Combinatoire des polytopes
DM 2 – Almost simplicial polytopes

Recall that a polytope is *simplicial* when all its facets are simplices. In this problem, we are interested in polytopes that are not simplicial, but almost. A d-dimensional polytope P is called

- k-*simplicial* if all its faces of dimension k are simplices,
- s-*almost simplicial* if all its facets are simplices, except one which has $d+s$ vertices.

Question 1. What is a d-simplicial polytope? Explain the equivalences:

$$P \text{ is simplicial } \iff P \text{ is } (d-1)\text{-simplicial } \iff P \text{ is } 0\text{-almost simplicial.}$$

The goal of the problem is to construct k-simplicial and s-almost simplicial polytopes with many faces, using constructions similar to that of the cyclic polytope seen in the course.

1 $(d-k)$-simplicial polytope

In this section, we construct a $(d-k)$-simplicial polytope with many faces (generalizing the cyclic polytope seen in the course).

Let $\mathbf{p} = (p_1, \dots, p_k)$ be a k-tuple of continuous functions $p_i : \mathbb{R} \to \mathbb{R}$. Define a curve $\chi_{\underline{p}} : \mathbb{R} \to \mathbb{R}^d$ by $\chi_{\underline{p}}(t) := \big(t, t^2, t^3, \dots, t^{d-k}, p_1(t), \dots, p_k(t)\big)$. We fix some numbers $t_1 < \dots < t_n$ and consider the polytope $Q := \operatorname{conv}(\{\chi_{\underline{p}}(t_1), \dots, \chi_{\underline{p}}(t_n)\})$.

Question 2. Show that any $d-k+1$ points on the curve $\chi_{\underline{p}}$ are affinely independent, and deduce that Q is $(d-k-1)$-simplicial.

[*Hint: compute the rank of the* $(d+1)\times(d-k+1)$-*matrix* $\begin{bmatrix} 1 & \dots & 1 \\ \chi_{\underline{p}}(t_1) & \dots & \chi_{\underline{p}}(t_{d-k+1}) \end{bmatrix}$ *and conclude.*]

Question 3. Show that any subset of at most $\lfloor (d-k)/2 \rfloor$ vertices of Q form a face of Q.

[*Hint: use a well choosen polynomial to define a supporting hyperplane of this face.*]

2 Almost simplicial polytope

In this section, we construct an s-almost simplicial polytope with many faces, using some results of the previous questions (which can now be admitted if needed).

We consider the real function $p(t) := (n-1)^{(t-1)(d-1)} t(t+1)\dots(t+d+s-1)$, we define the curve $\xi(t) := (t, t^2, ..., t^{d-1}, p(t))$, and we consider the polytope $Q := \operatorname{conv}(\{\xi(t_1), \dots, \xi(t_n)\})$, where we have chosen this time $t_i := -s-d+i$ for all $i \in [n]$.

To analyse this polytope, for any d-tuple of indices $\underline{i} = (i_1, \dots, i_d) \in [n]$ and for any d-tuple of variables $\underline{z} = (z_1, \dots, z_d)$, we define the determinant

$$D(\underline{i}, \underline{z}) := \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ \xi(t_{i_1}) & \xi(t_{i_2}) & \dots & \xi(t_{i_d}) & \underline{z} \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ t_{i_1} & t_{i_2} & \dots & t_{i_d} & z_1 \\ t_{i_1}^2 & t_{i_2}^2 & \dots & t_{i_d}^2 & z_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{i_1}^{d-1} & t_{i_2}^{d-1} & \dots & t_{i_d}^{d-1} & z_{d-1} \\ p(t_{i_1}) & p(t_{i_2}) & \dots & p(t_{i_d}) & z_d \end{bmatrix}.$$

and the half-space

$$H_{\underline{i}} := \left\{ \underline{z} \in \mathbb{R}^d \;\middle|\; D(\underline{i}, \underline{z}) \geq 0 \right\}.$$

We denote by $V(\underline{i})$ the Vandermonde determinant

$$V(\underline{i}) := \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_{i_1} & t_{i_2} & \dots & t_{i_d} \\ t_{i_1}^2 & t_{i_2}^2 & \dots & t_{i_d}^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_1}^{d-1} & t_{i_2}^{d-1} & \dots & t_{i_d}^{d-1} \end{bmatrix} = \prod_{k<\ell} (t_{i_\ell} - t_{i_k}),$$

Question 4. Observe that $p(t_1) = p(t_2) = \dots = p(t_{d+s}) = 0$ and $p(t_i) > 0$ for $d+s+1 \le i \le n$. Deduce that the hyperplane $H_{(1,\dots,d)}$ defines a facet of the polytope Q containing precisely the vertices $\xi(t_1), \dots, \xi(t_{d+s})$.

Question 5. Consider now $i_1 < i_2 < \dots < i_d < i_{d+1}$ with $i_{d+1} > d+s$. For any $j \in [d+1]$, we consider the Vandermonde determinant $W_j := V(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_{d+1})$. Show that

$$D(\underline{i}, \xi(t_{i_{d+1}})) = \sum_{j=1}^{d+1} (-1)^{d+1-j} p(t_{i_j}) W_j.$$

To evaluate this sum, we group terms two by two (leaving the first alone when $d+1$ is odd) and thus consider the term $p(t_{i_{d+1-2k}})W_{d+1-2k} - p(t_{i_{d-2k}})W_{d-2k}$ for any $0 \le k \le \lfloor (d+1)/2 \rfloor$. Observe that the definition of $t_i := -s - d + i$ implies that $1 \le t_{i_q} - t_{i_p} \le n-1$ for any $1 \le p < q \le d+1$. Use these inequalities to show that for any $1 < j \le d+1$, we have

- $p(t_{i_j})/p(t_{i_{j-1}}) \ge (n-1)^{d-1}$ with a strict inequality when $j = d+1$,
- $W_{j-1}/W_j \le (n-1)^{d-1}$,

and conclude that $D(\underline{i}, \xi(t_{i_{d+1}})) > 0$ for any choice of $i_1 < i_2 < \dots < i_d < i_{d+1}$ with $i_{d+1} > d+s$.

Question 6. Deduce from Question 5 that except the facet of Question 4, all other facets of the polytope Q are simplices, and conclude that the polytope Q is a s-almost simplicial polytope.

Question 7. Using the computation of determinant of Question 5, show that a subset $I := \{i_1 < \dots < i_d\}$ with $i_d > d+s$ defines a facet of Q if and only if the number of elements of I between any two elements of $[n] \smallsetminus I$ is even.