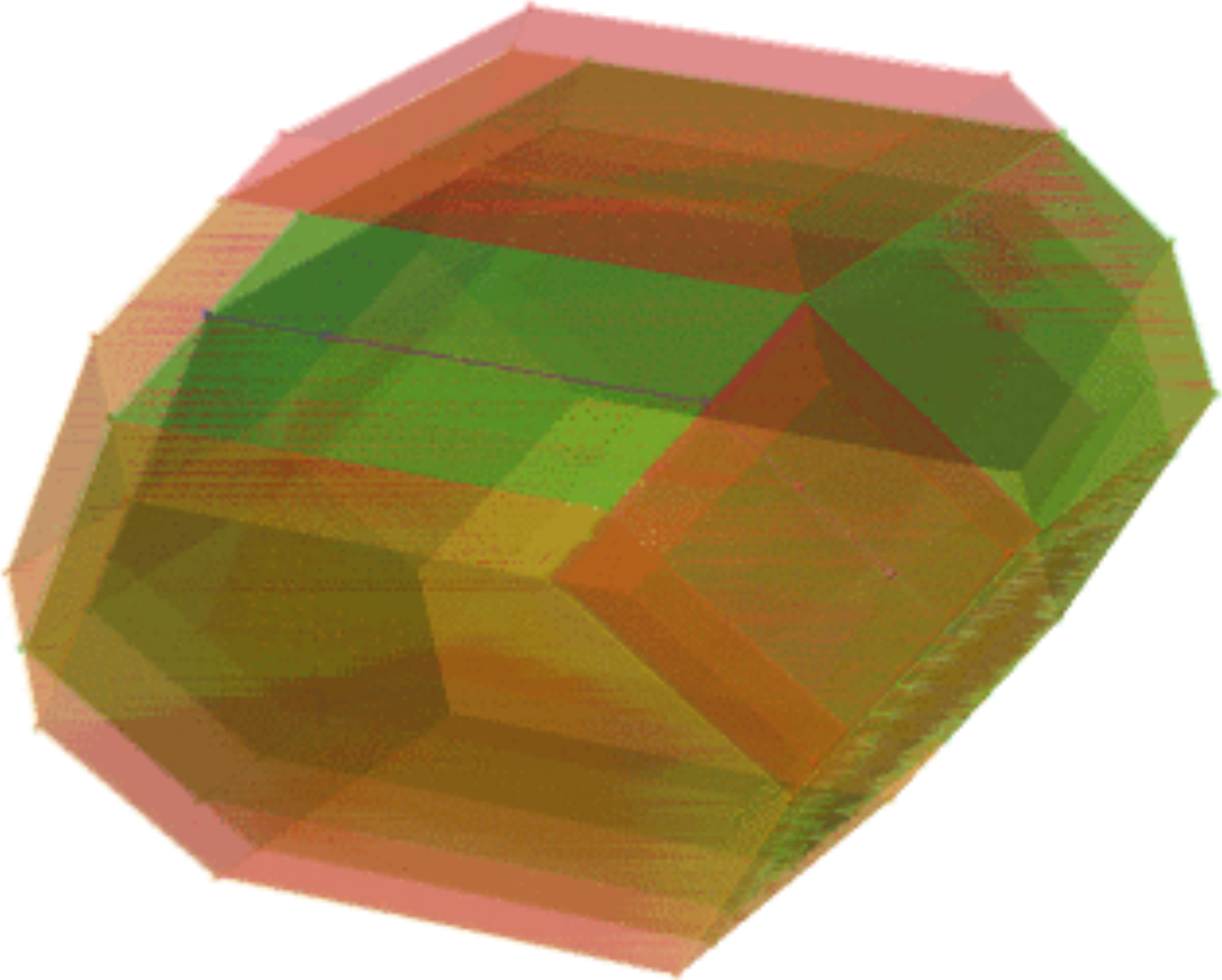
MORE FRIENDS FOR THE ASSOCIAHEDRON



V. PILAUD

(CNRS & LIX)

Geometry
and Combinatorics
of Associativity
October 23–27, 2017

PROGRAM

0. VARIOUS ASSOCIAHEDRA

- 3 constructions
- Loday's associahedron
- The Loday-Ronco Hopf algebra

I. PERMUTREEHEDRA

- Permutrees and permutree lattices
- Permutreehedra
- The permutree Hopf algebra

II. QUOTIENTOPES

- Lattice quotients and arc diagrams
- Quotientopes
- Hopf algebras on arc diagrams

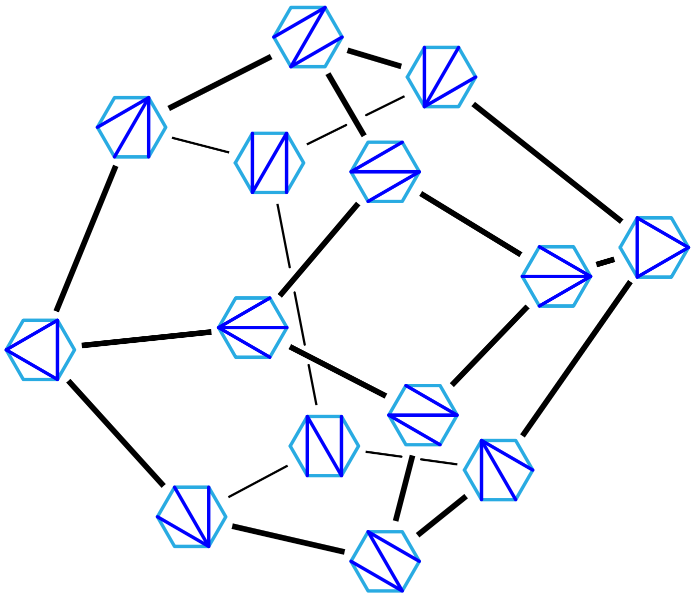
III. THE UNIVERSAL ASSOCIAHEDRON AND ITS PROJECTIONS

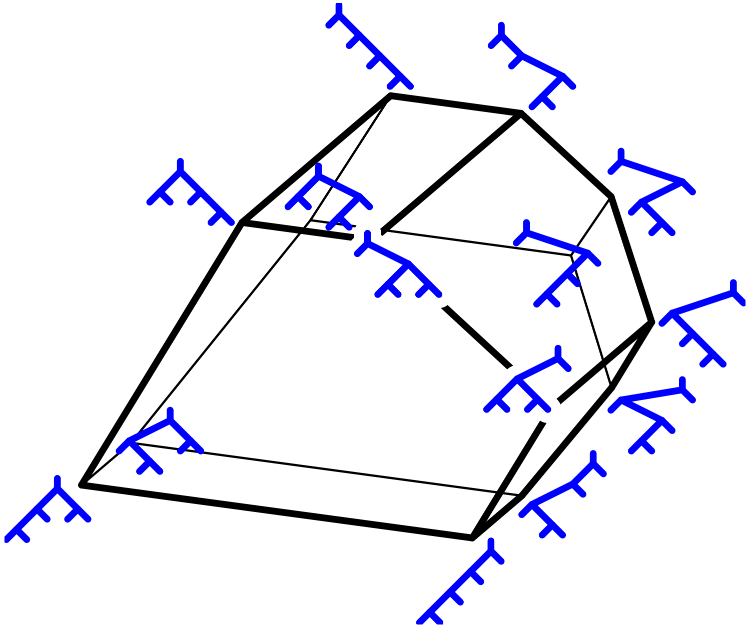
- $\mathbf{g}$- and $\mathbf{c}$-vectors
- The $\mathbf{g}$-vector fan is polytopal
- The universal associahedron
- Sections and projections
- Extensions to cluster algebras

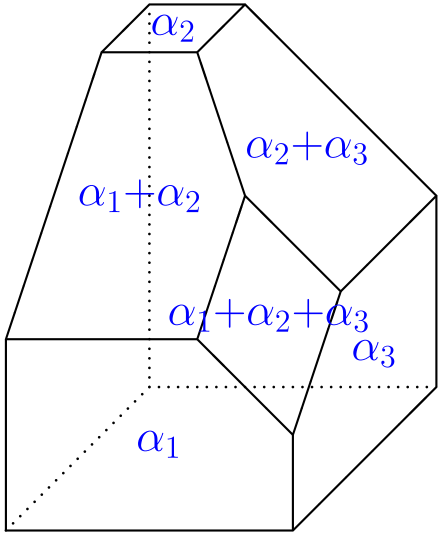
IV. NON-KISSING COMPLEXES AND GENTLE ASSOCIAHEDRA

- Non-kissing complexes
- Gentle associahedra
- Non-kissing lattice

0. VARIOUS ASSOCIAHEDRA







FANS & POLYTOPES

Ziegler, *Lectures on polytopes* ('95)
Matoušek, *Lectures on Discrete Geometry* ('02)

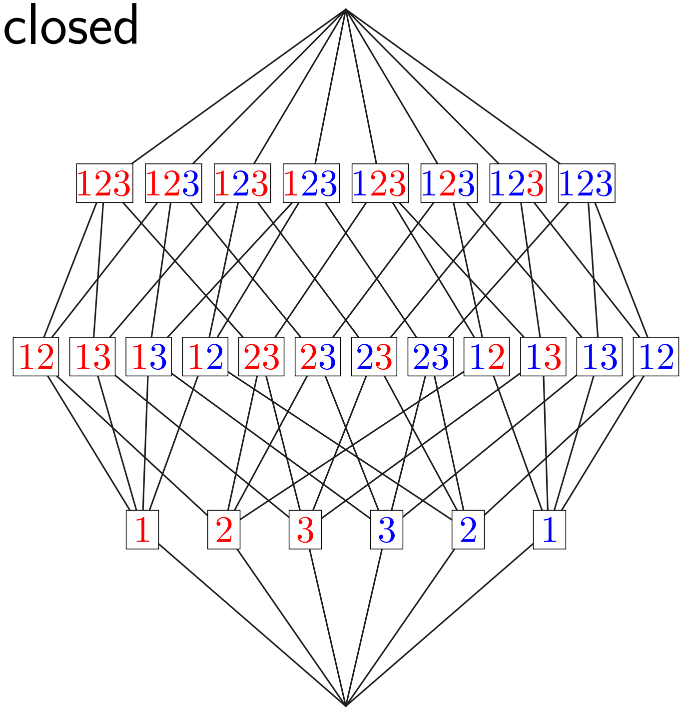
SIMPLICIAL COMPLEX

simplicial complex = collection of subsets of X downward closed

exm:

$$X = [n] \cup [n]$$

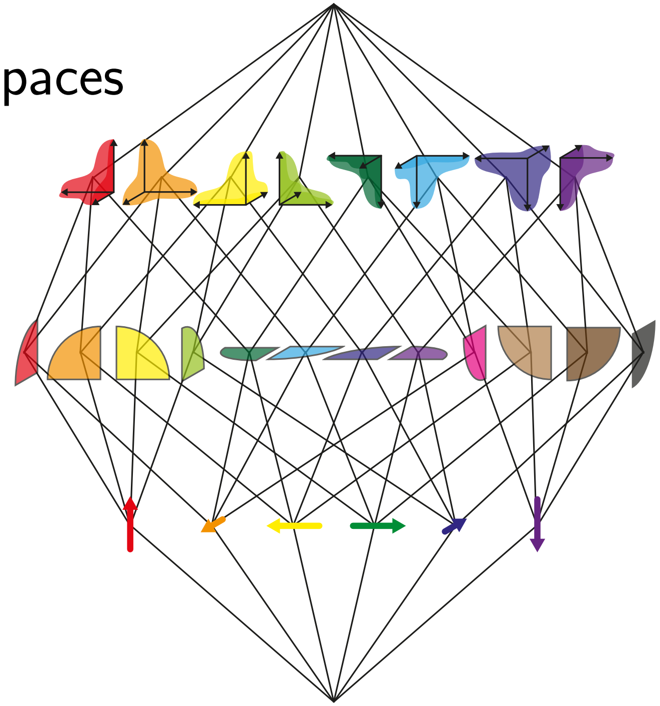
$$\Delta = \{I \subseteq X \mid \forall i \in [n],\ \{i, i\} \not\subseteq I\}$$

