The greedy flip tree of a subword complex

Vincent PILAUD

CNRS - LIX, École Polytechnique

REDUCED EXPRESSIONS
& SUBWORD COMPLEXES

REDUced EXPRESSIONS
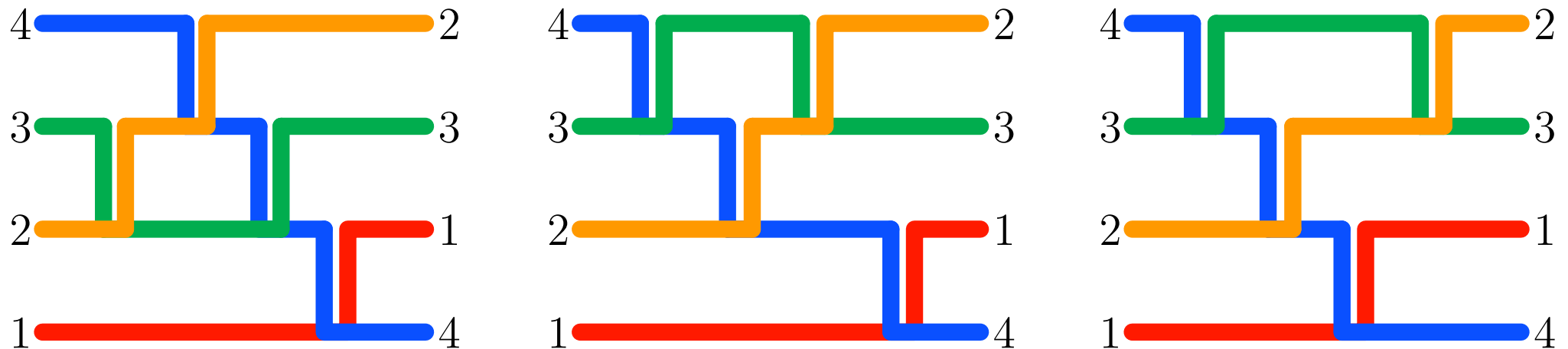
$\mathfrak{S}_n$ = symmetric group
$S = \{\tau_i \mid i \in n-1\}$ set of simple transpositions $\tau_i = (i \ \ i+1)$

ρ permutation of $\mathfrak{S}_n$
reduced expression of ρ = minimal length expression $\rho = s_1 \cdots s_\ell$ with $s_i \in S$

Count and enumerate reduced expressions of ρ

Example. $\rho = [4,1,3,2] = \tau_2 \ \tau_3 \ \tau_2 \ \tau_1 \ = \ \tau_3 \ \tau_2 \ \tau_3 \ \tau_1 \ = \ \tau_3 \ \tau_2 \ \tau_1 \ \tau_3$
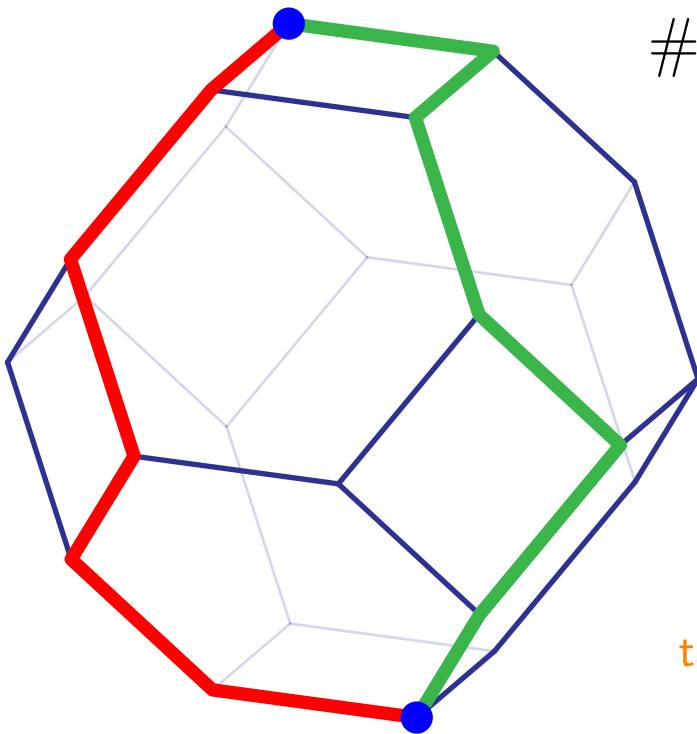
REDUCED EXPRESSIONS

$\mathfrak{S}_n =$ symmetric group
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ρ permutation of $\mathfrak{S}_n$
reduced expression of $\rho =$ minimal length expression $\rho = s_1 \cdots s_\ell$ with $s_i \in S$

Count and enumerate reduced expressions of ρ

\# reduced expressions of $w_\circ =$

$$\frac{\binom{n}{2}!}{1^{n-1}2^{n-2}\cdots(2n-3)^1}$$

Stanley.
On the number of reduced decompositions
of elements of Coxeter groups. 1984

Edelmann & Greene.
Combinatorial correspondences for Young tableaux, balanced
tableaux, and maximal chains in the Bruhat order of $\mathfrak{S}_n$. 1984

$\mathfrak{S}_n$ = symmetric group
$S = \{\tau_i \mid i \in n-1\}$ set of simple transpositions $\tau_i = (i \ \ i+1)$
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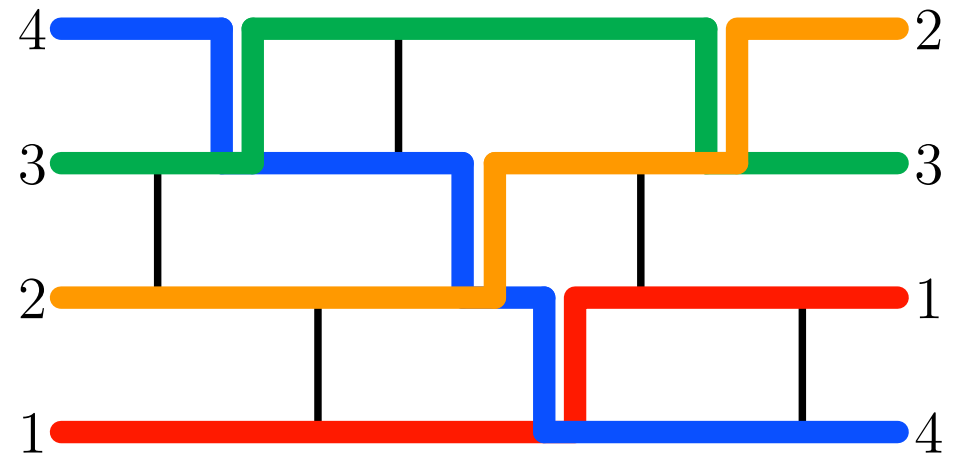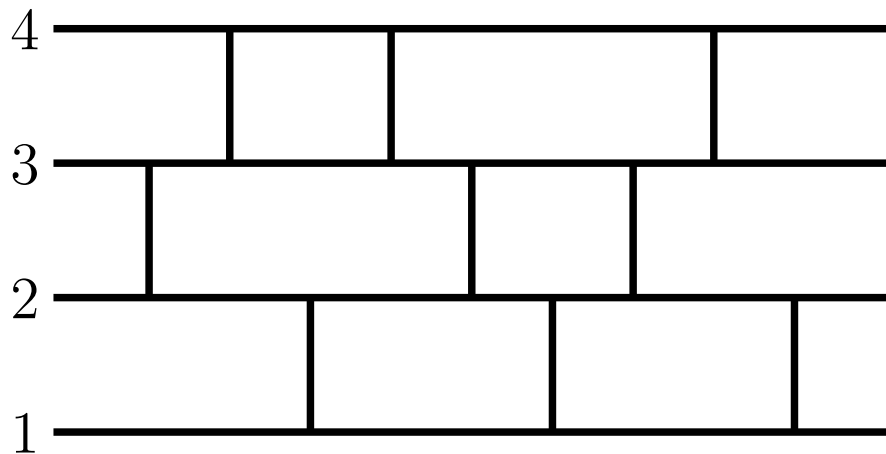$\mathbb{Q} = q_1 \, q_2 \, \cdots \, q_m$ word on the alphabet S

Enumerate subwords of $\mathbb{Q}$ which are reduced expressions for ρ

Example. $\rho = [4, 1, 3, 2] = \tau_2 \, \tau_3 \, \tau_2 \, \tau_1 = \tau_3 \, \tau_2 \, \tau_3 \, \tau_1 = \tau_3 \, \tau_2 \, \tau_1 \, \tau_3$
$\mathbb{Q} = \tau_2 \, \tau_3 \, \tau_1 \, \tau_3 \, \tau_2 \, \tau_1 \, \tau_2 \, \tau_3 \, \tau_1$

Possible subwords:

$$\tau_2 \ \ \tau_3 \ \ \cdot \ \ \cdot \ \ \tau_2 \ \ \tau_1 \ \ \cdot \ \ \cdot \ \ \cdot \quad \longrightarrow \quad 34789$$
$$\tau_2 \ \ \tau_3 \ \ \cdot \ \ \cdot \ \ \cdot \ \ \cdot \ \ \tau_2 \ \ \cdot \ \ \tau_1 \quad \longrightarrow \quad 34568$$
$$\cdot \ \ \tau_3 \ \ \cdot \ \ \cdot \ \ \tau_2 \ \ \cdot \ \ \cdot \ \ \tau_3 \ \ \tau_1 \quad \longrightarrow \quad 13467$$
$$\cdot \ \ \tau_3 \ \ \cdot \ \ \cdot \ \ \tau_2 \ \ \tau_1 \ \ \cdot \ \ \tau_3 \ \ \cdot \quad \longrightarrow \quad 13479$$

etc

$\mathfrak{S}_n$ = symmetric group
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$\mathbb{Q} = \tau_2 \, \tau_3 \, \tau_1 \, \tau_3 \, \tau_2 \, \tau_1 \, \tau_2 \, \tau_3 \, \tau_1$

