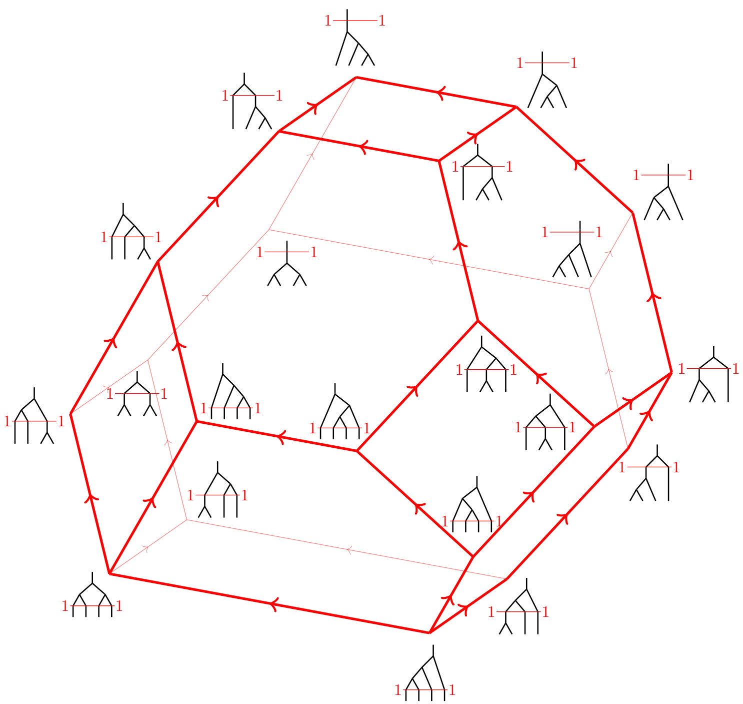
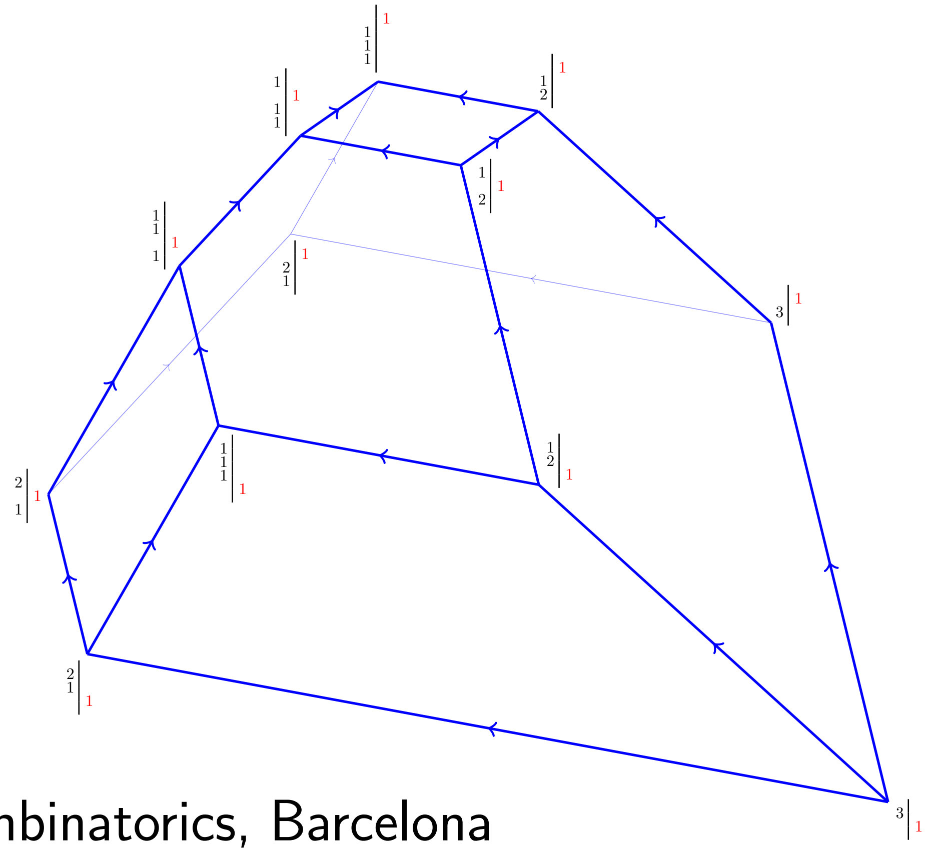
# Deformed permutahedra, quotientopes, and beyond

V. PILAUD (CNRS & LIX, École Polytechnique ⟶ Universitat de Barcelona)





3 talks in combinatorics, Barcelona

Thursday September 14th, 2023

slides: http://www.lix.polytechnique.fr/~pilaud/documents/presentations/freehedra.pdf

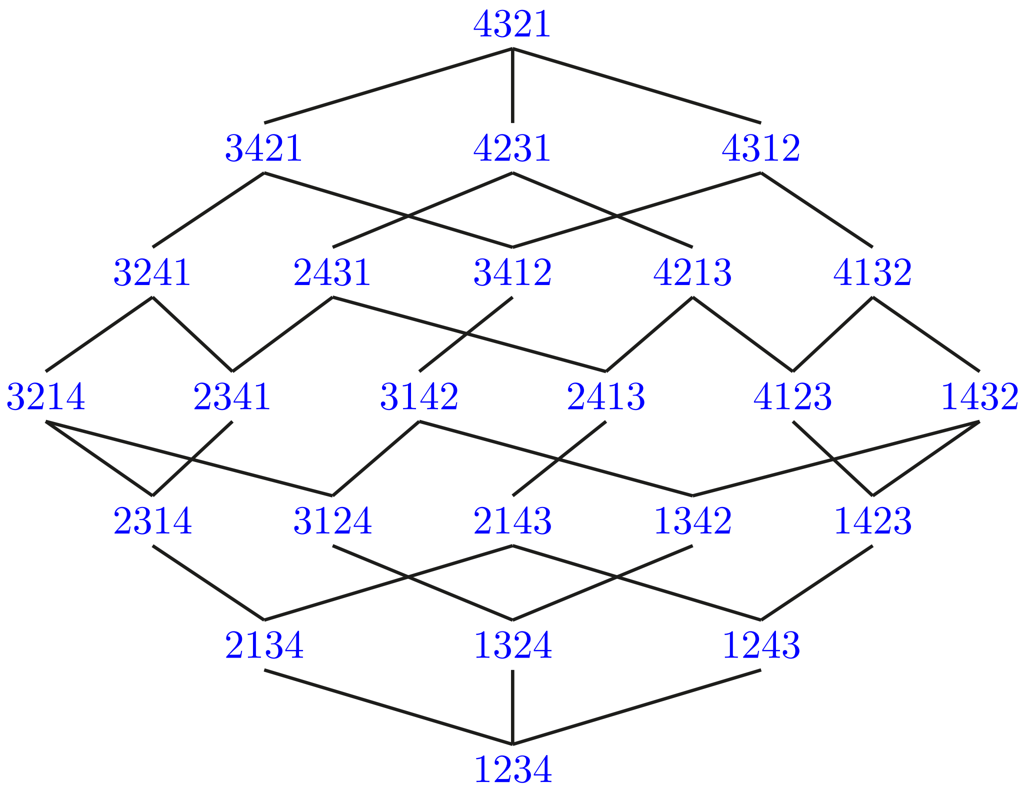
related papers: arXiv:2305.08471 — arXiv:1711.05353 — arXiv:2007.01008 — arXiv:2201.06896 — arXiv:2307.05940

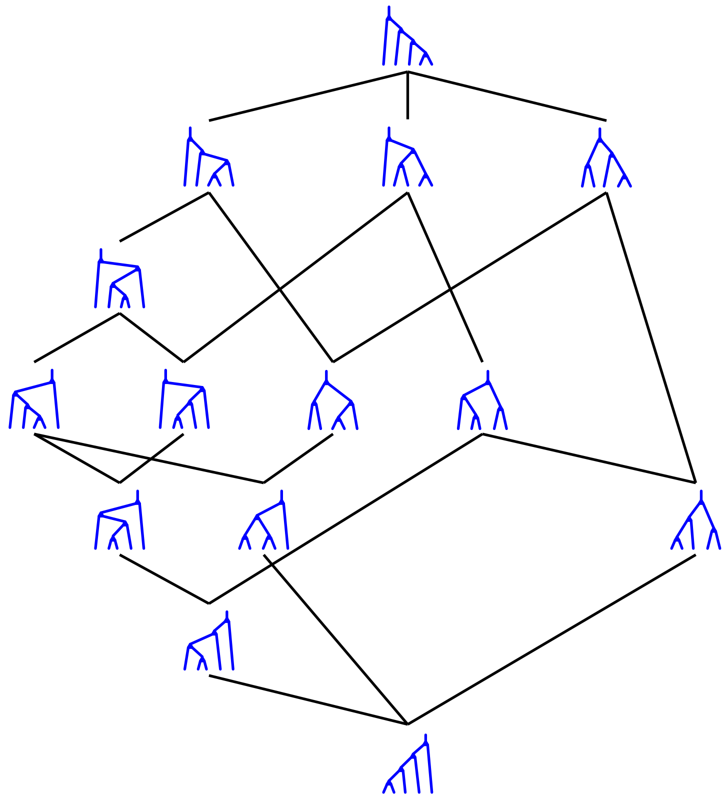
# PERMUTAHEDRA & ASSOCIAHEDRA

P.–Santos–Ziegler ('23)

# LATTICES: WEAK ORDER & TAMARI LATTICE

lattice = partially ordered set $L$ where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$



weak order = permutations of $[n]$
ordered by paths of simple transpositions

Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

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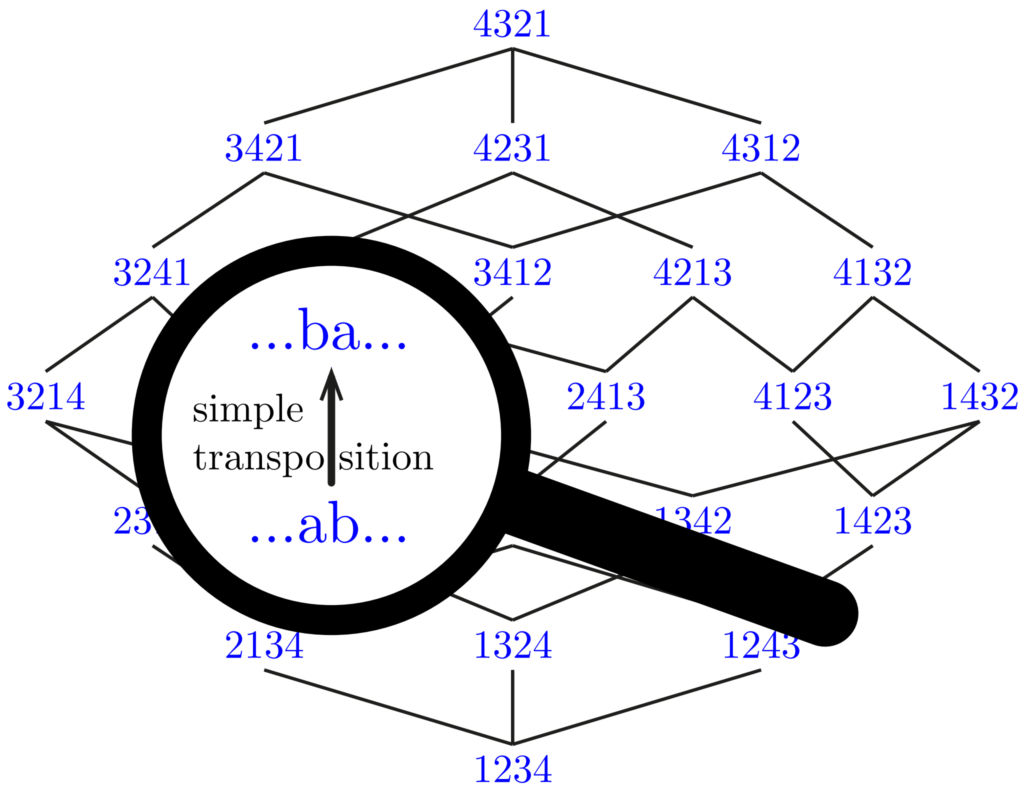


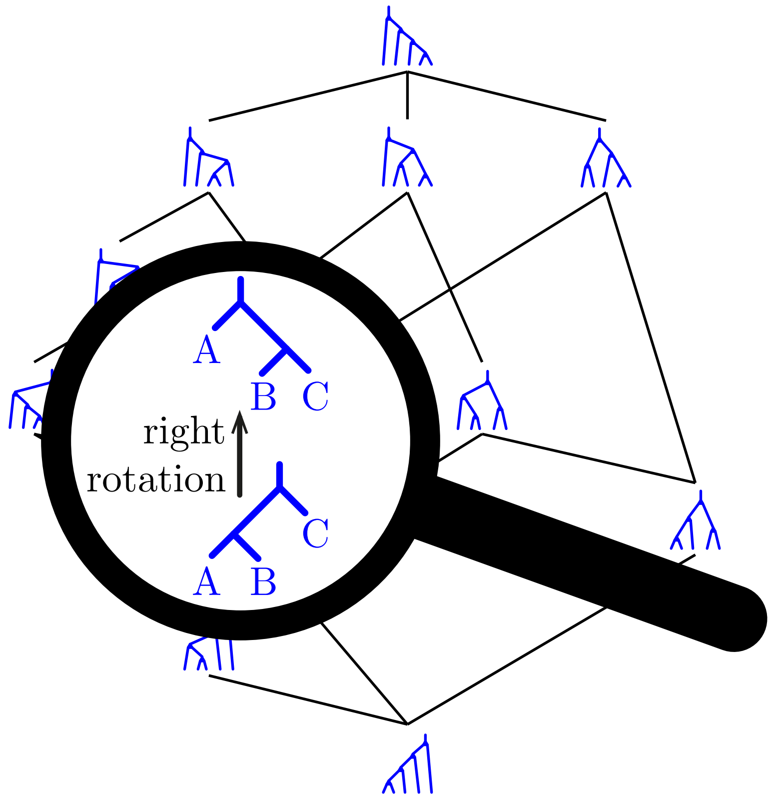


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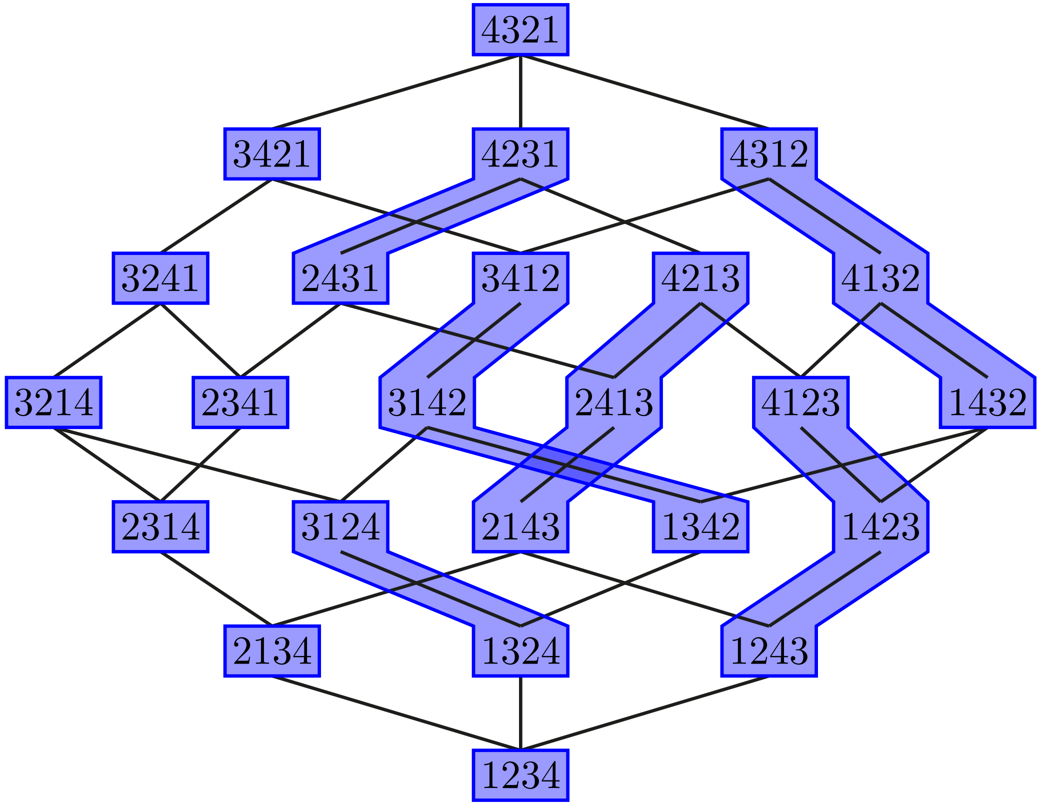


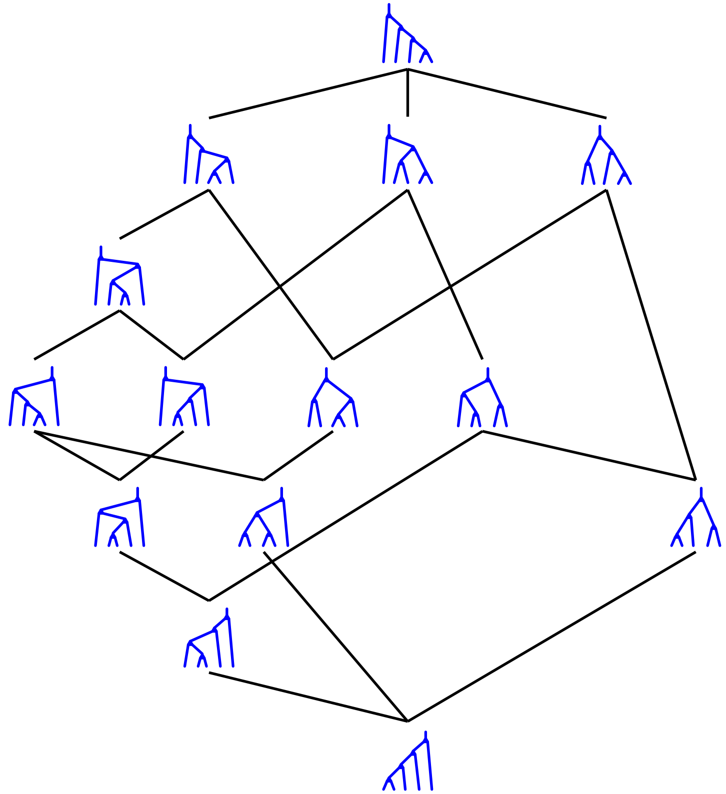

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sylvester congruence $=$ equivalence classes are sets of linear extensions of binary trees
$=$ equivalence classes are fibers of BST insertion
$=$ rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$

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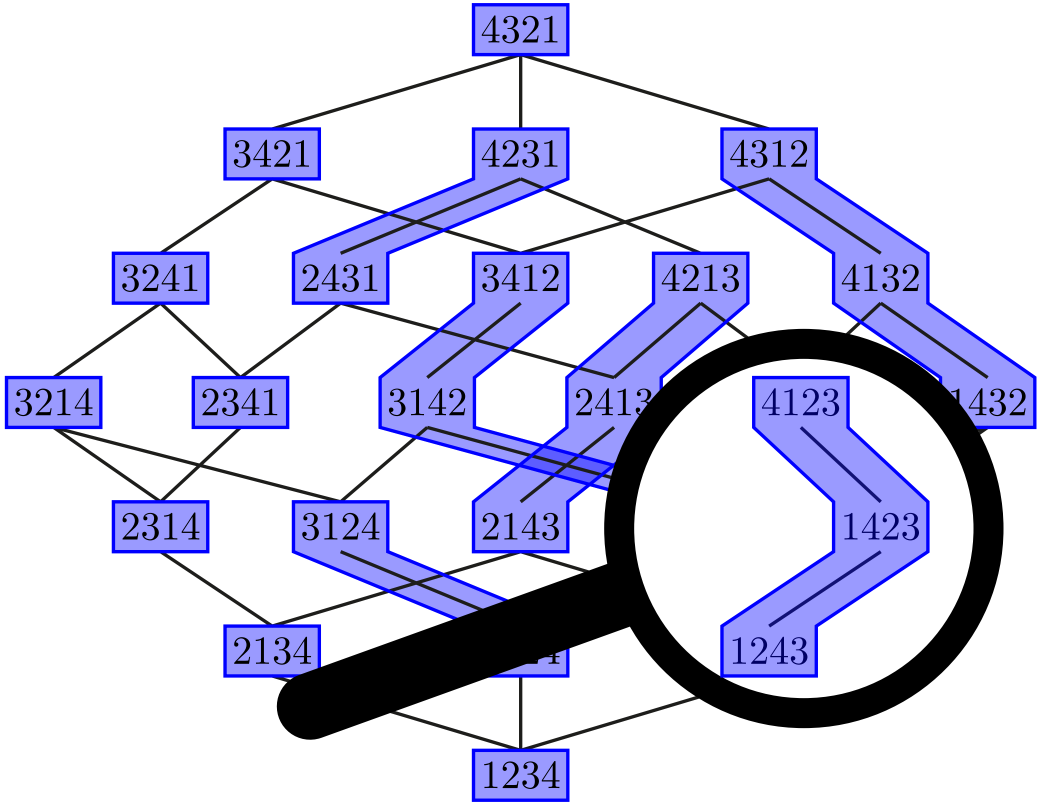