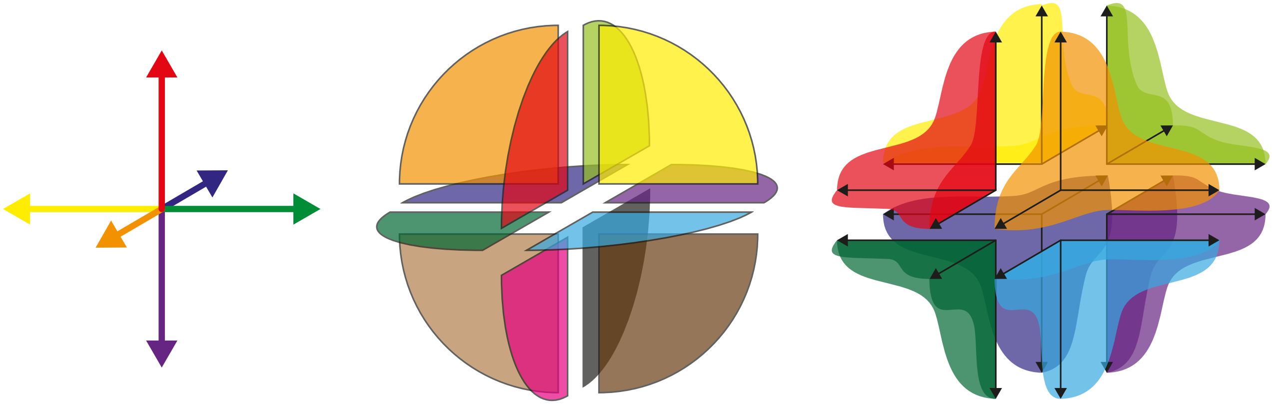


FANS

polyhedral cone = positive span of a finite set of $\mathbb{R}^d$
= intersection of finitely many linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face





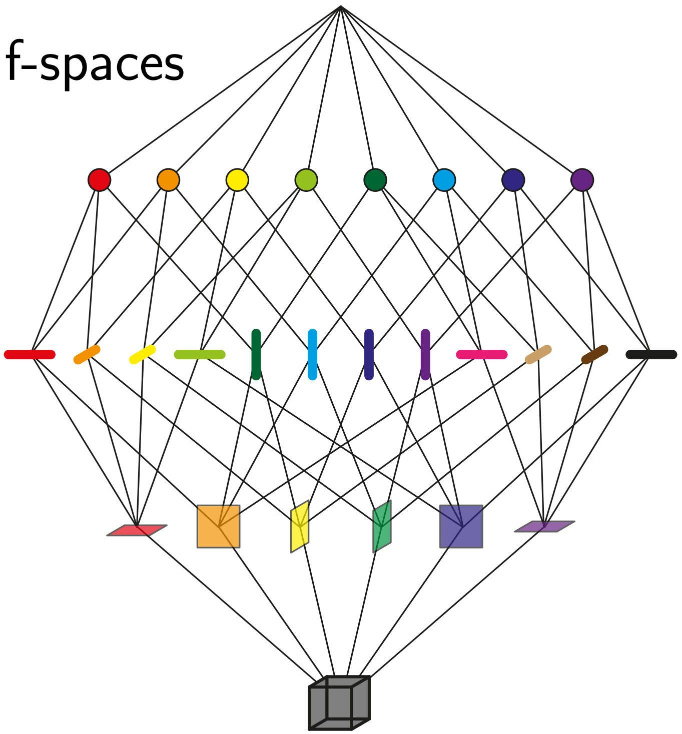
simplicial fan = maximal cones generated by d rays

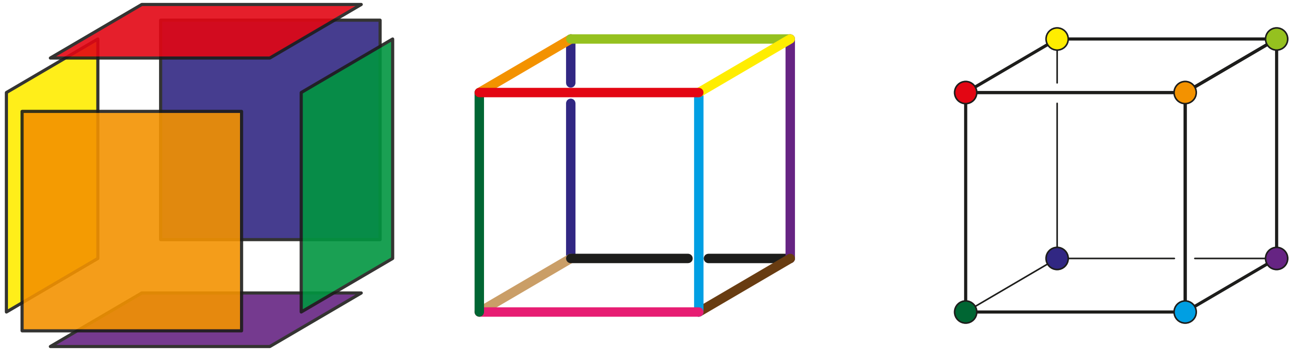
POLYTOPES

polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane

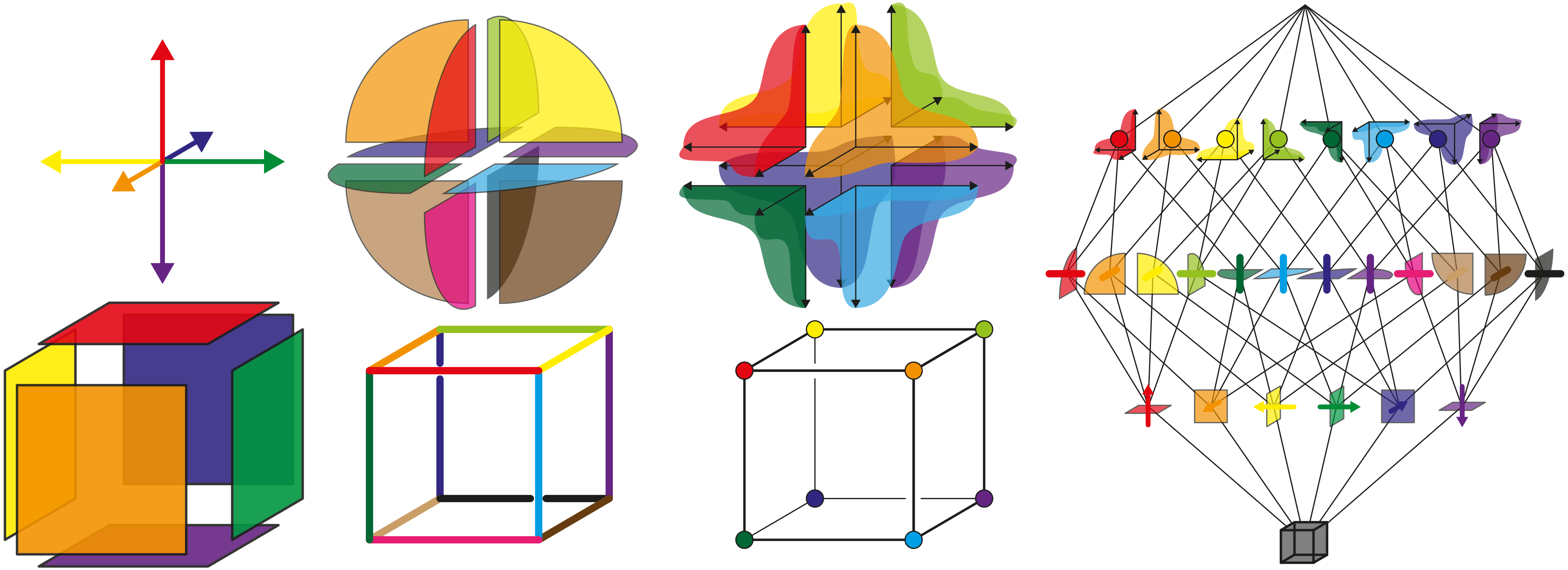
face lattice = all the faces with their inclusion relations





simple polytope = facets in general position = each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