GENERALIZATION TO COXETER GROUPS

W = finite Coxeter group
S = simple system of generators for W
ρ element of W
$\mathbb{Q} = q_1 \, q_2 \, \cdots \, q_m$ word on the alphabet S
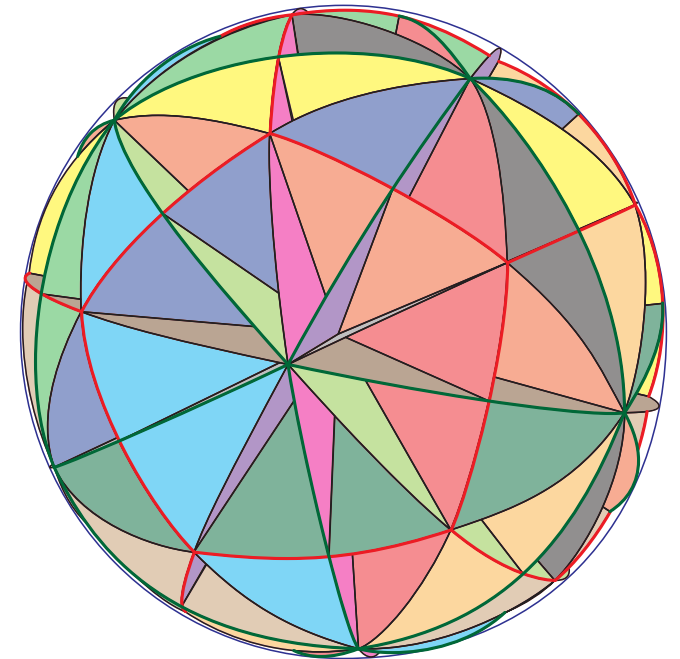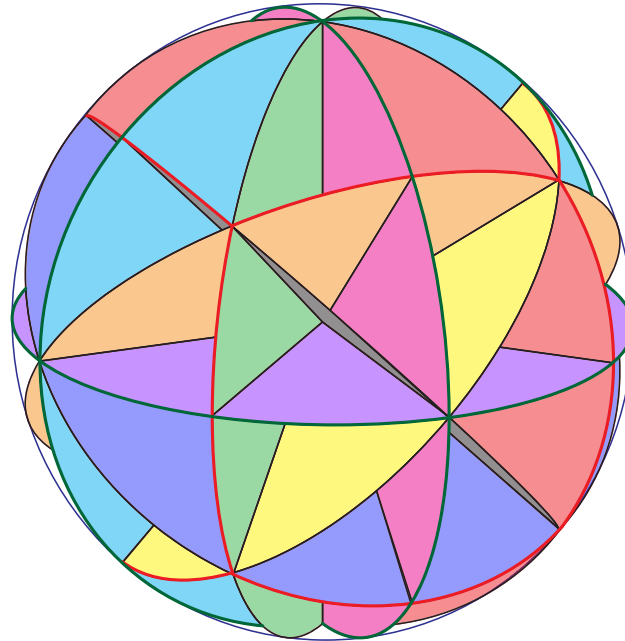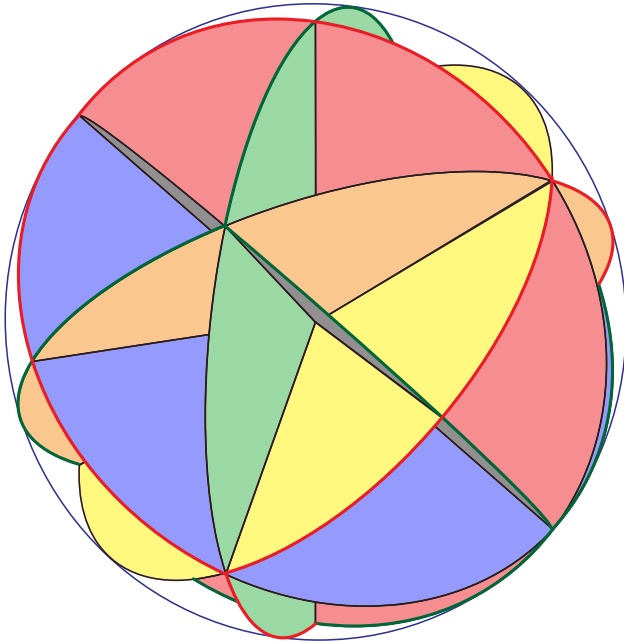
Enumerate subwords of $\mathbb{Q}$ which are reduced expressions for ρ

$\mathfrak{S}_n =$ symmetric group
$S = \{\tau_i \mid i \in n-1\}$ set of simple transpositions $\tau_i = (i \ i+1)$
ρ permutation of $\mathfrak{S}_n$
$Q = q_1 \, q_2 \, \cdots \, q_m$ word on the alphabet S

Subword complex $\mathcal{SC}(Q, \rho) =$ simplicial complex with

- vertices $= [m] =$ positions in the word Q

- facets $= \mathcal{F}(Q, \rho) =$ complements in $[m]$ of position sets of reduced expressions of ρ in Q

Knutson & Miller. Subword complexes in Coxeter groups. 2004.

