUASI-METRICS

DEF. quasi-metric on Q = function $\delta : Q^2 \to \mathbb{R}_{\geq 0}$ st:

- separability: $\delta(p,q) = 0 \iff p = q$,
- triangular inequality: $\delta(p,q) + \delta(q,r) \geq \delta(p,r)$.

DEF. $P \subseteq Q$ a set of sites of Q.

δ-Voronoi region $\mathrm{Vor}_\delta(p,P) = \{r \in Q \mid \delta(p,r) \leq \delta(q,r)$ for all $q \in P\}$.

δ-Voronoi diagram $\mathrm{Vor}_\delta(P)$ = partition of Q formed by $\mathrm{Vor}_\delta(p,P)$ for $p \in P$.

DEF. δ-Delaunay complex $\mathrm{Del}_\delta(P)$ = intersection complex of $\mathrm{Vor}_\delta(P)$

$$\mathrm{Del}_\delta(P) = \Big\{ X \subseteq P \;\Big|\; \bigcap_{p \in X} \mathrm{Vor}_\delta(p,P) \neq \varnothing \Big\} \subseteq 2^P.$$

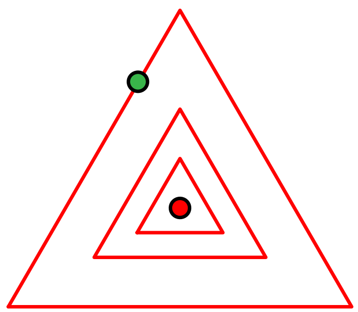
TRIANGULAR DISTANCE

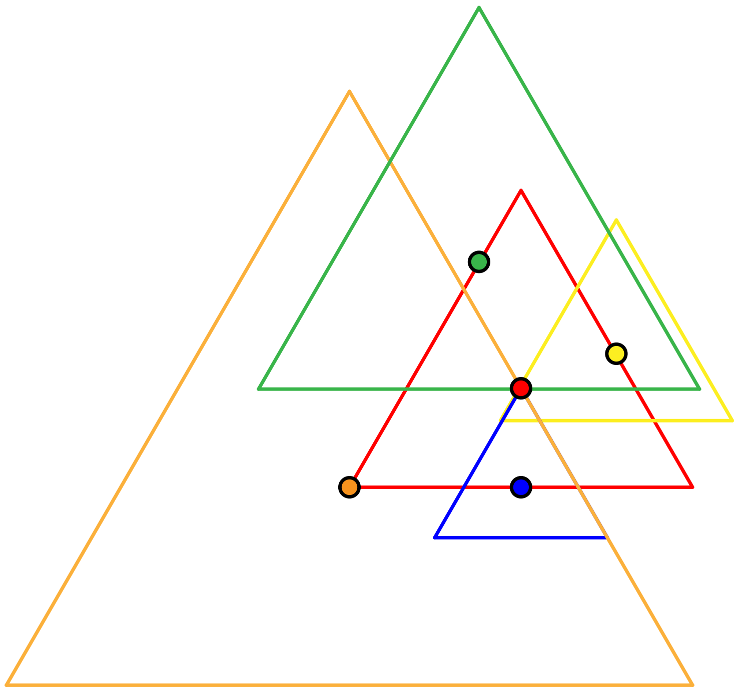
Fix $c \in \mathbb{R}_{\geq 0}$, and consider the hyperplane $\boldsymbol{H} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 = c \right\}$.
and its standard equilateral triangle $\triangle = \operatorname{conv}(c\boldsymbol{e}_1, c\boldsymbol{e}_2, c\boldsymbol{e}_3)$

DEF. triangular distance between $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{H}$ =

$$\mathrm{TD}(\boldsymbol{x}, \boldsymbol{y}) = \min \left\{ \lambda \in \mathbb{R}_{\geq 0} \;\middle|\; \boldsymbol{x} \in \boldsymbol{y} + \lambda(\triangle - c\mathbb{1}/3) \right\}.$$

remark: intuitively, $\mathrm{TD}(\boldsymbol{x}, \boldsymbol{y})$ is obtained by dilating a standard equilateral triangle $\triangle$ centered at $\boldsymbol{y}$ until it reaches $\boldsymbol{x}$.