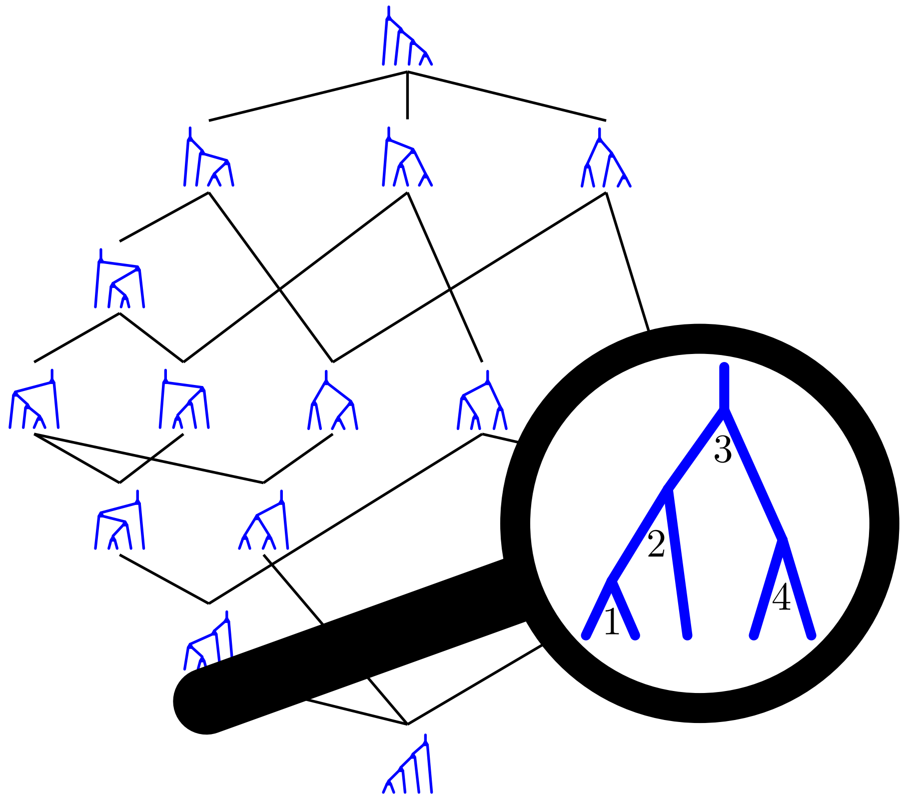


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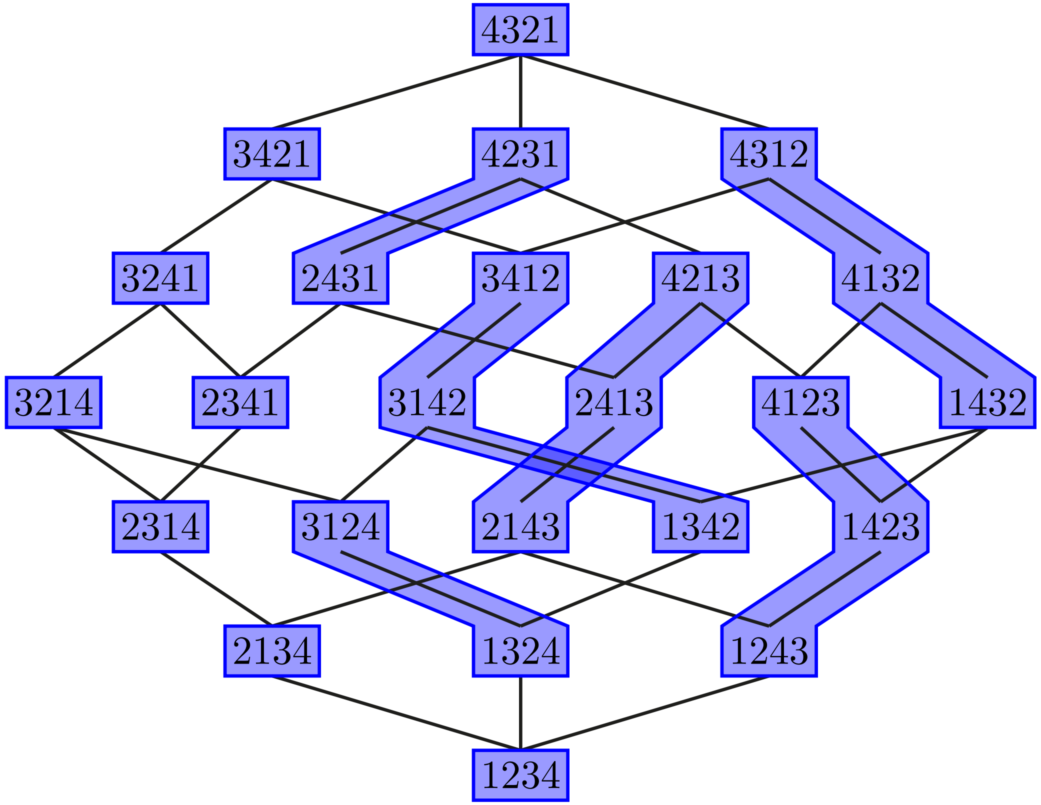


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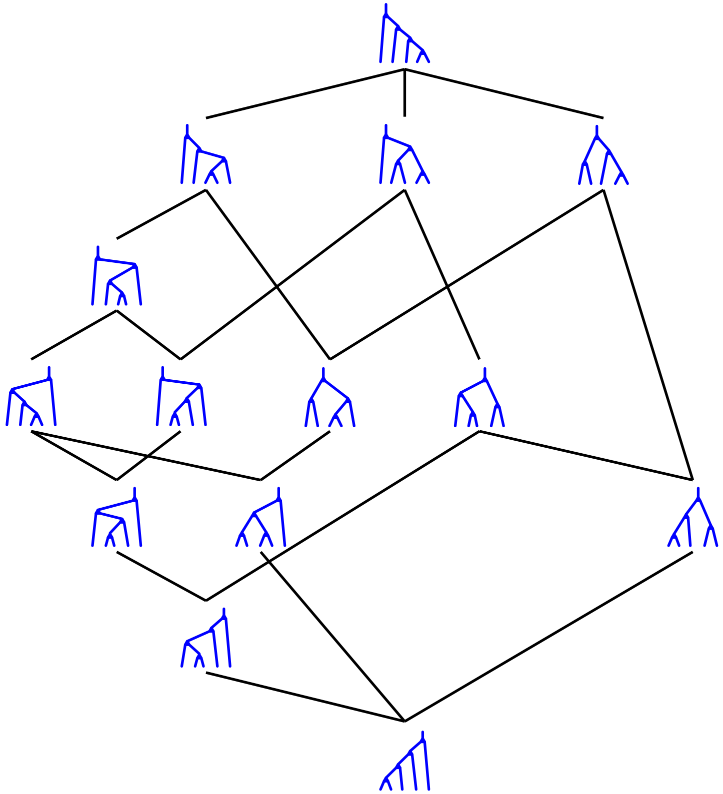


Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

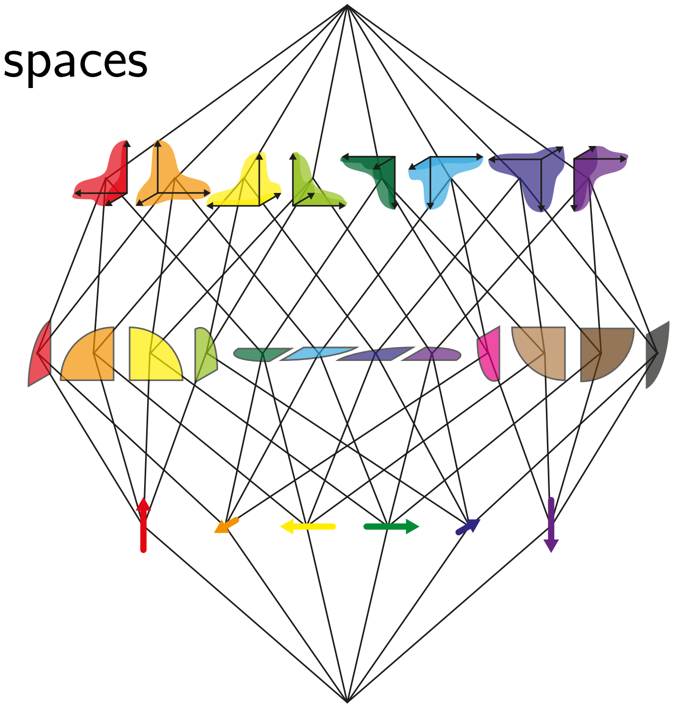
lattice congruence = equivalence relation $\equiv$ which respects meets and joins

$x \equiv x'$ and $y \equiv y' \implies x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$

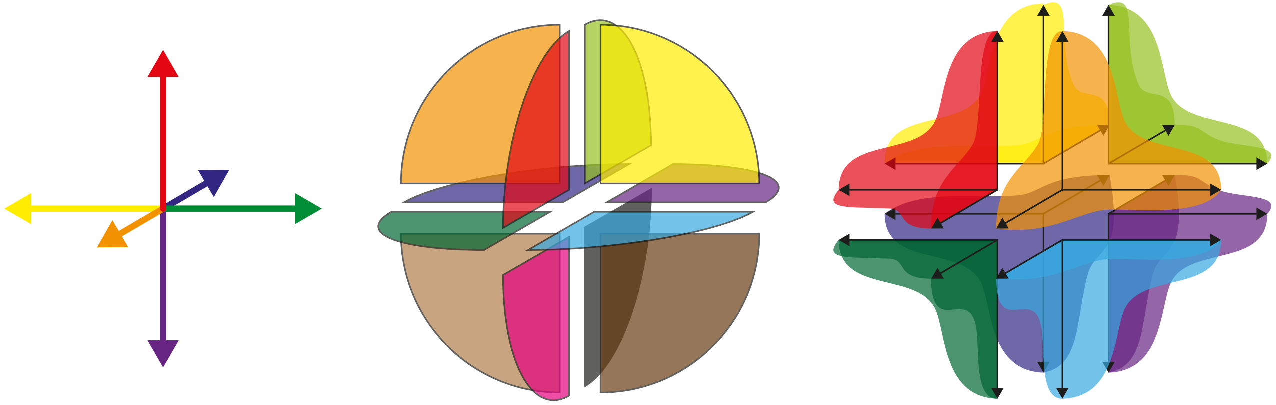
quotient lattice = lattice on classes with $X \le Y \iff \exists\, x \in X,\ y \in Y, x \le y$

# FANS: BRAID FAN & SYLVESTER FAN

<u>polyhedral cone</u> = positive span of a finite set of vectors
= intersection of a finite set of linear half-spaces

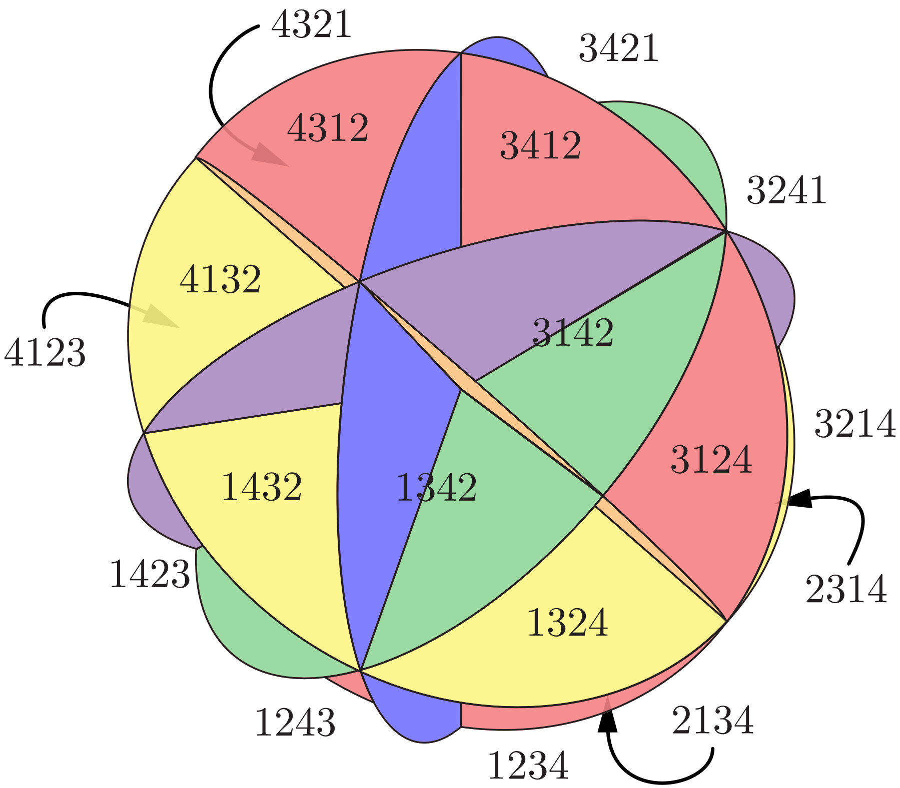


<u>fan</u> = collection of polyhedral cones closed by faces
and where any two cones intersect along a face



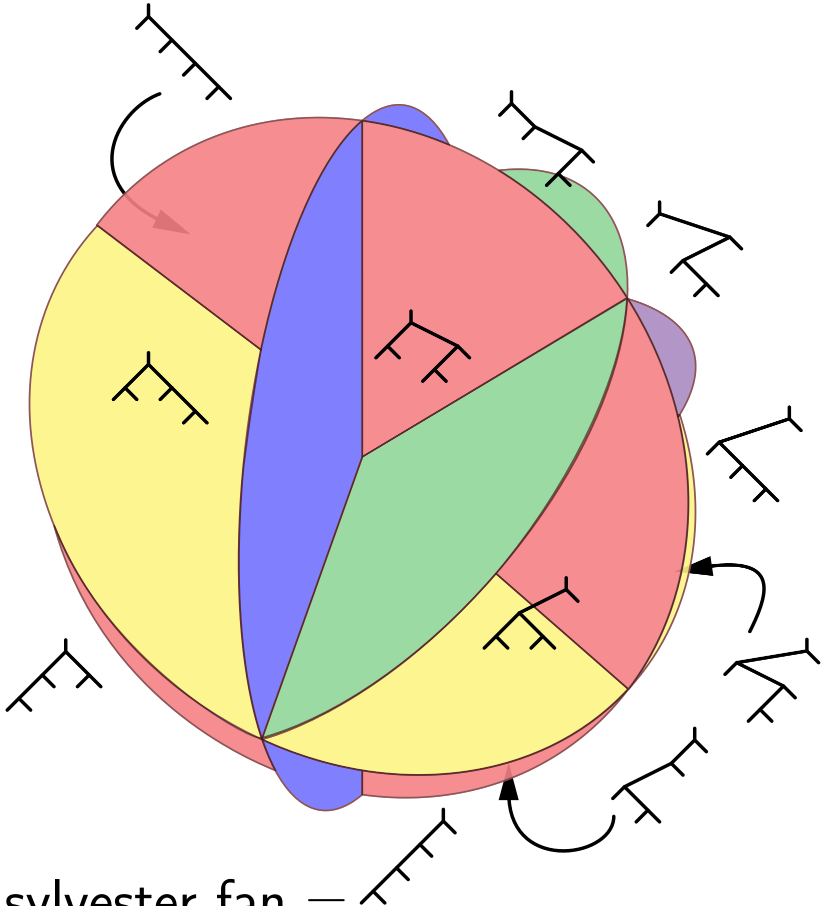
# FANS: BRAID FAN & SYLVESTER FAN

fan = collection of polyhedral cones closed by faces and intersecting along faces




braid fan =

$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$



sylvester fan =

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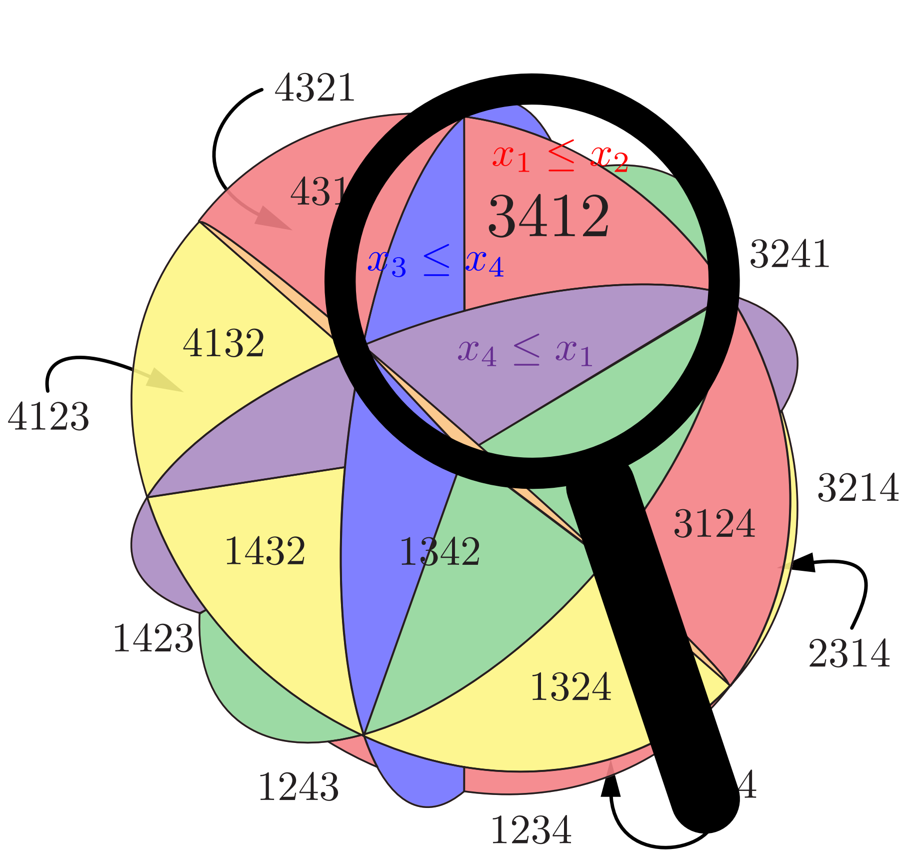


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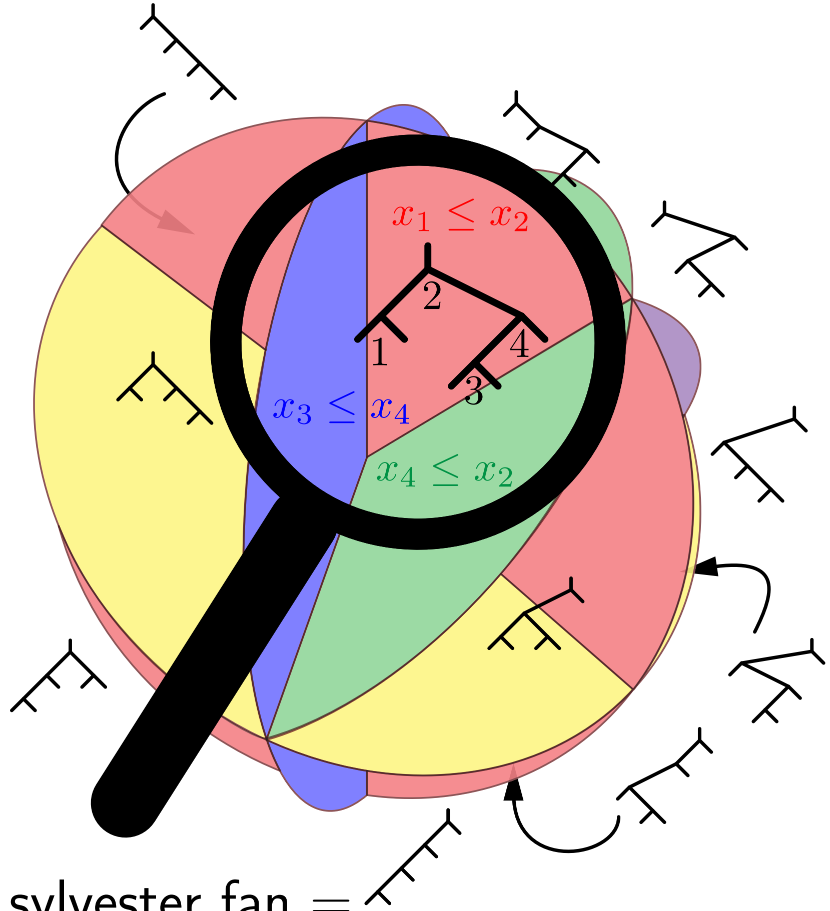


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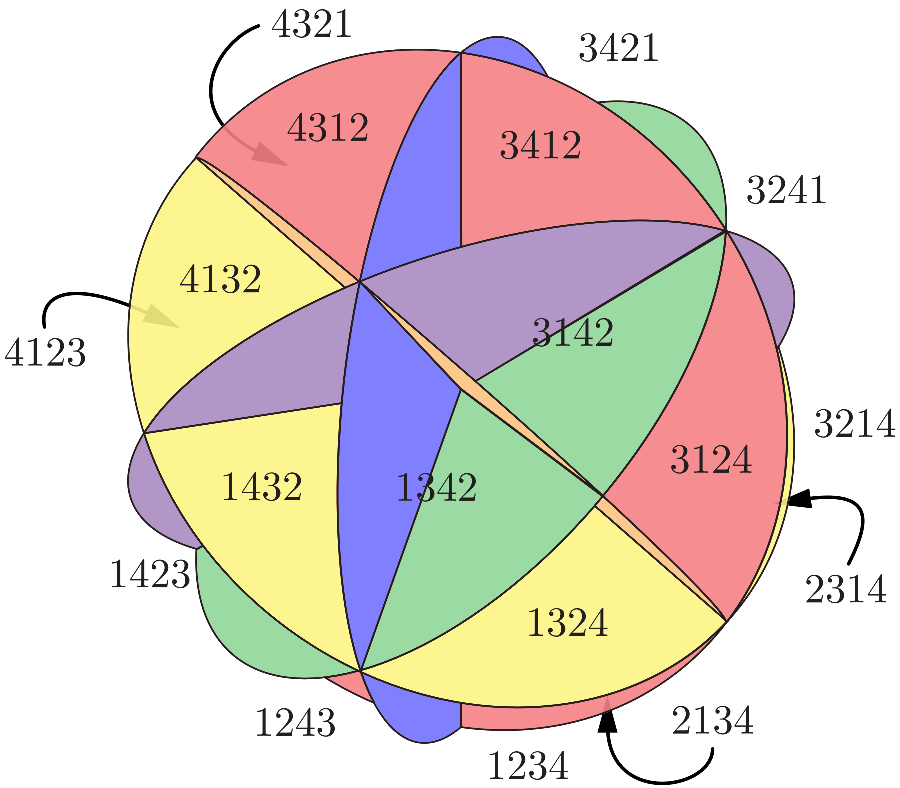


