Combinatoire des polytopes
TD A – Basic notions

1 High dimension is counter-intuitive

Exercice 1 (Cochonnet paradox)**.** Consider a box to store your “pétanque” blue balls with a place in the middle for the red “cochonnet”, as illustrated in Figure 1.

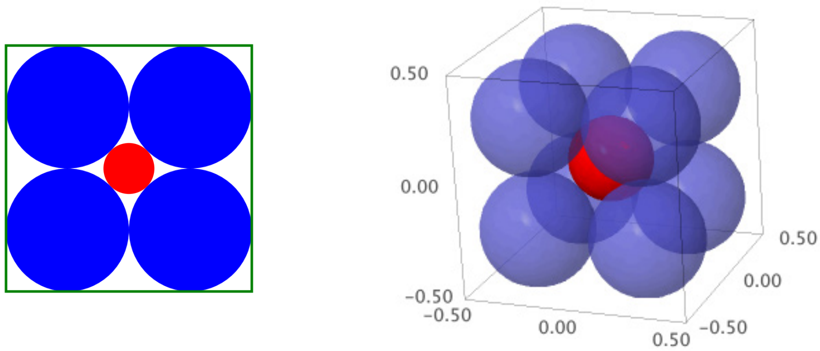

Figure 1: Placing the pétanque balls and cochonnet into the box in dimension 2 and 3.

(1) Compute the radius and area of the red cochonnet.

(2) What would be the radius and volume of the red cochonnet in dimension d?
[*Hint: Along the long diagonal, one can fit 2 blue balls and 2 red cochonnets. The volume* V_d *of the d-dimensional unit ball is given by*

$$V_{2\delta} = \frac{\sqrt{\pi}^{\delta}}{\delta!} \qquad \textit{and} \qquad V_{2\delta+1} = \frac{\sqrt{\pi}^{\delta} \cdot 2^{2\delta+1} \cdot \delta!}{(2\delta+1)!}.$$

If you never did this computation, consider the functions

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt \qquad \textit{and} \qquad B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dx dy,$$

show that $\Gamma(x+1) = x\Gamma(x)$*, that* $\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x,y)$*, that* $\Gamma(1/2) = \sqrt{\pi}$*, and that the volume* V_d *satisfies the recurrence relation* $V_{d+1} = V_d \cdot B(d/2+1, 1/2)$ *and conclude.*]

(3) What happens in dimension 10?

2 Convexity

Exercice 2 (Three convexity theorems)**.**

(1) (Radon's theorem). Show that any set A of $d+2$ points in $\mathbb{R}^d$ admits two disjoint subsets $A_1, A_2 \subset A$ such that

$$\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \varnothing.$$

(2) (Helly's theorem). Let $C_1, \dots, C_n$ be n convex sets in $\mathbb{R}^d$, with $n \geq d+1$. Show that if the intersection of every $d+1$ of these sets is non-empty, then the intersection of all the C_i is non-empty.
[*Hint: Use induction on n and Radon's theorem.*]

(3) (Centerpoint theorem). Let $X \subset \mathbb{R}^d$ be a set of n points. A point $\overline{x} \in \mathbb{R}^d$ is a *centerpoint* of X if each closed half-space containing $\overline{x}$ contains at least $\frac{n}{d+1}$ points of X. Prove that each finite point set in $\mathbb{R}^d$ has at least one centerpoint.