flip = two subwords of $\mathbb{Q}$ which differ at precisely two positions
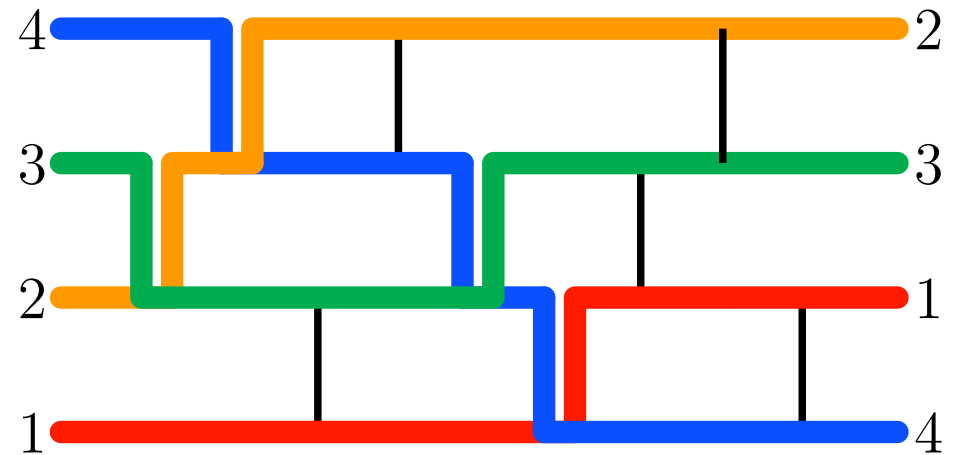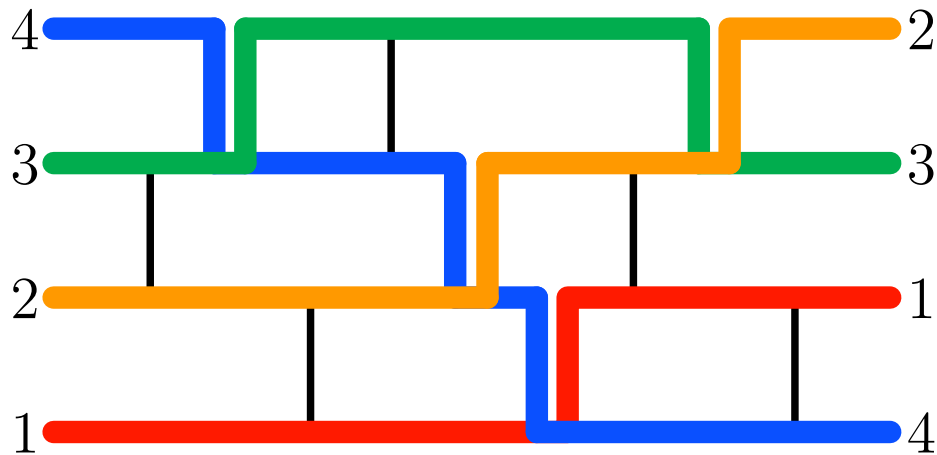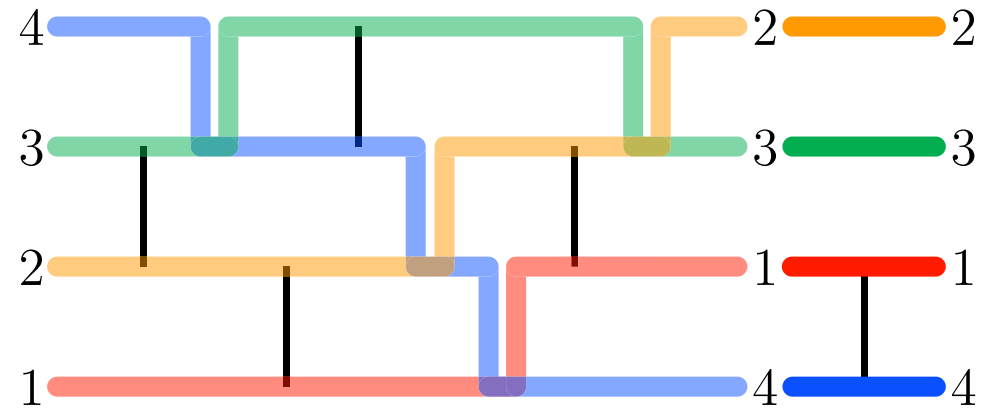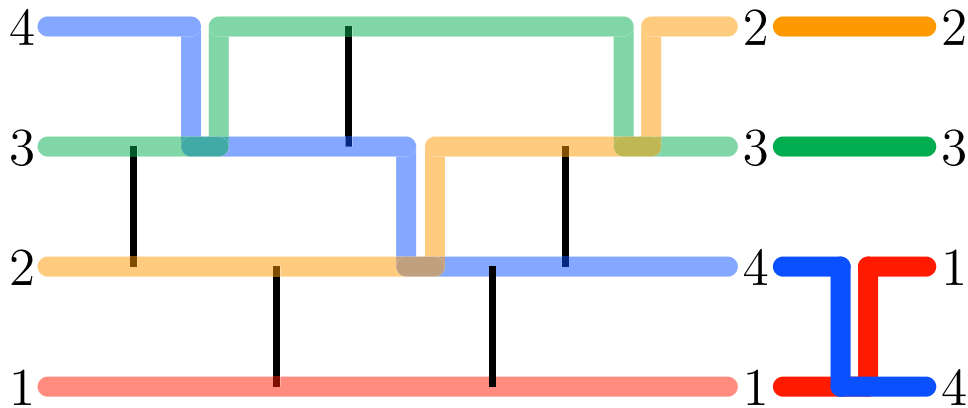
FLIP
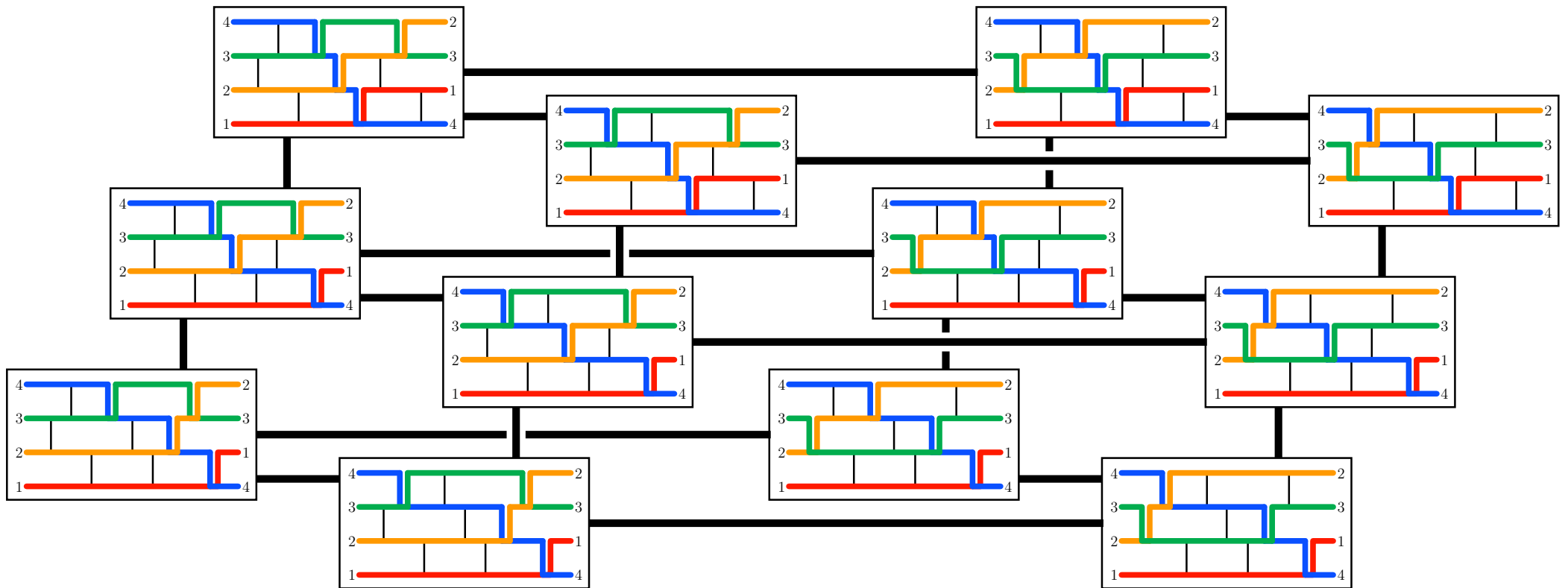
The flip graph is connected
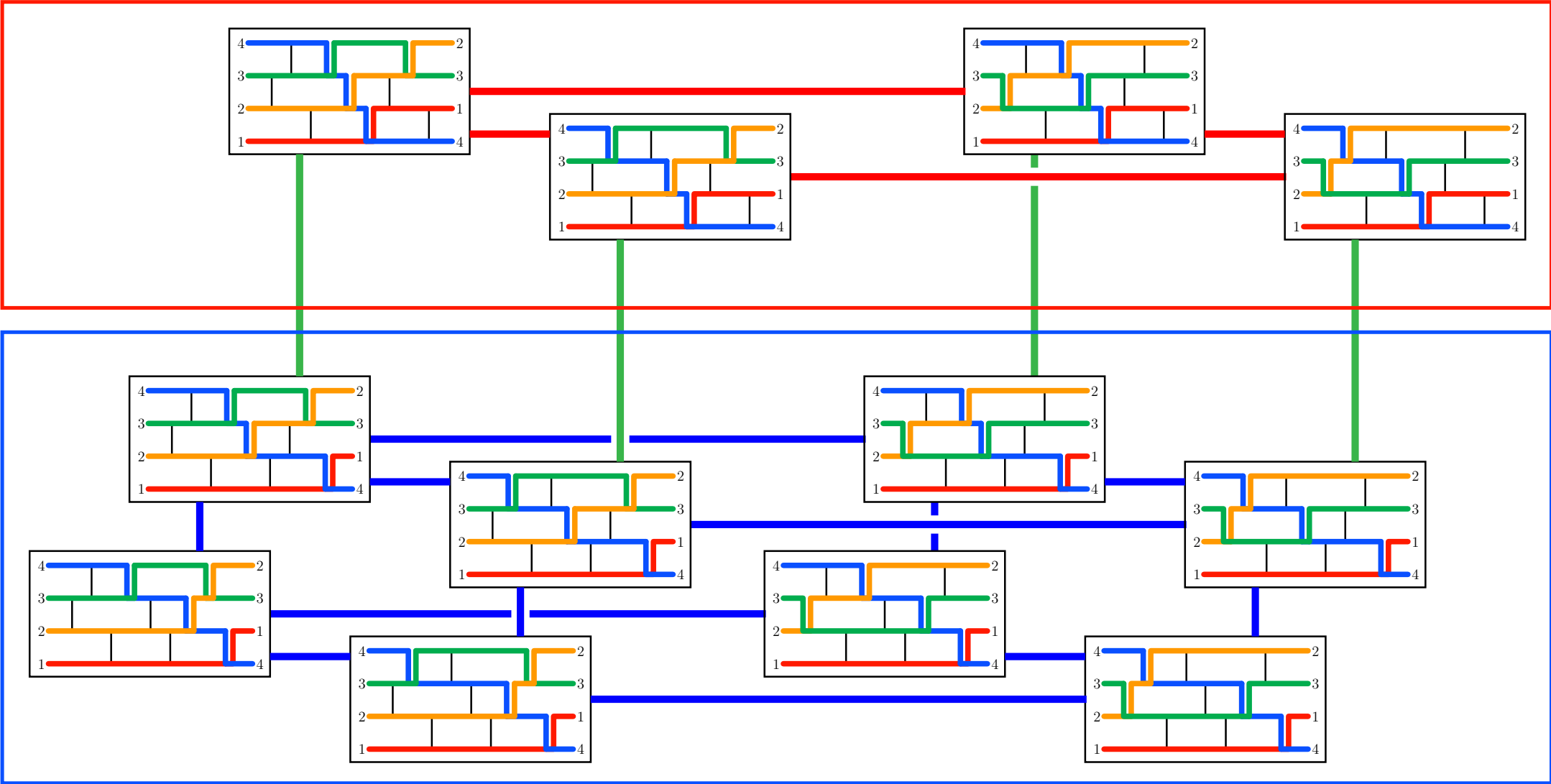
GOAL: Find a natural spanning tree of the flip graph

$Q = q_1 \, q_2 \, \cdots \, q_{m-1} \, q_m$ and $Q_{\dashv} = q_1 \, q_2 \, \cdots \, q_{m-1}$

$\mathcal{F}(Q, \rho) =$ facets of $\mathcal{SC}(Q, \rho) =$ complements of reduced expressions of ρ in Q

$$\mathcal{F}(Q, \rho) \;=\; \mathcal{F}(Q_{\dashv}, \rho q_m) \;\sqcup\; \big(\mathcal{F}(Q_{\dashv}, \rho) \star m\big)$$

INDUCTIVE STRUCTURE

$$\mathcal{F}(\mathrm{Q}, \rho) \;=\; \mathcal{F}(\mathrm{Q}_{\dashv}, \rho q_m) \;\sqcup\; \big(\mathcal{F}(\mathrm{Q}_{\dashv}, \rho) \star m\big)$$