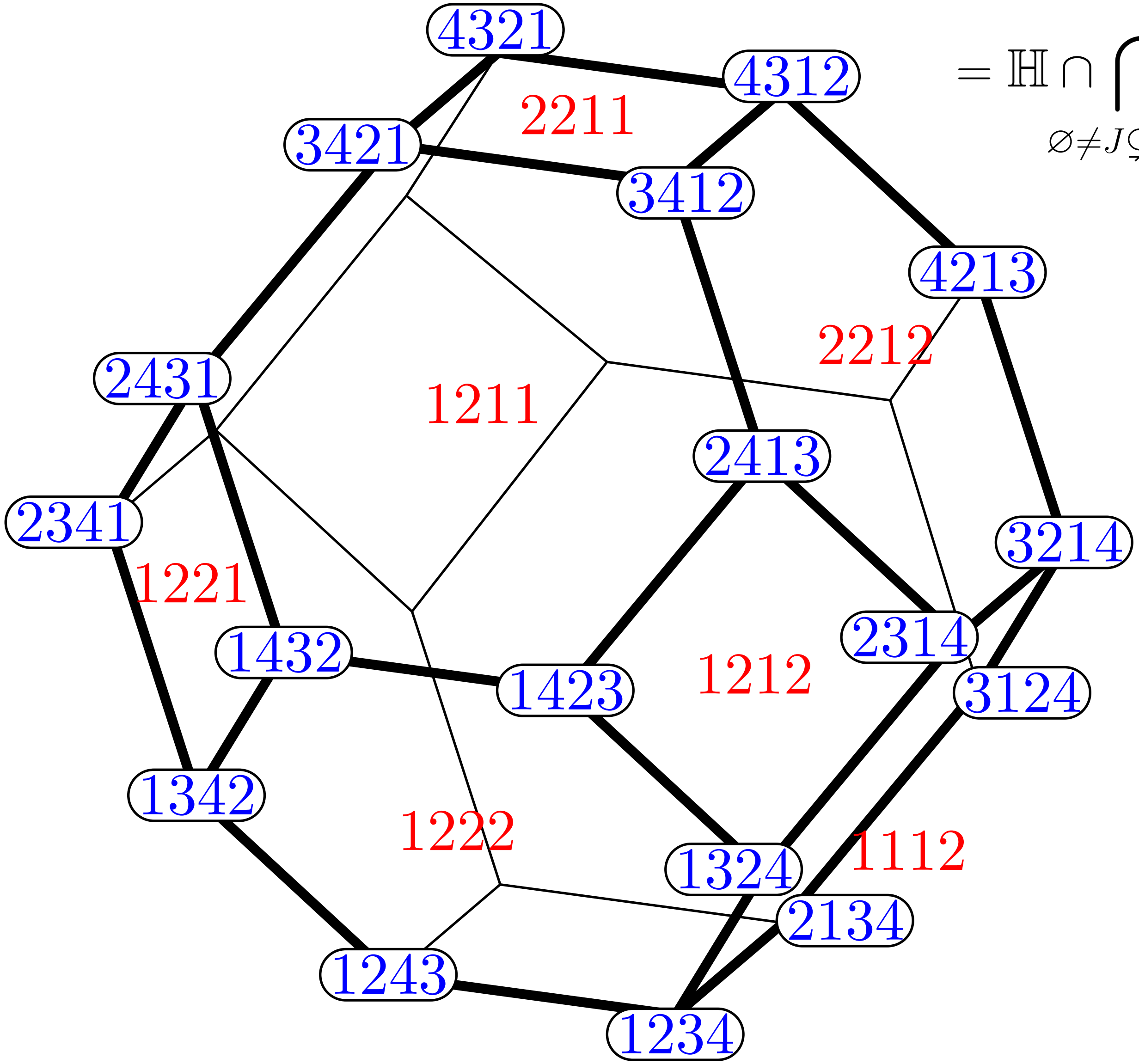
normal fan of P = $\{$ normal cone of F $\mid$ F face of P $\}$

simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

EXM: PERMUTAHEDRON

Hohlweg, *Permutahedra and associahedra* ('12)

PERMUTAHEDRON



Permutohedron Perm(n)

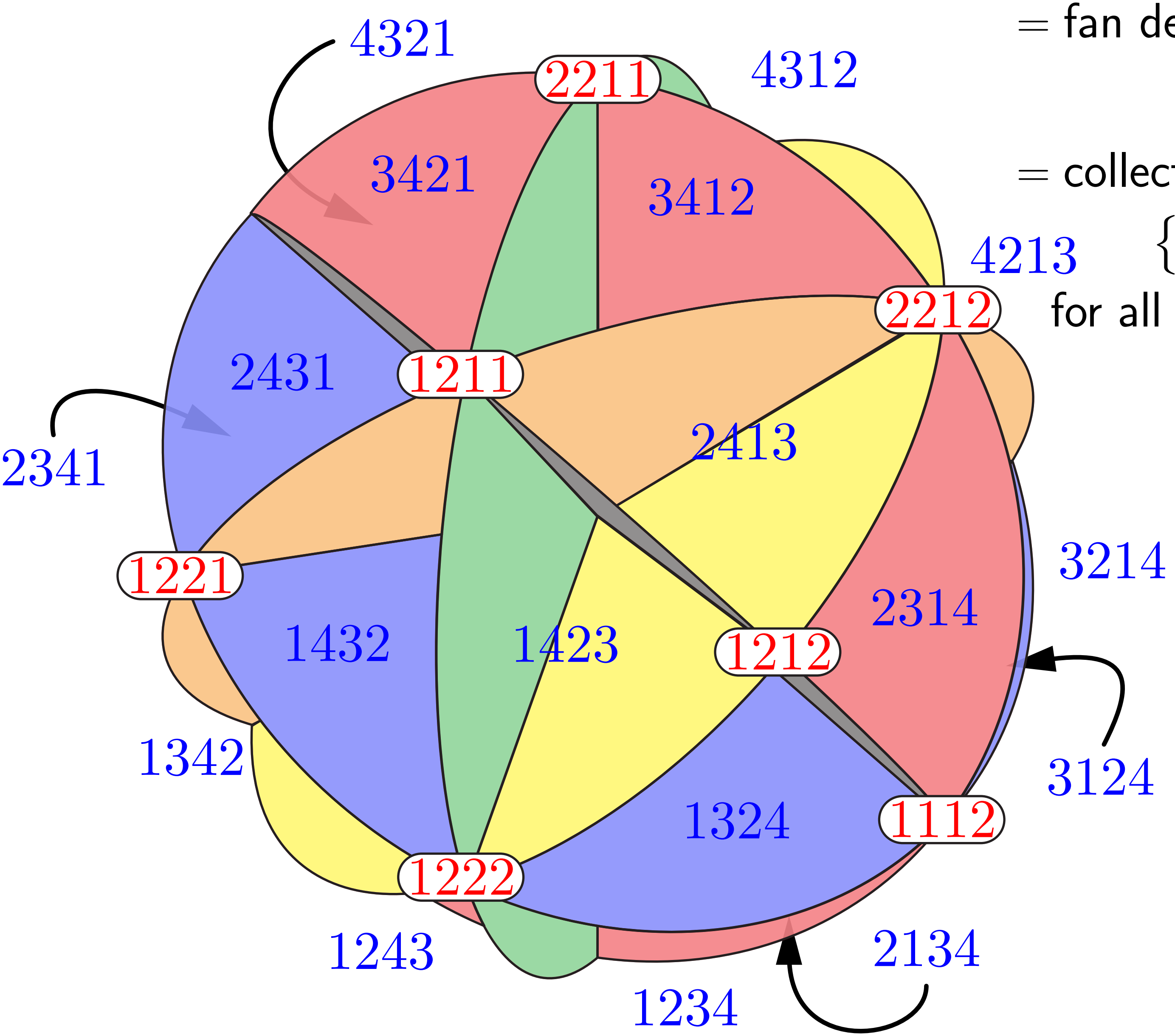
$$= \operatorname{conv}\left\{(\sigma(1),\dots,\sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

COXETER ARRANGEMENT



Coxeter fan

= fan defined by the hyperplane arrangement

$$\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_i = x_j\}_{1 \le i < j \le n+1}$$

= collection of all cones

$$\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_i < x_j \text{ if } \pi(i) < \pi(j)\}$$

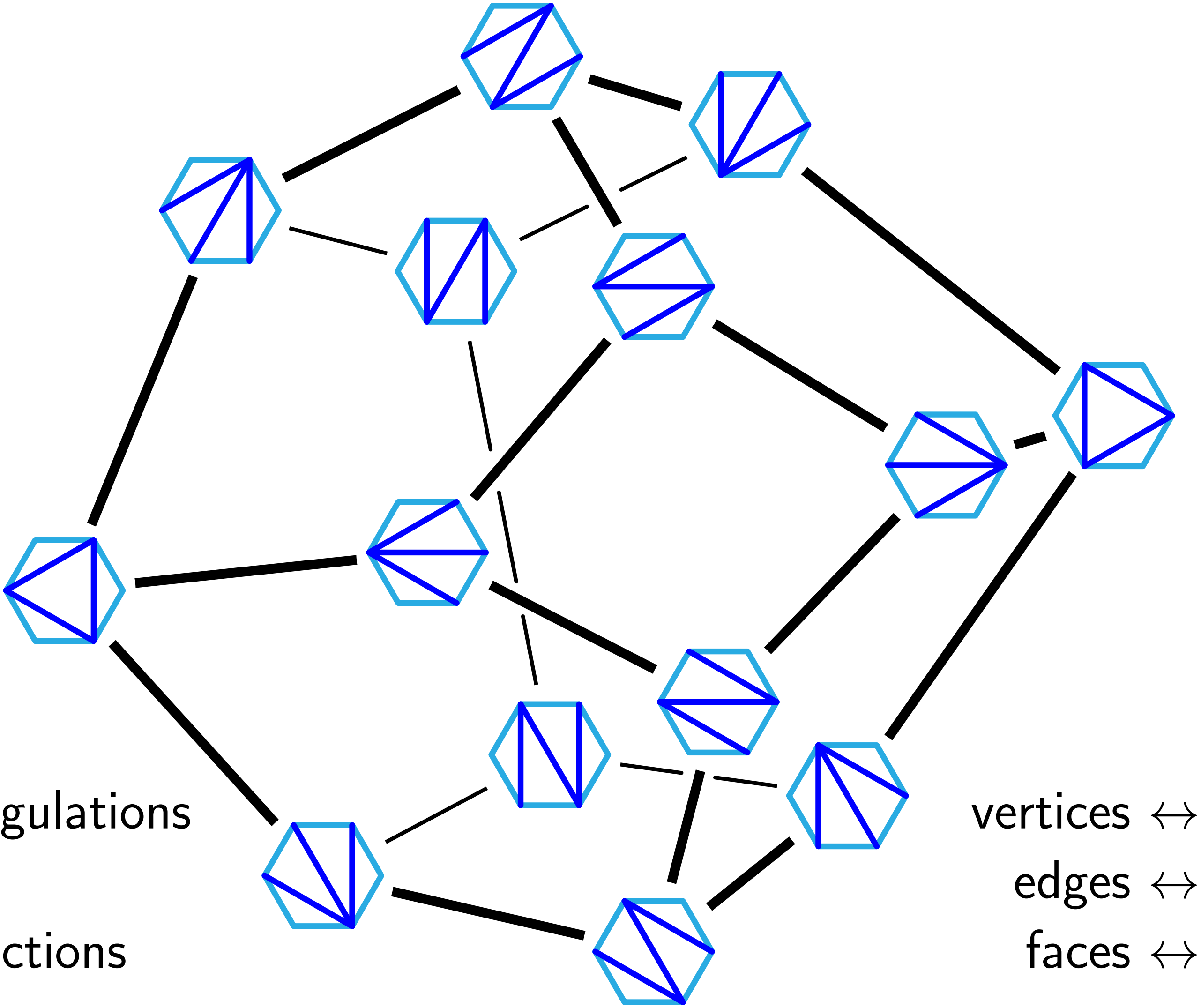
for all surjections $\pi : [n+1] \to [n+1-k]$

ASSOCIAHEDRA

Ceballos-Santos-Ziegler,
Many non-equivalent realizations of the associahedron ('11)

ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

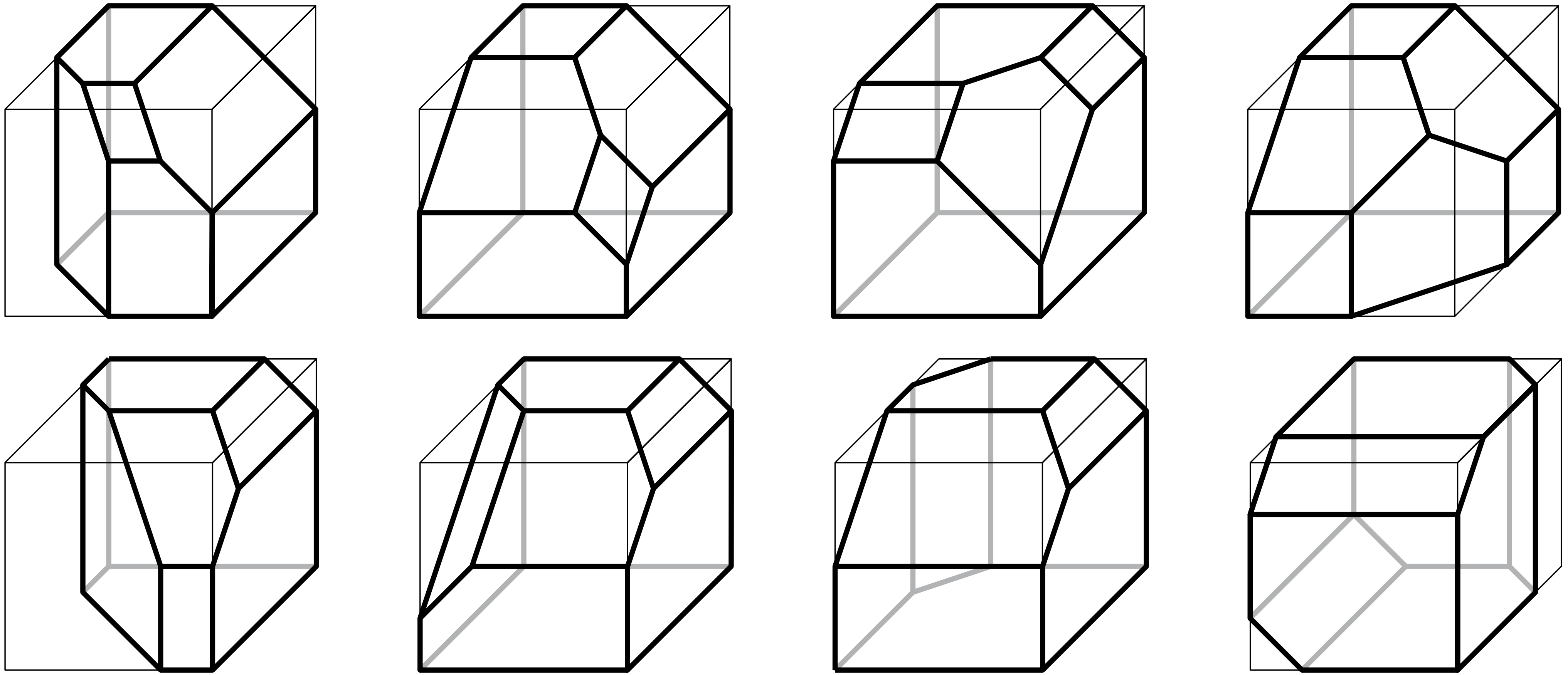


vertices ↔ triangulations
edges ↔ flips
faces ↔ dissections

vertices ↔ binary trees
edges ↔ rotations
faces ↔ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion



(Pictures by Ceballos-Santos-Ziegler)

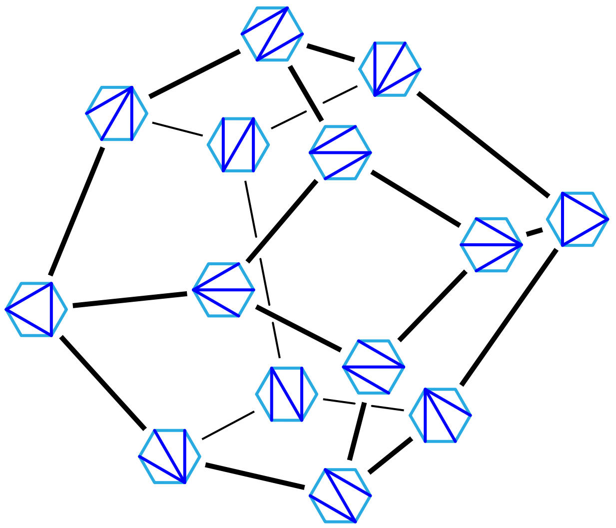
Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

... — Gel'fand-Kapranov-Zelevinski ('94) — ... — Chapoton-Fomin-Zelevinsky ('02) — ... — Loday ('04) — ...

— Ceballos-Santos-Ziegler ('11)

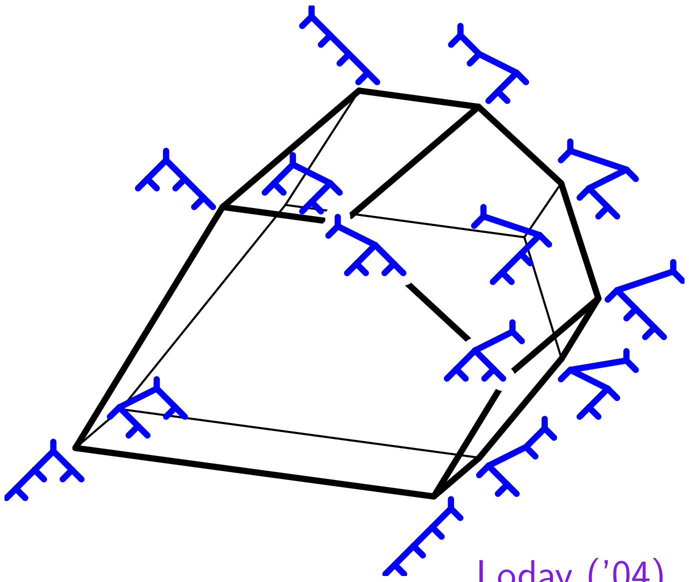
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

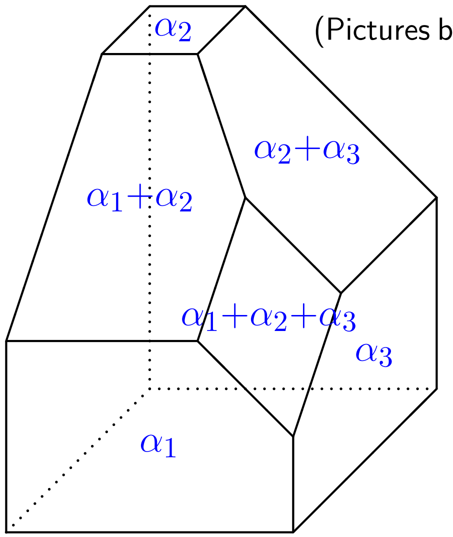
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

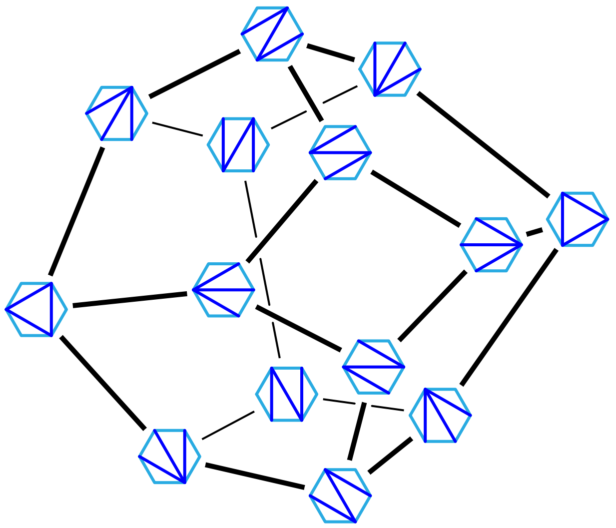
(Pictures by CFZ)



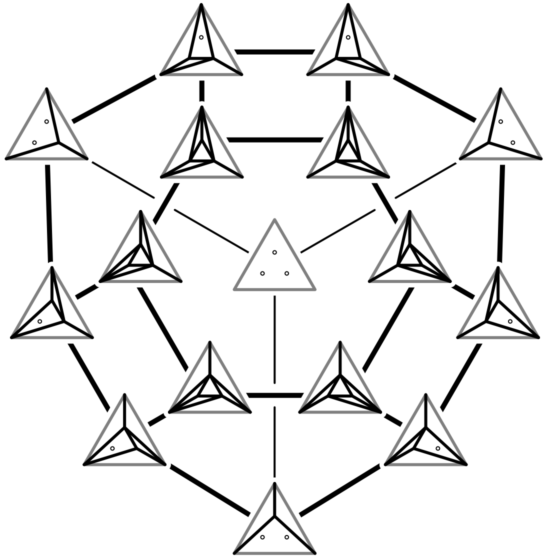
Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

THREE FAMILIES OF REALIZATIONS

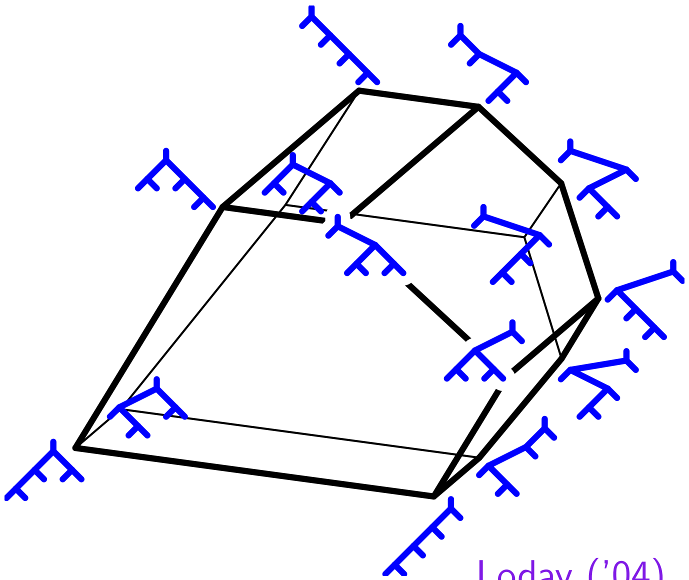
SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)



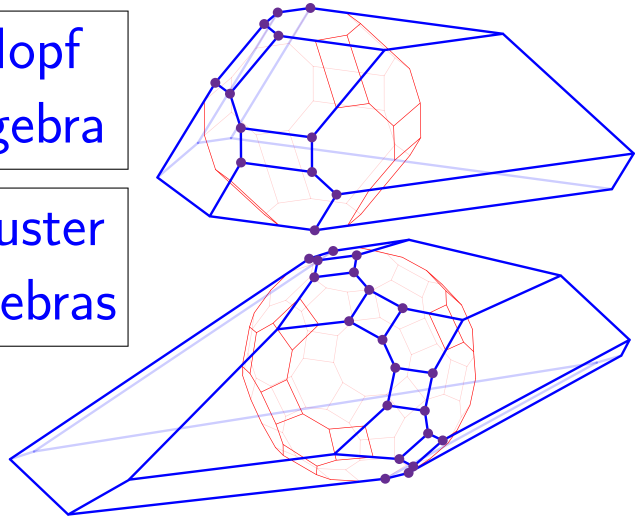
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