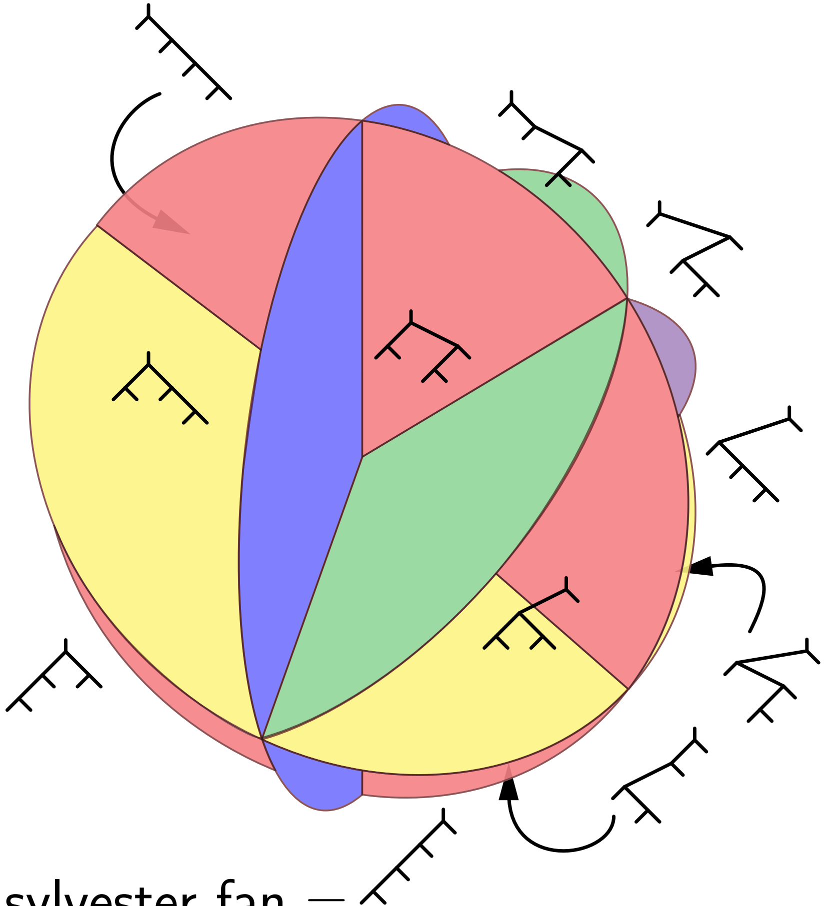

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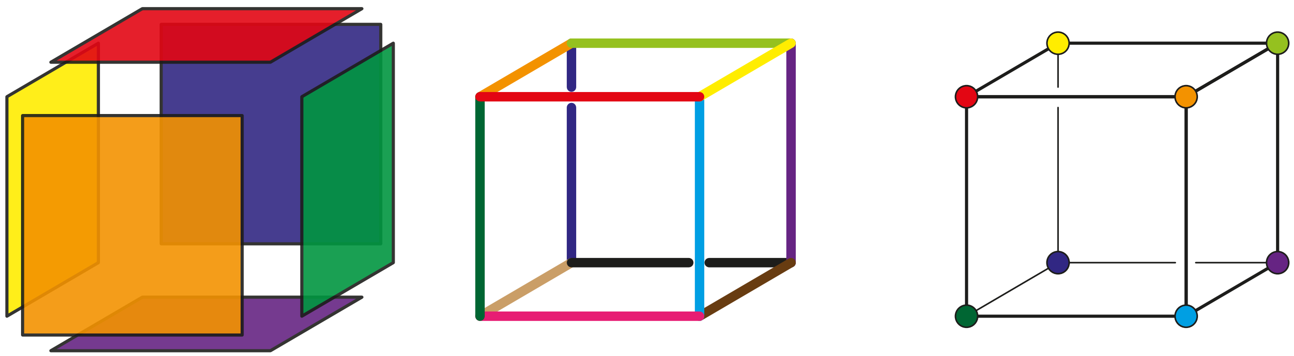
quotient fan $=$ $\mathbb{C}(T)$ is obtained by glueing $\mathbb{C}(\sigma)$ for all linear extensions $\sigma$ of $T$

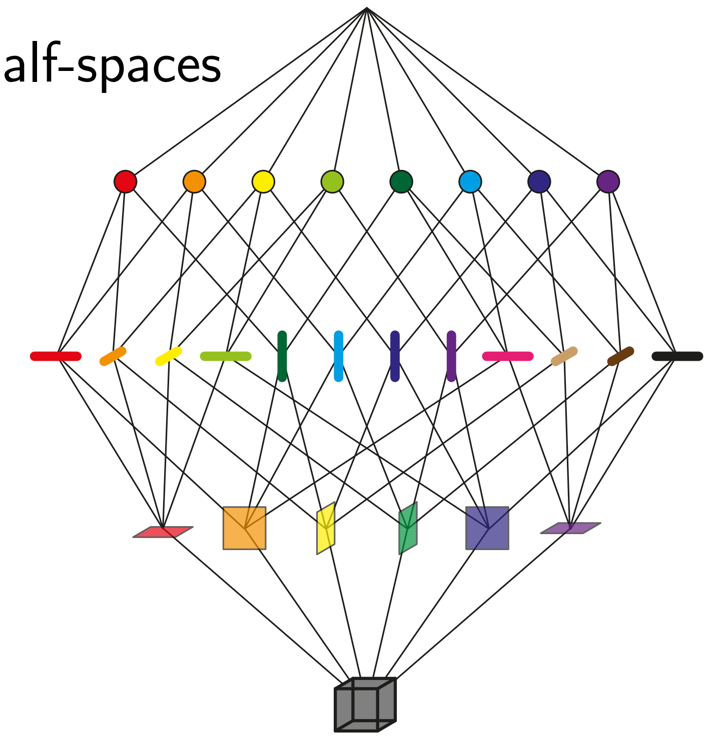
# POLYTOPES: PERMUTAHEDRON & ASSOCIAHEDRON

polytope = convex hull of a finite set of points
= bounded intersection of a finite set of affine half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations





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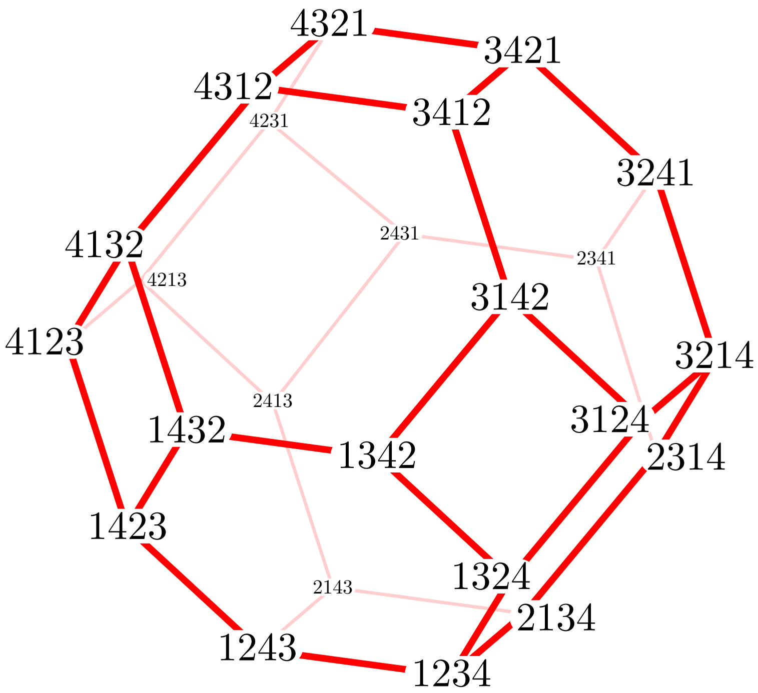


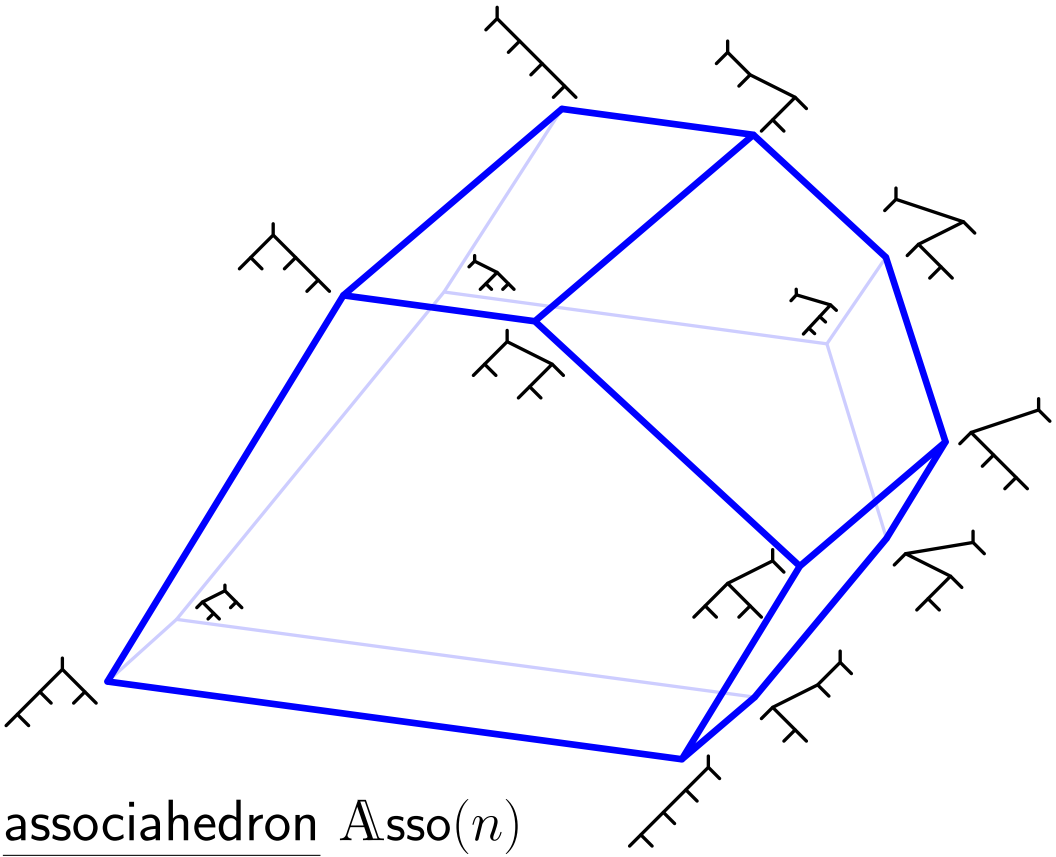

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$= \mathrm{conv}\left\{[\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n\right\}$

$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \mathbb{H}_J$

where $\mathbb{H}_J = \left\{\boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \geq \binom{|J|+1}{2}\right\}$

associahedron $\mathbb{A}\mathrm{sso}(n)$

$= \mathrm{conv}\left\{[\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; T \text{ binary tree}\right\}$

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Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)

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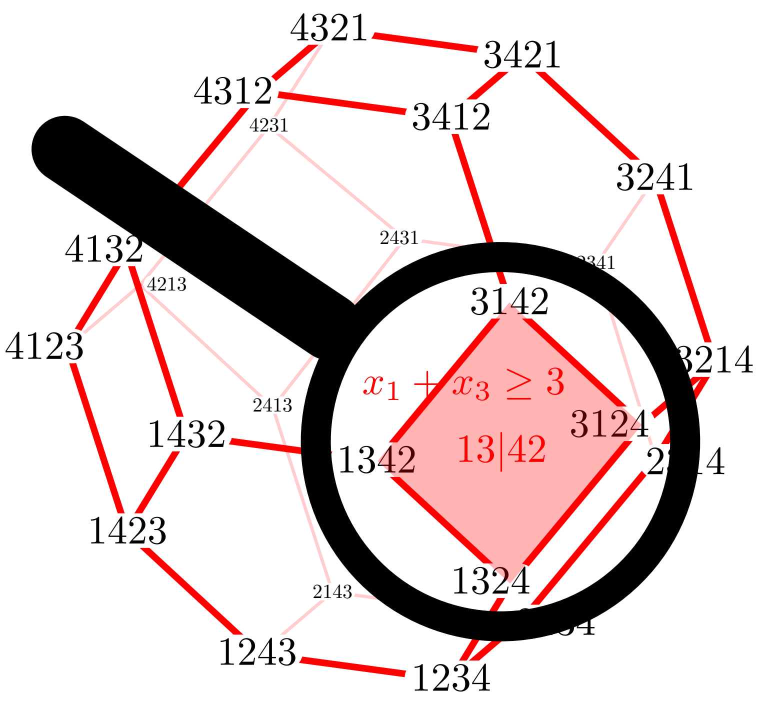


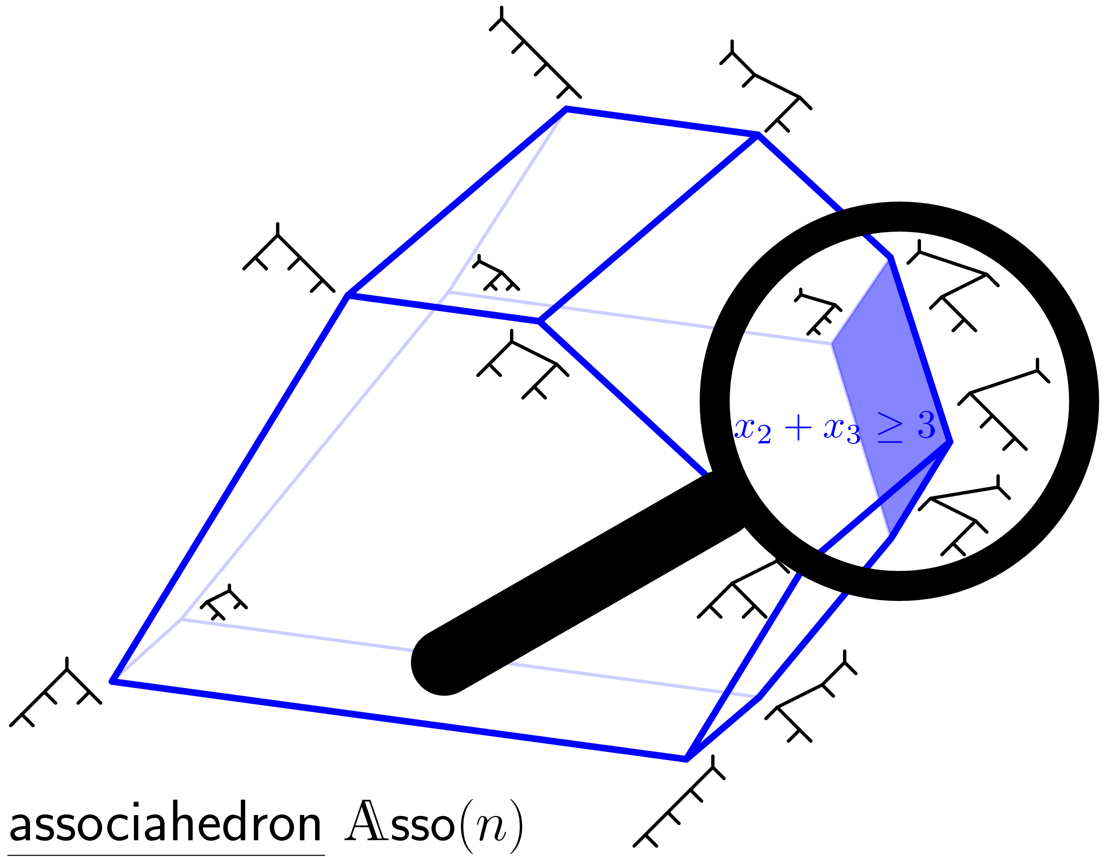


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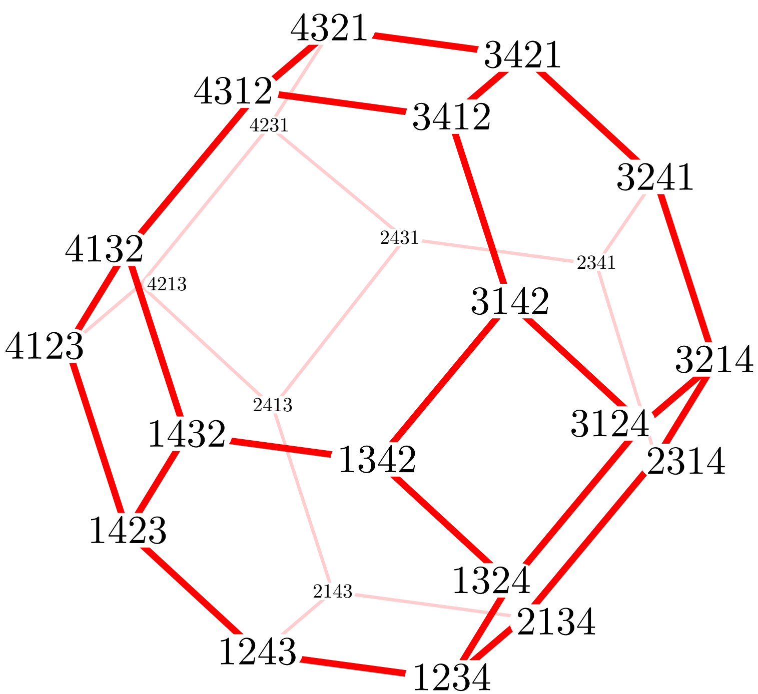


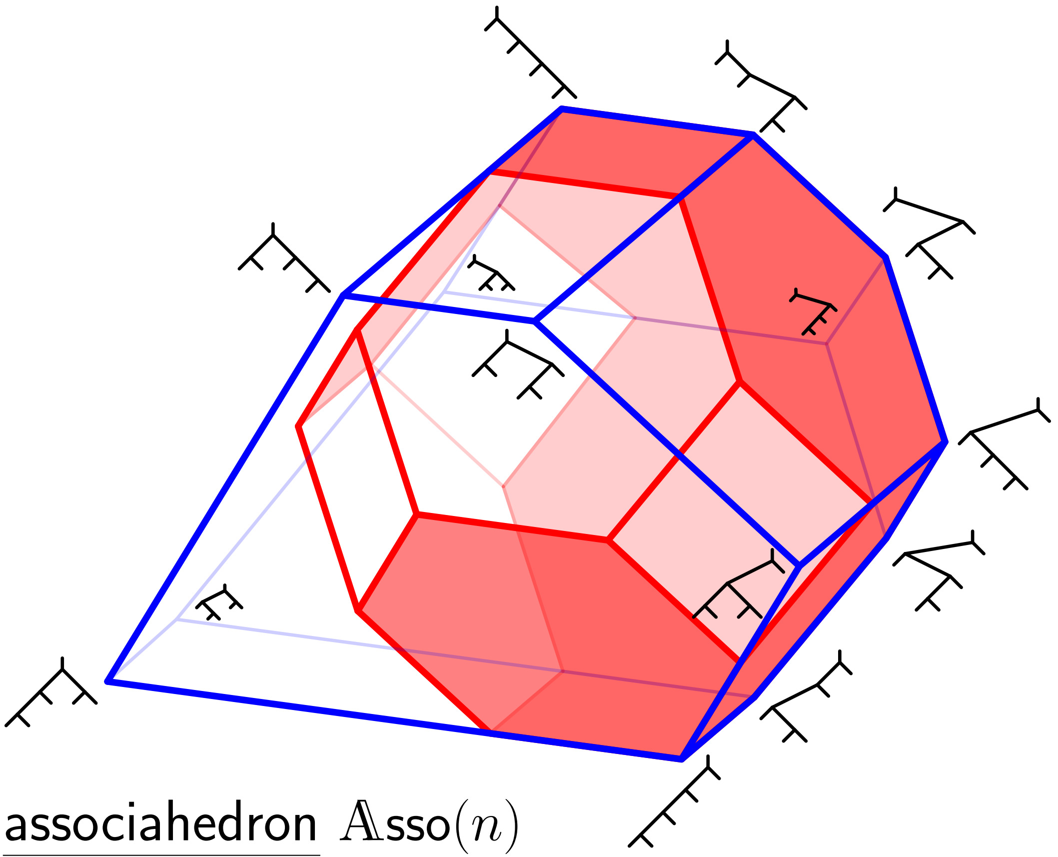

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POLYWOOD

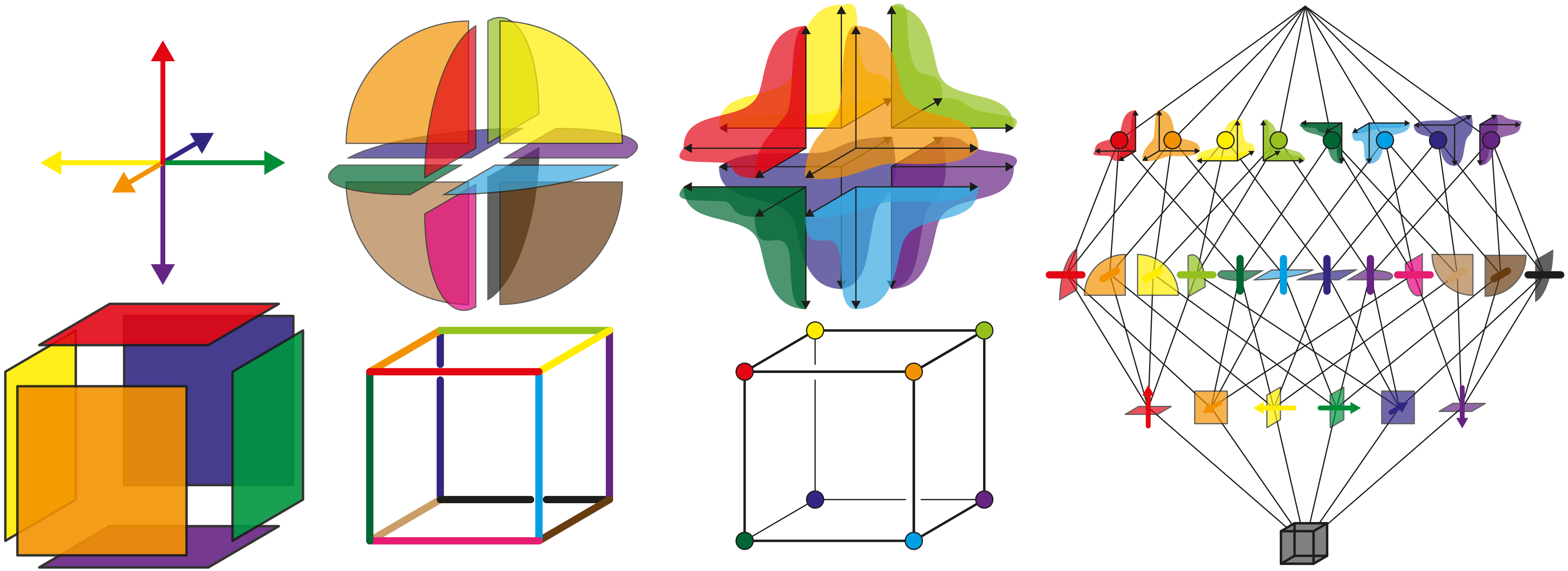
# LATTICES – FANS – POLYTOPES

permutahedron $\mathbb{P}\mathrm{erm}(n)$

$\Longrightarrow$ braid fan

associahedron $\mathbb{A}\mathrm{sso}(n)$

$\Longrightarrow$ Sylvester fan



face $\mathbb{F}$ of polytope $\mathbb{P}$

normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ = { normal cone of $\mathbb{F}$ | $\mathbb{F}$ face of $\mathbb{P}$ }