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$\mathcal{F}(Q, \rho) =$ facets of $\mathcal{SC}(Q, \rho) =$ complements of reduced expressions of ρ in Q

$$
\mathcal{F}(Q, \rho) = \begin{cases} \mathcal{F}(Q_{\dashv}, \rho q_m) & \text{if } \rho \not\preceq Q_{\dashv} \\ \mathcal{F}(Q_{\dashv}, \rho) \star m & \text{if } \ell(\rho q_m) > \ell(\rho) \\ \mathcal{F}(Q_{\dashv}, \rho q_m) \sqcup \big(\mathcal{F}(Q_{\dashv}, \rho) \star m \big) & \text{otherwise} \end{cases}
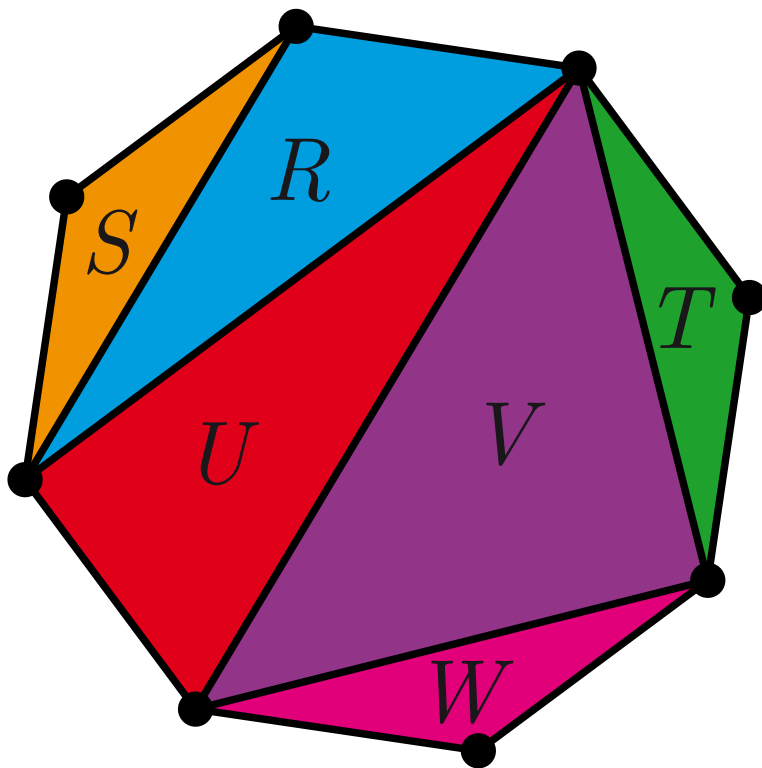$$
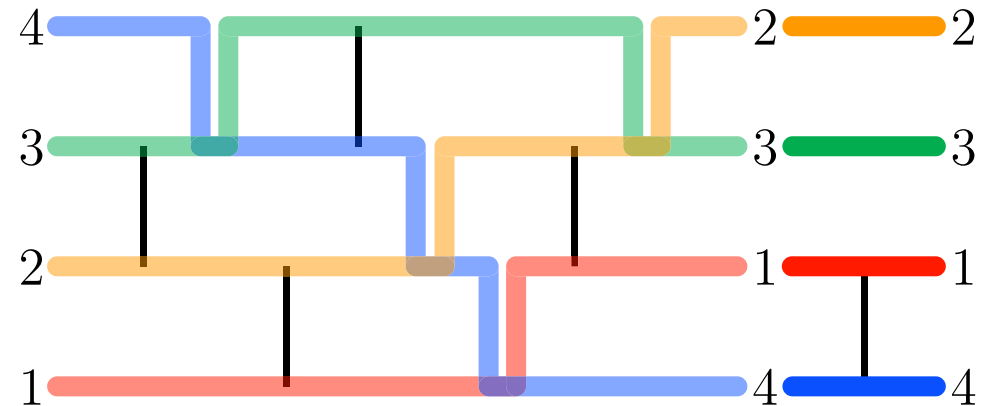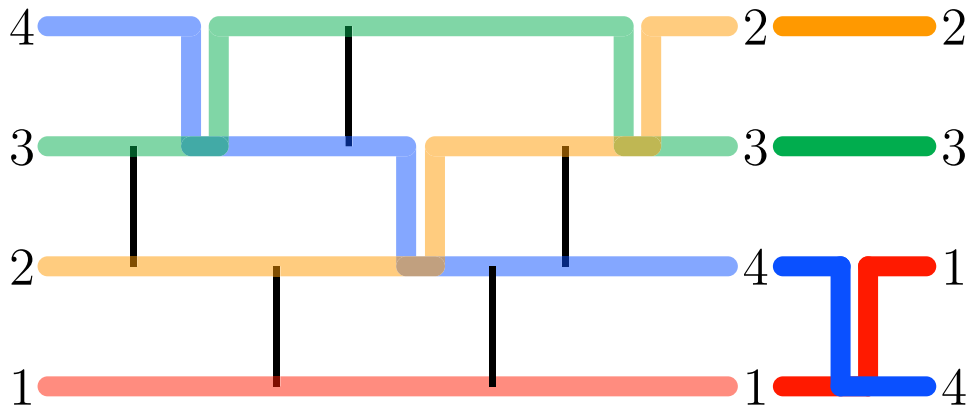
$\Rightarrow$ Inductive enumeration of $\mathcal{F}(Q, \rho)$ with complexity $O(m^2 n)$ per facet
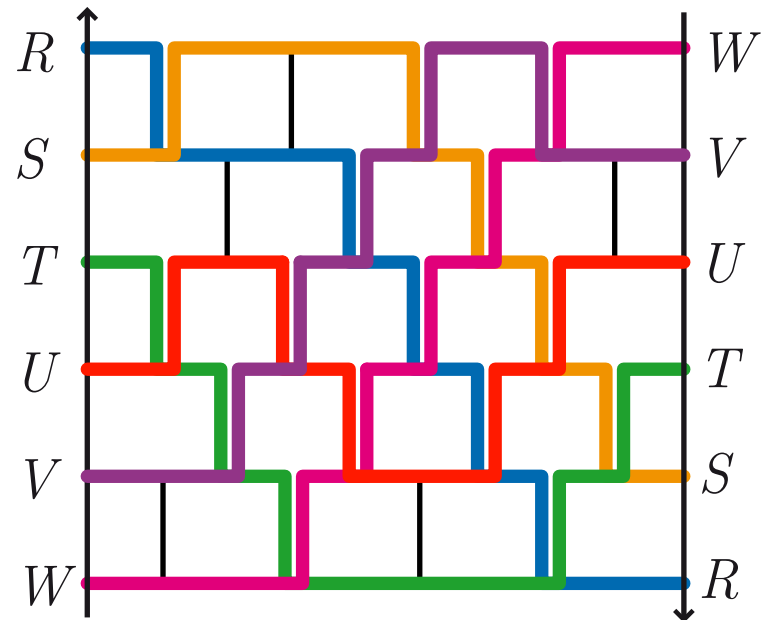
COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS

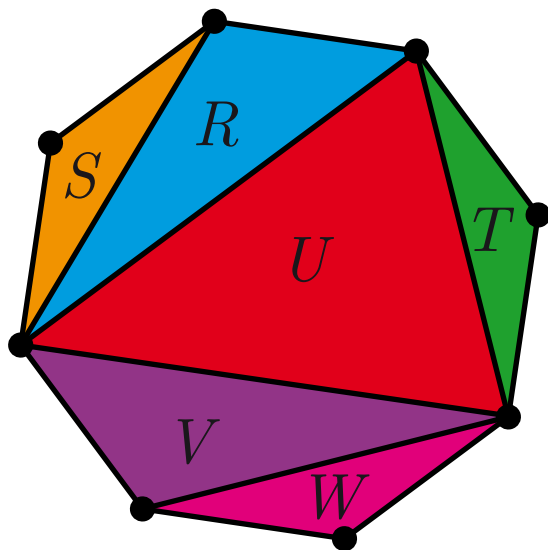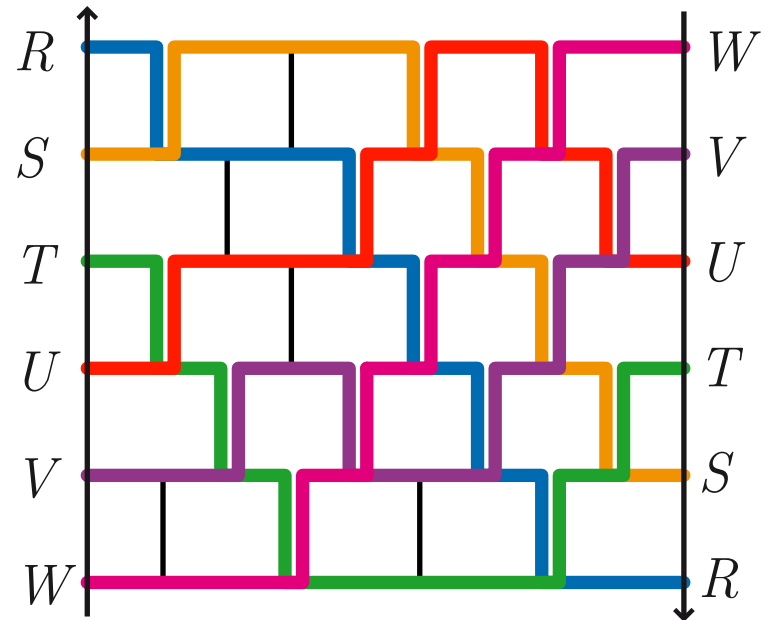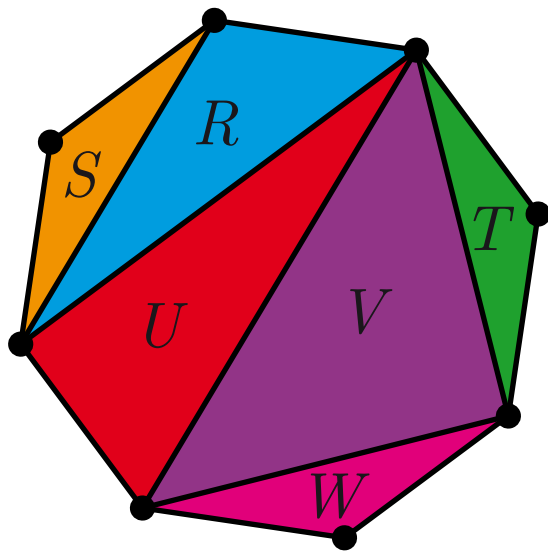bijection between