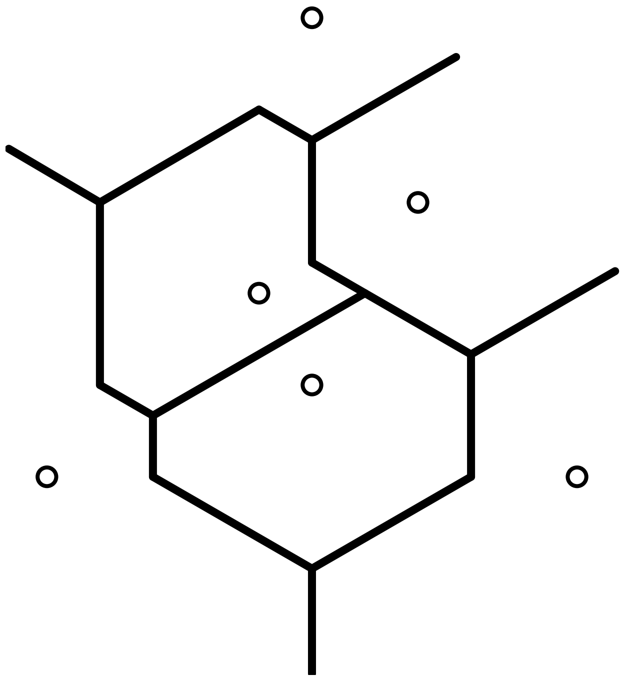


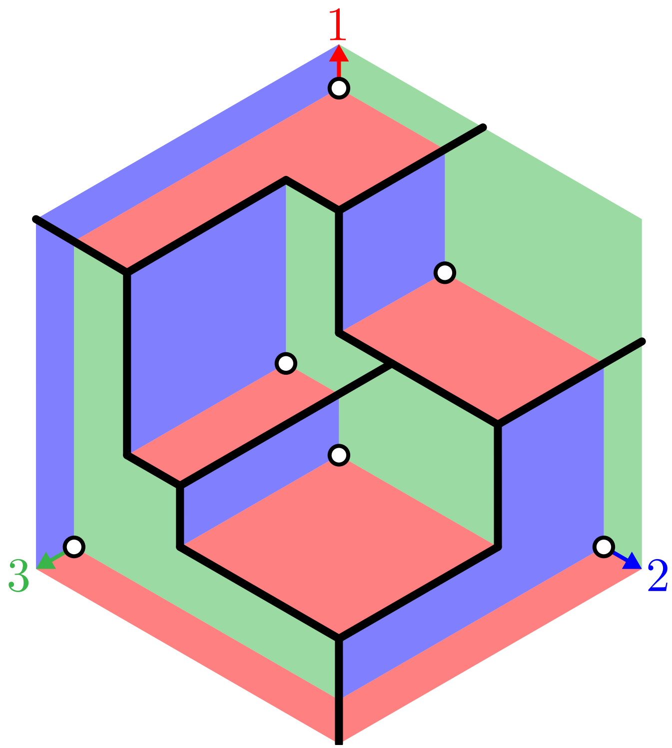


remark: TD is a quasi-distance, but is not symmetric.

GEODESIC EMBEDDINGS VS TD-DELAUNAY REALIZATIONS

PROP. Given a Schnyder wood W on a planar map M, the region vectors of the vertices of M with respect to W define a point-set whose TD-Delaunay triangulation is isomorphic to M.





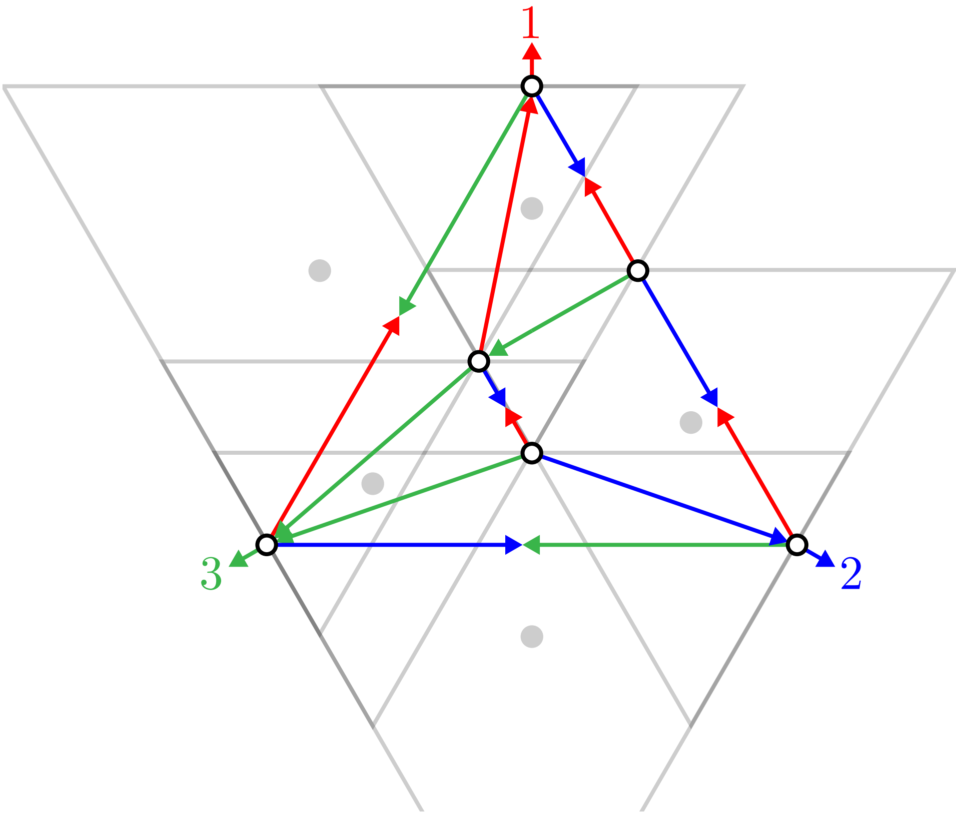
CORO. Any 3-connected planar graph admits a TD-Delaunay realization.