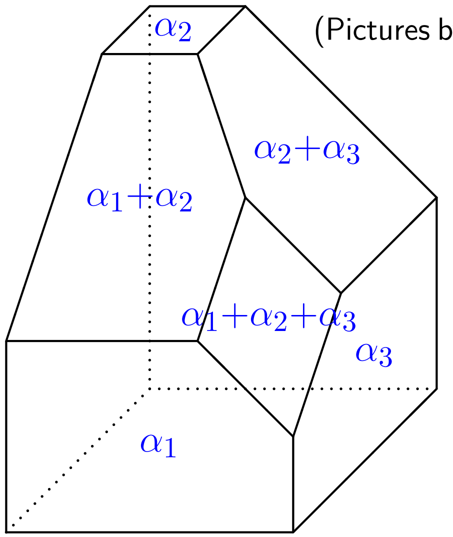
Hopf algebra

Cluster algebras



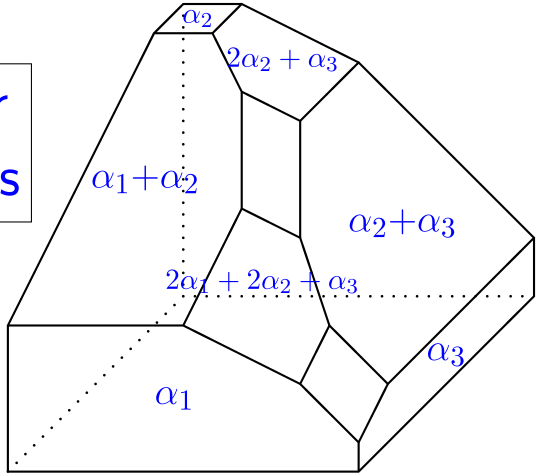
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



LODAY'S ASSOCIAHEDRON

Shnider-Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)
Loday, *Realization of the Stasheff polytope* ('04)

LODAY'S ASSOCIAHEDRON

$$\mathsf{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1\le i\le j\le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i)\cdot r(\mathrm{T},i)\big]_{i\in[n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{ \mathbf{x}\in\mathbb{R}^{n+1} \;\middle|\; \sum_{i\le k\le j} x_i \ge \binom{j-i+2}{2} \right\}$$

Shnider-Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)
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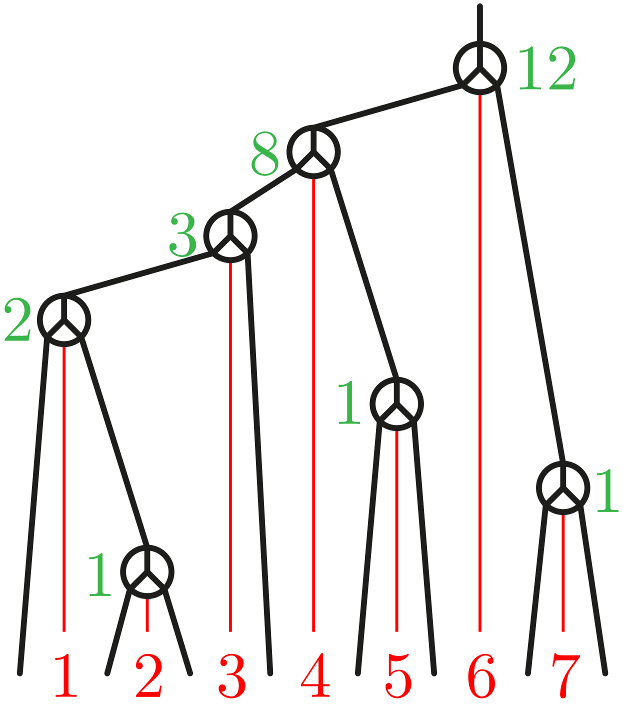

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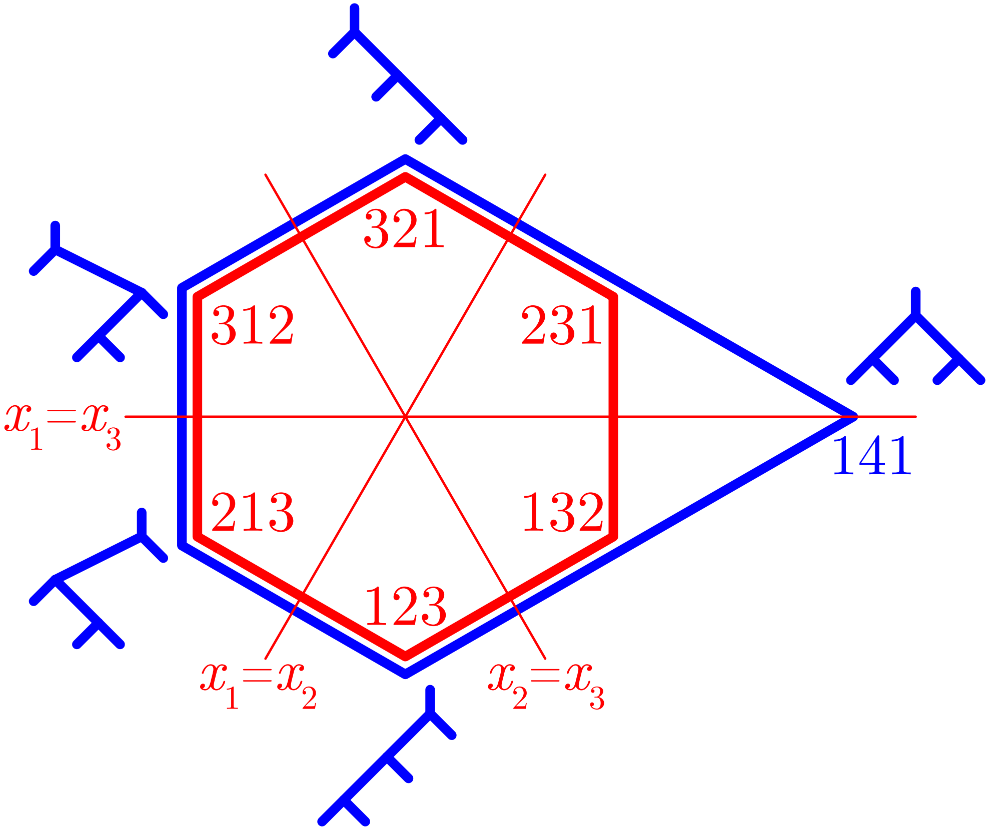
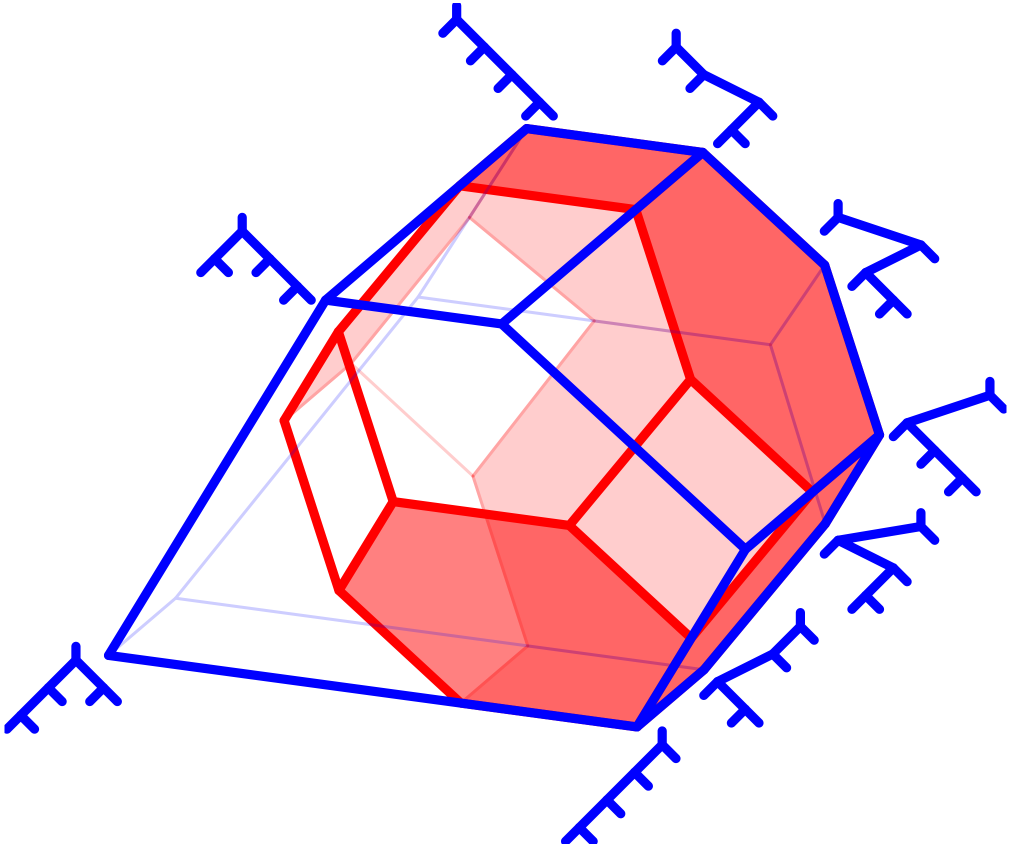
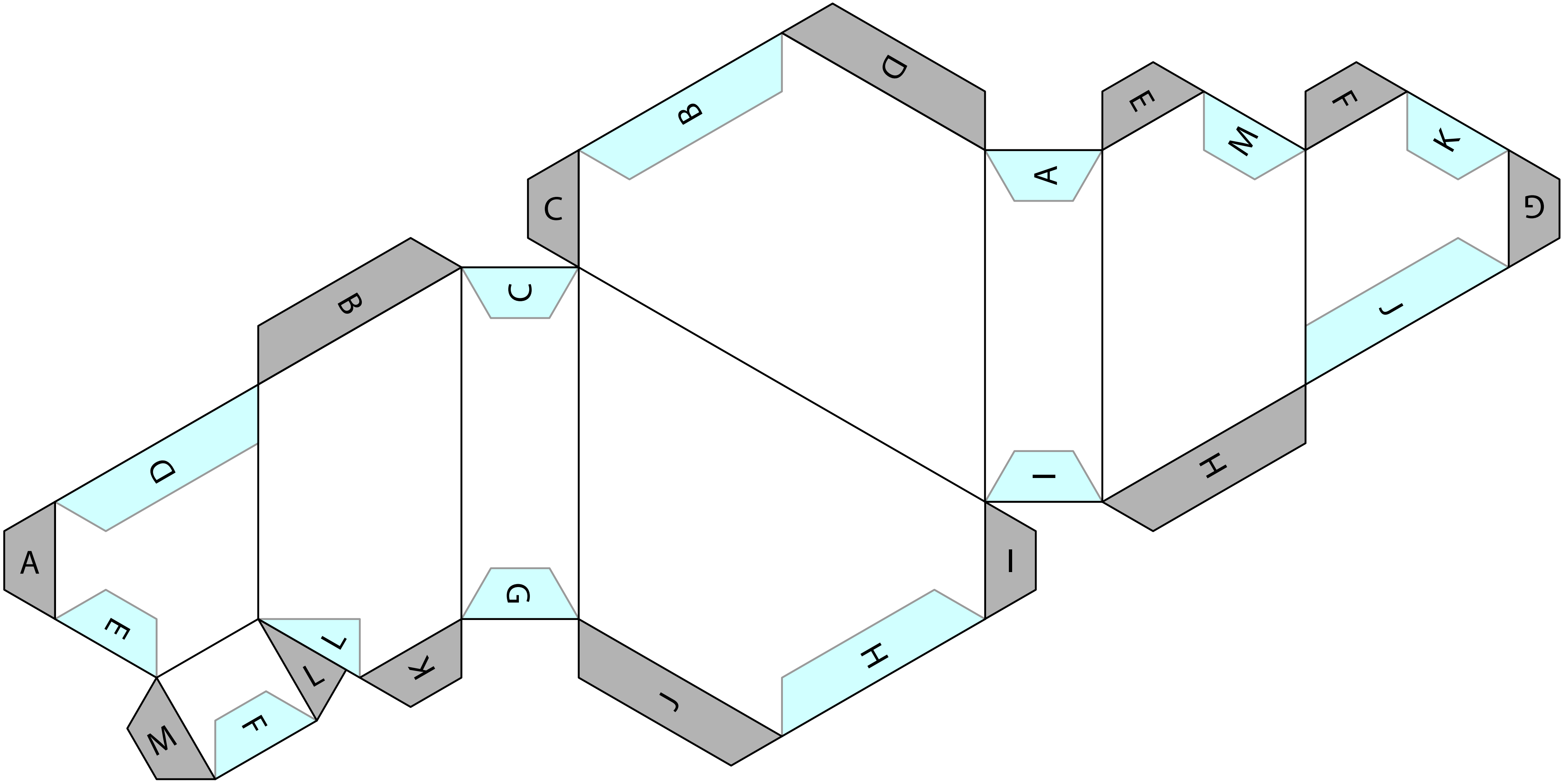