- triangulations of a convex $(n+2)$-gon

- subwords of the odd-even word $Q = \left(\prod_{i \in \left[\frac{n}{2} \right]} \tau_{2i+1} \cdot \prod_{i \in \left[\frac{n}{2} \right]} \tau_{2i} \right)^{\frac{n}{2}}$
  which are reduced expressions for the longest element $w_\circ = [n, n-1, \ldots, 2, 1]$
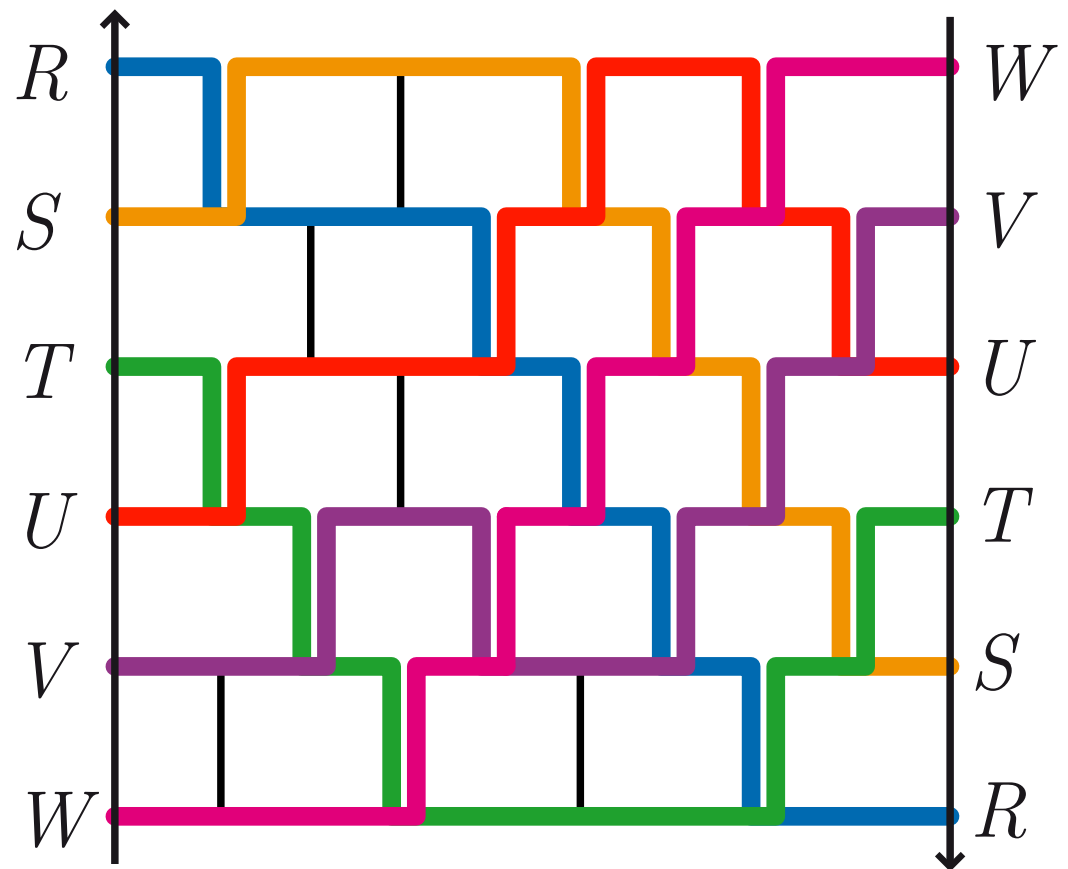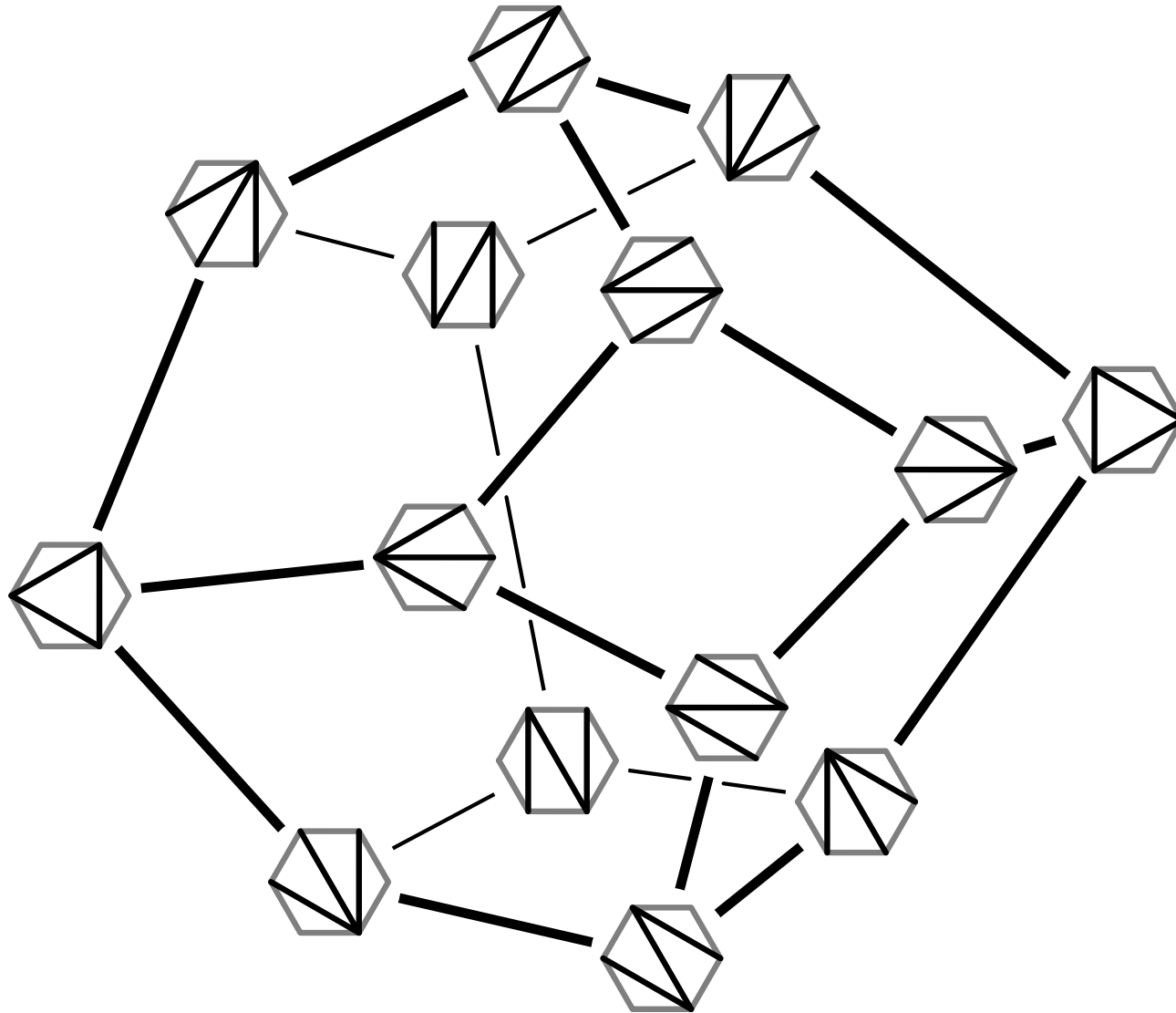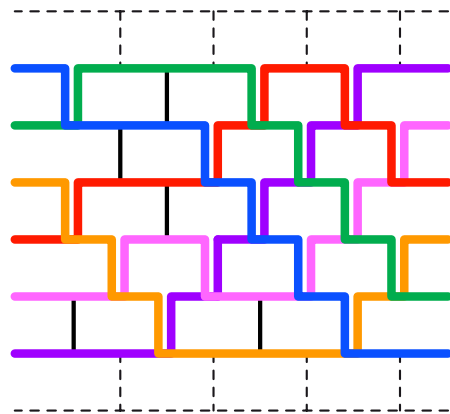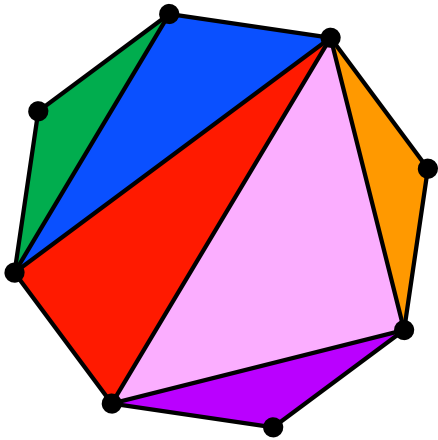
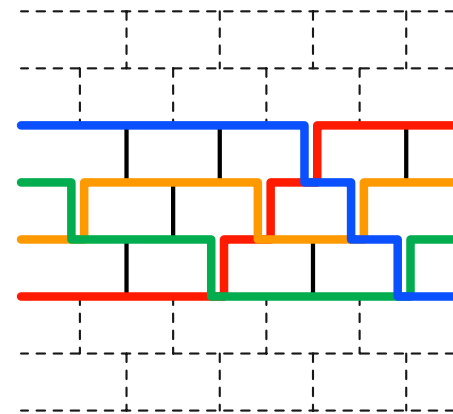# FLIP IN TRIANGULATIONS

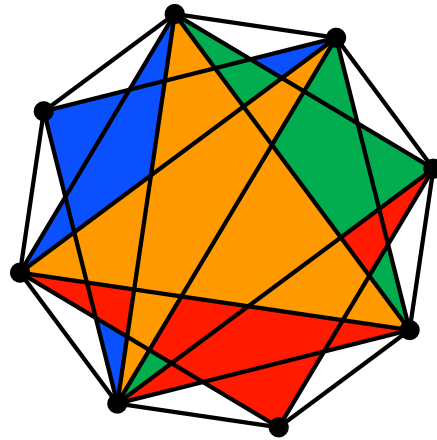The flip graph is the 1-skeleton of the associahedron

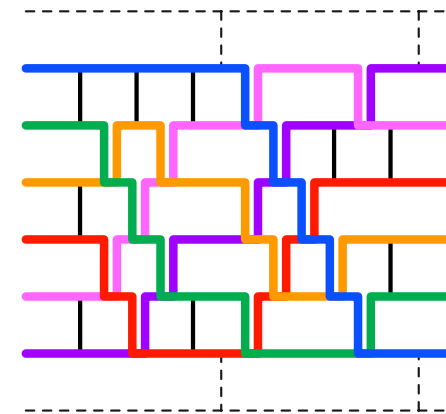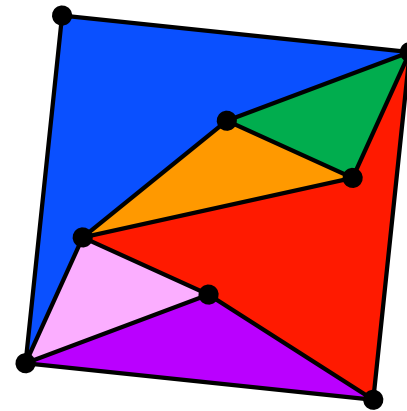# COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS

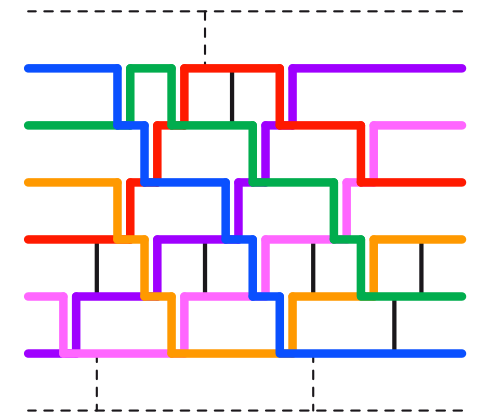triangulations
of convex polygons,

multitriangulations
of convex polygons,

pseudotriangulations
of point sets in
general position,

pseudotriangulations
of sets of disjoint
convex bodies.
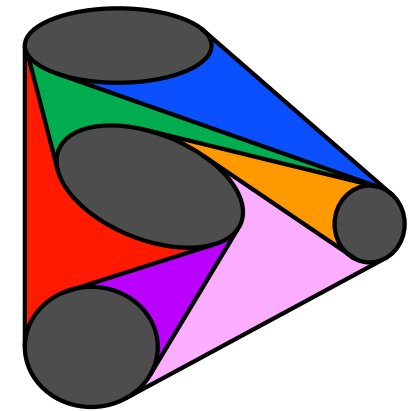
P. & Pocchiola, Pseudotriangulations, multitriangulations, and primitive sorting networks, 2012.

Stump, A new perspective on multitriangulations, 2011.

GREEDY FLIP ALGORITHM

increasing flip = flip from I to J with $I \smallsetminus i = J \smallsetminus j$ and $i < j$
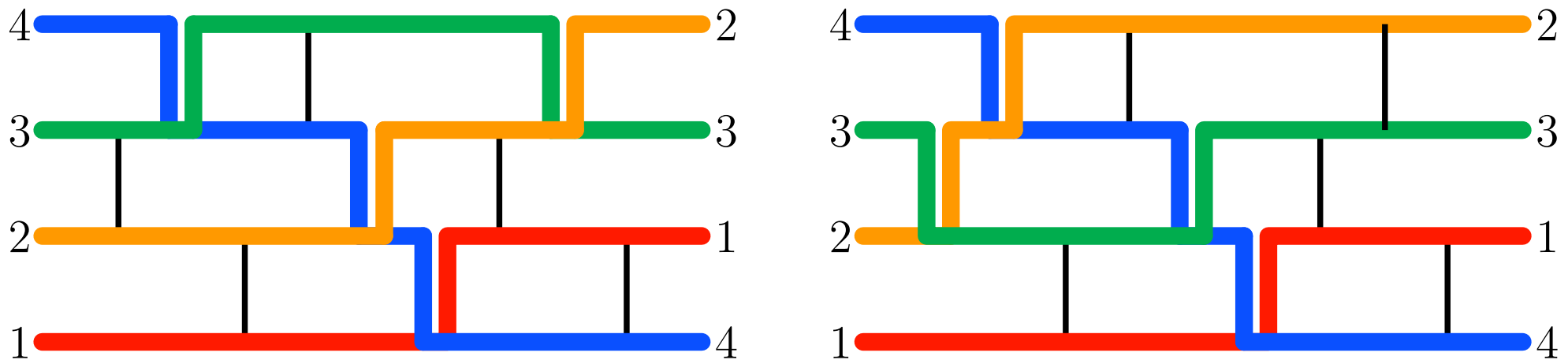
The increasing flip graph is acyclic, connected, and has a unique sink

greedy facet $\mathsf{G}(\mathbb{Q}, \rho)$ = unique sink of the increasing flip graph
= lexicographically maximal facet of $\mathcal{SC}(\mathbb{Q}, \rho)$

The greedy facet $\mathsf{G}(\mathbb{Q}, \rho)$ can be constructed inductively from $\mathsf{G}(\varepsilon, e) = \emptyset$ using the following formulas:

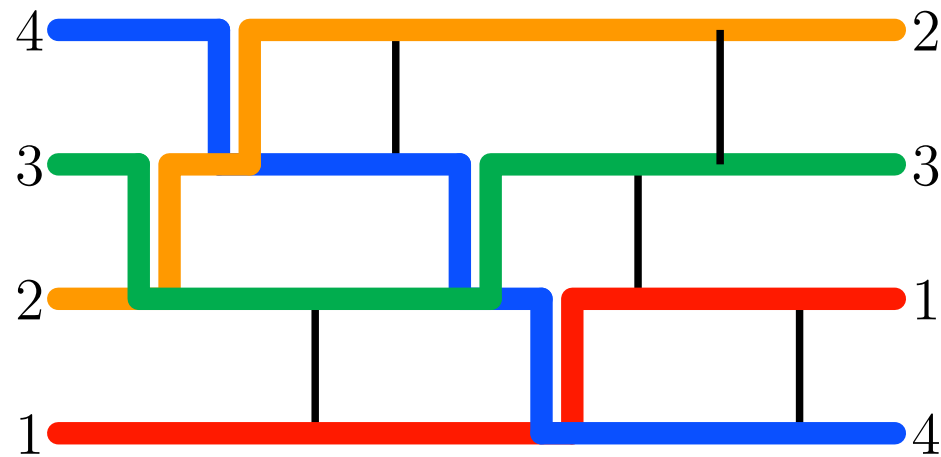$$\mathsf{G}(\mathbb{Q}, \rho) = \begin{cases} \mathsf{G}(\mathbb{Q}_\dashv, \rho) \cup m & \text{if } \rho \prec \mathbb{Q}_\dashv \\ \mathsf{G}(\mathbb{Q}_\dashv, \rho q_m) & \text{otherwise} \end{cases}$$

$$\mathsf{G}(\mathbb{Q}, \rho) = \begin{cases} \mathsf{G}(\mathbb{Q}_\vdash, q_1 \rho)^{\rightarrow} & \text{if } \ell(q_1 \rho) < \ell(\rho) \\ 1 \cup \mathsf{G}(\mathbb{Q}_\vdash, \rho)^{\rightarrow} & \text{otherwise} \end{cases}$$

where $\mathbb{Q}_\dashv = q_1 q_2 \cdots q_{m-1}$, $\mathbb{Q}_\vdash = q_2 \cdots q_{m-1} q_m$ and $X^{\rightarrow} = \{x + 1 \mid x \in X\}$