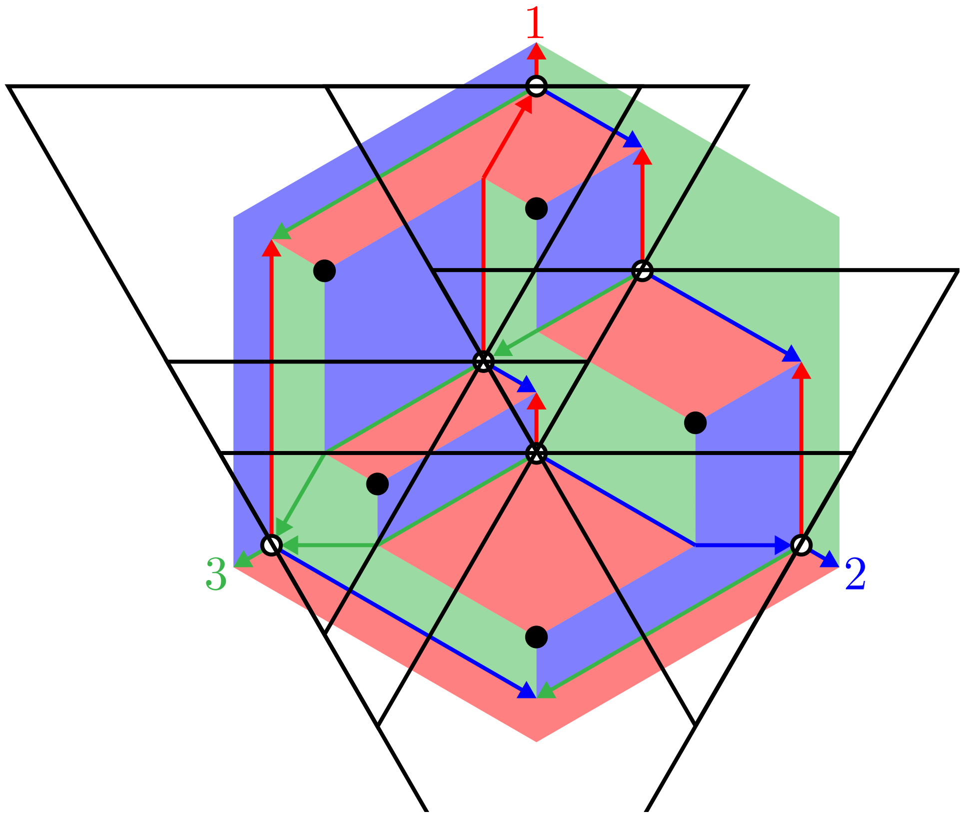
EMPTY REVERSED EQUILATERAL TRIANGLES

anti-standard equilateral triangle $\nabla = -\triangle$

PROP. For any points $\boldsymbol{p}$, $\boldsymbol{q}$ of $\boldsymbol{P}$ and any $\boldsymbol{Q} \subseteq \boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ there is an empty ∇ passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{Q}$ belongs to a face of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ the circumscribed ∇ of $\boldsymbol{Q}$ is empty.





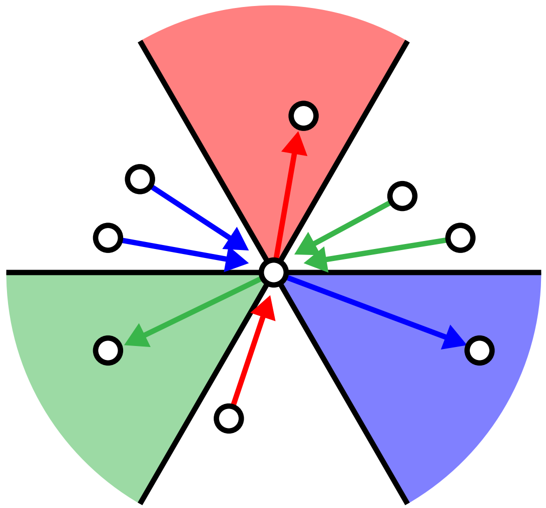
EMPTY REVERSED EQUILATERAL TRIANGLES

anti-standard equilateral triangle $\nabla = -\triangle$

PROP. For any points $\boldsymbol{p}$, $\boldsymbol{q}$ of $\boldsymbol{P}$ and any $\boldsymbol{Q} \subseteq \boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ there is an empty ∇ passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{Q}$ belongs to a face of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ the circumscribed ∇ of $\boldsymbol{Q}$ is empty.

PROP. In a TD-triangulation, the edges around a vertex look geometrically like



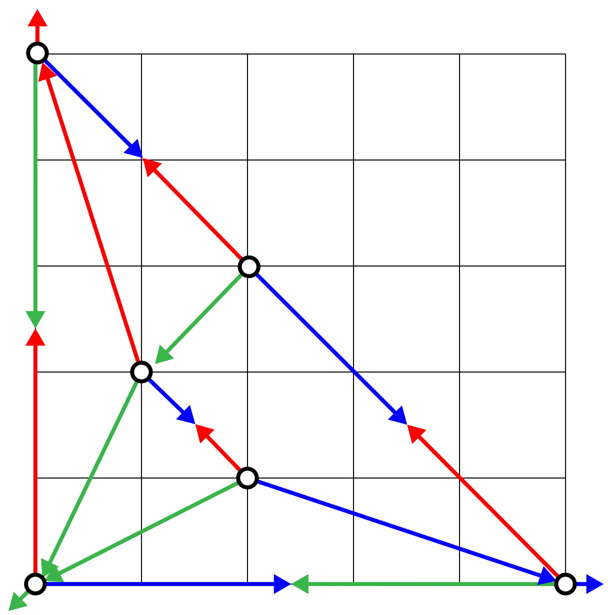
In particular, the paths $P_1(v)$, $P_1(v)$ and $P_1(v)$ stay in the red, blue and green angles.

EXISTENCE OF SCHNYDER WOODS

EXISTENCE

THM. Any 3-connected planar map admits a Schnyder wood.

CORO. Any 3-connected planar map with f faces admits a straight line embedding with vertices located on a $(f-1) \times (f-1)$ grid.



remark: Original proofs of Schnyder (for triangulations) and Felsner (for maps) based on edge contractions (difficult since contractions do not preserve 3-connectedness).
Here, proof for triangulations based on canonical orderings (a similar proof for arbitrary 3-connected planar maps is possible but more difficult).

CANONICAL ORDERING

M = triangulated planar map (except the external face)

DEF. canonical ordering of M = order on the vertices $v_1, \ldots, v_n$ such that for all $k \geq 3$, the submap M_k of M induced by $\{v_1, \ldots, v_k\}$ satisfies:

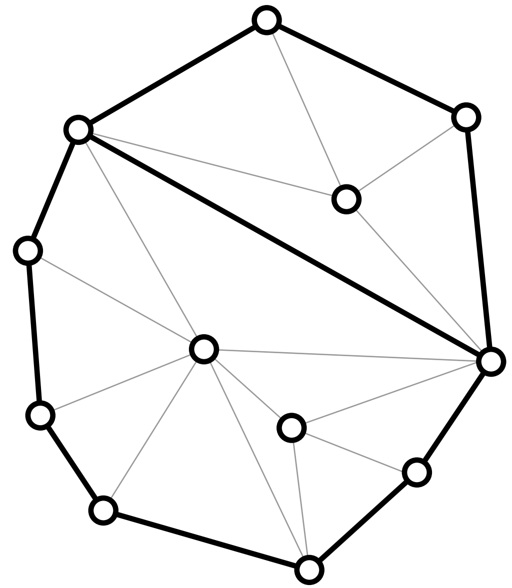
- M_k is connected and its boundary is a simple cycle,
- M_k is triangulated,
- v_{k+1} is in the outer face of M_k.

PROP. Any triangulated map admits a canonical ordering.

proof idea: start from M and delete a vertex on the outer face incident to only two other vertices of the outer face.