# LATTICES – FANS – POLYTOPES

permutahedron $\mathbb{P}\mathrm{erm}(n)$

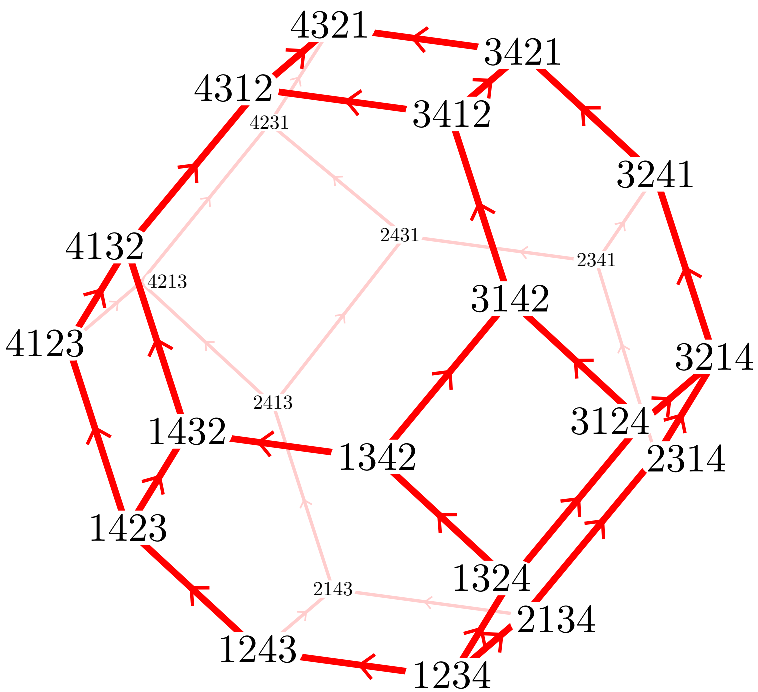
$\Longrightarrow$ braid fan

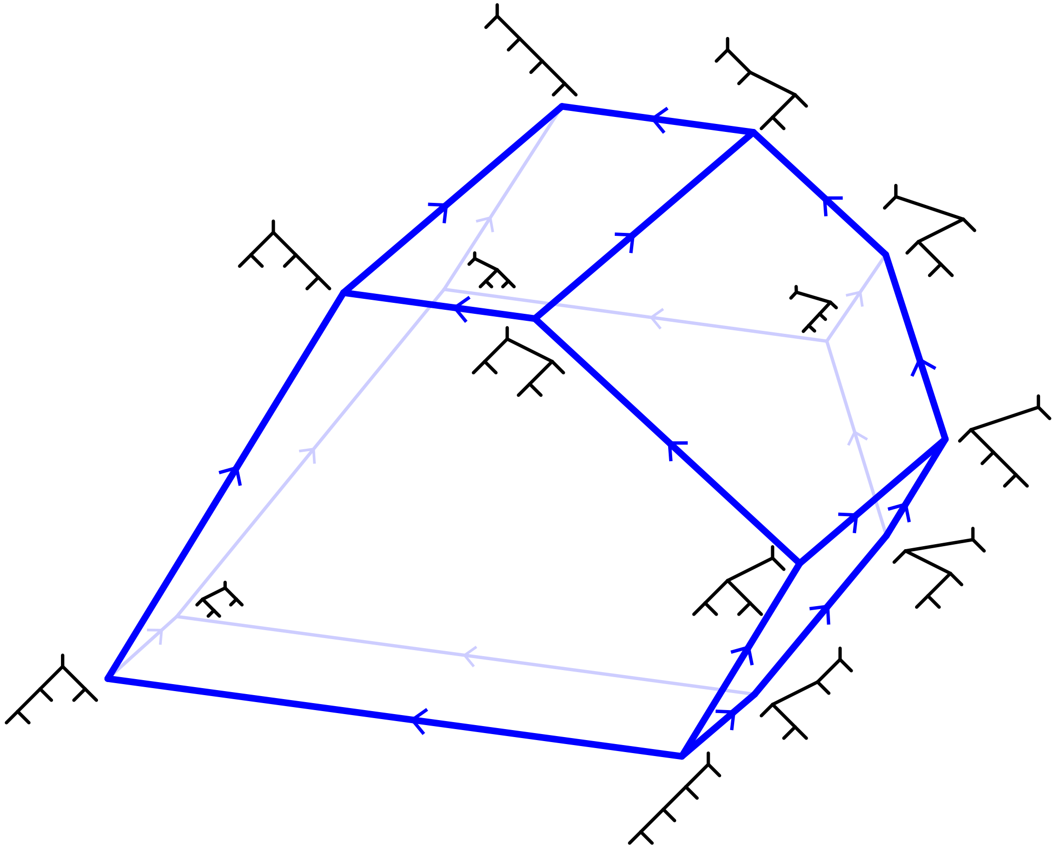
$\Longrightarrow$ weak order on permutations

associahedron $\mathbb{A}\mathrm{sso}(n)$

$\Longrightarrow$ Sylvester fan

$\Longrightarrow$ Tamari lattice on binary trees



# HOPF ALGEBRAS: MALVENUTO–REUTENAUER & LODAY–RONCO

product = linear map $\cdot : V \otimes V \to V$ = a tool to combine two elements (glue)
coproduct = linear map $\triangle : V \to V \otimes V$ = a tool to decompose an element (scisors)
Hopf algebra = $(V, \cdot, \triangle)$ such that $\triangle(a \cdot b) = \triangle(a) \cdot \triangle(b)$

Two operations on permutations:

shuffle $12 \sqcup\!\sqcup 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$
convol. $12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$

| | Malvenuto–Reutenauer | $\supseteq$ | Loday–Ronco |
|---|---|---|---|
| vector space | $\langle\, \mathbb{F}_\sigma \mid \sigma$ permutation of any size $\rangle$ | | $\langle\, \mathbb{P}_T \mid T$ binary tree of any size $\rangle$ |
| product | $\mathbb{F}_\rho \cdot \mathbb{F}_\sigma = \sum_{\tau \in \rho \sqcup\!\sqcup \sigma} \mathbb{F}_\tau = \sum_{\rho\backslash\sigma \le \tau \le \rho/\sigma} \mathbb{F}_\tau$ | | $\mathbb{P}_R \cdot \mathbb{P}_S = \sum_{R\backslash S \le \tau \le R/S} \mathbb{P}_T$ |
| coproduct | $\triangle(\mathbb{F}_\tau) = \sum_{\tau \in \rho \star \sigma} \mathbb{F}_\rho \otimes \mathbb{F}_\sigma$ | | $\triangle(\mathbb{P}_T) = \sum_{\substack{R_1 \cdots R_k \| S \\ \text{cut of } T}} \big(\prod_{i \in [k]} \mathbb{P}_{R_i}\big) \otimes \mathbb{P}_S$ |

Hopf subalgebra = define $\mathbb{P}_T = \sum_\tau \mathbb{F}_\tau$ over all permutations $\tau$ in the BST fiber of $T$

# OPEN PROBLEM: COMPUTING SHORTEST ROTATION PATHS

REM. The diameter of the permutahedron $\mathbb{P}\mathsf{erm}(n)$ is $\binom{n}{2}$.
The simple transposition distance between two permutations $\sigma, \tau$ of $[n]$ is the number of inversions in $\sigma\tau^{-1}$.

THM. The diameter of the associahedron $\mathbb{A}\mathsf{sso}(n)$ is precisely $2n-6$ when $n > 10$.

Sleator–Tarjan–Thurston ('88) — Dehornoy ('10) — Pournin ('14)

QU. Is it polynomial to determine the rotation distance between two binary trees?

Hanke–Ottmann–Schuierer ('96)

The flip distance problem is NP-complete on

- triangulations of polygons with holes
- triangulations of planar point sets

Lubiw–Pathak ('12) — Aichholzer–Mulzer–Pilz ('12)

# QUOTIENTOPES

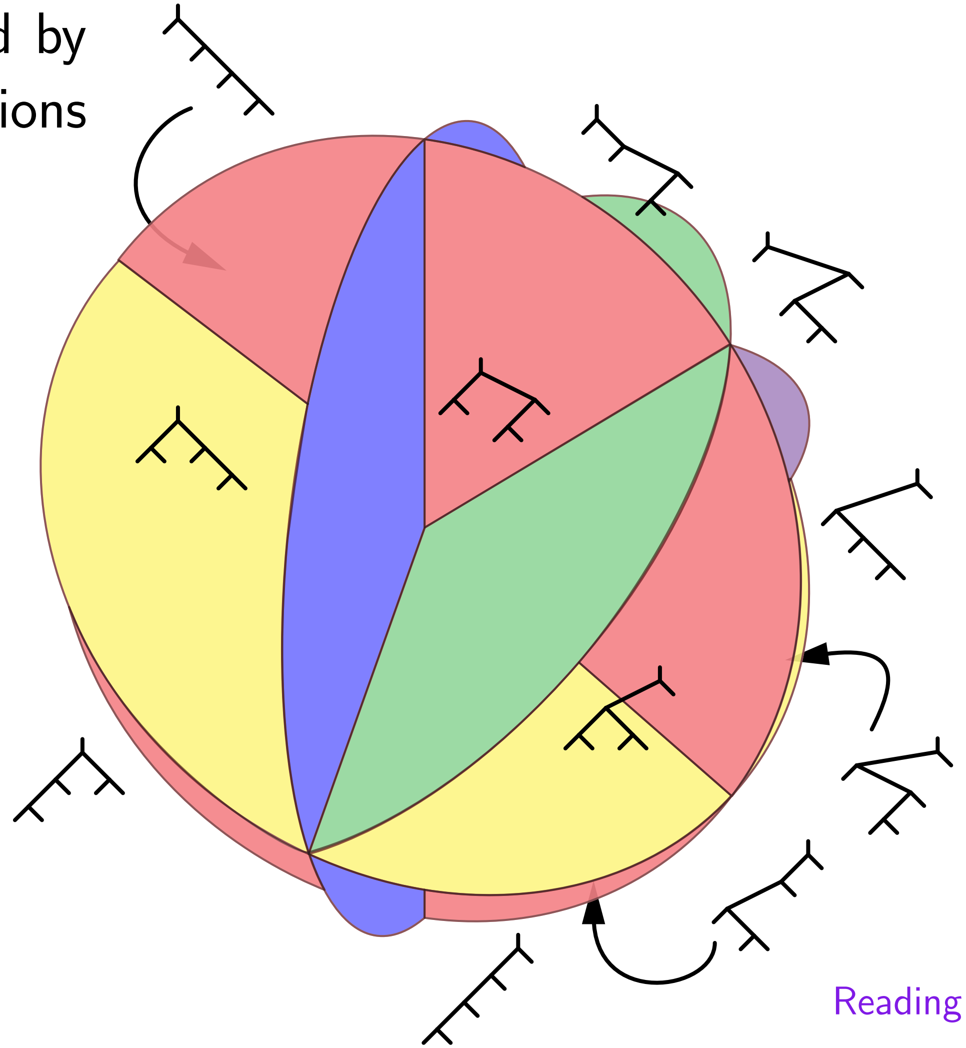
Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('23)

# QUOTIENT FAN

lattice congruence = equivalence relation on $L$ compatible with meets and joins:

$$x \equiv x' \text{ and } y \equiv y' \text{ implies } x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

quotient fan $\mathcal{F}_{\equiv}$ = chambers are obtained by glueing the chambers $\mathbb{C}(\sigma)$ of the permutations $\sigma$ in the same congruence class of $\equiv$



Reading ('05)

# QUOTIENT FANS & QUOTIENTOPES

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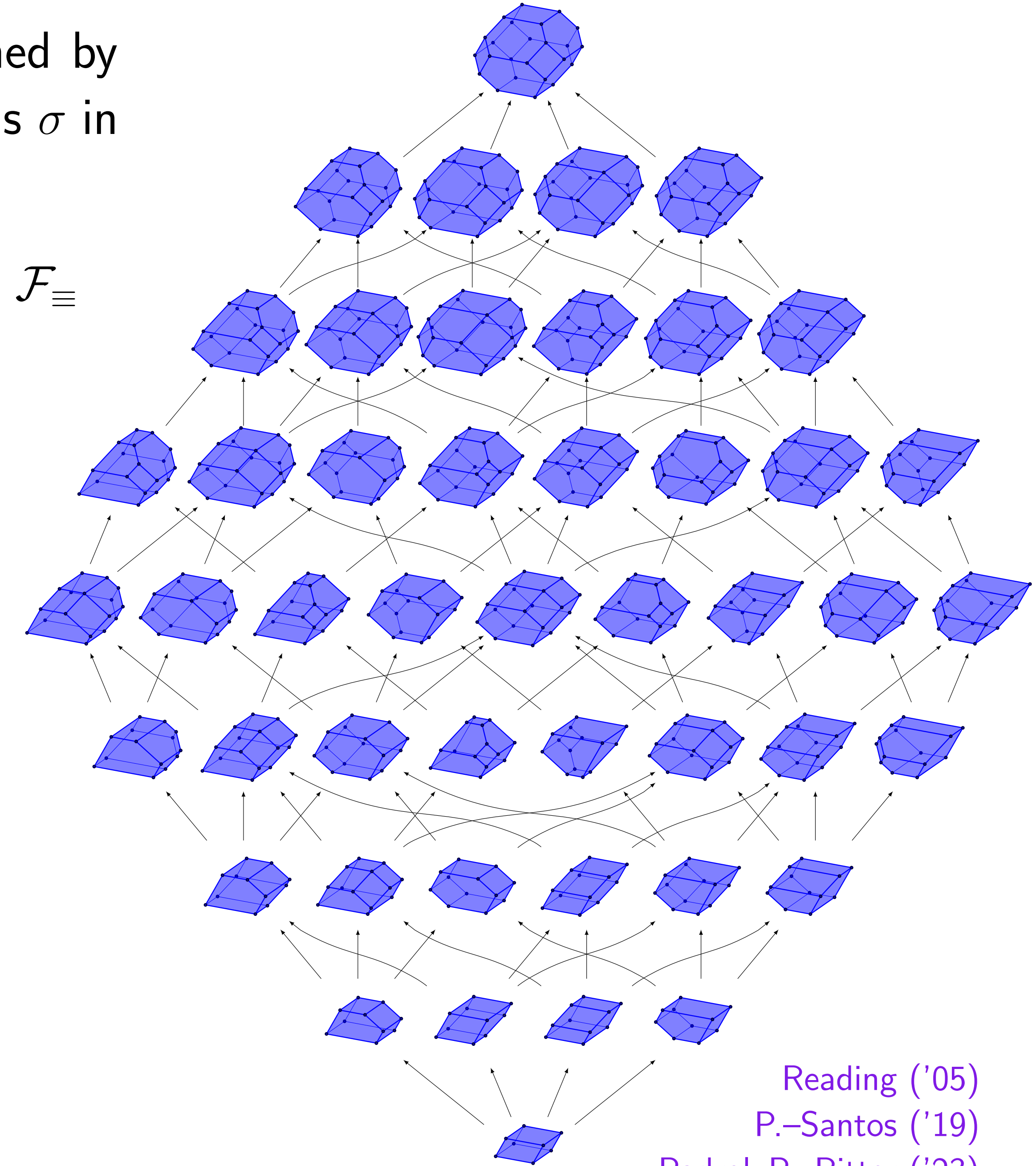
quotientope = polytope with normal fan $\mathcal{F}_{\equiv}$



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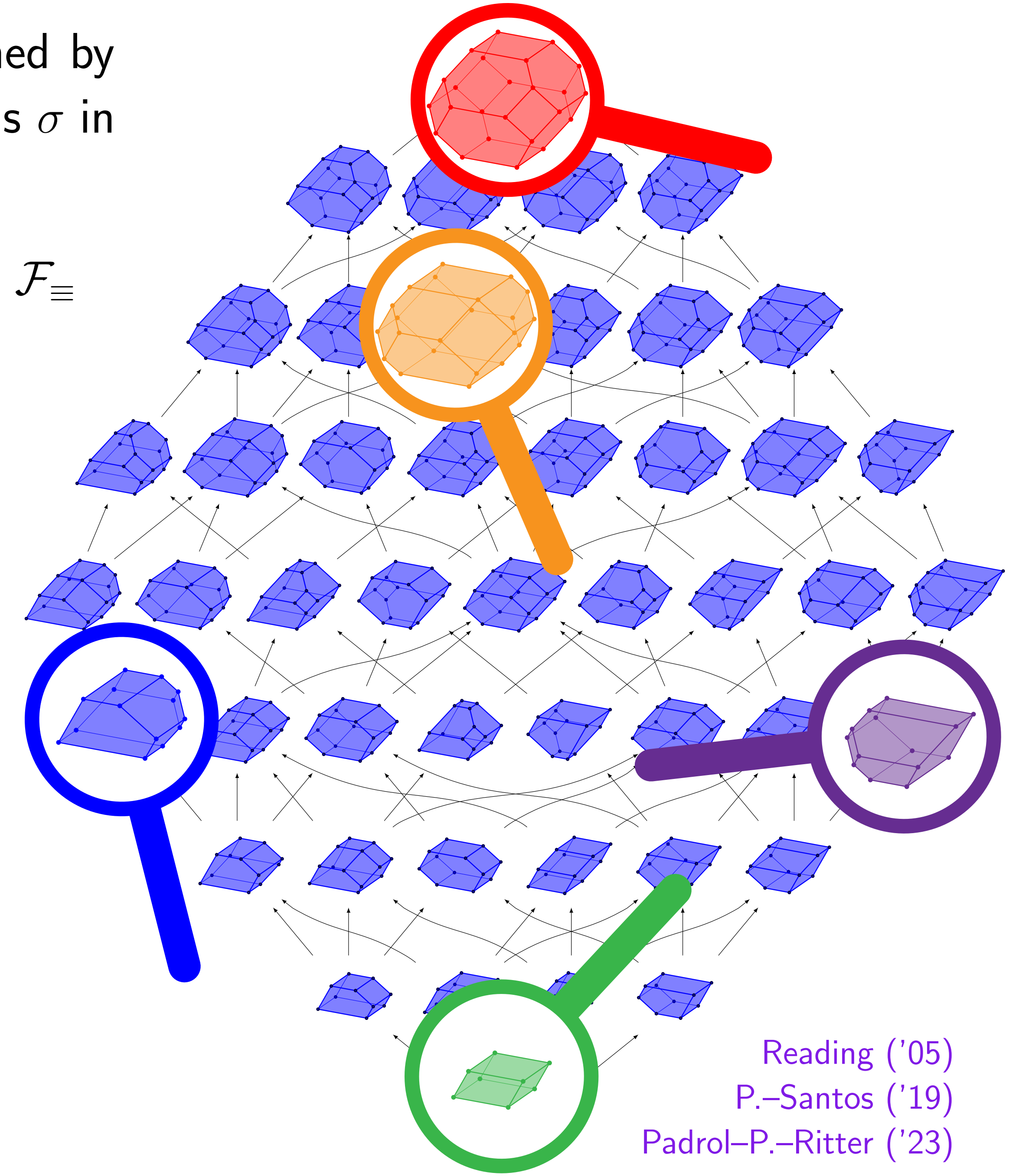

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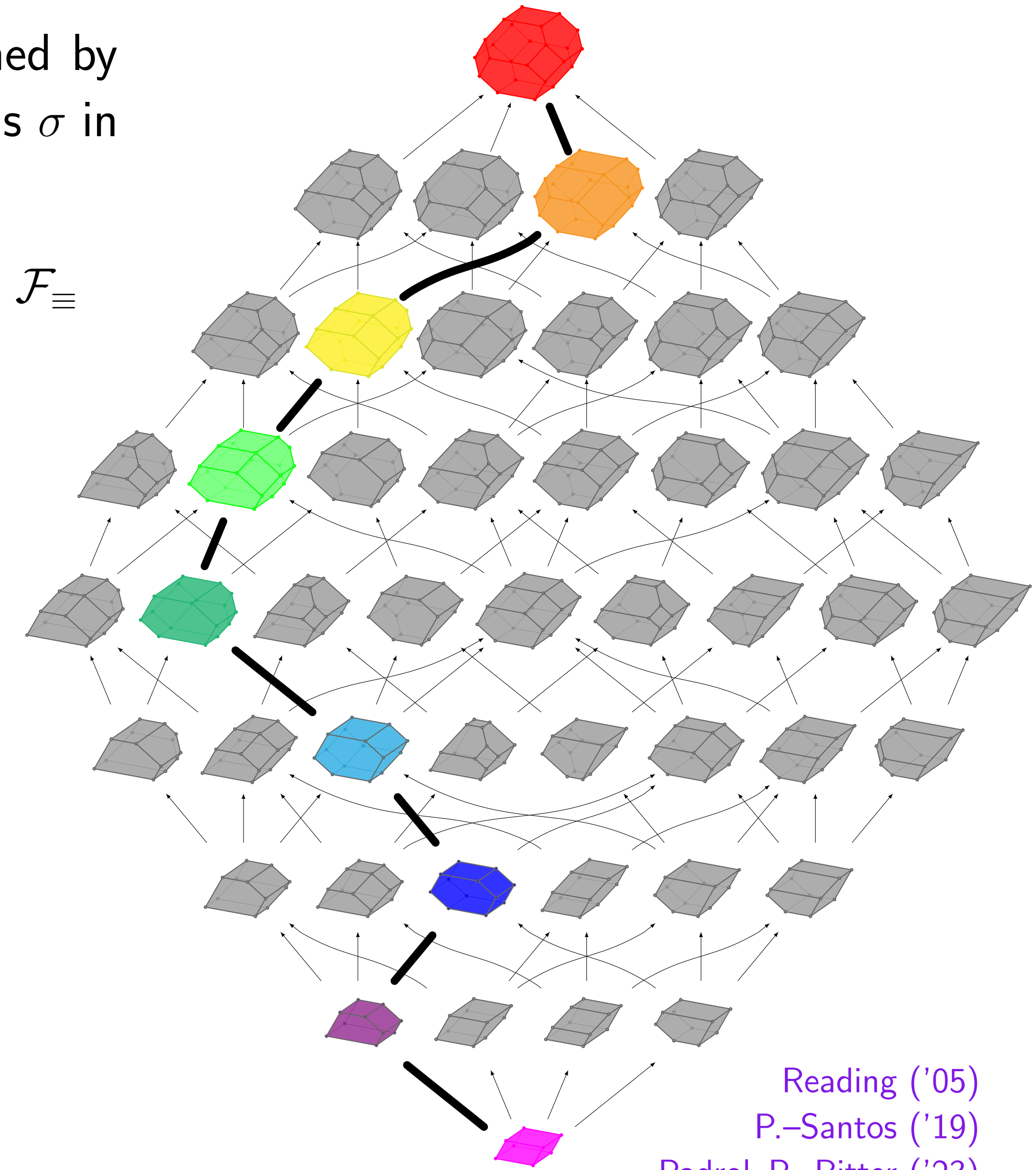

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POLYWOOD

Reading ('05)
P.–Santos ('19)
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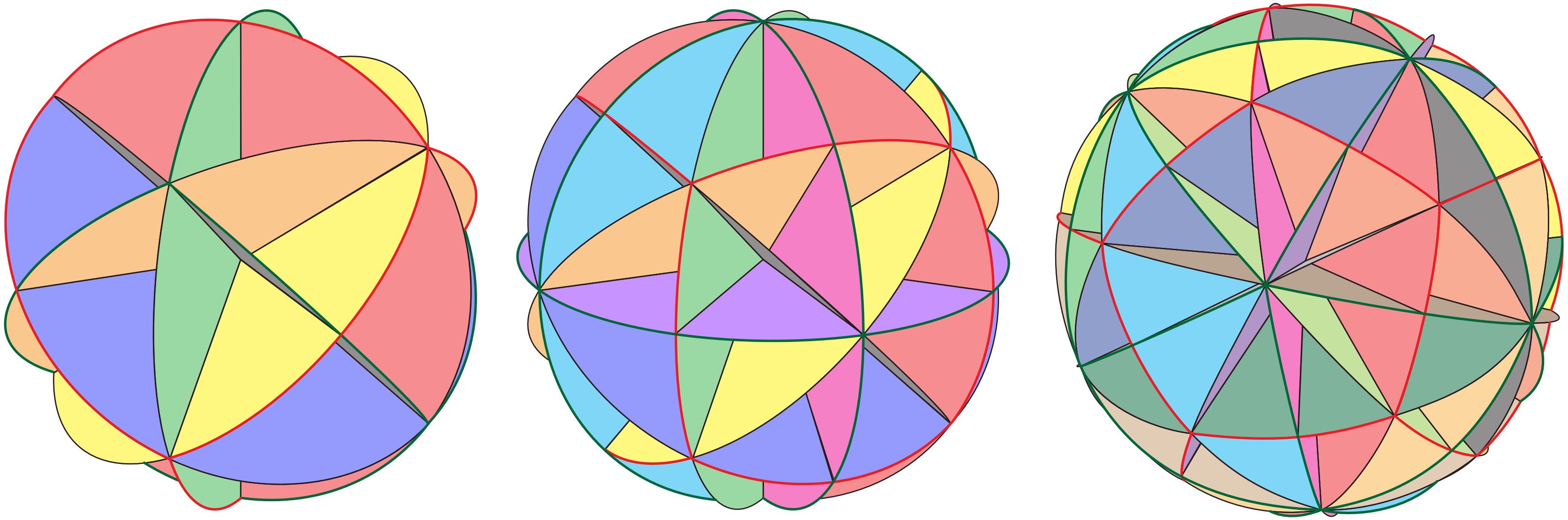
# OPEN PROBLEM: QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$

base region $B$ = distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$

inversion set of a region $C$ = set of hyperplanes of $\mathcal{H}$ that separate $B$ and $C$

poset of regions $\mathrm{PR}(\mathcal{H}, B)$ = regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets



THM. If $\mathrm{PR}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a congruence of $\mathrm{PR}(\mathcal{H}, B)$, the cones obtained by glueing the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan $\mathcal{F}_\equiv$

Reading ('05)

QU. Is the quotient fan $\mathcal{F}_\equiv$ always polytopal?

