Combinatoire des polytopes
TD C – Relations on f-vectors and extremal polytopes

1 Relations between f-vectors and h-vectors

Consider a d-dimensional polytope P. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a linear functional such that $\phi(v) \neq \phi(w)$ for any adjacent vertices v, w of P, and orient the edges of P in increasing values of P. The *h-vector* of P is the vector $(h_0(P), h_1(P), \dots, h_d(P))$ where $h_j(P)$ is the number of vertices of indegree j in the graph of P oriented by ϕ. Its *h-polynomial* is the polynomial $h(P, x) \coloneqq \sum_{j=0}^{d} h_j(P) x^j$.

Exercice 1 (h-vectors of the simplex and the cube)**.** What are the h-vectors and h-polynomials of the d-dimensional simplex and cube?

Exercice 2 (f- to h-vector transformation)**.** Show that for two sequences $(f_i)_{0 \le i \le d}$ and $(h_j)_{0 \le j \le d}$,

$$\forall i, \quad f_i = \sum_{j=0}^{d} \binom{j}{i} h_j \qquad \Longleftrightarrow \qquad \forall j, \quad h_j = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i.$$

In other words, the matrix $\left[\binom{j}{i}\right]_{0 \le i,j \le d}$ is invertible and its inverse is $\left[(-1)^{i+j}\binom{i}{j}\right]_{0 \le i,j \le d}$.

[*Hint: Consider the corresponding counting polynomials $f(x) \coloneqq \sum_{i=0}^{d} f_i x^i$ and $h(x) \coloneqq \sum_{j=0}^{d} h_j x^j$ and show the relation $f(x) = h(x+1)$.*]

2 Simple polytopes

Exercice 3 (Simple and simplicial)**.** Show that a polytope that is both simple and simplicial is either a simplex or a polygon.

Exercice 4 (Simple polytope from its graph)**.** The objective of this exercice is to show the following statement due to Blind and Mani-Levitska, using the elegant proof of Kalai:

Two simple polytopes with isomorphic graphs are combinatorially equivalent.

In other words, we want to prove that the graph of a simple polytope P gives enough information to determine which subsets of vertices define faces of P. For this, we will use certain acyclic orientations of the graph of P. We say that an acyclic orientation of the graph of P is *good* if the graph of each face of P has a unique sink. The proof is decomposed into two steps:

(1) Good acyclic orientations of P can be recognized from the graph of P. For any acyclic orientation O of the graph of P, denote by $h_j(O)$ the number of vertices of P with indegree j for O, and define $F(O) \coloneqq h_0(O) + 2\, h_1(O) + \cdots + 2^d\, h_d(O)$. Prove that good acyclic orientations are precisely those which minimize $F(O)$.

(2) Faces of P can be determined from all good acyclic orientations of P. Prove that a regular induced subgraph of the graph of P is the graph of a face of P if and only if its vertices are initial with respect to some good acyclic orientation of P.

Exercice 5 (Induced cycles and faces)**.** Prove that all induced cycles of length 3, 4 and 5 in the graph of a simple d-dimensional polytope P are graphs of 2-dimensional faces of P. Is it still true for cycles of length 6? [*Hint: 3-dimensional cube.*]

3 Neighborly polytopes

Exercice 6 (A small neighborly polytope)**.** Let $P := (\triangle_2 \times \triangle_2)^\diamond = \triangle_2 \oplus \triangle_2$. Describe the vertices and the edges of P. Deduce that P is 2-neighborly.

Exercice 7 (Subgraphs of 4-polytopes)**.** Show that every graph is an induced subgraph of the graph of a 4-dimensional polytope.

[*Hint: start from a cyclic polytope and stack vertices on undesired edges.*]

Exercice 8 (Gale's evenness criterion)**.** Consider the moment curve $\mu_d : t \mapsto (t, t^2, t^3, \dots, t^d)$ and the cyclic polytope $C_d(n) = \operatorname{conv}\{\mu_d(t_i) \mid i \in [n]\}$ for some fixed $t_1 < t_2 < \cdots < t_n$. We identify a d-subset $F \subset [n]$ with the point set $\{\mu_d(t_i) \mid i \in F\}$. Call *block* of $F \in [n]$ the intervals of F, and say that a block is *internal* if it does not contain 1 or n.

(1) Show that a point $\mu_d(t_k)$ is located on one or the other side of the affine hyperplane passing through $\{\mu(t_{i_1}), \dots, \mu(t_{i_d})\}$ according to the sign of the Vandermonde determinant

$$\det \begin{bmatrix} 1 & \dots & 1 & 1 \\ t_{i_1} & \dots & t_{i_d} & t_k \\ \vdots & \ddots & \vdots & \vdots \\ t_{i_1}^d & \dots & t_{i_d}^d & t_k^d \end{bmatrix}.$$

(2) Remind and prove the product formula for this determinant.

(3) Deduce that a d-subset F of $[n]$ defines a facet of $C_d(n)$ if and only if all internal blocks have even size (*Gale's evenness criterion*).

(4) Deduce the following facts on cyclic polytopes:
   (a) $C_d(n)$ is neighborly.
   (b) All cyclic polytopes of dimension d with n vertices are combinatorially equivalent.
   (c) The number of facets of $C_d(n)$ is

$$f_{d-1}\big(C_d(n)\big) = \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor}.$$

   [*Hint: Prove first that the number of ways to choose a $2k$-subset of $[n]$ such that all blocks are even is $\binom{n-k}{k}$. To obtain the formula, distinguish the cases when the first block is even or odd.*]