Such a vertex exists since:

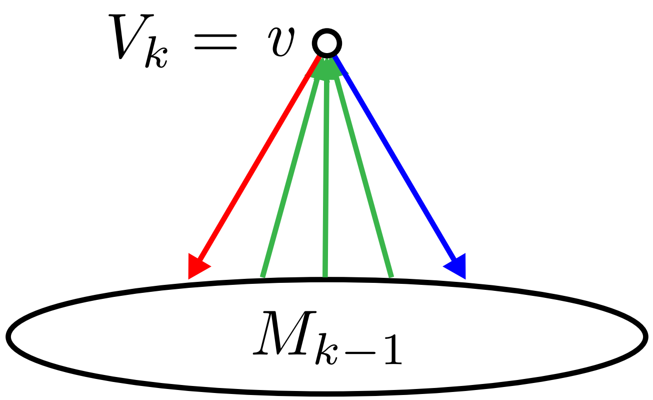
- either all vertices are valid,
- or there is a minimal length chord, separating at least a valid vertex.

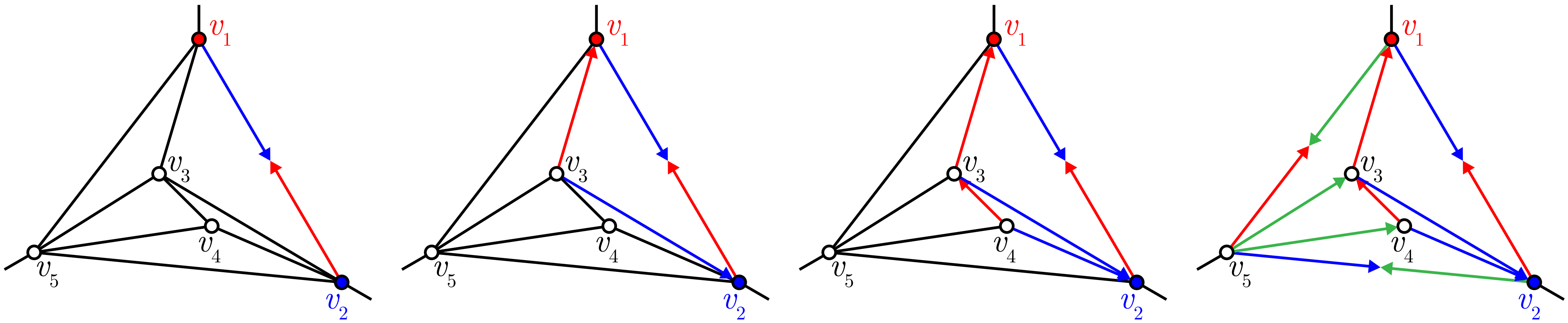


EXISTENCE FROM A CANONICAL ORDERING

PROP. Any triangulated map admits a canonical ordering.

PROP. A canonical ordering defines a Schnyder woods, using the local rule





THREE APPLICATIONS OF SCHNYDER WOODS

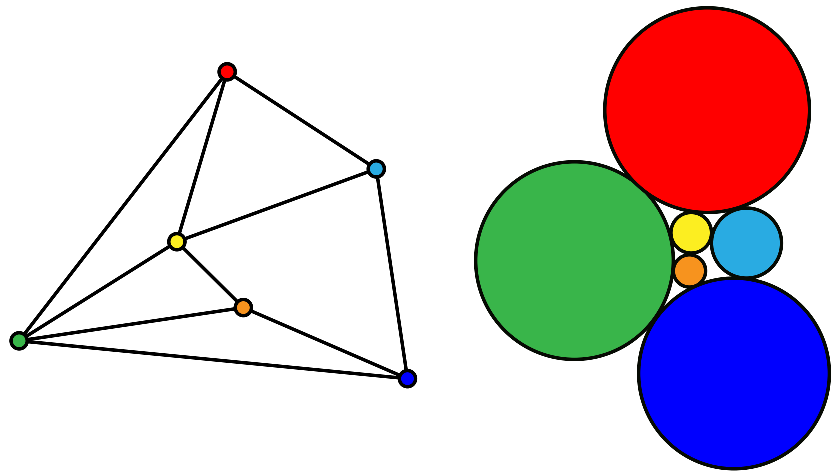
CONTACT REPRESENTATIONS

DEF. $\boldsymbol{X}$ = set of compact bodies whose interiors are pairwise disjoint.

contact graph of $\boldsymbol{X}$ = graph with

- vertices = bodies of $\boldsymbol{X}$
- edges = contacts between the bodies of $\boldsymbol{X}$.

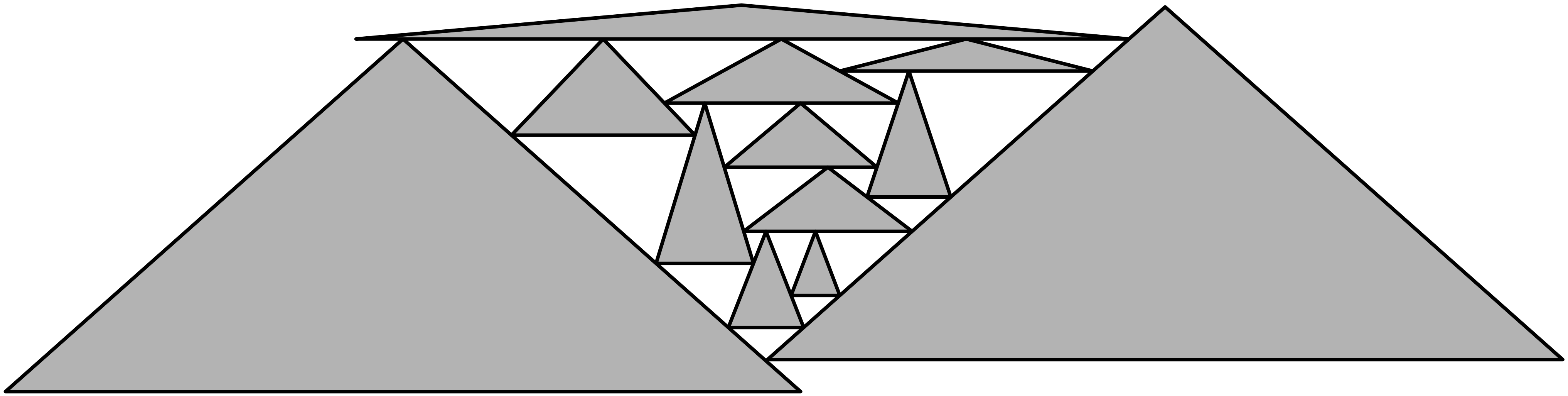
contact representation of G = set $\boldsymbol{X}$ whose contact graph is isomorphic to G.