[*Hint: For each closed half-space* $\bar{H}^+$ *such that* $|\bar{H}^+ \cap X| > \frac{d}{d+1}n$, *consider* $\operatorname{conv}(\bar{H}^+ \cap X)$, *and finish using Helly's theorem.*]

3 Fourier-Motzkin elimination

Exercice 3 (Fourier-Motzkin elimination for polyhedra)**.** The objective of this exercise is to provide an algorithmic proof that an affine projection of a polyhedron is a polyhedron. This enables to show that a V-polyhedron is an H-polyhedron since a V-polyhedron

$$\operatorname{conv}(V) + \operatorname{cone}(Y) = \left\{ \overline{x} \in \mathbb{R}^d \;\middle|\; \exists\, \overline{t} \in \mathbb{R}^n, \in \mathbb{R}^m \text{ such that } \overline{t} = 1,\, \overline{t} \geq \overline{0},\, \overline{u} \geq \overline{0} \text{ and } \overline{x} = V\overline{t} + Y\overline{u} \right\}$$

can be interpreted as the projection of the H-polyhedron

$$\left\{ (\overline{x}, \overline{t}, \overline{u}) \in \mathbb{R}^{d+n+m} \;\middle|\; \overline{t} = 1,\, \overline{t} \geq \overline{0},\, \overline{u} \geq \overline{0} \text{ and } \overline{x} = V\overline{t} + Y\overline{u} \right\}.$$

(1) Let $Q = \{t \in \mathbb{R} \mid a_i t \leq b_i \text{ for } i \in [m]\}$ be a polyhedron on the real line with $a_i, b_i \in \mathbb{R}$ for $i \in [m]$. Give a constructive way to check if $Q = \varnothing$.
(2) Let $\pi_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the coordinate projection $\pi_d(x_1, \dots, x_{d-1}, x_d) = (x_1, \dots, x_{d-1})$. Let $Q = \left\{ \overline{x} \in \mathbb{R}^d \;\middle|\; \langle\, \overline{a}_i \mid \overline{x} \,\rangle \leq b_i \text{ for } i \in [m] \right\}$ be a polyhedron, with $\overline{a}_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$ for $i \in [m]$. For $\overline{y} \in \mathbb{R}^{d-1}$ define $Q_{\overline{y}} \coloneqq \{\overline{x} \in \mathbb{R} \mid (\overline{y}, x) \in Q\}$. Show that for all $\overline{y} \in \mathbb{R}^{d-1}$, the set $Q_{\overline{y}}$ is a polyhedron and give an explicit inequality description in terms of the inequality description of Q.
(3) Argue (using (1)) that the image $\pi_d(Q) = \left\{ \overline{y} \in \mathbb{R}^{d-1} \;\middle|\; Q_{\overline{y}} \neq \varnothing \right\}$ is a polyhedron.
(4) Conclude that the image of a polyhedron by an affine map is a polyhedron.

4 Examples of polyhedral cones

Exercice 4 (Incidence configuration of an directed graph)**.** The *incidence configuration* of a directed graph $G = (V, E)$ is the vector configuration $I(G) \coloneqq \{\overline{e}_w - \overline{e}_v \mid (v, w) \in E\} \subset \mathbb{R}^V$. Show that
(1) $I(G)$ is independent if and only if G has no (not necessarily oriented) cycle, that is, if G is a forest,
(2) $I(G)$ spans the hyperplane $\mathbb{H} \coloneqq \left\{ \overline{x} \in \mathbb{R}^V \;\middle|\; \langle\, \overline{1} \mid \overline{x} \,\rangle = 0 \right\}$ if and only if G is connected,
(3) $I(G)$ forms a basis of the hyperplane $\mathbb{H}$ if and only if G is a spanning tree.

Exercice 5 (Cones from directed graphs)**.** The *incidence cone* of a directed graph $G = (V, E)$ is the polyhedral cone $C(G) \coloneqq \mathbb{R}_{\geq 0} I(G) = \mathbb{R}_{\geq 0} \{\overline{e}_w - \overline{e}_v \mid (v, w) \in E\} \subset \mathbb{R}^V$.
(1) What is the polar cone of $C(G)$?
(2) What is the dimension of $C(G)$?
(3) When is $C(G)$ is a pointed cone?
(4) When $C(G)$ is pointed, describe the rays of $C(G)$. When is $C(G)$ a simplicial cone?
(5) Show that the facets of $C(G)$ correspond to minimal directed cuts of G.
(6) More generally, show that the k-dimensional faces of $C(G)$ correspond to subgraphs H of G with $|V| - k$ connected components and such that the quotient directed graph G/H is acyclic.