# DEFORMED PERMUTAHEDRA

Edmonds ('70)
Postnikov ('09)

# DEFORMED PERMUTAHEDRA

deformation of a polytope $\mathbb{P}$ = polytope $\mathbb{Q}$ such that

- $\mathbb{Q}$ is obtained from $\mathbb{P}$ by moving its vertices such that edge directions are preserved
- $\mathbb{Q}$ is obtained from $\mathbb{P}$ by translating its inequalities without passing through a vertex
- the normal fan of $\mathbb{P}$ refines the normal fan of $\mathbb{Q}$
- $\mathbb{Q}$ is a weak Minkowski summand of $\mathbb{P}$, i.e. there is $\mathbb{R}$ and $\lambda > 0$ such that $\lambda\mathbb{P} = \mathbb{Q} + \mathbb{R}$

POLYWOOD

deformed permutahedron = polymatroid = generalized permutahedron

[Edmonds ('70)] [Postnikov ('09)]

# REMOVAHEDRA VS. DEFORMED PERMUTAHEDRA

deformation of $\mathbb{P}$ = obtained by translating inequalities in the facet description of $\mathbb{P}$
removahedron of $\mathbb{P}$ = obtained by removing inequalities in the facet description of $\mathbb{P}$

outsidahedra
removahedra
permutrees

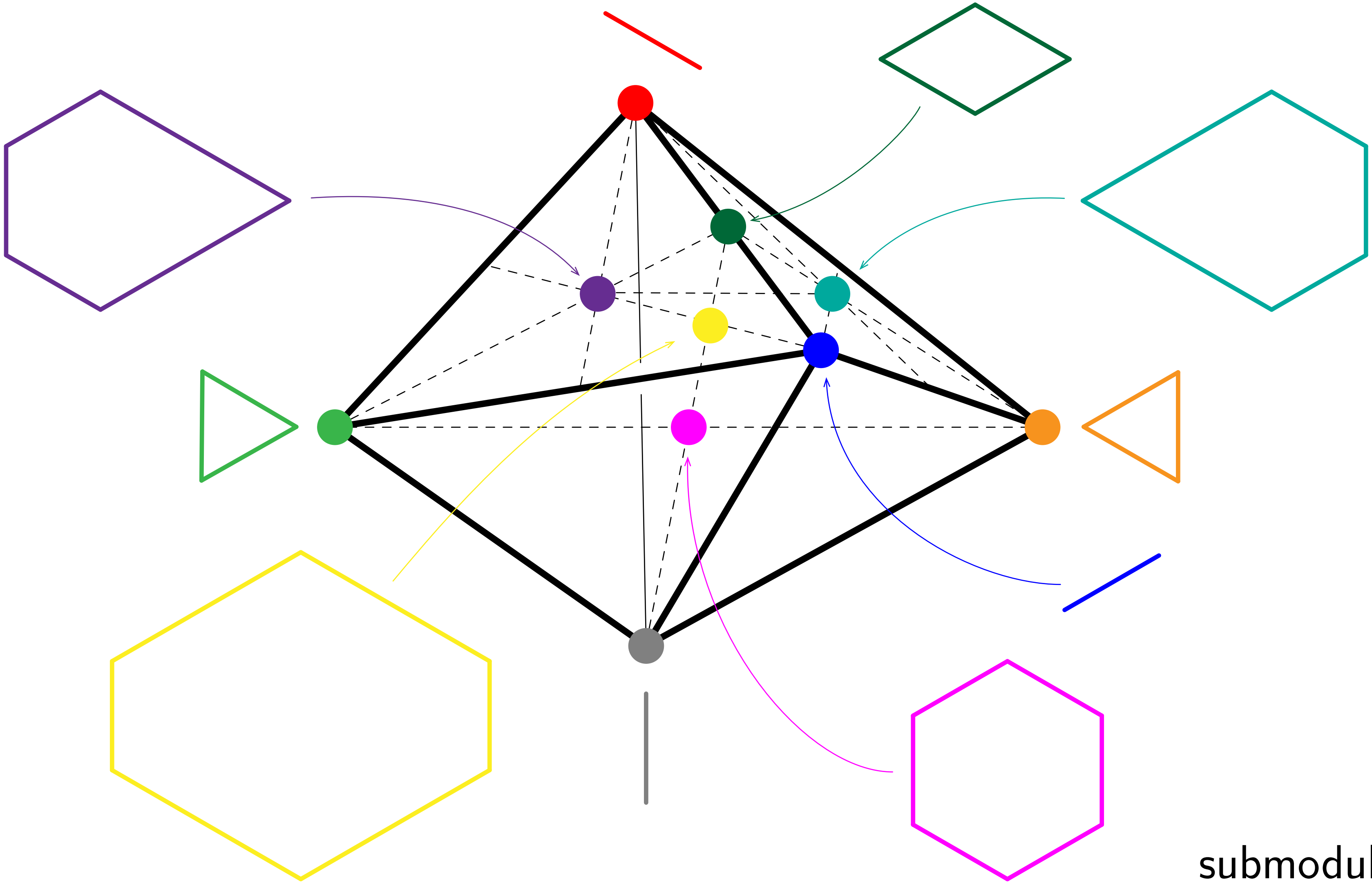
insidahedra
deformed permutahedra
quotientopes

POLYWOOD

# DEFORMATION CONE

deformation of a polytope $\mathbb{P}$ = polytope $\mathbb{Q}$ such that $\lambda\mathbb{P} = \mathbb{Q} + \mathbb{R}$ for some $\mathbb{R}$ and $\lambda > 0$

deformation cone of $\mathbb{P}$ = all deformations of $\mathbb{P}$ (under dilations and Minkowski sums)



submodular cone

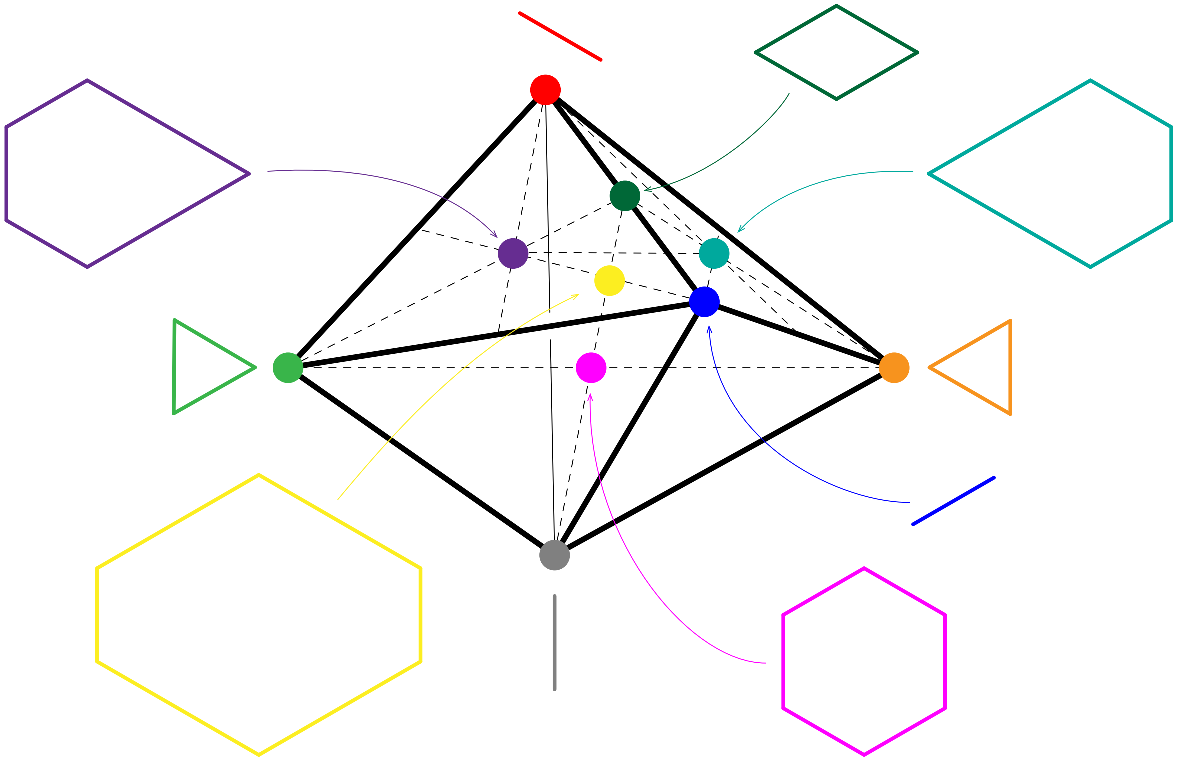
= deformation cone of $\mathbb{P}\mathrm{erm}(3)$

# OPEN PROBLEM: RAYS OF THE DEFORMATION CONE

THM. The deformation cone of the permutahedron $\mathbb{P}\mathrm{erm}(n)$ is (isomorphic to) the set of submodular functions $h : 2^{[n]} \to \mathbb{R}_{\geq 0}$ satisfying $h(\varnothing) = h([n]) = 0$ and the submodular inequalities $h(I) + h(J) \geq h(I \cap J) + h(I \cup J)$ for all $I, J \subseteq [n]$.

THM. The facets correspond to submodular inequalities where $|I \smallsetminus J| = |J \smallsetminus I| = 1$.

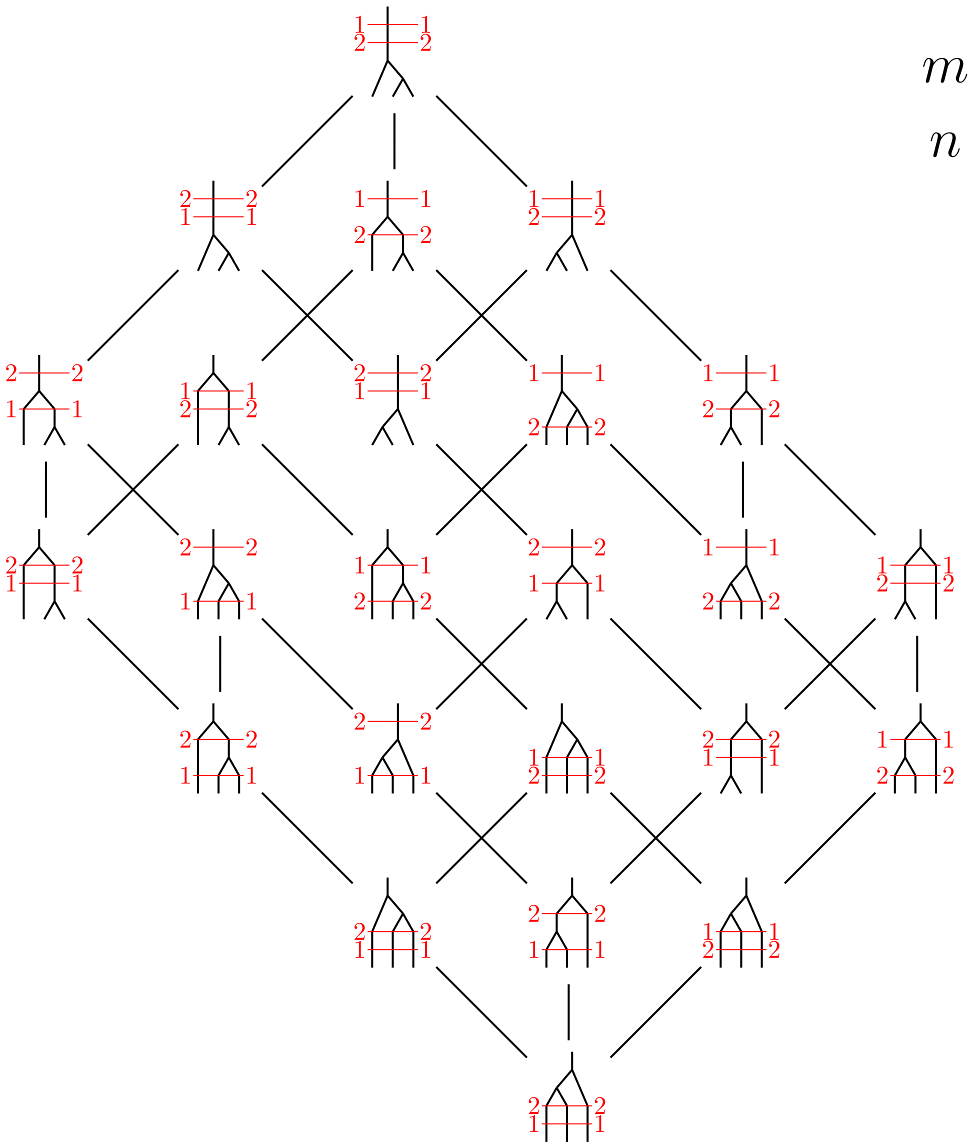
QU. Describe (or count) the rays of the submodular cone. Edmonds ('70)



# MULTIPLIHEDRA & HOCHSCHILD POLYTOPES

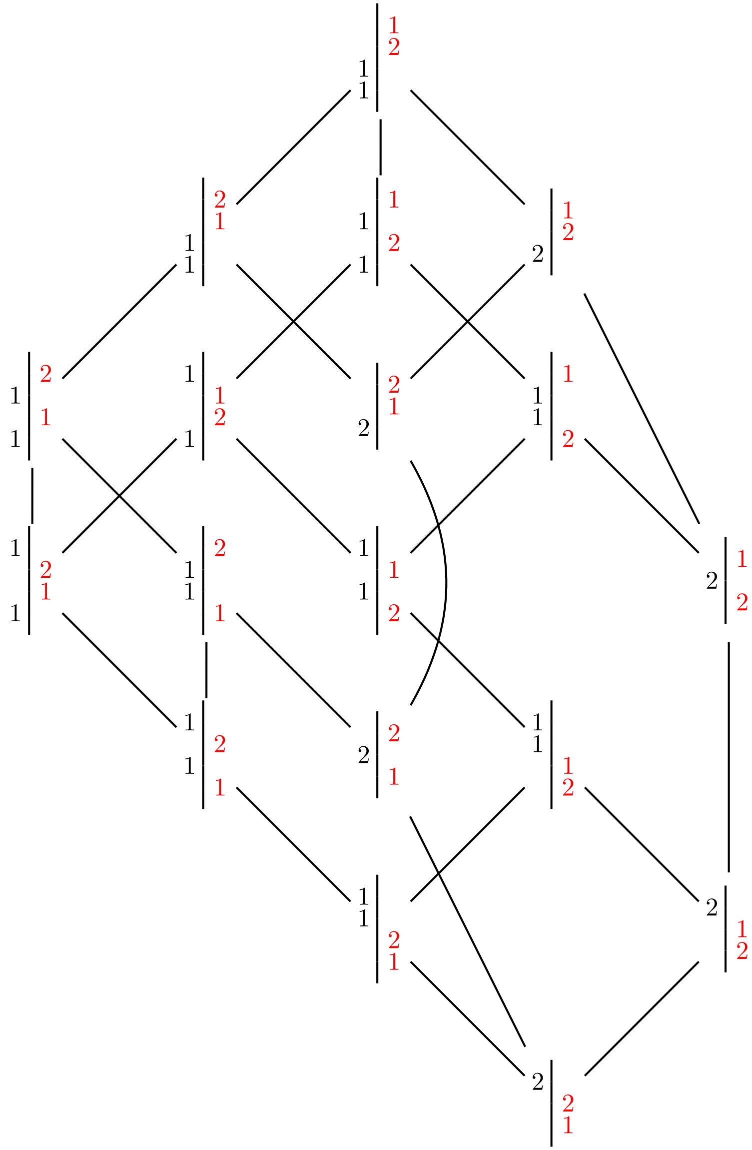
P.–Polyakova ('23$^+$)

# LATTICES: PAINTED TREES & LIGHTED SHADES



$m = 2$

$n = 2$



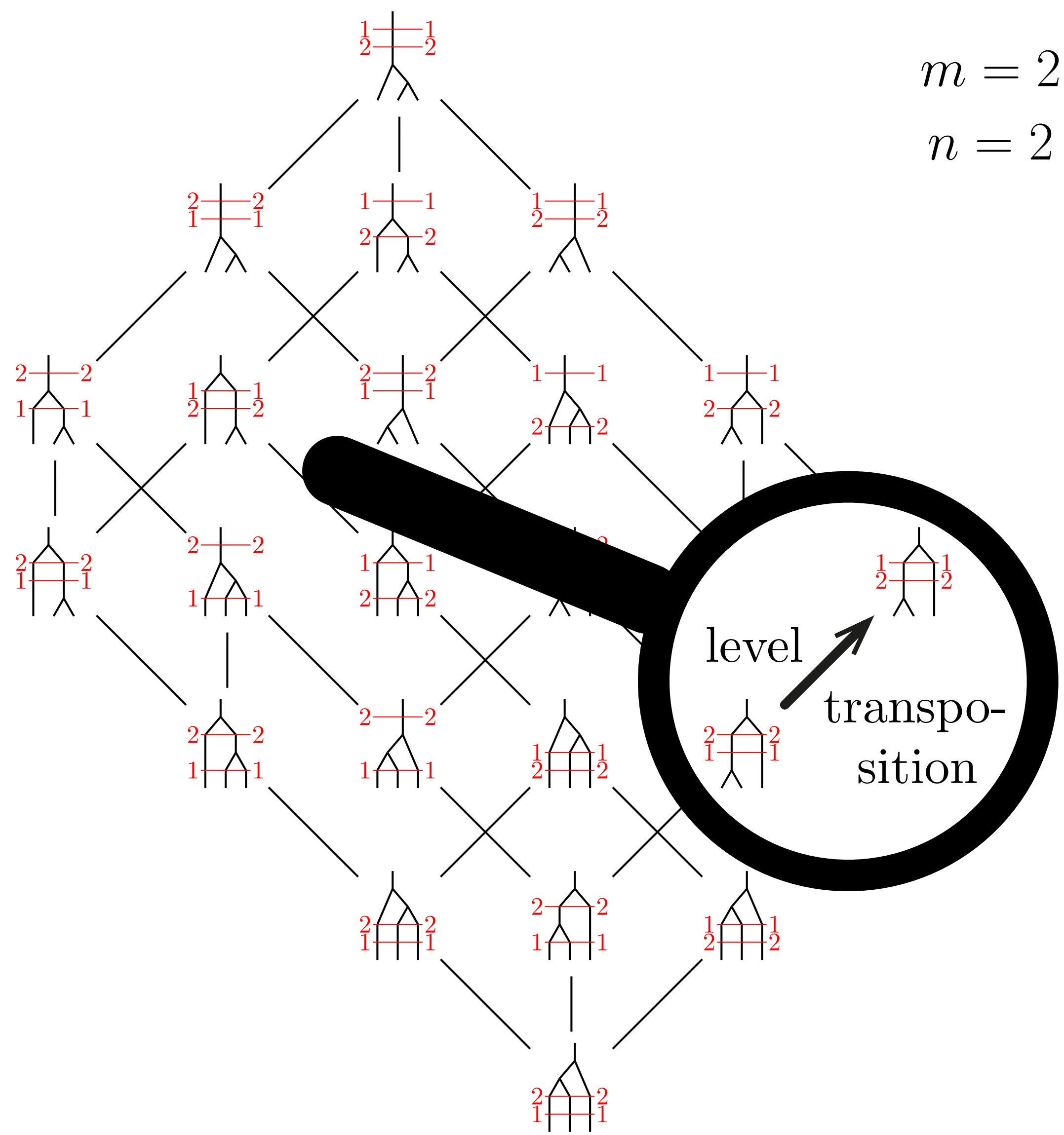
$m$-painted $n$-tree $=$ binary tree with $n$ nodes and $m$ levels

