Combinatoire des polytopes
Examen du 21/02/2019

Exercice 1 (Trivalent vertices and faces)**.**
(1) Show that for any 3-dimensional polytope with v_3 vertices of degree 3 and f_3 facets of degree 3 (*i.e.* triangles), we have the inequality $v_3 + f_3 \geq 8$.
(2) Give examples of 3-dimensional polytopes with $(v_3, f_3) = (8, 0), (6, 2), (4, 4), (2, 6)$ and $(0, 8)$.
(3) Can the other pairs (v_3, f_3) with $v_3 + f_3 = 8$ be reached?

Exercice 2 (Gram formula for angles)**.** Consider a d-dimensional polytope P, a face F of P, and a sufficiently small ball B_F centered at a point in the relative interior of F. We call the *solid angle* of P at F the fraction α_F of B_F that is contained in P. We denote by α_i the sum of the solid angles of P at its i-dimensional faces. We want to prove the following analogue of Euler's formula for solid angles:

$$\sum_{i=0}^{d} (-1)^i \alpha_i = 0.$$

(1) Show that this formula is equivalent to

$$\sum_{i=0}^{d-2} (-1)^i \alpha_i = (-1)^d \big(f_{d-1}/2 - 1 \big)$$

where f_{d-1} is the number of $(d-1)$-dimensional faces of P.

(2) Show the result for a 2-dimensional polytope P.

(3) Consider now a 3-dimensional polytope P. Choose a random direction $\overline{u}$ on the 2-dimensional sphere and project P orthogonally to this direction $\overline{u}$ to a polygon $P_{\overline{u}}$.
- What is the probability that a vertex v of P does not project to a vertex in the projected polygon $P_{\overline{u}}$ in terms of the solid angle of P at v?
- Deduce the expected number of vertices of the projected polygon $P_{\overline{u}}$.
- What is the expected number of edges of the projected polygon $P_{\overline{u}}$?
- Using these expectations, show that $\alpha_0 - \alpha_1 = -f_2/2 + 1$.

(4) Extend this method to any dimension d.

Exercice 3 (A 4-dimensional polytope with a non-prescribable 2-face)**.**

(1) Consider a polytope P with *vertex-facet incidence graph* $\mathcal{I}$. In other words, $\mathcal{I}$ is the bipartite graph whose nodes are the vertices of P and the facets of P, and with an arc from a vertex v to a facet F if and only if v belongs to F.
- Show that the faces of P are in bijection with the maximal complete bipartite subgraphs of $\mathcal{I}$, *i.e.* with inclusion maximal pairs $(\mathcal{V}, \mathcal{F})$ where $\mathcal{V}$ is a subset of vertices of P and $\mathcal{F}$ is a subset of facets of P such that $v \in F$ for any $v \in \mathcal{V}$ and $F \in \mathcal{F}$.
- Deduce that the face lattice of P is completely determined by the vertex-facet incidences of P.

(2) Consider the Schlegel diagram of a 4-dimensional polytope Q on the left of Figure 1.
- What is the number of vertices and facets of Q?
- List all facets of Q (for each facet F, just list the vertices of F in alphabetical order). Label these facets from 1 to 8 in lexicographic order.

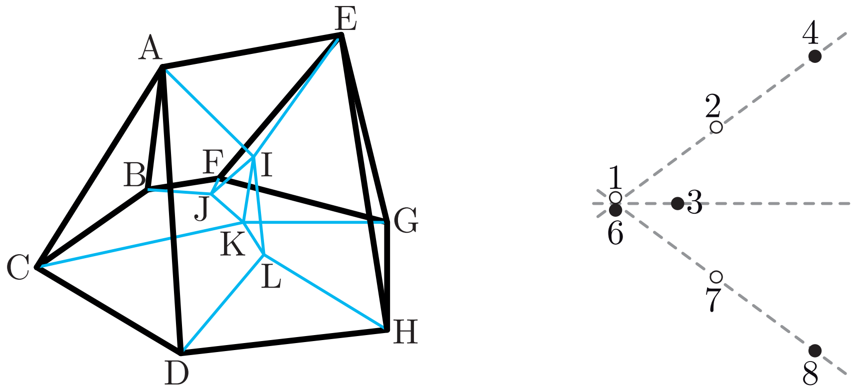

Figure 1: A Schlegel diagram (left) and a Gale diagram (right).

(3) Consider the planar affine Gale diagram G of a polytope R on the right of Figure 1.

- What is the dimension and the number of vertices of R?
- List all circuits C of G for which $C_4 \neq 0$ and $C_6 \neq 0$.
- List all cocircuits X of G for which $X_1 = 0$.
- What are the facets of R?

(4) Show that the face lattices of the polytopes Q and R are opposite.