GREEDY FLIP PROPERTY

If m is a flippable element of $\mathsf{G}(\mathbb{Q}, \rho)$,
then $\mathsf{G}(\mathbb{Q}_{\dashv}, \rho q_m)$ is obtained from $\mathsf{G}(\mathbb{Q}, \rho)$ flipping m
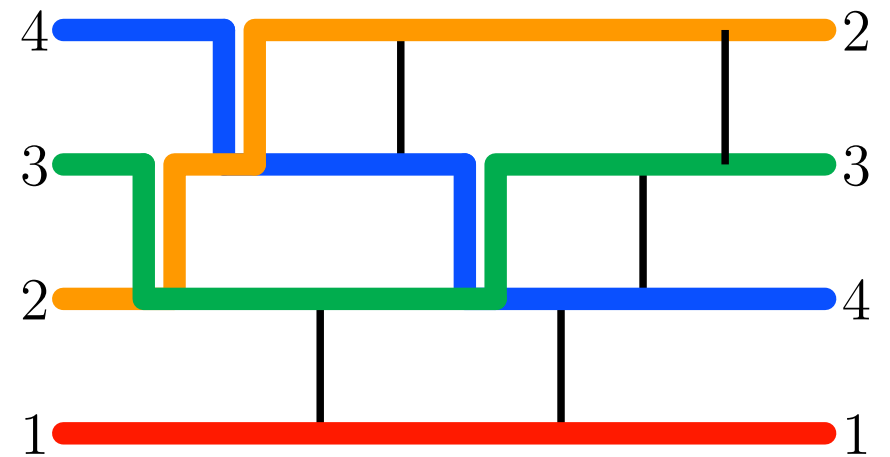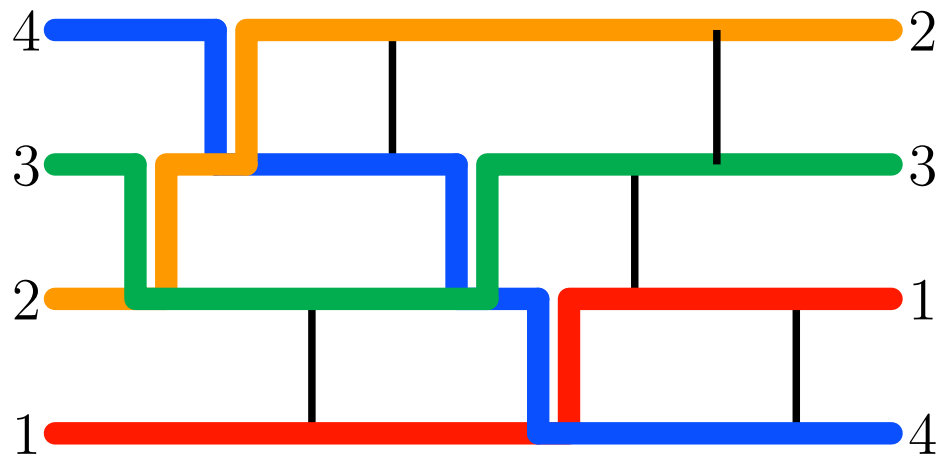
GREEDY FLIP TREE — INDUCTIVE DEFINITION
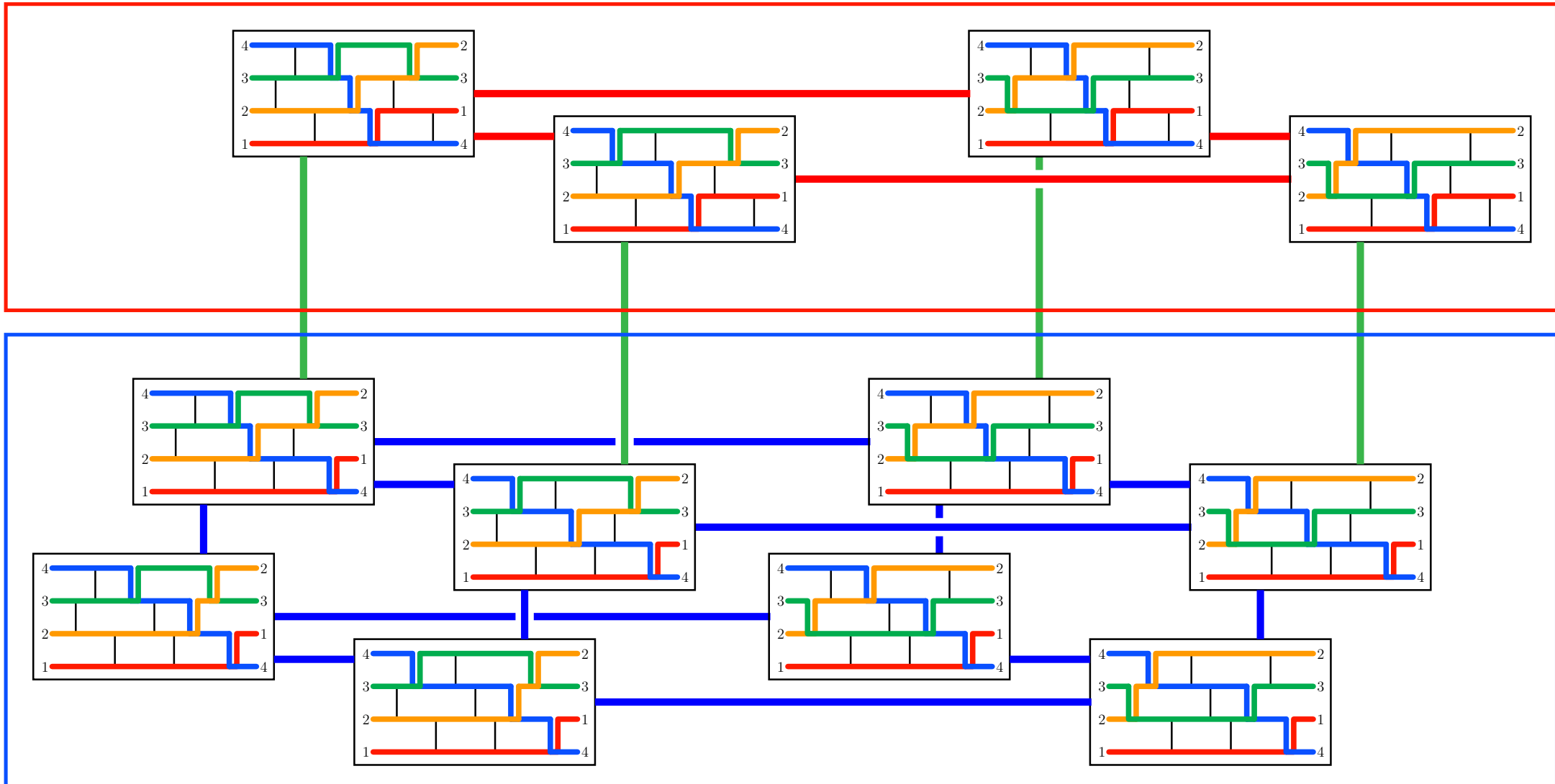
$$\mathcal{F}(Q, \rho) = \mathcal{F}(Q_{\dashv}, \rho q_m) \sqcup \left(\mathcal{F}(Q_{\dashv}, \rho) \star m \right)$$

GREEDY FLIP TREE — INDUCTIVE DEFINITION

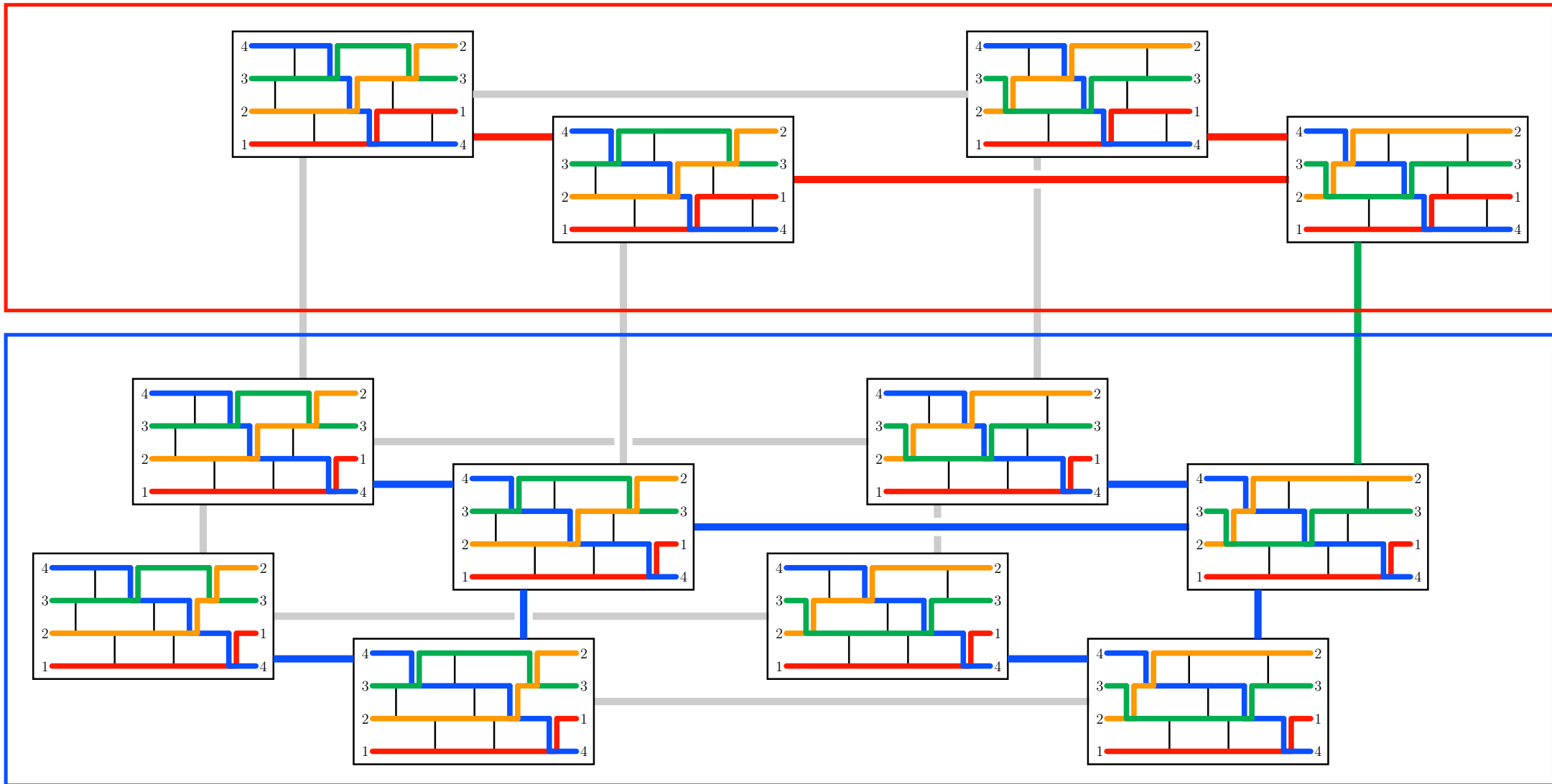$$\mathcal{G}(Q, \rho) = \mathcal{G}(Q_{\dashv}, \rho q_m) \sqcup \left(\mathcal{G}(Q_{\dashv}, \rho) \star m\right) \sqcup \left\{ \begin{array}{l} \text{arc from } G(Q_{\dashv}, \rho q_m) \text{ to} \\ G(Q, \rho) = G(Q_{\dashv}, \rho) \cup m \end{array} \right\}$$

Inductive structure of the facets $\mathcal{F}(Q, \rho)$ of the subword complex $\mathcal{SC}(Q, \rho)$:

$$\mathcal{F}(Q, \rho) = \begin{cases} \mathcal{F}(Q_\dashv, \rho q_m) & \text{if } \rho \not\preceq Q_\dashv \\ \mathcal{F}(Q_\dashv, \rho) \star m & \text{if } \ell(\rho q_m) > \ell(\rho) \\ \mathcal{F}(Q_\dashv, \rho q_m) \sqcup \big(\mathcal{F}(Q_\dashv, \rho) \star m\big) & \text{otherwise} \end{cases}$$

Inductive definition of the greedy flip tree $\mathcal{G}(Q, \rho)$:

$$\mathcal{G}(Q, \rho) = \begin{cases} \mathcal{G}(Q_\dashv, \rho q_m) & \text{if } \rho \not\preceq Q_\dashv \\ \mathcal{G}(Q_\dashv, \rho) \star m & \text{if } \ell(\rho q_m) > \ell(\rho) \\ \mathcal{G}(Q_\dashv, \rho q_m) \sqcup \big(\mathcal{G}(Q_\dashv, \rho) \star m\big) \\ \quad \sqcup \{\text{arc from } \mathsf{G}(Q_\dashv, \rho q_m) \text{ to} & \text{otherwise} \\ \quad\quad \mathsf{G}(Q, \rho) = \mathsf{G}(Q_\dashv, \rho) \cup m\} \end{cases}$$