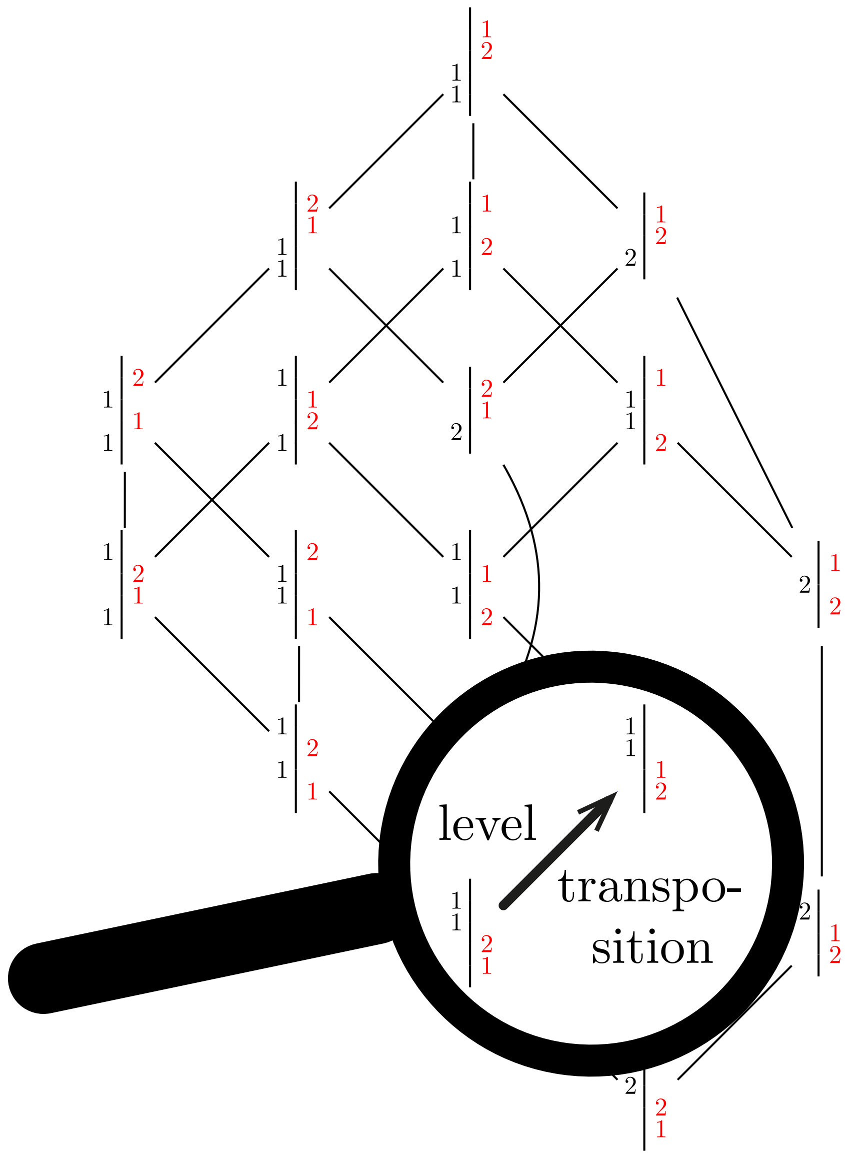
Chapoton–P. ('22$^+$)

$m$-lighted $n$-shade $=$ partition of $[n]$ with $m$ levels

P.–Polyakova ('23$^+$)

# LATTICES: PAINTED TREES & LIGHTED SHADES

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Chapoton–P. ('22$^{+}$)

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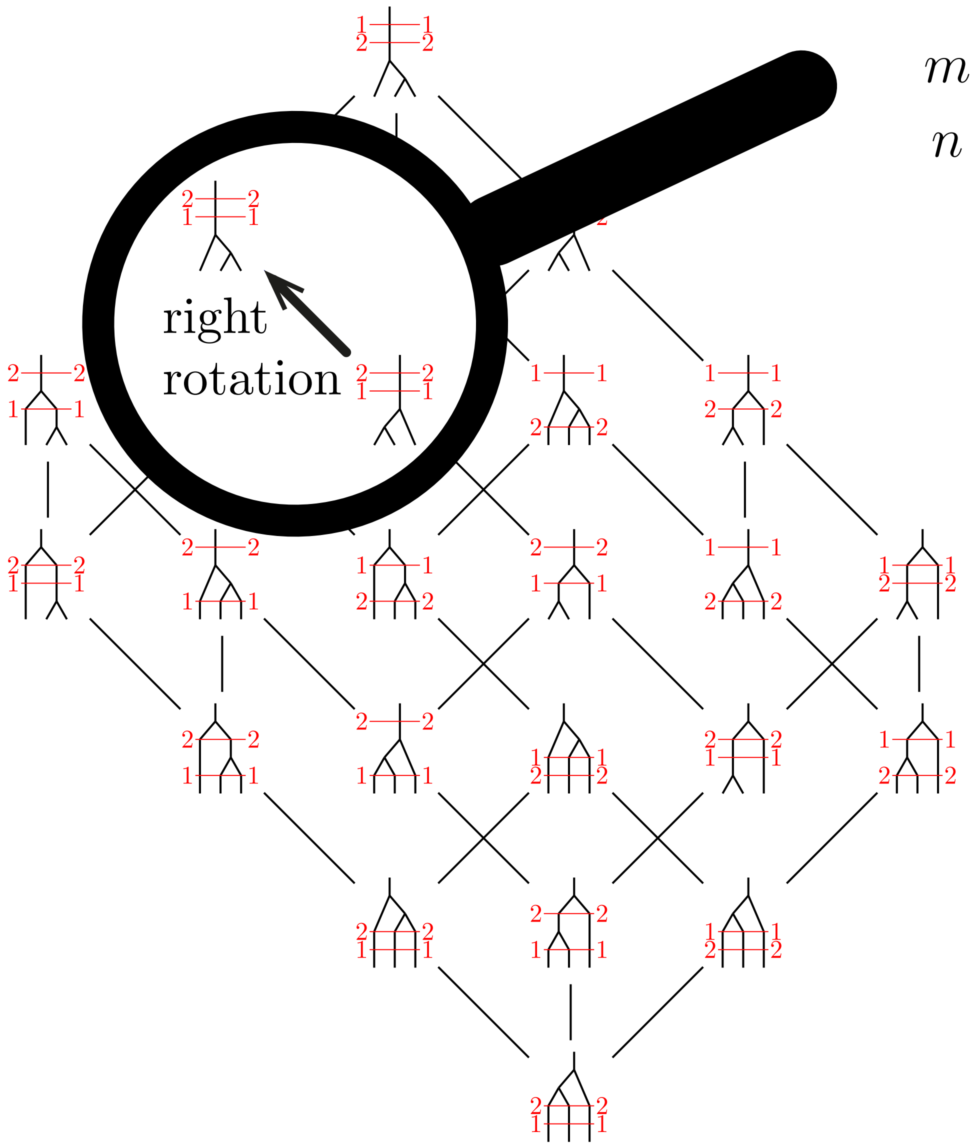
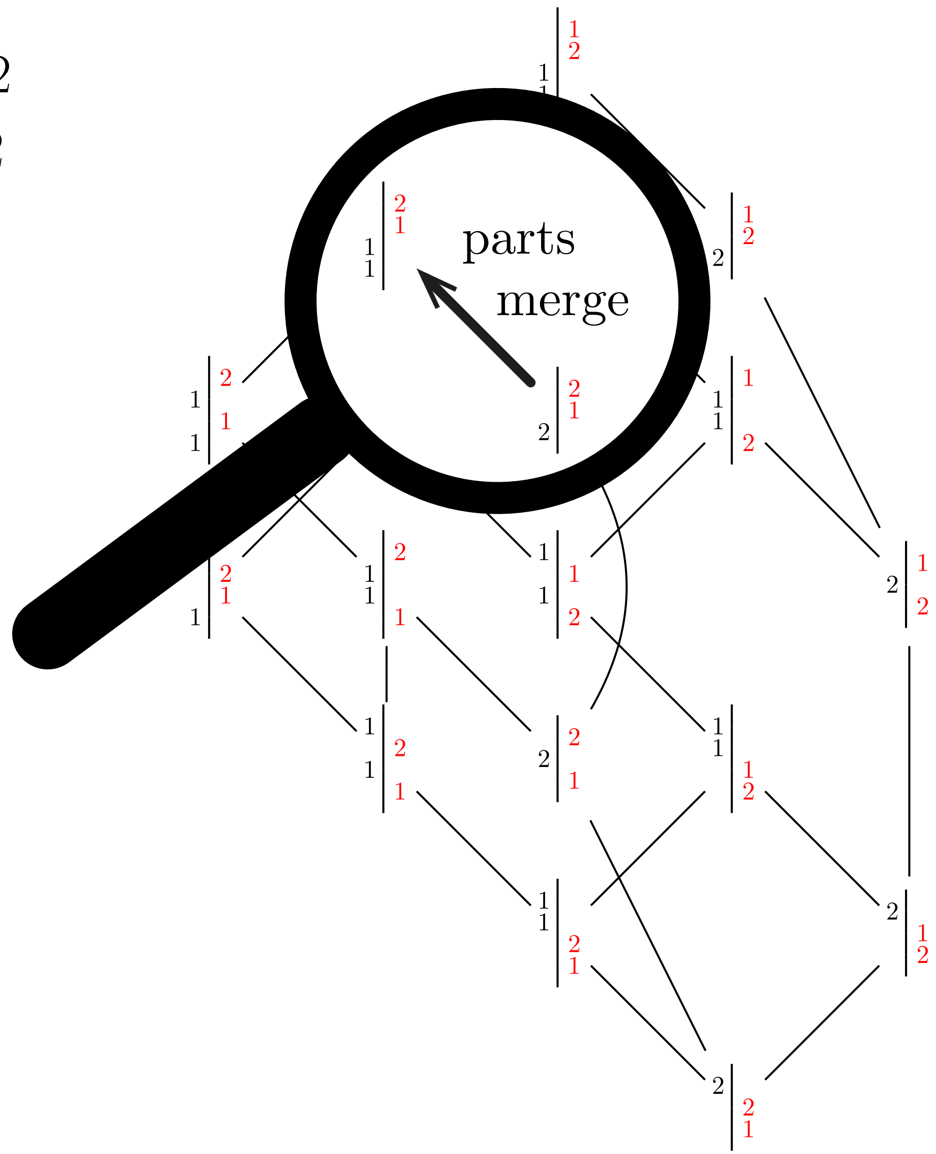



$m = 2$

$n = 2$

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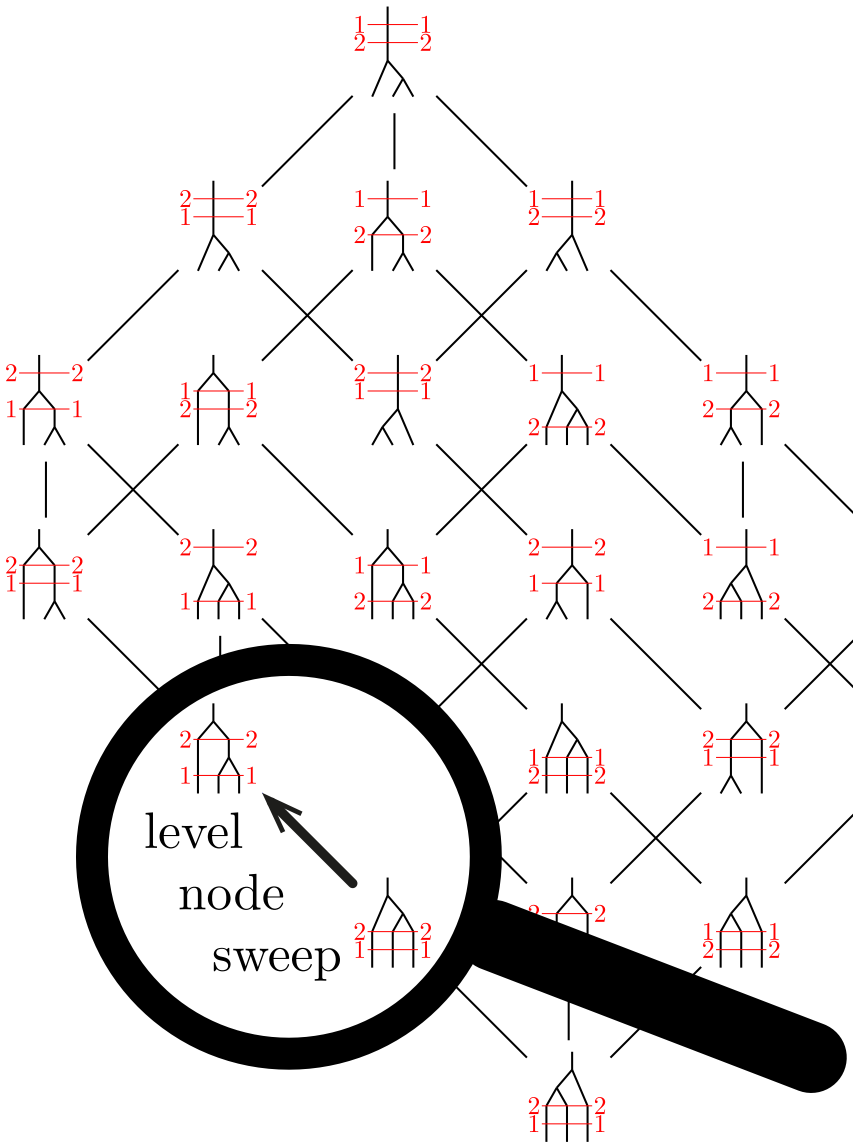
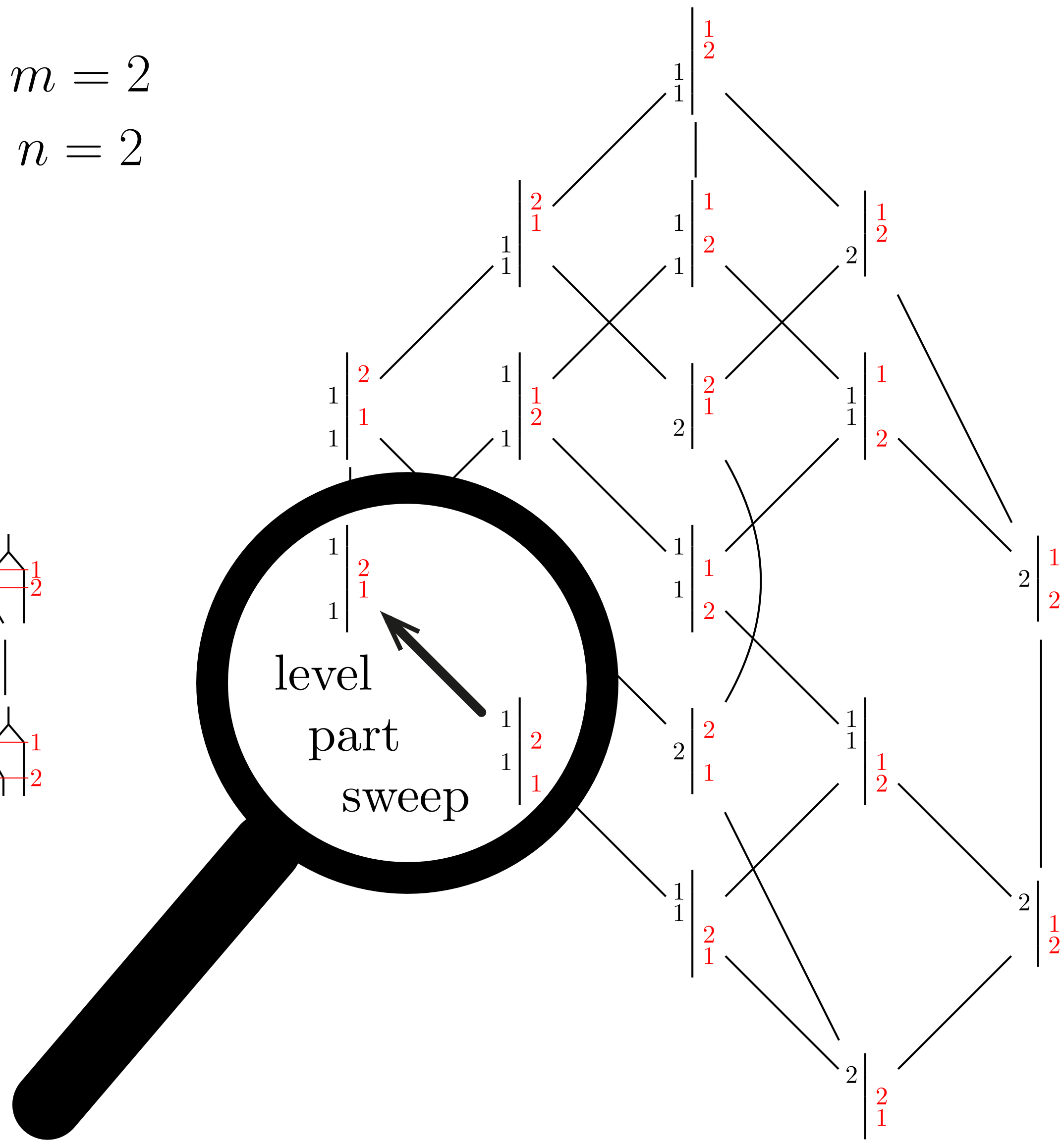



$m = 2$

$n = 2$

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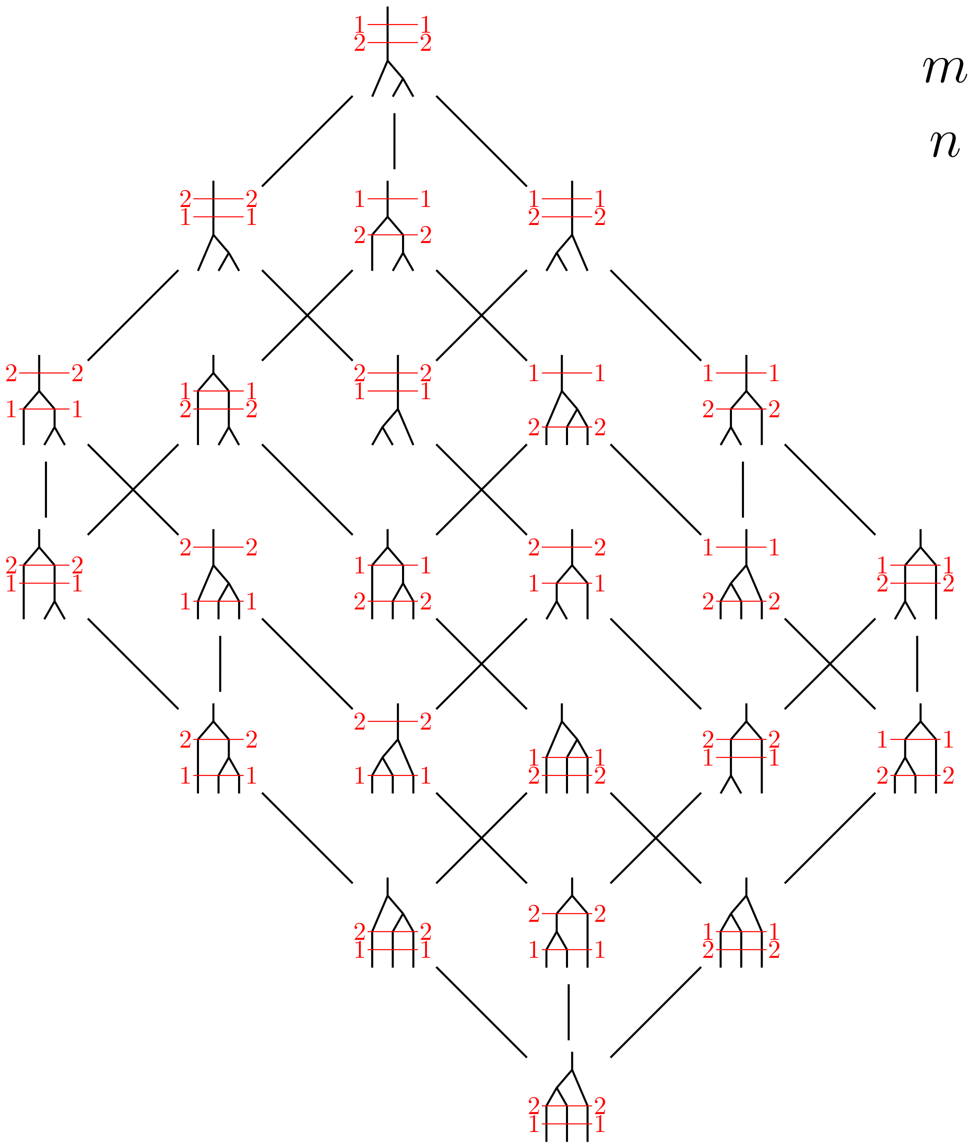
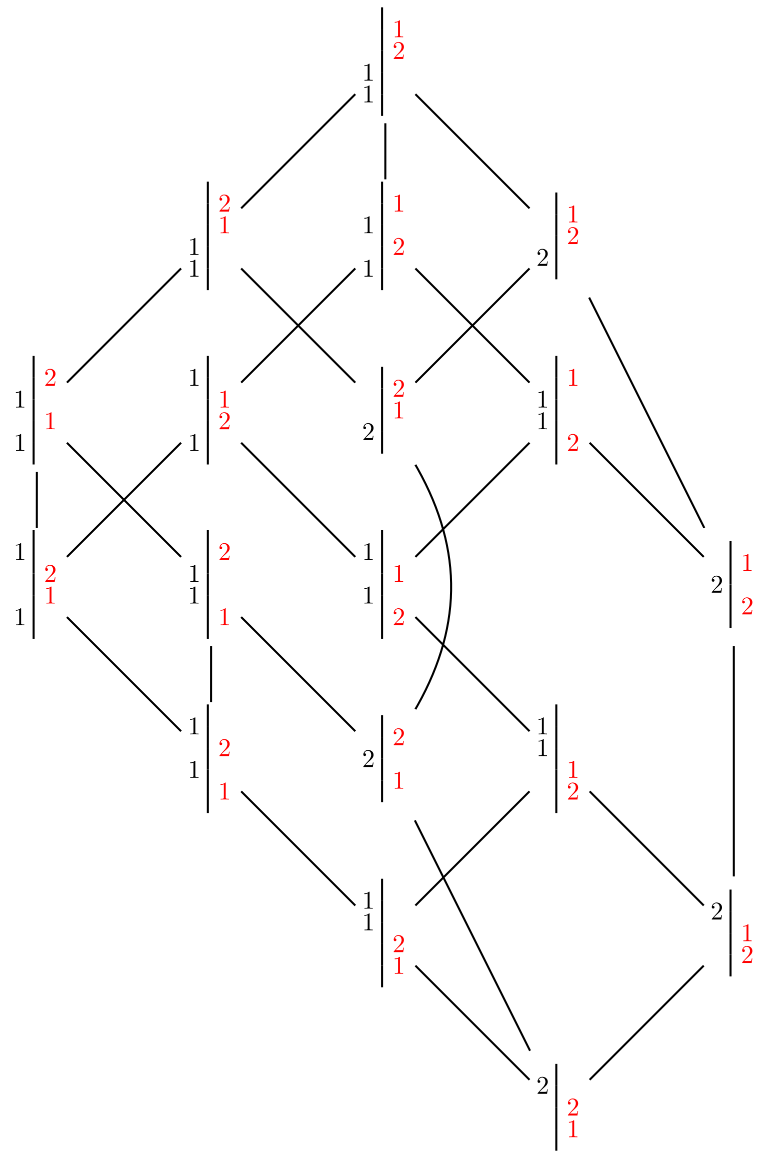