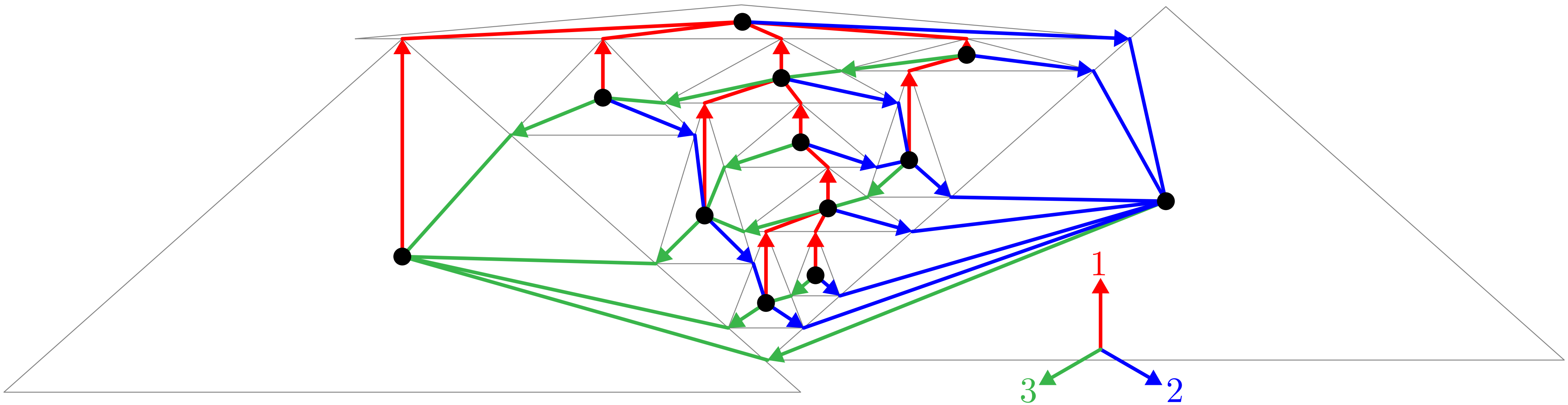


(img src: Wikipedia)

THM. (Circle packing) Any planar simple graph has a circle contact representation.

remark: in fact, the Koebe–Andreev–Thurston theorem says that this circle contact representation is unique up to MÃ¶bius transformations and reflections in lines.

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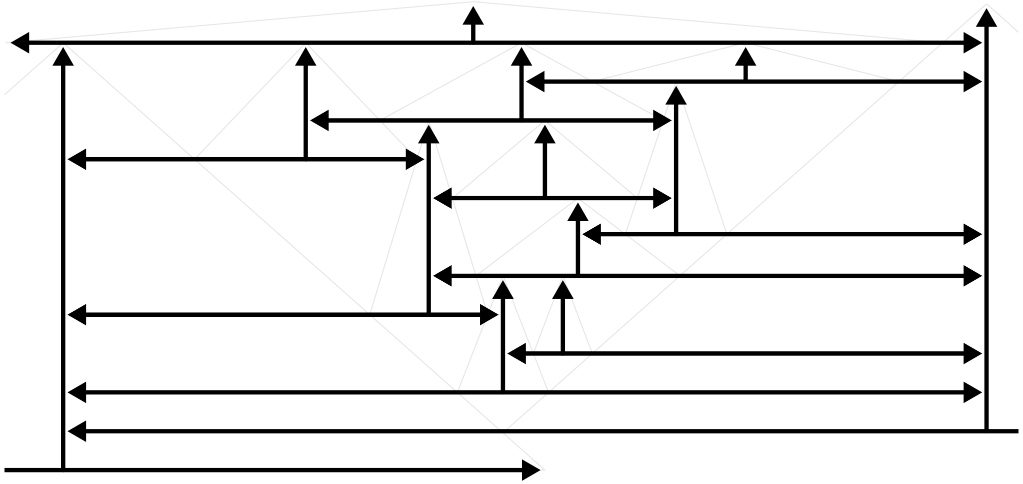
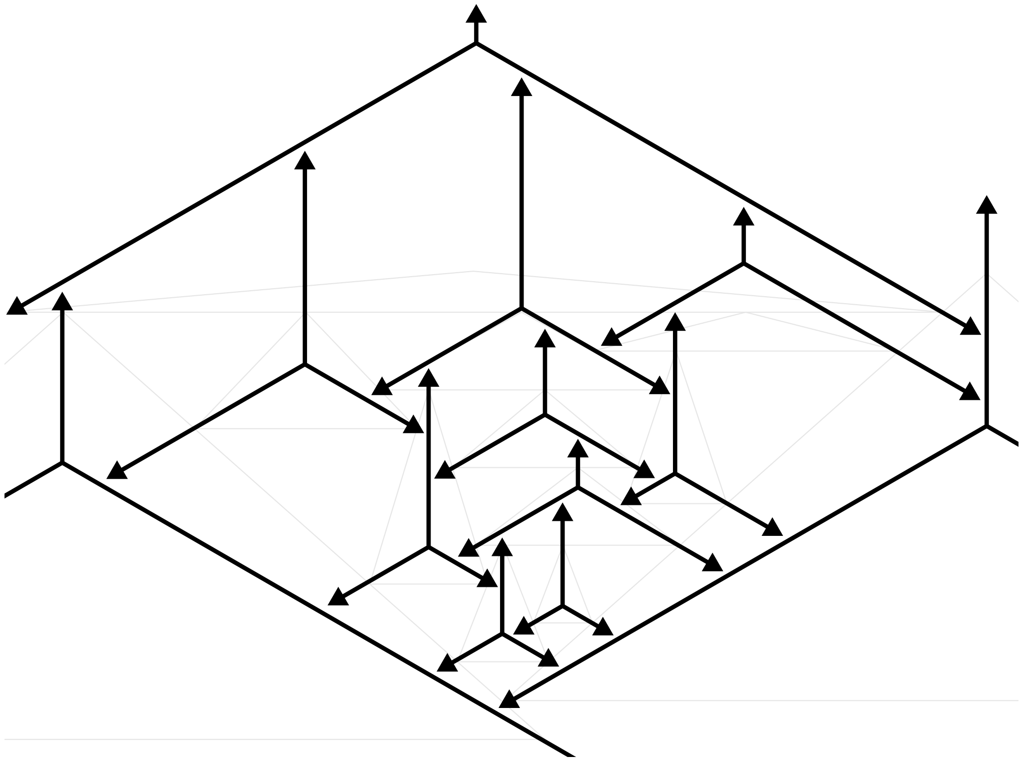
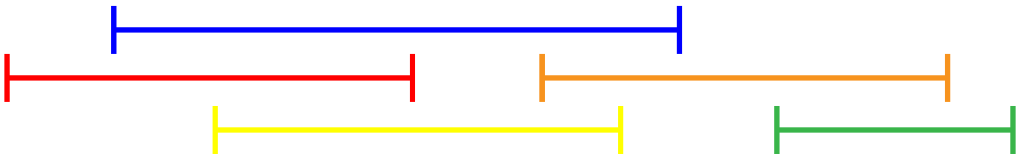
TRIANGLE-CONTACT REPRESENTATIONS