POLYWOOD

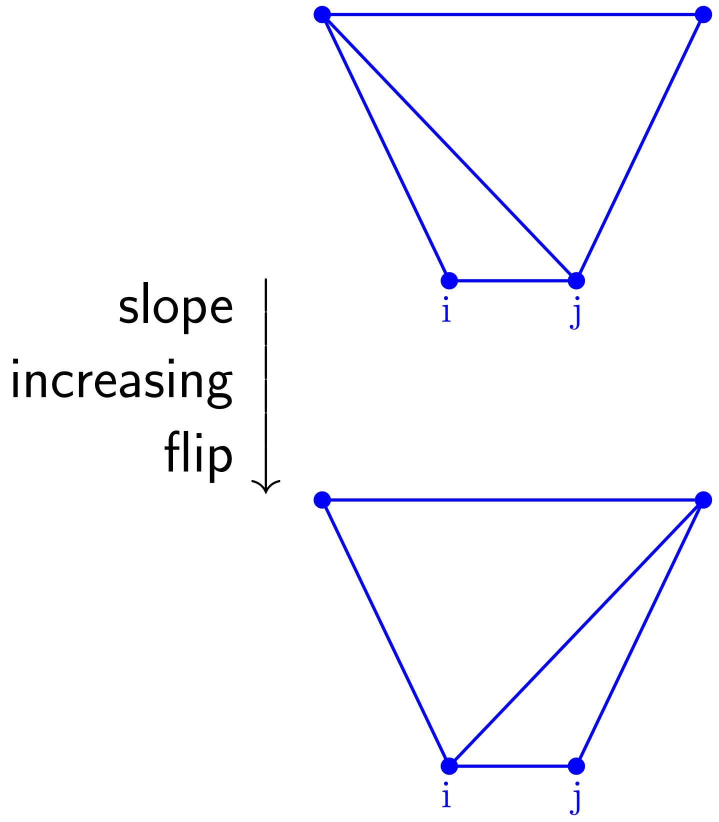
POLYWOOD



LODAY'S ASSOCIAHEDRON

TAMARI LATTICE

Tamari lattice $=$ slope increasing flips on triangulations

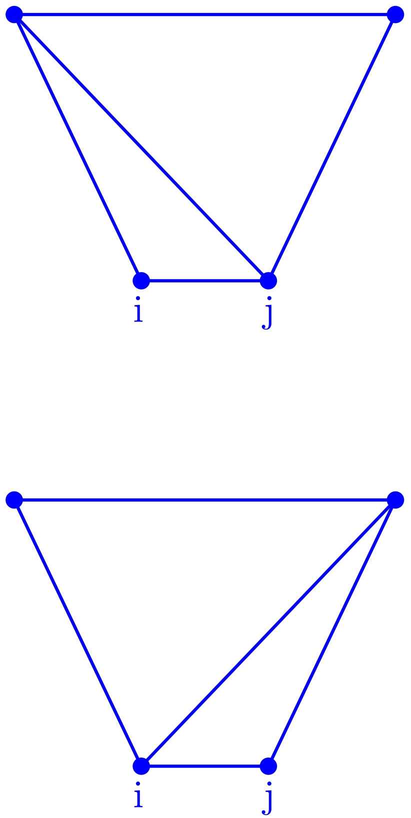


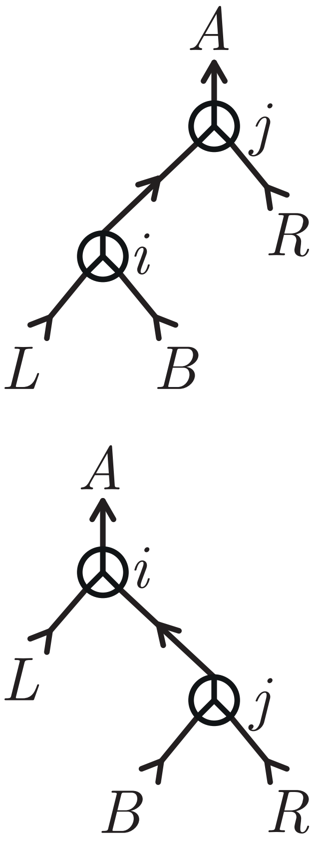
TAMARI LATTICE

Tamari lattice = slope increasing flips on triangulations

= right rotations on binary trees

slope increasing flip ↓



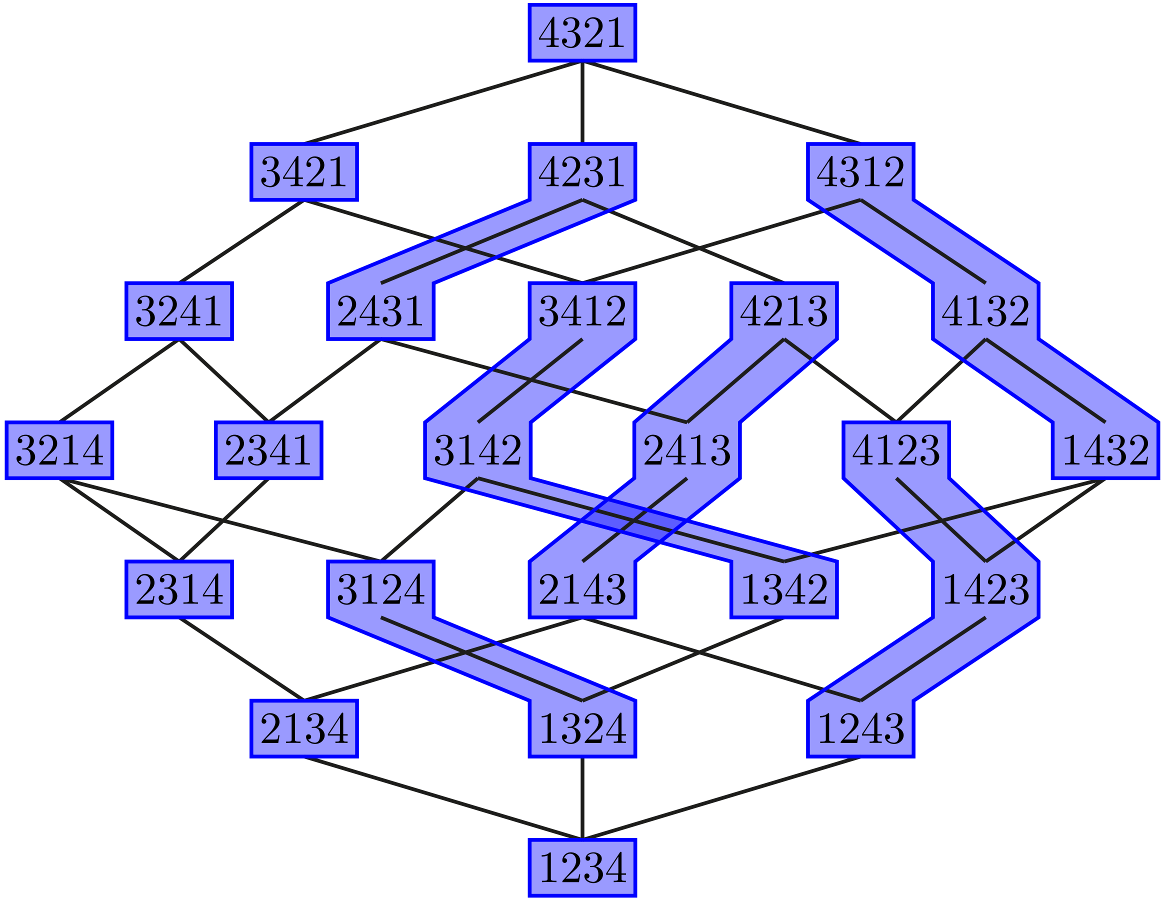


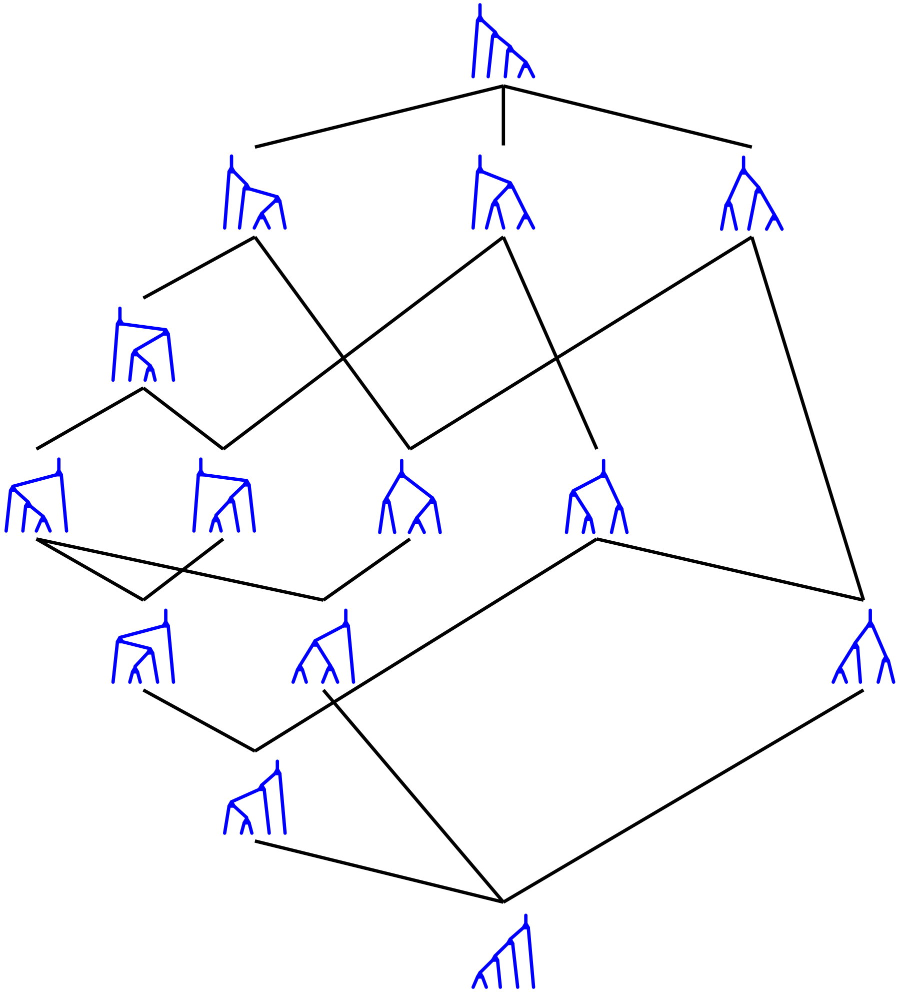
↓ right rotation

Tamari Festschrift ('12)

TAMARI LATTICE

Tamari lattice = slope increasing flips on triangulations
= right rotations on binary trees
= lattice quotient of the weak order by the sylvester congruence

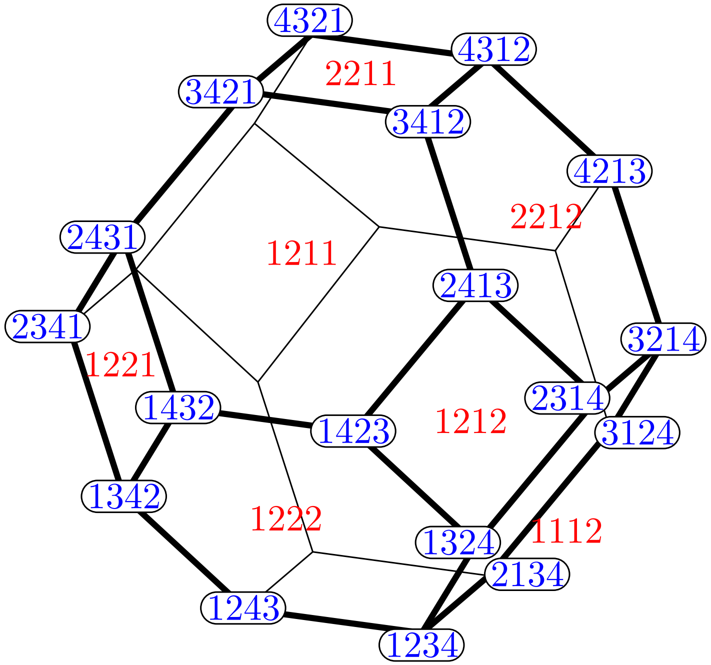


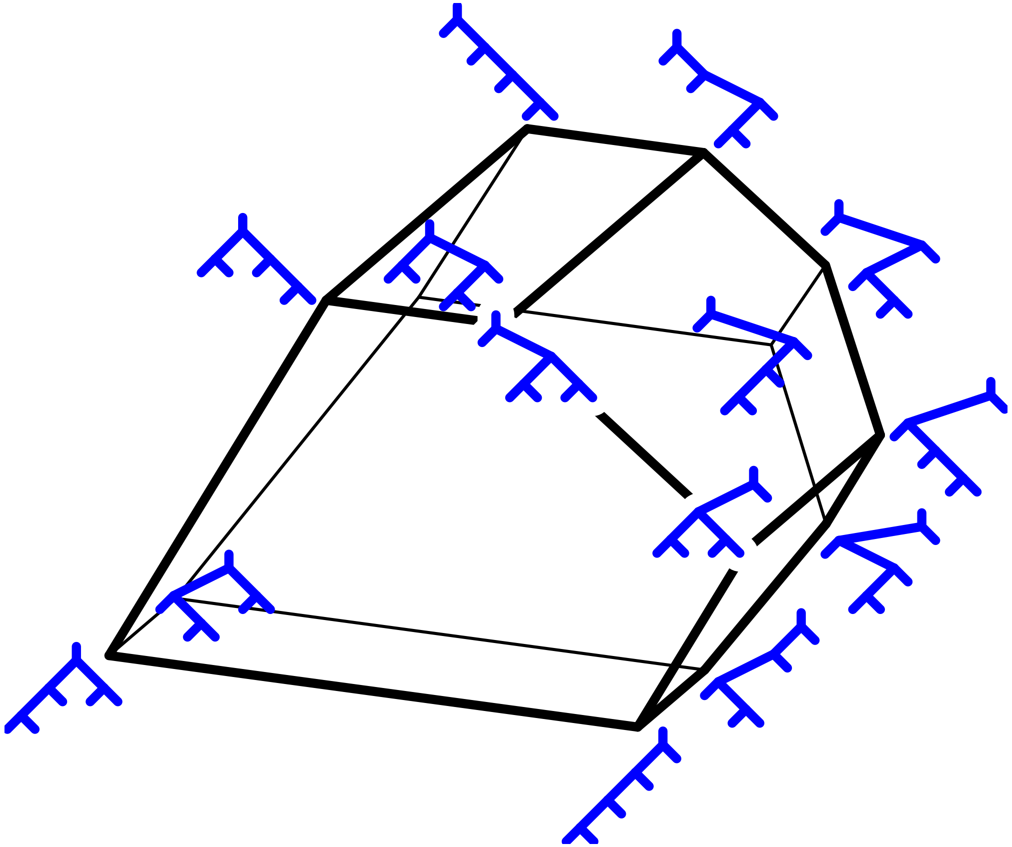


Tamari Festschrift ('12)

TAMARI LATTICE

Tamari lattice $=$ slope increasing flips on triangulations
$=$ right rotations on binary trees
$=$ lattice quotient of the weak order by the sylvester congruence
$=$ orientation of the graph of the associahedron in direction $e \to w_\circ$





Tamari Festschrift ('12)

LODAY-RONCO HOPF ALGEBRA

Malvenuto-Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)
Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)

SHUFFLE AND CONVOLUTION

For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n:

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \dots < \tau(n) \text{ and } \tau(n+1) < \dots < \tau(n+n')\}$$

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

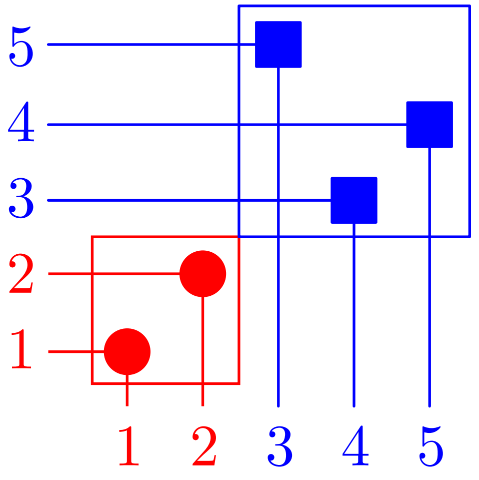
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1) + n, \dots, \tau'(n') + n] \in \mathfrak{S}_{n+n'}$