GREEDY FLIP TREE — DIRECT DEFINITION

$\mathsf{g}(I) = $ greedy index of a facet $I \in \mathcal{F}(\mathbb{Q}, \rho) = $
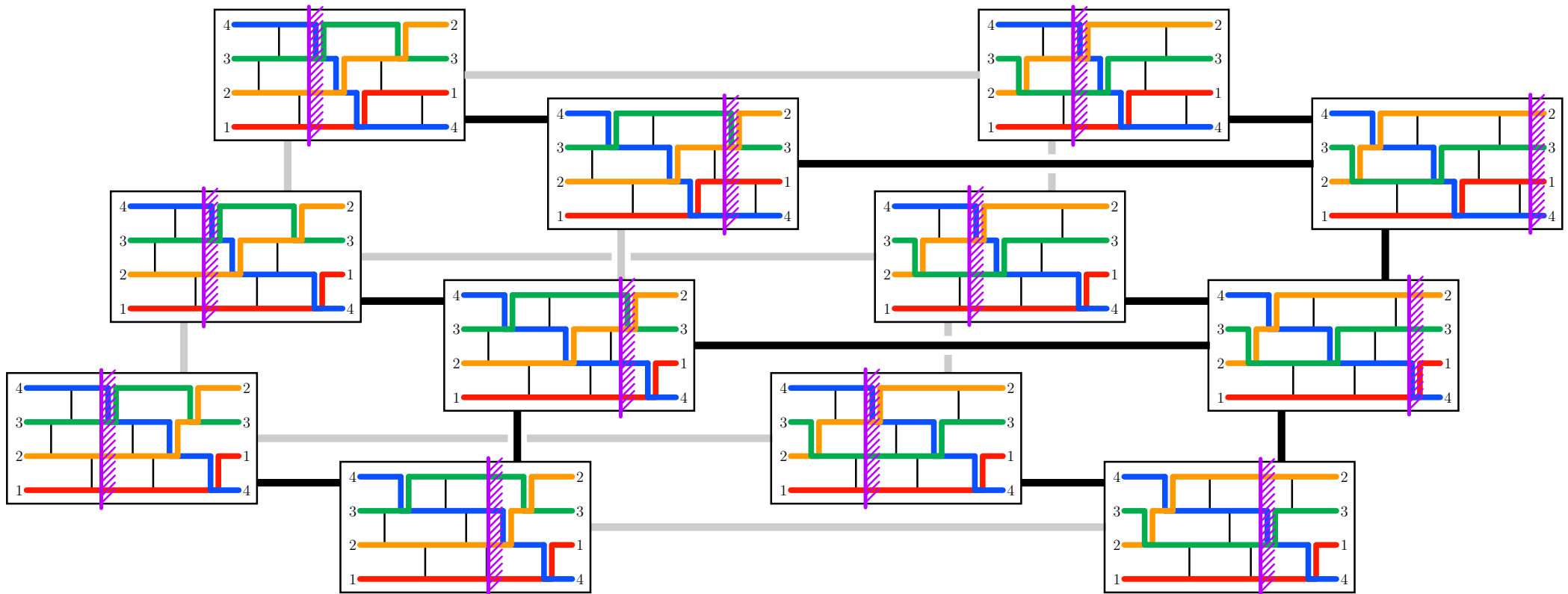last position $x \in [m]$ such that $I \cap [x] = \mathsf{G}(q_1 \cdots q_x, \sigma_{[x] \smallsetminus I})$
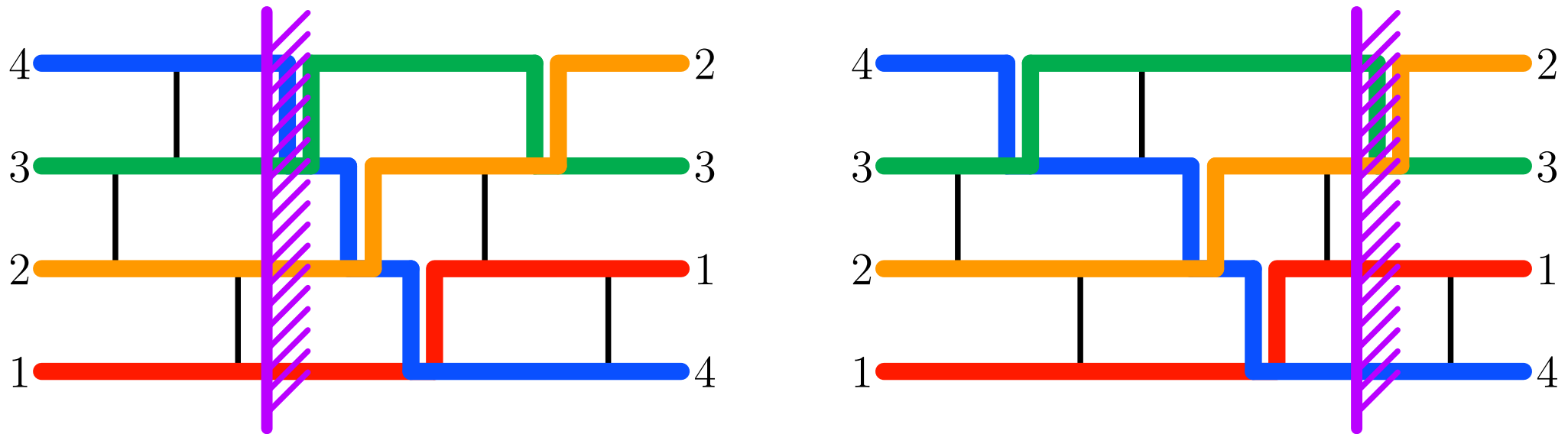
If $I, J \in \mathcal{F}(\mathbb{Q}, \rho)$ with $I \smallsetminus i = J \smallsetminus j$ and $i < j \leq \mathsf{g}(J)$, then $\mathsf{g}(I) = j - 1$

GREEDY FLIP TREE — DIRECT DEFINITION

The greedy flip tree $\mathcal{G}(\mathbb{Q}, \rho)$ has

- nodes $= \mathcal{F}(\mathbb{Q}, \rho) =$ complements of reduced expressions of ρ in $\mathbb{Q}$
- arcs $=$ flip (I, J) such that $I \smallsetminus i = J \smallsetminus j$ with $i < j \leq \mathsf{g}(J)$.

GREEDY FLIP ALGORITHM

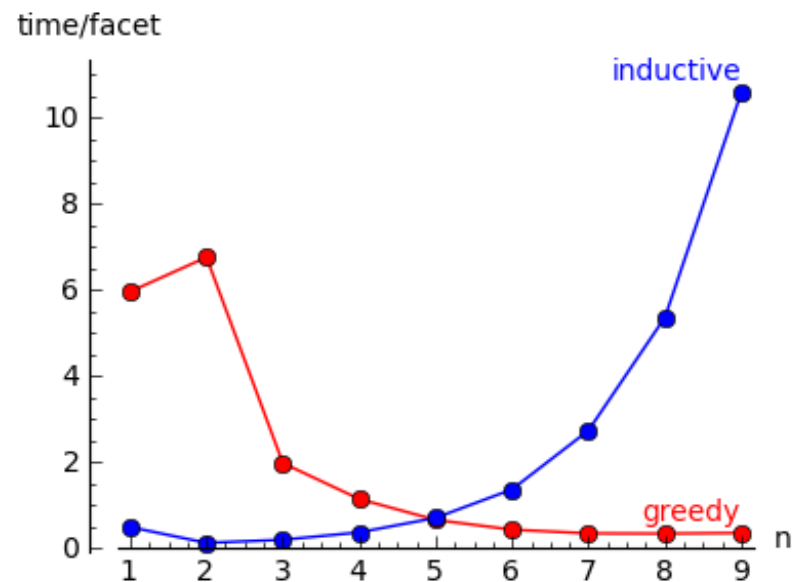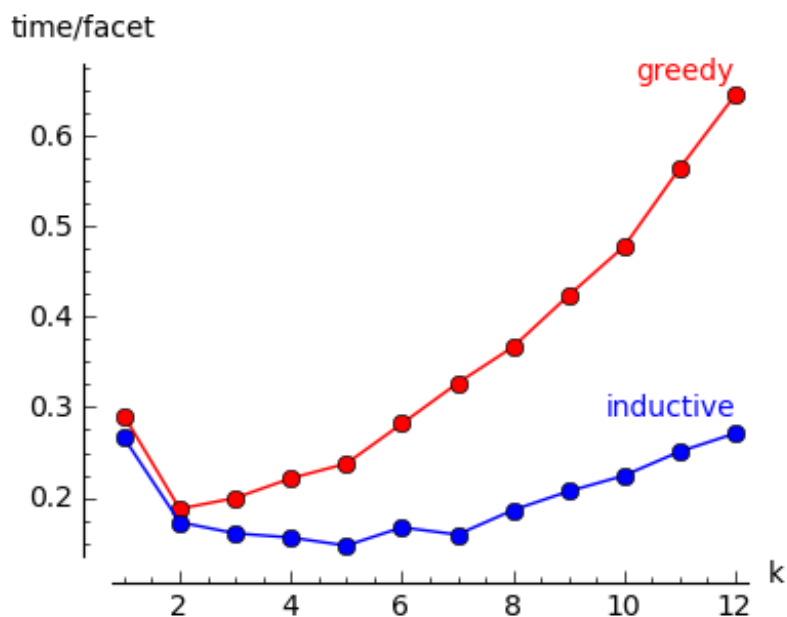Greedy Flip Algorithm $=$ Depth first search generation on the greedy flip tree

Preorder traversal provides an iterator on the reduced expressions of ρ in Q

Working space in $O(mn)$
Running time in $O(m^2n)$ per facet $\longrightarrow$ similar to the inductive algorithm

Implemented in Sage (Stump's combinat patch on subword complexes)
Experimental time comparison to generate the k-triangulations of the n-gon:

Thank you