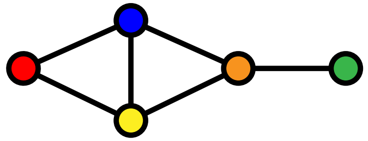




INTERVAL GRAPHS

DEF. interval graph $=$ intersection graph of intervals.



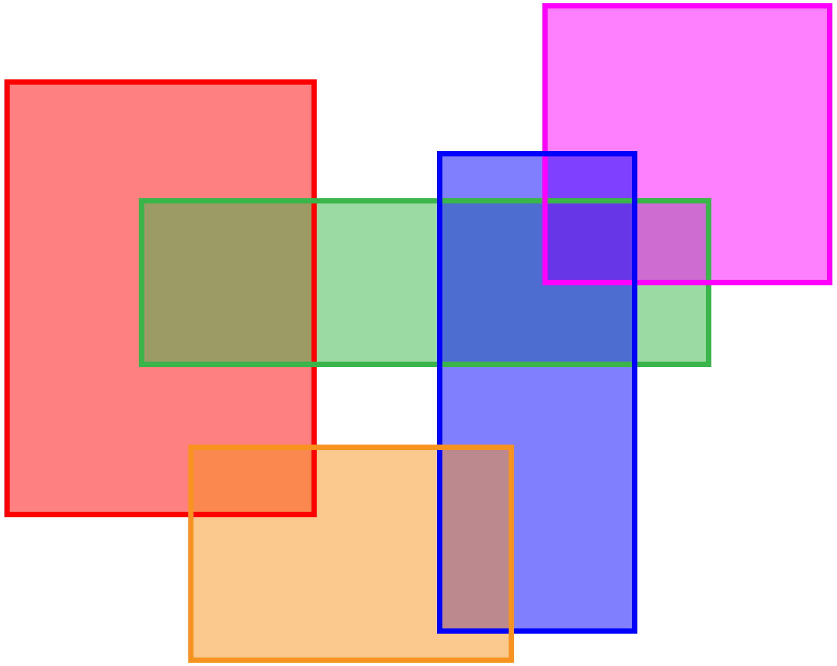


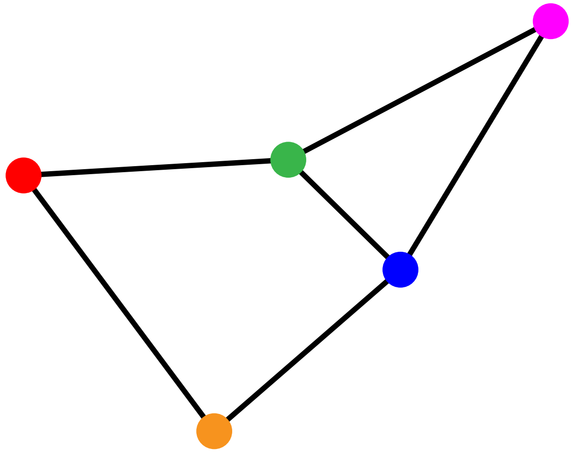
PROP. A graph $G = (V, E)$ is an interval graph if and only if

- all induced cycles are triangles,
- there is a partial order on V whose comparability graph is the complement of G.

BOXICITY

DEF. boxicity of G = smallest d such that there exists axis-parallel boxes in $\mathbb{R}^d$ whose intersection graph is isomorphic to G.