Exercice 6 (Half-space containement)**.** Let $P \coloneqq \left\{ \overline{x} \in \mathbb{R}^d \;\middle|\; \langle\, \overline{a}_i \mid \overline{x} \,\rangle \leq b_i \text{ for } i \in [m] \right\}$ be a non-empty polyhedron, where $\overline{a}_i \in (\mathbb{R}^d)^*$ and $b_i \in \mathbb{R}$, for $i \in [m]$. Show that, for $\overline{a} \in (\mathbb{R}^d)^*$ and $b \in \mathbb{R}$, the inequality $\langle\, \overline{a} \mid \overline{x} \,\rangle \leq b$ holds for each $\overline{x} \in P$ if and only if there are reals $\lambda_i \geq 0$, for $i \in [m]$, such that $\overline{a} = \sum_{i \in [m]} \lambda_i \overline{a}_i$ and $b \geq \sum_{i \in [m]} \lambda_i b_i$.

5 Examples of polytopes

Exercice 7 (Matching polytope)**.** The *matching polytope* $M(G)$ of a graph $G = (V, E)$ is defined as the convex hull of the characteristic vectors $\chi_M \in \mathbb{R}^E$ of all matchings M on G.

(1) Show that the matching polytope is contained in the polytope $N(G)$ defined by

$$x_e \geq 0 \quad \text{for all } e \in E, \qquad \text{and} \qquad \sum_{e \ni v} x_e \leq 1 \quad \text{for all } v \in V.$$

(2) If G is bipartite, show that the polytopes $M(G)$ and $N(G)$ coincide.

[*Hint: Consider a point* $\overline{x} \in N(G)$. *If* $\overline{x}$ *has integer coordinates, show that it is the characteristic vector of a matching on* G. *Otherwise, show that one can slightly perturb the coordinates of* $\overline{x}$ *that are not integer, and conclude that* $\overline{x}$ *is not a vertex of* $N(G)$.]

(3) Show that the result fails when G is not bipartite.

Exercice 8 (Transportation polytope)**.** Given a supply function $\mu : M \to \mathbb{R}_{\geq 0}$ on a source set M and a demand function $\nu : N \to \mathbb{R}_{\geq 0}$ on a sink set N, the *transportation polytope* $P(\mu, \nu)$ is the polytope of $\mathbb{R}^{M \times N}$ defined by:

$$\forall m \in M,\ \forall n \in N, \quad x_{m,n} \geq 0, \quad \sum\nolimits_{n' \in N} x_{m,n'} = \mu(m), \quad \text{and} \quad \sum\nolimits_{m' \in M} x_{m',n} = \nu(n).$$

Call *support* of a point $\overline{x} \in P(\mu, \nu)$ the subgraph of $K_{M,N}$ consisting of the edges (m, n) for which $x_{m,n} > 0$. Show the following properties:

(1) $P(\mu, \nu)$ is non-empty if and only if $\sum_{m \in M} \mu(m) = \sum_{n \in N} \nu(n)$.

(2) Provided it is non-empty, $P(\mu, \nu)$ has dimension $(|M| - 1)(|N| - 1)$.

(3) A point of $P(\mu, \nu)$ is a vertex of $P(\mu, \nu)$ if and only if its support is a forest (*i.e.* contains no cycle). Moreover, a vertex of $P(\mu, \nu)$ is determined by its support.

(4) The supports of two adjacent vertices of $P(\mu, \nu)$ differ by a cycle.

The *Birkhoff polytope* of size m is a particular example of transportation polytope, whose supply and demand functions are both constant to m. Its vertices are precisely the permutation matrices.