(5) Show that the polytope Q has an hexagonal 2-dimensional face whose geometry cannot be prescribed, meaning that there are hexagons which cannot appear as a 2-face of any polytope combinatorially equivalent to Q. For this, prove that

- any convex hexagon with alternating black and white vertices is the affine Gale diagram of a polytope,
- for any polytope combinatorially equivalent to R, the three lines passing through the vertices 2 and 4, through the vertices 3 and 5, and through the vertices 7 and 8, of the Gale diagram G must be concurrent,
- the iterated vertex figure $(R/v_1)/v_2$ cannot be prescribed for R,
- and conclude by polarity.

(6) We recall from TD F that a polytope is neighborly if and only if its Gale diagram is balanced, meaning that there are at least $\lfloor \frac{n-r+1}{2} \rfloor$ vectors on each side of any hyperplane spanned by $r-1$ vectors (where r is the dimension of G). Show that any convex polygon with alternating black and white vertices is the affine Gale diagram of a neighborly polytope. What are the dimension, the number of vertices and the number of facets of this polytope?

Exercice 4 (Realization space of a polytope)**.** Let $\overline{v}_1, \dots, \overline{v}_d$ be d affinely independent points in $\mathbb{R}^d$, and H be the hyperplane they span.

(1) Given a point $\overline{p} \in \mathbb{R}^d$, how can you check (algebraically) whether $\overline{p} \in H$, and if $\overline{p} \notin H$, in which of the open halfspaces defined by H does $\overline{p}$ lie?

(2) Given $\overline{p}, \overline{q} \in \mathbb{R}^d$, how can you check (algebraically) whether $\overline{p}$ and $\overline{q}$ lie in the same open halfspace defined by H?

(3) Prove that the realization space of a polytope is a primary basic semialgebraic set.

(4) (If you did not already do it in the previous point.) Prove that the realization space of a (simplicial) d-dimensional polytope is a primary basic semialgebraic set defined by polynomials of degree at most d.

(Do it only for simplicial polytopes if you find it easier.)

Exercice 5 (Beneath-beyond and realization spaces of stacked polytopes)**.**

(1) Let F be a face of a d-dimensional polytope $P \subseteq \mathbb{R}^d$. Consider the set $\mathcal{N}_F$ of the vectors $(\overline{a}, b) \in \mathbb{R}^{d+1}$ such that $\langle \overline{a} \mid \overline{x} \rangle = b$ for all $\overline{x} \in F$ and $\langle \overline{a} \mid \overline{x} \rangle \leq b$ for all $\overline{x} \in P$. Show that $\mathcal{N}_F$ is a polyhedral cone. What are its generating rays?

(2) Let P be a d-dimensional polytope, let $\overline{q} \in \mathbb{R}^d \smallsetminus P$ and let $Q := \operatorname{conv}(P \cup \{\overline{q}\})$. Prove that every face G of Q is either a face of P or the convex hull of the union of a face of P with $\{\overline{q}\}$.

(3) Let P be a d-dimensional polytope, let $\overline{q} \in \mathbb{R}^d \smallsetminus P$ and let $Q := \operatorname{conv}(P \cup \{\overline{q}\})$. Let H be a supporting hyperplane such that $P \subset \overline{H}^-$. We say that $\overline{q}$ is beneath / on / beyond H if $\overline{q}$ is in H^- / H / H^+, respectively. If F is a facet of P, we say that $\overline{q}$ is beneath / on / beyond F if it is beneath / on / beyond its supporting hyperplane H (oriented so that $P \subset \overline{H}^-$).

Prove that a facet F of P is also a facet of Q if and only if $\overline{q}$ is beneath F.

(4) Let P be a d-dimensional polytope, let $\overline{q} \in \mathbb{R}^d \smallsetminus P$, let $Q := \operatorname{conv}(P \cup \{\overline{q}\})$, and let G be a face of P. Prove that

- G is a face of Q if and only if there is a facet F of P, with $G \subseteq F$, such that $\overline{q}$ is beneath F.
- $\operatorname{conv}(G \cup \{\overline{q}\})$ is a face of Q if and only if
  (i) either $\overline{q} \in \operatorname{aff}(G)$ (equivalently, $\overline{q}$ is on every facet of P containing G),
  (ii) or $\overline{q}$ is beneath at least one of the facets of P containing F and beyond at least one of the facets of P containing F.

(5) Let G be a face of P. Show that there is a point $\overline{q}$ beyond all the facets of P containing G and beneath all the facets of P not containing G. We then say that the polytope $\operatorname{conv}(P \cup \{\overline{q}\})$ is obtained from P by *stacking* a vertex over G.

(6) Let P be a d-dimensional polytope with vertex set $V = \{\overline{v}_1, \dots, \overline{v}_n\}$. Is the combinatorial type of $\operatorname{conv}(V \smallsetminus \{\overline{v}_n\})$ always determined by the combinatorial type of P?

(7) A *stacked* polytope is a polytope obtained from a simplex by iterative stacking operations over arbitrary facets. Show that the realization spaces of stacked polytopes are trivial (stably equivalent to a point).