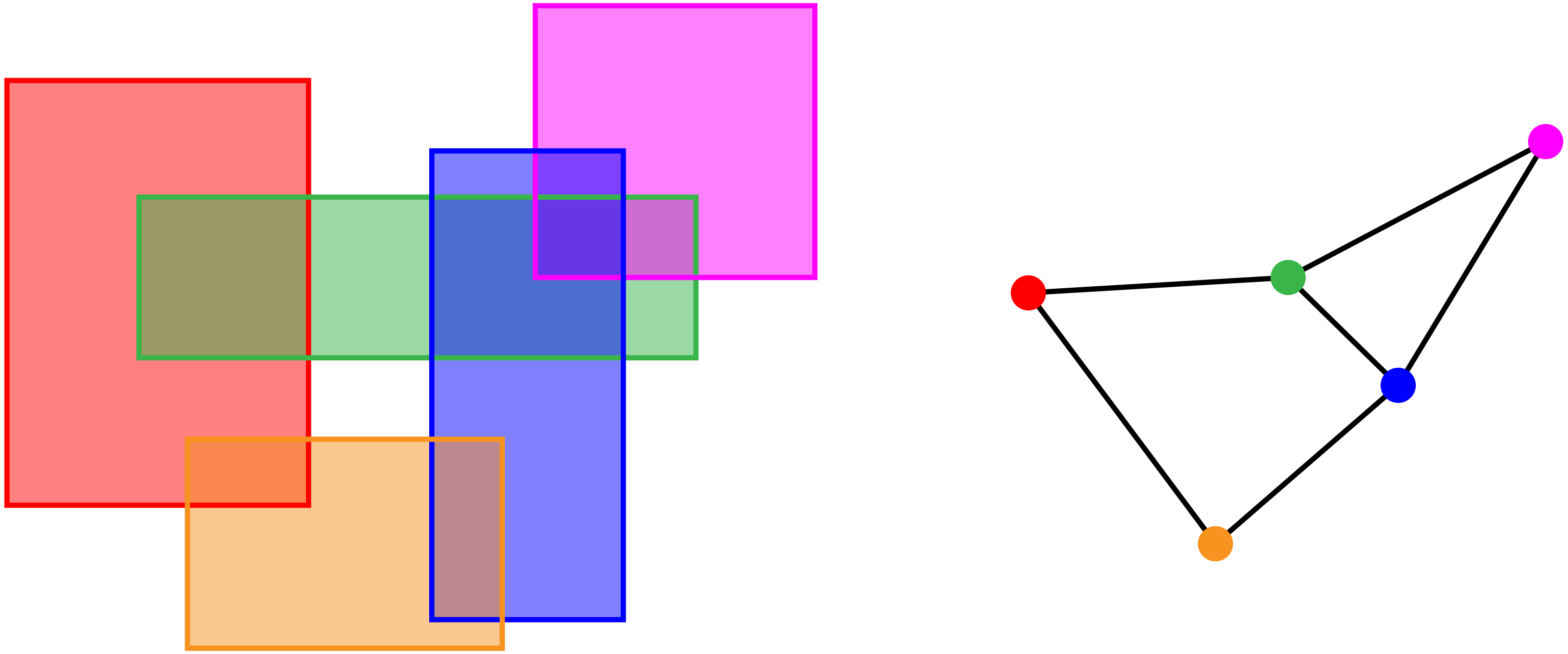


QU. What is the boxicity of

- a complete graph?
- a cycle of length at least 4?

BOXICITY

DEF. <u>boxicity</u> of G = smallest d such that there exists axis-parallel boxes in $\mathbb{R}^d$ whose intersection graph is isomorphic to G.



PROP. The boxicity of $G = (V, E)$ is the smallest d such that there exists d interval graphs $G_1 = (V, E_1), \ldots, G_d = (V, E_d)$ such that $E = E_1 \cap \cdots \cap E_d$.

PROP. The boxicity of $G = (V, E)$ is at most $|V|/2$.

BOXICITY

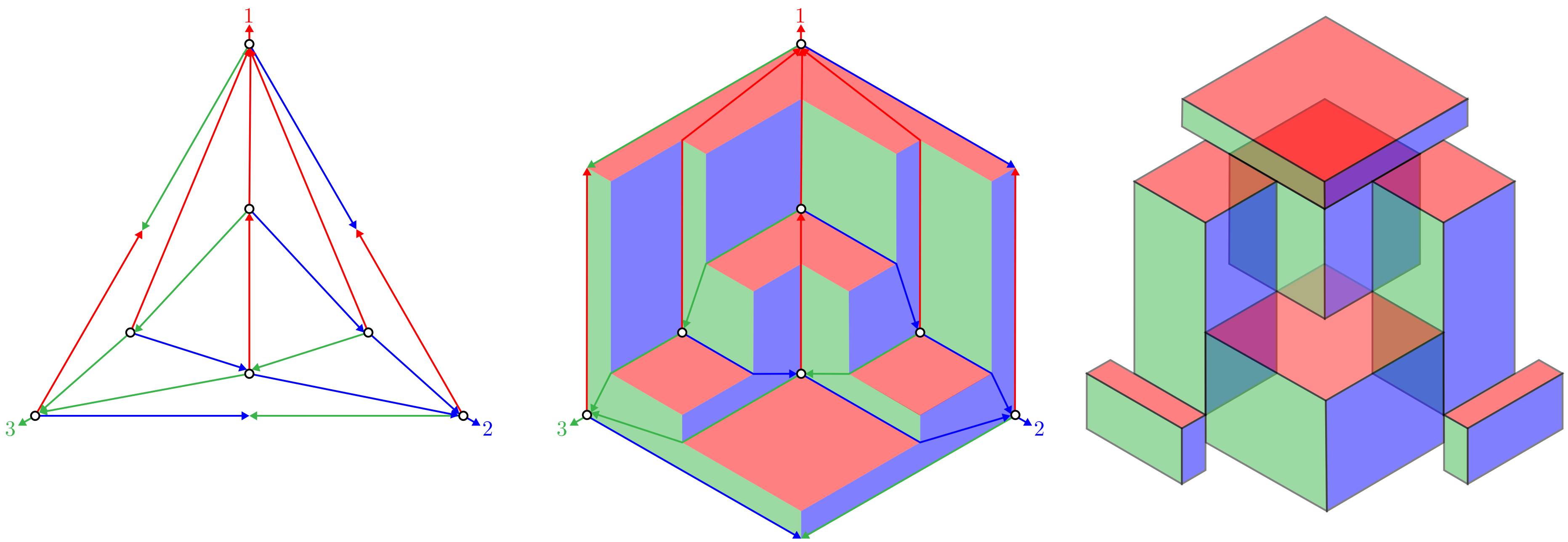
DEF. <u>boxicity</u> of G = smallest d such that there exists axis-parallel boxes in $\mathbb{R}^d$ whose intersection graph is isomorphic to G.