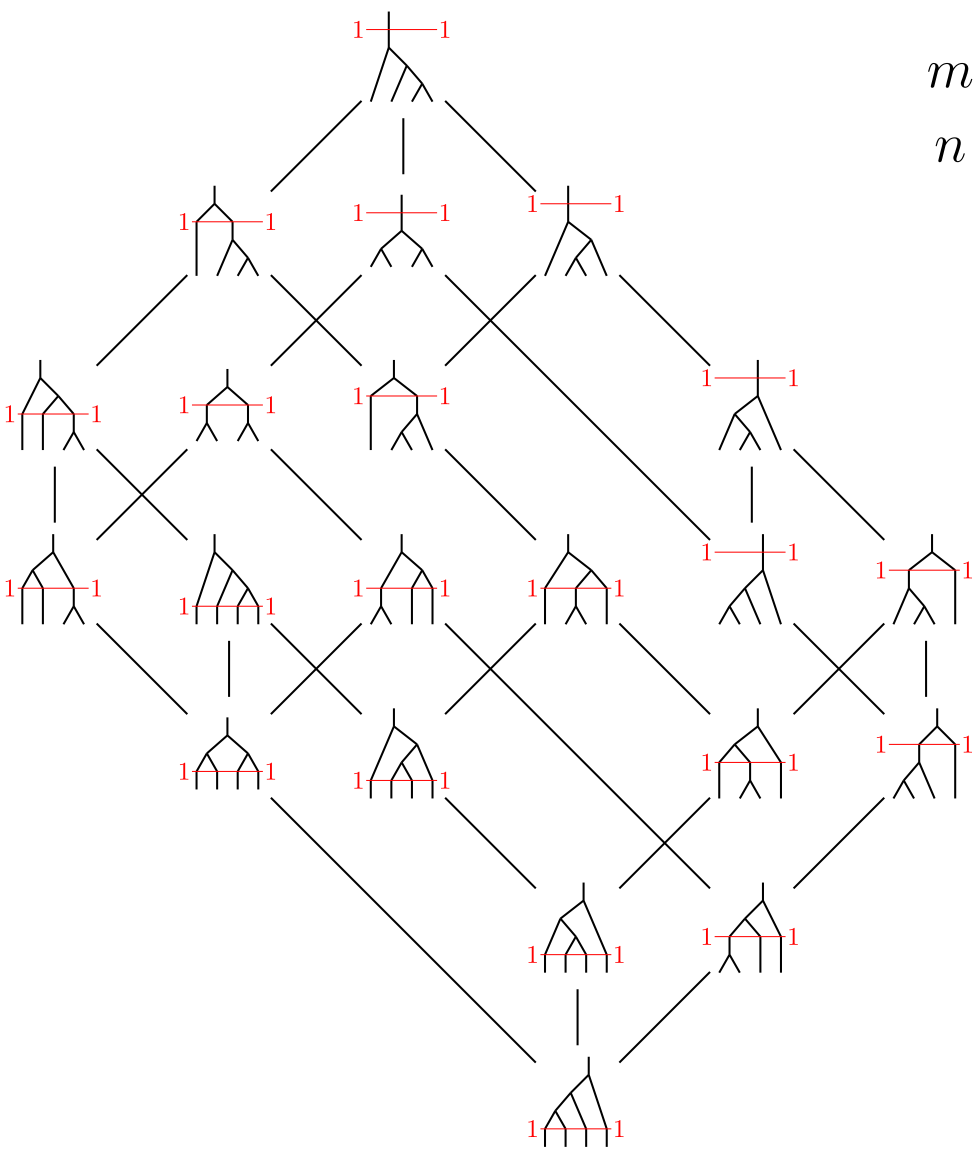
$$m = 2$$
$$n = 2$$



shadow map = arity sequence on the right branch

meet semilattice morphism, but not lattice morphism

# LATTICES: PAINTED TREES & LIGHTED SHADES

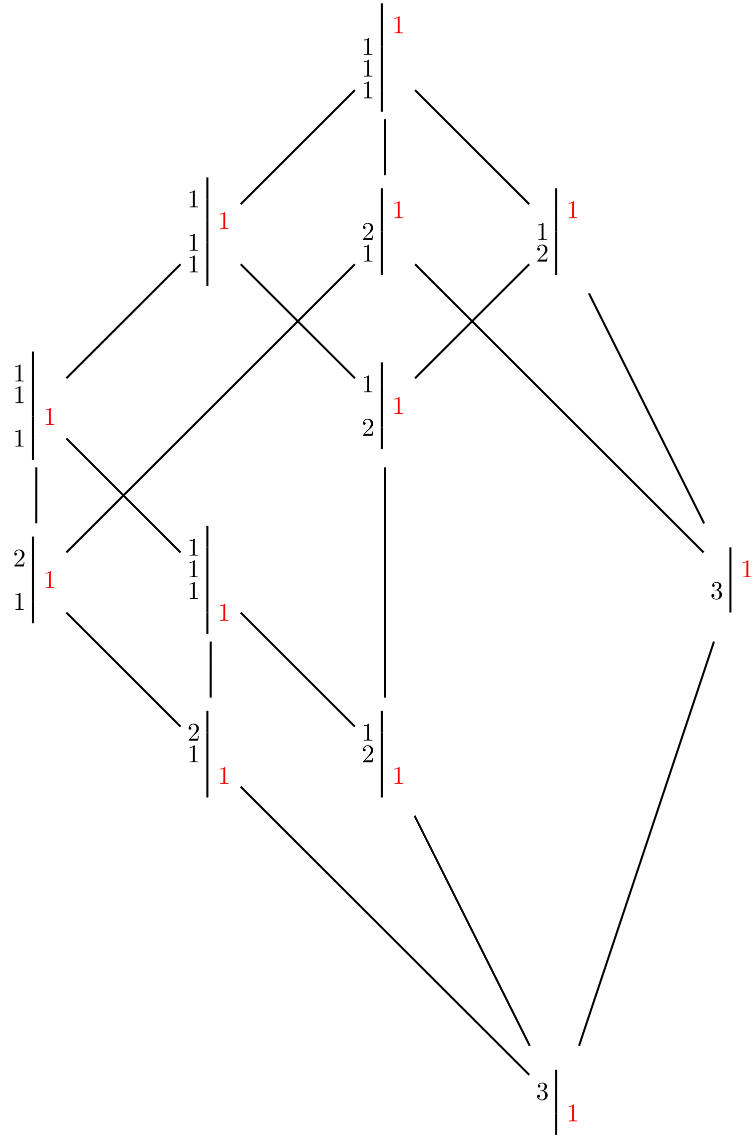


$m = 1$

$n = 3$

multiplihedron lattice

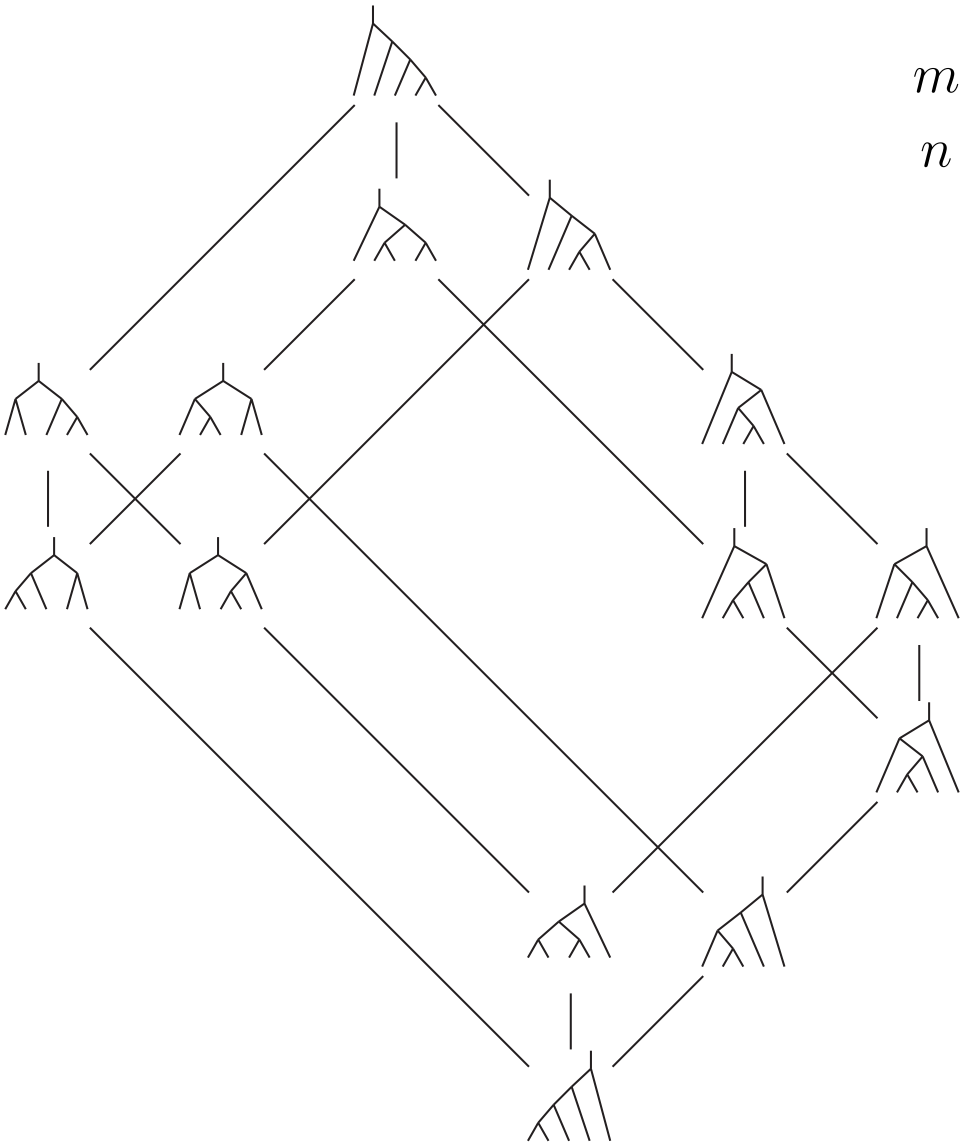
Stasheff ('63) — Forcey ('08) — Ardila–Doker ('13)



Hochschild lattice

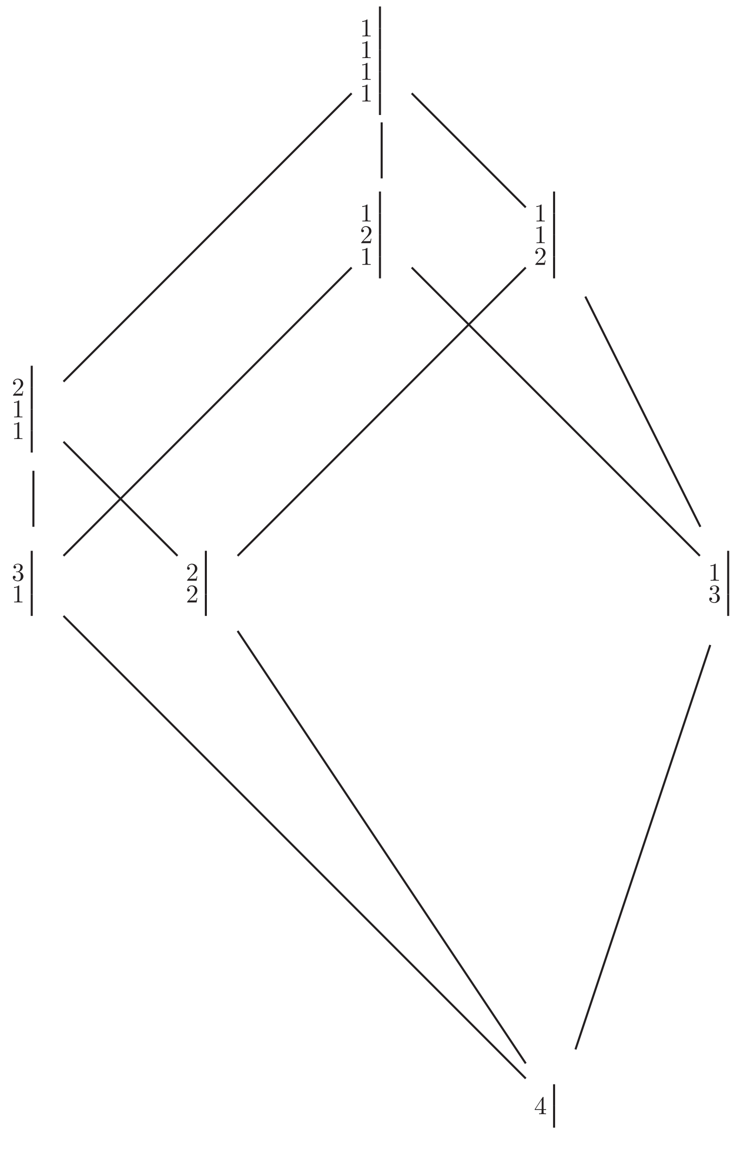
Abad–Crainic–Dherin ('11) — Poliakova ('20$^+$)
Chapoton ('20) — Combe ('21) — Mühle ('22)

# LATTICES: PAINTED TREES & LIGHTED SHADES



$m = 0$

$n = 4$

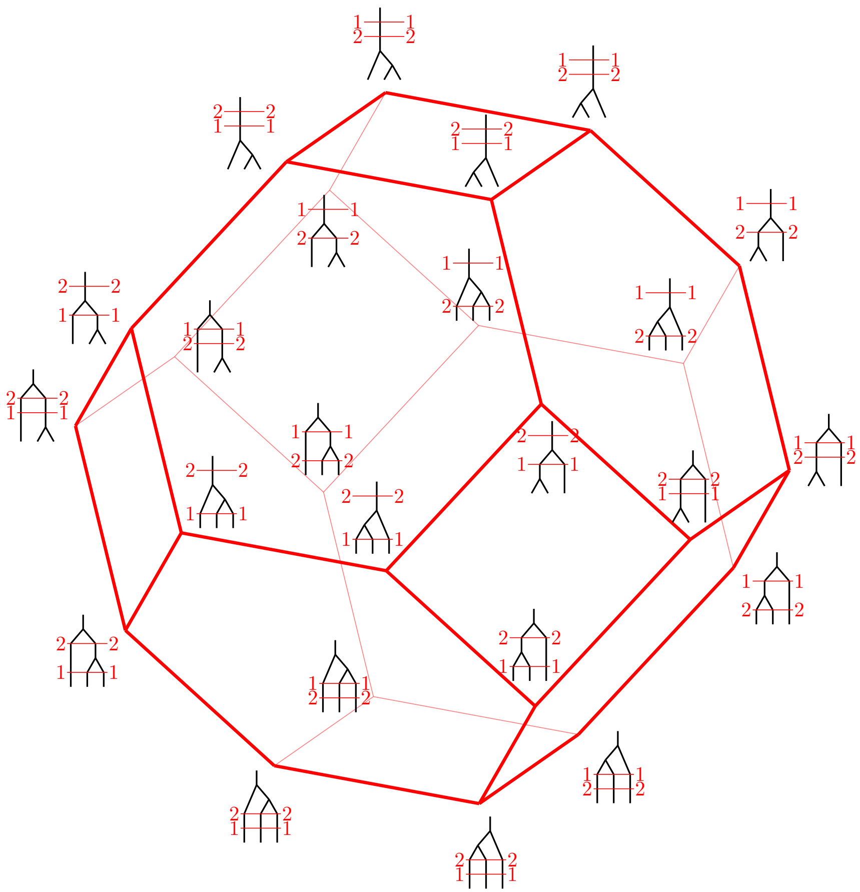



Tamari lattice

Tamari ('51)

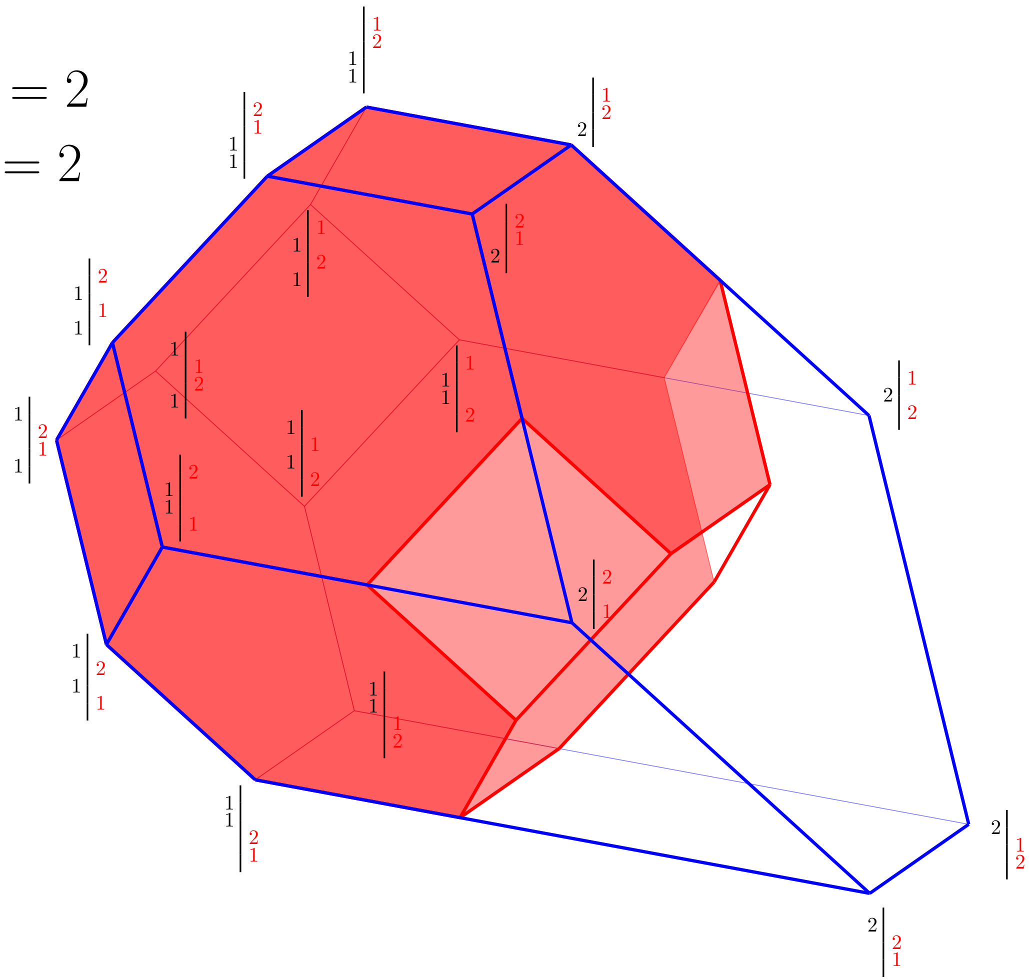
boolean lattice

# POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE



$m = 2$

$n = 2$



$(m, n)$-multiplihedron

$= $ shuffle of $\mathbb{P}\mathrm{erm}(m)$ and $\mathbb{A}\mathrm{sso}(n)$

$$= \mathbb{P}\mathrm{erm}(m) \times \mathbb{A}\mathrm{sso}(n) + \sum_{i \in [m], j \in [n]} [\boldsymbol{e}_i, \boldsymbol{e}_{m+j}]$$

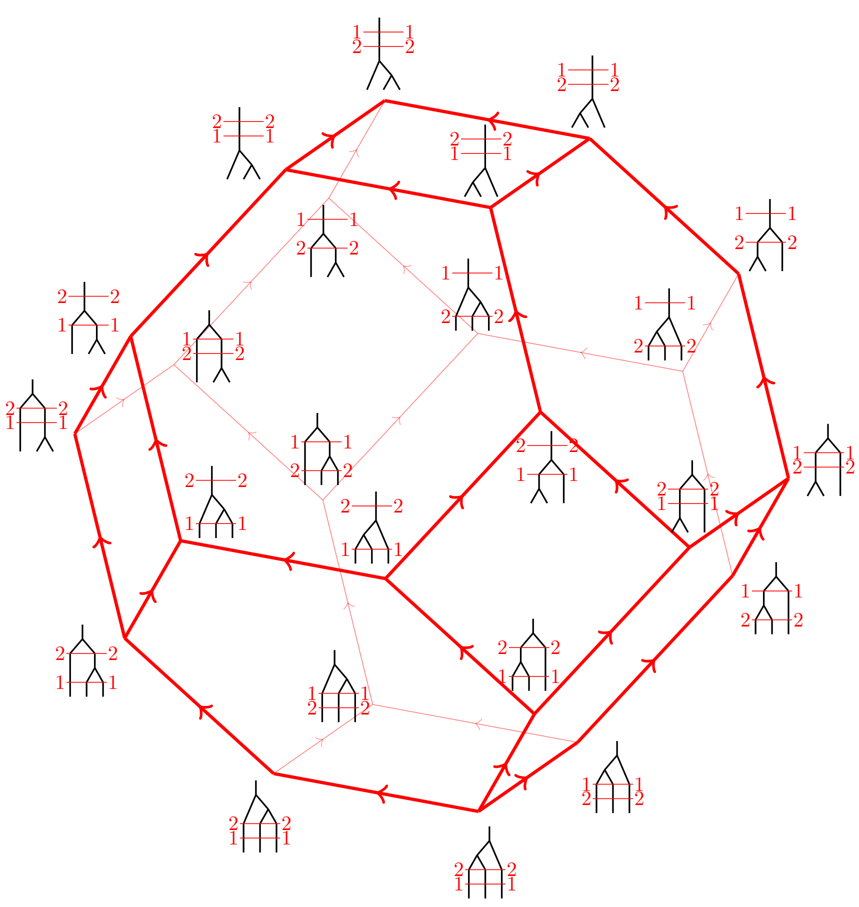
Chapoton–P. ('22$^+$)