shifted shuffle product $\tau \,\bar{\sqcup\!\sqcup}\, \tau' = \left\{ (\tau\bar{\tau}') \circ \pi^{-1} \;\middle|\; \pi \in \mathfrak{S}^{(n,n')} \right\} \subset \mathfrak{S}_{n+n'}$

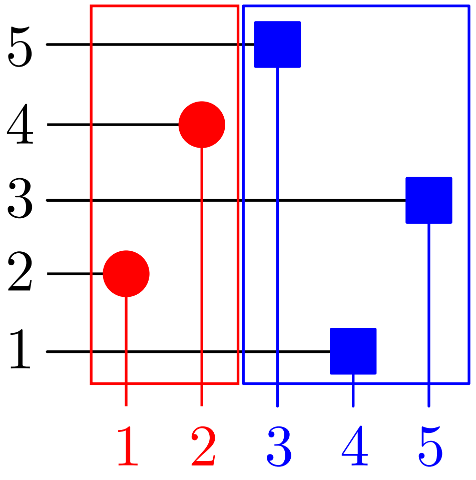
convolution product $\tau \star \tau' = \left\{ \pi \circ (\tau\bar{\tau}') \;\middle|\; \pi \in \mathfrak{S}^{(n,n')} \right\} \subset \mathfrak{S}_{n+n'}$

Exm: $12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

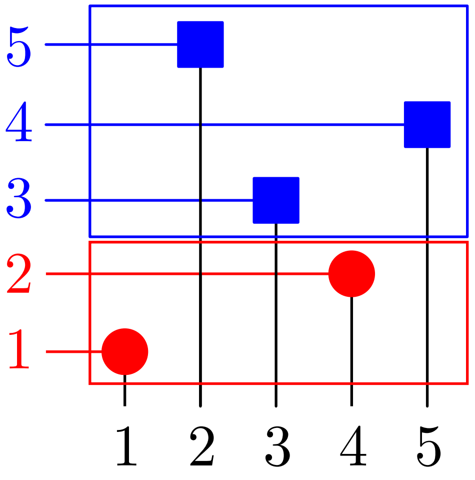
$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



concatenation



shuffle



convolution

MALVENUTO-REUTENAUER ALGEBRA

Combinatorial Hopf Algebra $=$ combinatorial vector space $\mathcal{B}$ endowed with

$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$$

$$\text{coproduct } \triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

which are “compatible”, ie.

$$\begin{tikzcd}
\mathcal{B} \otimes \mathcal{B} \arrow[r, "\cdot"] \arrow[d, "\triangle \otimes \triangle"'] & \mathcal{B} \arrow[r, "\triangle"] & \mathcal{B} \otimes \mathcal{B} \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[rr, "I \otimes \mathrm{swap} \otimes I"'] & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[u, "\cdot \otimes \cdot"']
\end{tikzcd}$$

THM. The vector space $\mathbf{k}\mathfrak{S} = \bigoplus_{n \in \mathbb{N}} \mathbf{k}\mathfrak{S}_n$ with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ endowed with

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\shuffle}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

is a combinatorial Hopf algebra.