THM. Any planar graph has boxicity 3.

<u>remark:</u> initially proved by Thomassen with a different method.

<u>proof idea:</u>

- enough to consider triangulations,
- use Schnyder woods and geodesic embeddings.



GEOMETRIC SPANNERS

$G = (V, E)$ graph weighted by $\omega : E \to \mathbb{R}_{>0}$.
weight of a path $e_1, \dots, e_k = \sum_{i \in [k]} \omega(e_i)$.
$d_G(u, v) =$ minimum weight of a path between u and v in G.

exm: $G = (V, E)$ geometric graph and $\omega(u, v) = \|u - v\|$.

DEF. t-spanner of $G =$ subgraph H of G such that $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$.
stretch factor of $H =$ smallest factor t such that H is a t-spanner of G.
geometric spanner $=$ spanner of the complete geometric graph.

GEOMETRIC SPANNERS

DEF. t-spanner of G = subgraph H of G such that $d_H(u,v) \leq t \cdot d_G(u,v)$ for all $u, v \in V$.
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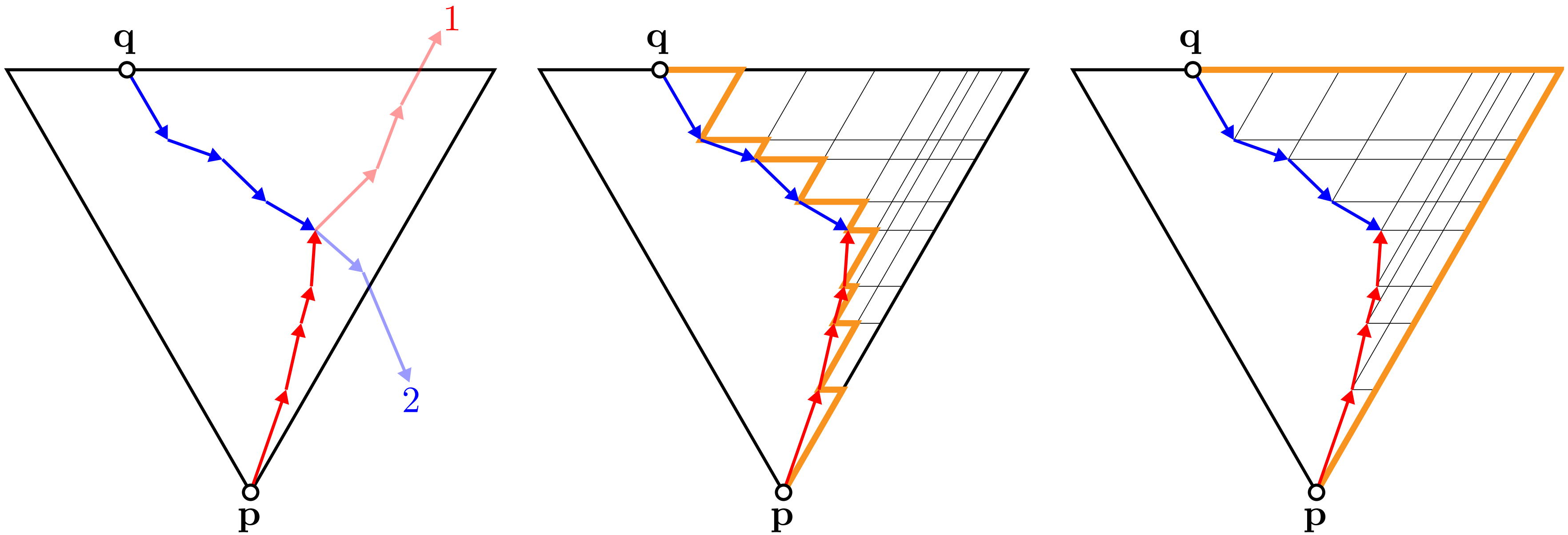
geometric spanner = spanner of the complete geometric graph.

THM.

- The complete geometric graph is a 1-spanner.
- The Delaunay triangulation is a t-spanner for $(\pi/2 <)\ 1.5846 < t < 1.998\ (< 2)$.
- The TD-Delaunay is a 2-spanner.

proof idea: for the TD-triangulation



GEOMETRIC SPANNERS

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- The complete geometric graph is a 1-spanner.
- The Delaunay triangulation is a t-spanner for $(\pi/2 <)\ 1.5846 < t < 1.998\ (< 2)$.
- The TD-Delaunay is a 2-spanner.

DEF. For $i \in [3]$ and $\boldsymbol{p} \in \boldsymbol{P}$, denote by

- parent$_i(\boldsymbol{p})$ = target of the unique outgoing edge of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P})$ colored by i.
- children$_i(\boldsymbol{p})$ = all points $\boldsymbol{q} \in \boldsymbol{P}$ such that $\boldsymbol{p} =$ parent$_i(\boldsymbol{q})$.
- closest$_i(\boldsymbol{p})$ = point of children$_i(\boldsymbol{p})$ closest to $\boldsymbol{p}$ for the triangular distance.
- first$_i(\boldsymbol{p})$ and last$_i(\boldsymbol{p})$ = first and last points of children$_i(\boldsymbol{p})$ clockwise around $\boldsymbol{p}$.

THM. (Bonichon, Gavoille, Hanusse, and Perkovic)
The subgraph of the TD-Delaunay triangulation $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P})$ obtained by erasing at each vertex $\boldsymbol{p}$ all incoming arcs except the arcs first$_i(\boldsymbol{p})$, last$_i(\boldsymbol{p})$ and closest$_i(\boldsymbol{p})$ for $i \in [3]$ (if they exist) is a planar 6-spanner with degree at most 12.

SOME REFERENCES

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- Stefan Felsner. *Geometric graphs and arrangements*.
  Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Wiesbaden, 2004.
- Stefan Felsner. *Lattice Structures from Planar Graphs*.
  Electron. J. Combin. 11, #R15, 2004.
- Walter Schnyder. *Planar graphs and poset dimension*.
  Order, 5(4):323–343, 1989.