$(m, n)$-Hochschild polytope

$= $ removahedron of $\mathbb{M}\mathrm{ul}(m, n)$

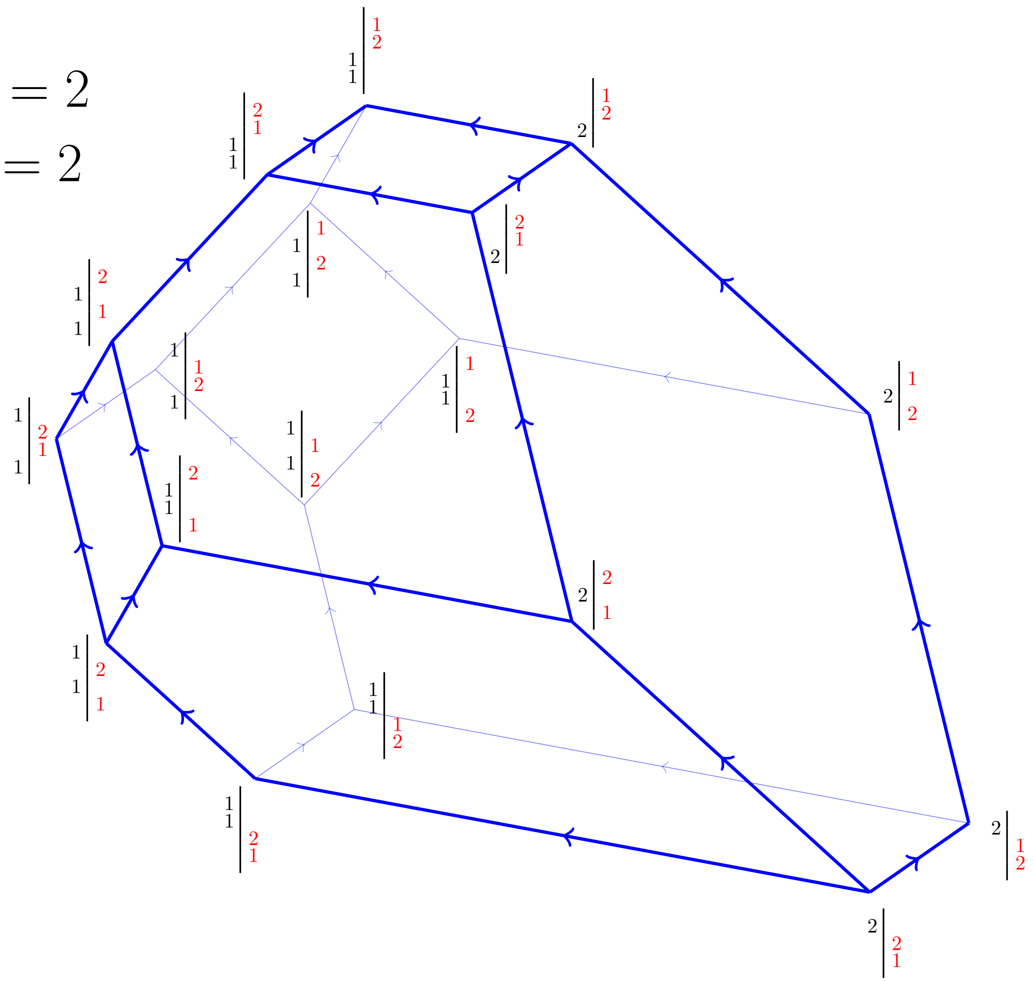
P.–Polyakova ('23$^+$)

# POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE



$m = 2$

$n = 2$



$(m, n)$-multiplihedron

$\Longrightarrow (m, n)$-multiplihedron lattice

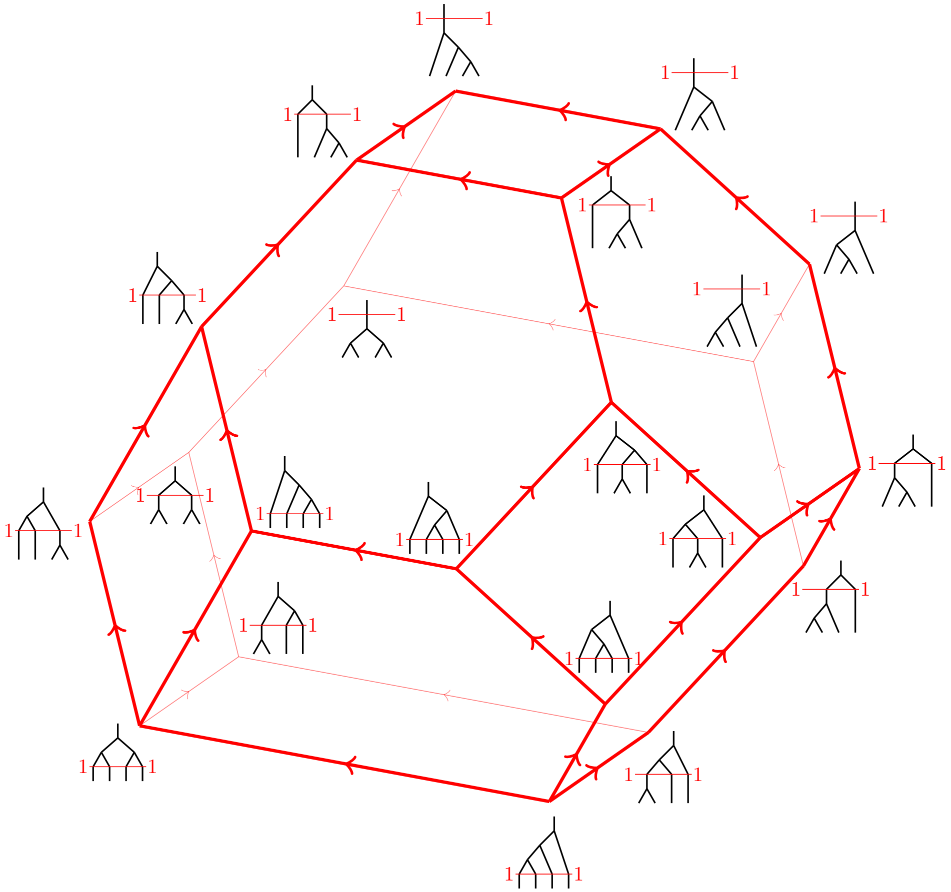
Chapoton–P. ('22$^+$)

$(m, n)$-Hochschild polytope

$\Longrightarrow (m, n)$-Hochschild lattice

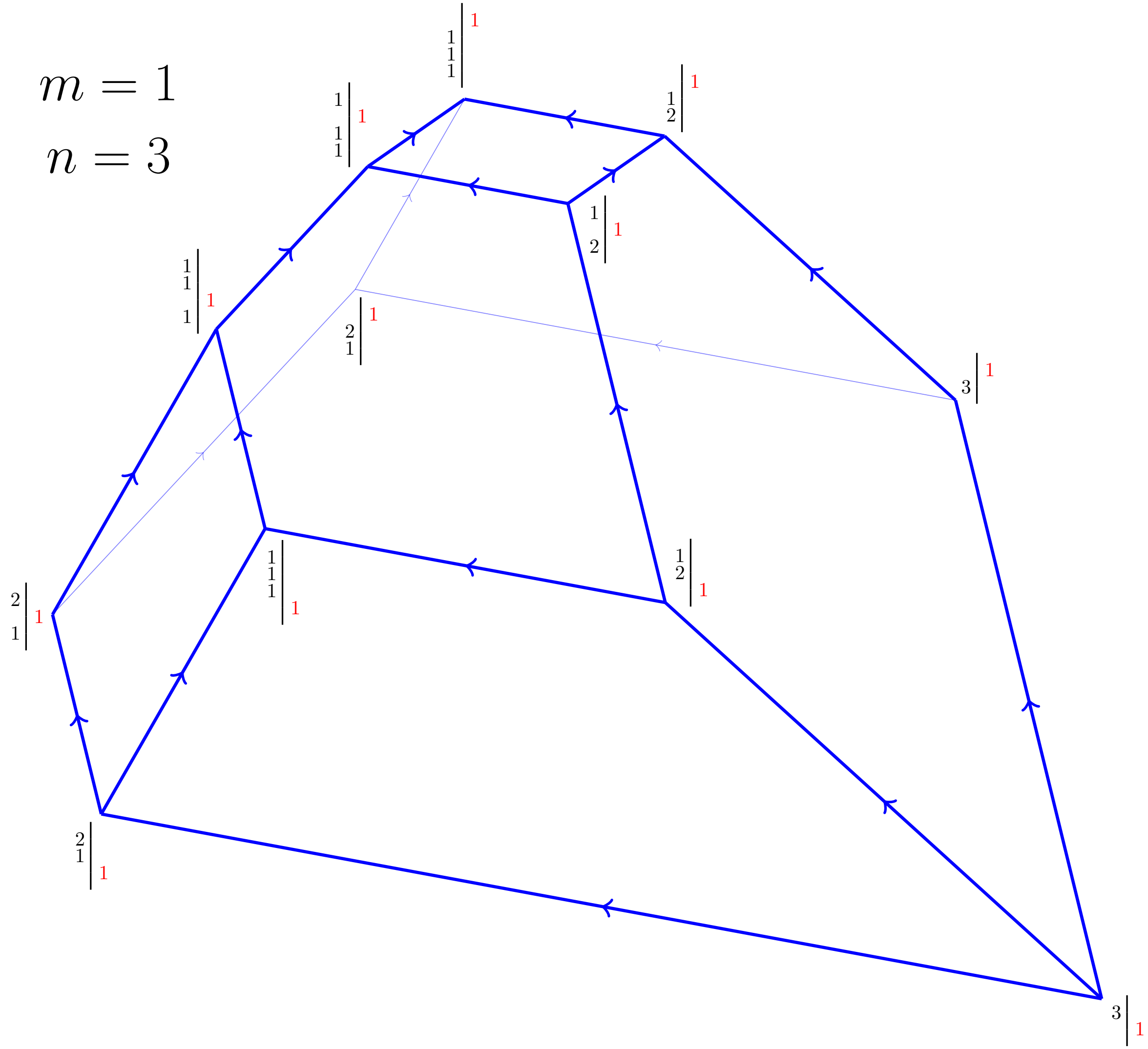
P.–Polyakova ('23$^+$)

# POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE



$m = 1$

$n = 3$



$(m, n)$-multiplihedron

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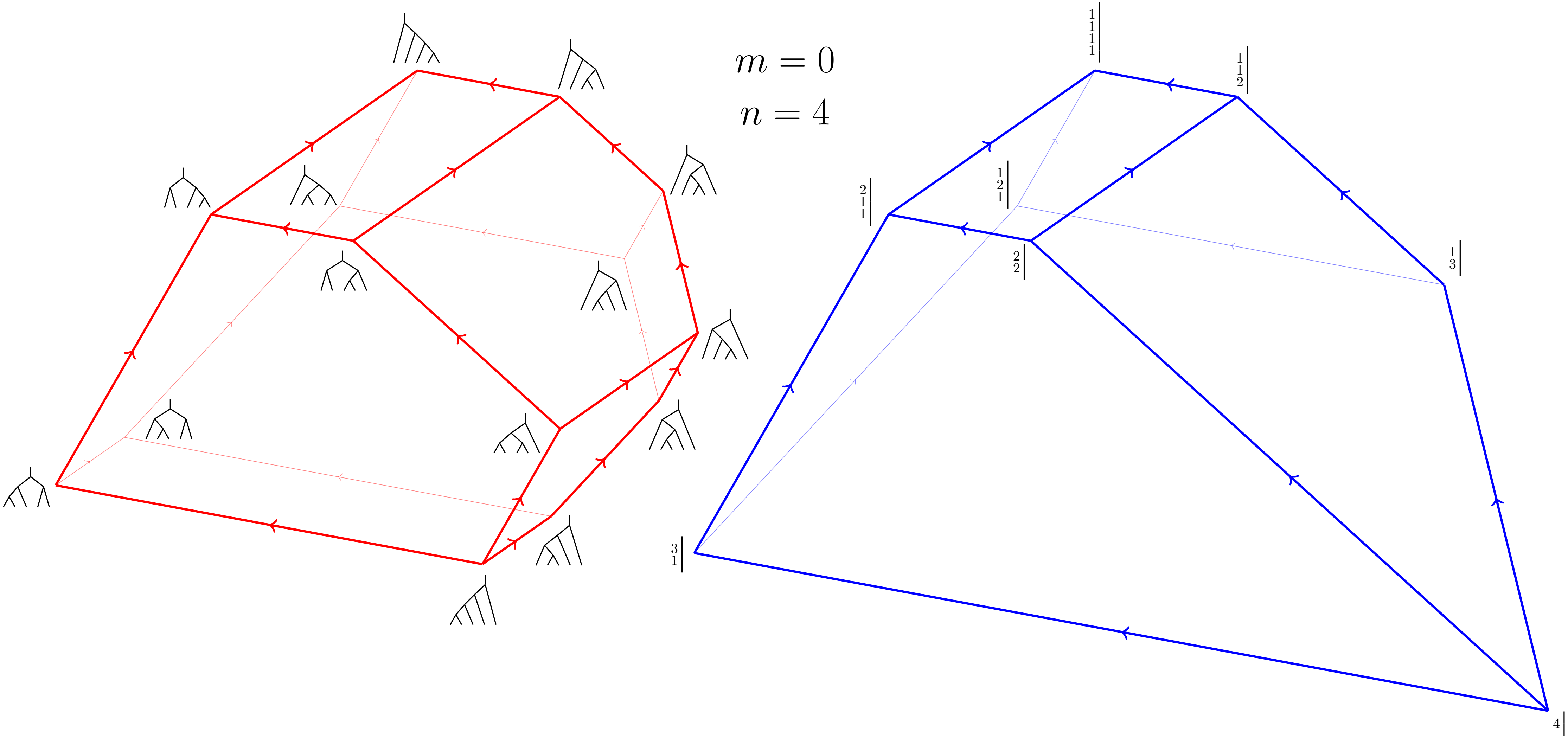
Chapoton–P. ('22$^+$)

$(m, n)$-Hochschild polytope

$\Longrightarrow (m, n)$-Hochschild lattice

P.–Polyakova ('23$^+$)

# POLYTOPES: MULTIPLIHEDRON & HOCHSCHILD POLYTOPE




$(m, n)$-multiplihedron

$\Longrightarrow (m, n)$-multiplihedron lattice

Chapoton–P. ('22$^+$)

$(m, n)$-Hochschild polytope

$\Longrightarrow (m, n)$-Hochschild lattice

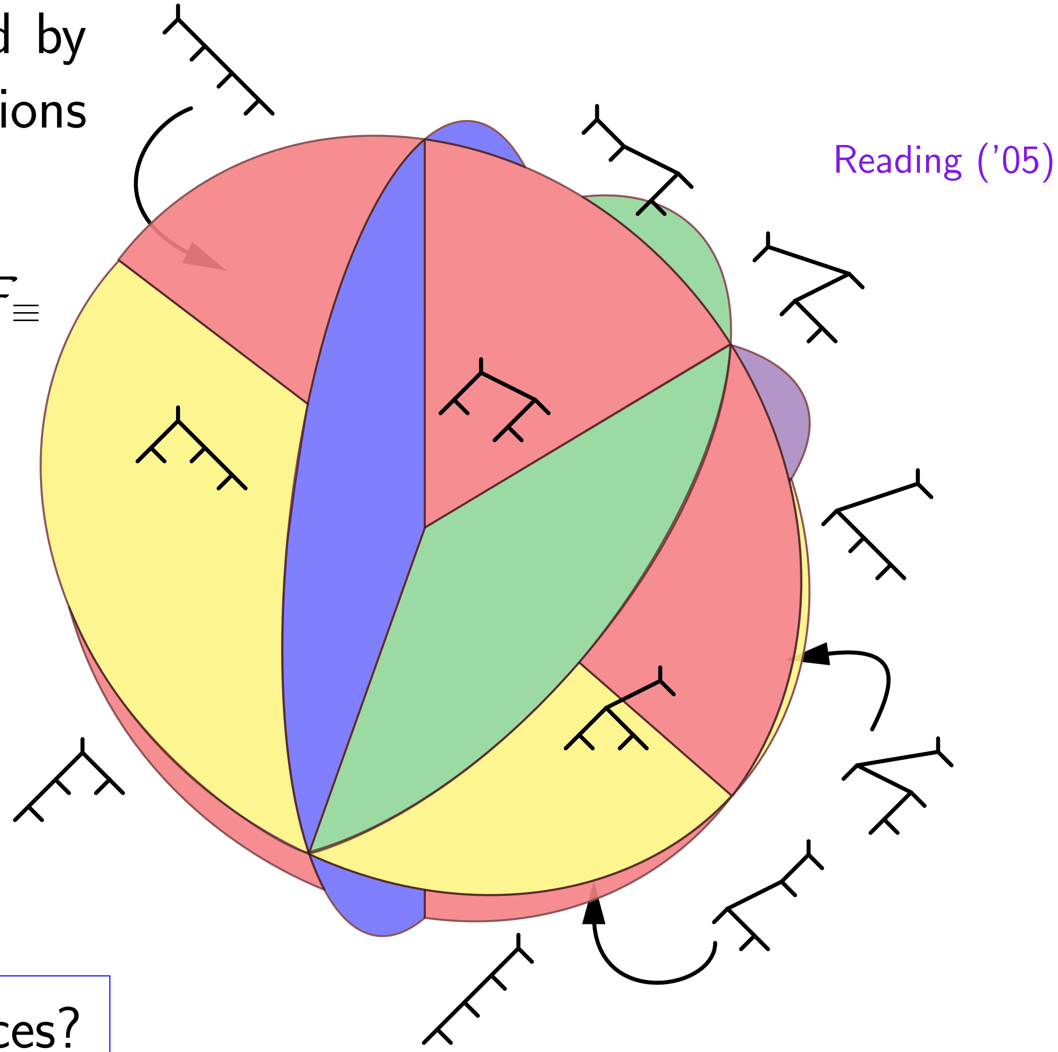
P.–Polyakova ('23$^+$)

# OPEN PROBLEM: SEMIQUOTIENTOPES

lattice congruence = equivalence relation on $L$ compatible with meets and joins:
$x \equiv x'$ and $y \equiv y'$ implies $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$

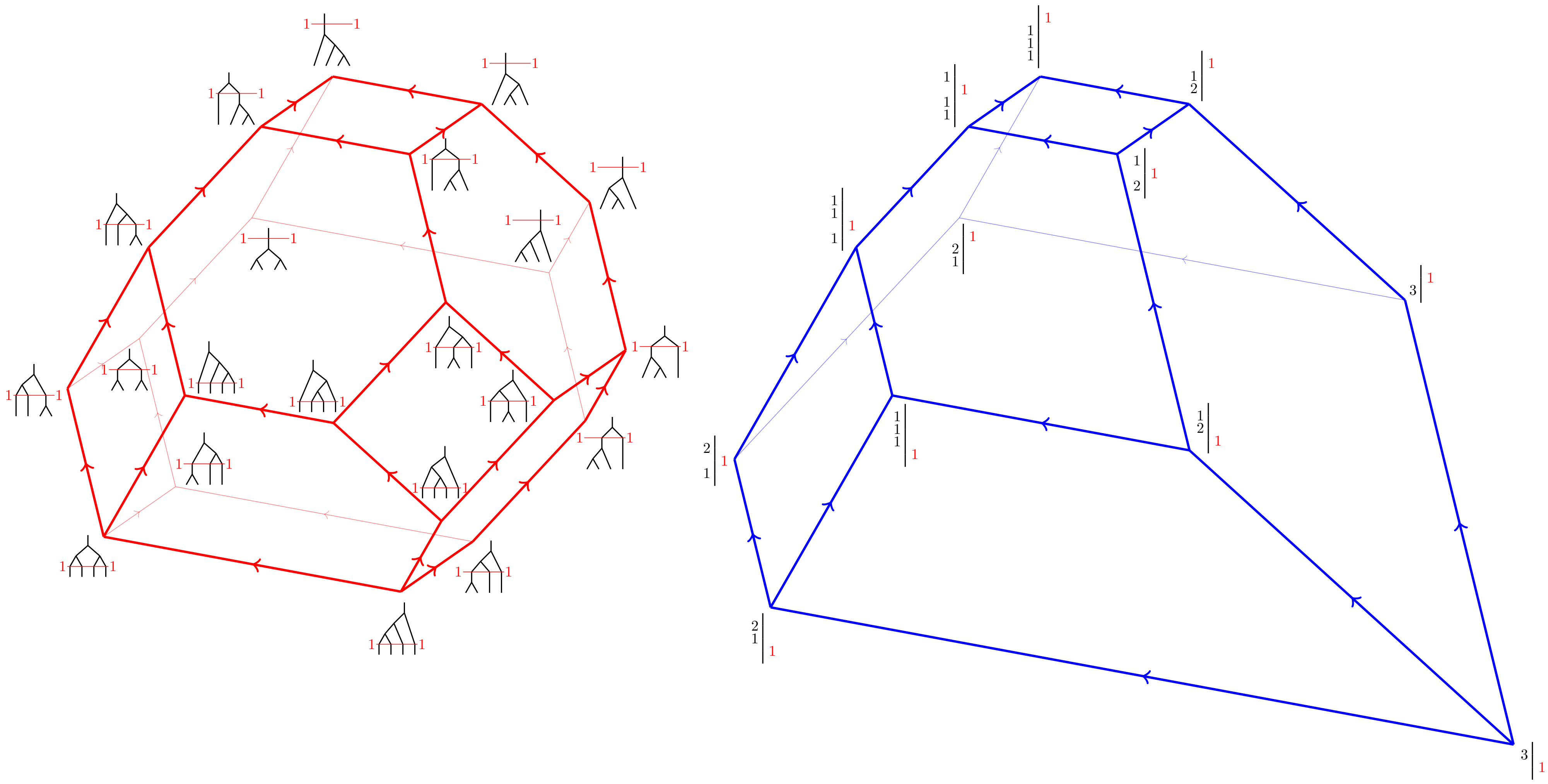
quotient fan $\mathcal{F}_\equiv$ = chambers are obtained by glueing the chambers $\mathbb{C}(\sigma)$ of the permutations $\sigma$ in the same congruence class of $\equiv$

quotientope = polytope with normal fan $\mathcal{F}_\equiv$



Reading ('05)

QU. Extends to meet semilattice congruences?



# THANK YOU