Malvenuto-Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)

MALVENUTO-REUTENAUER ALGEBRA

THM. The vector space $\mathbf{k}\mathfrak{S} = \bigoplus_{n\in\mathbb{N}} \mathbf{k}\mathfrak{S}_n$ with basis $(\mathbb{F}_\tau)_{\tau\in\mathfrak{S}}$ endowed with

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma\in\tau \,\overline{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle\mathbb{F}_\sigma = \sum_{\sigma\in\tau\star\tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

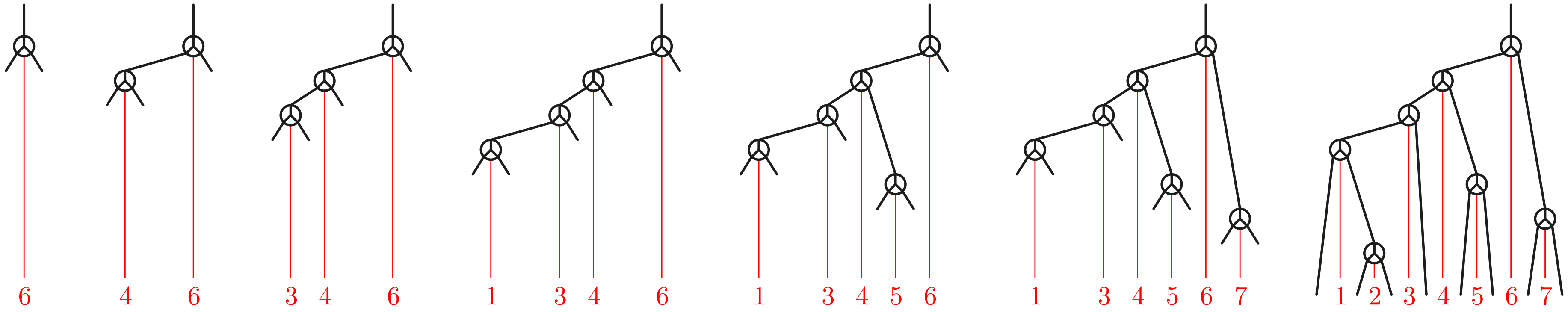
is a combinatorial Hopf algebra.

Malvenuto-Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)

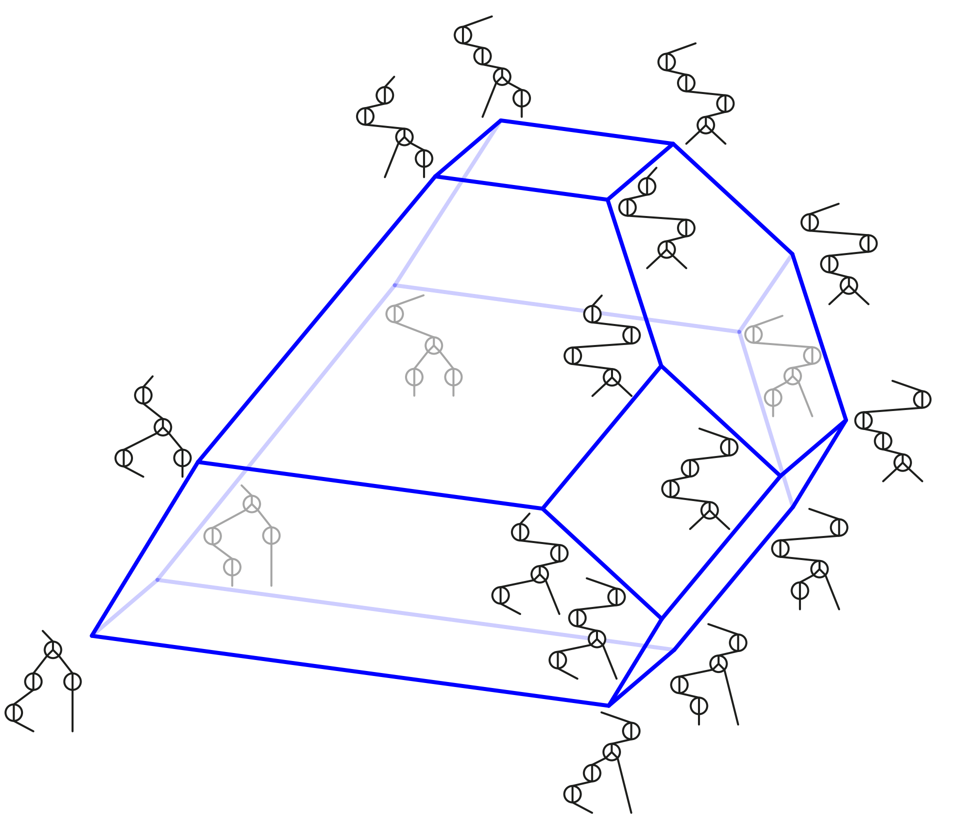
THM. For a binary search tree T, consider the element $\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau\in\mathfrak{S}\\ \mathrm{BST}(\tau)=\mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau\in\mathcal{L}(\mathrm{T})} \mathbb{F}_\tau$.
These elements generate a Hopf subalgebra $\mathbf{k}\mathfrak{T}$ of $\mathbf{k}\mathfrak{S}$.

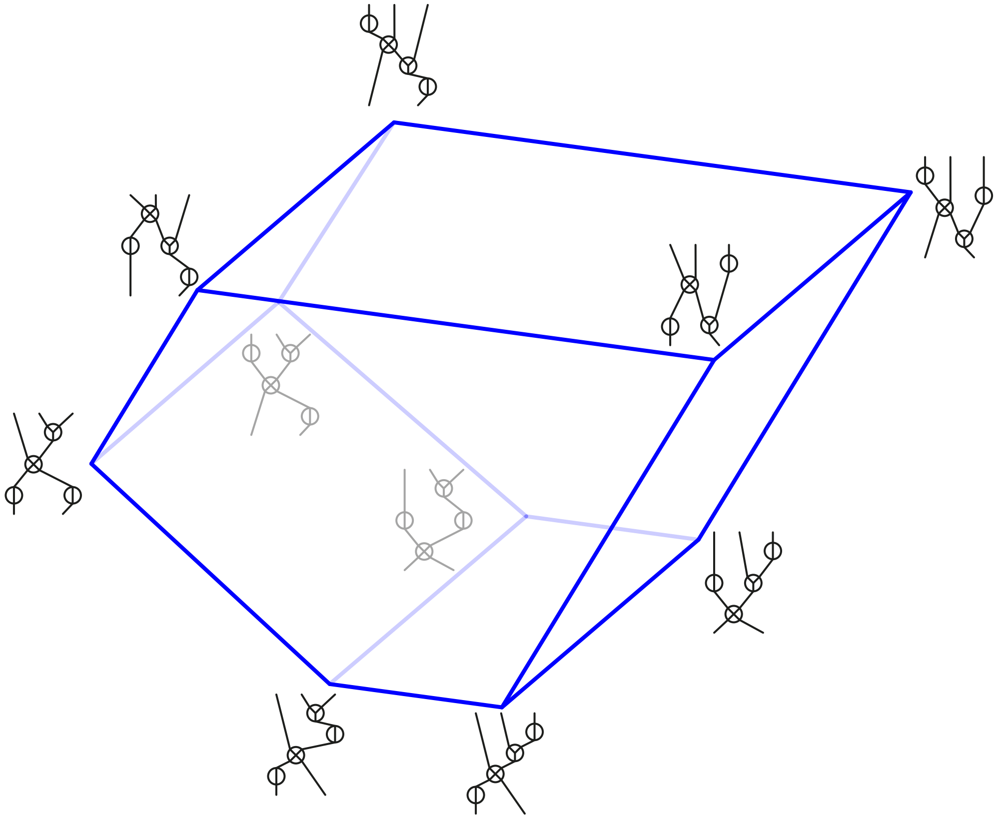
Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)

binary search tree insertion of 2751346



I. PERMUTREEHEDRA





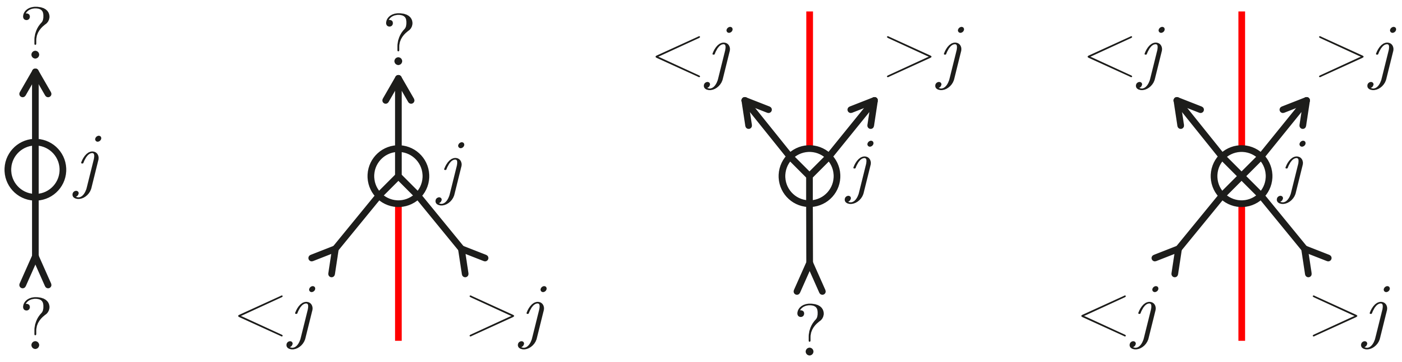
Chatel-P., *Cambrian Hopf Algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREES

Chatel-P., *Cambrian Hopf Algebras* ('17)
P.-Pons, *Permutrees* ('17)

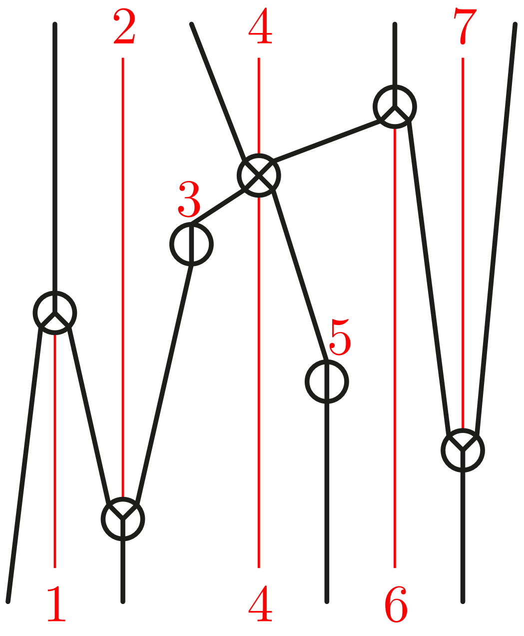
PERMUTREES

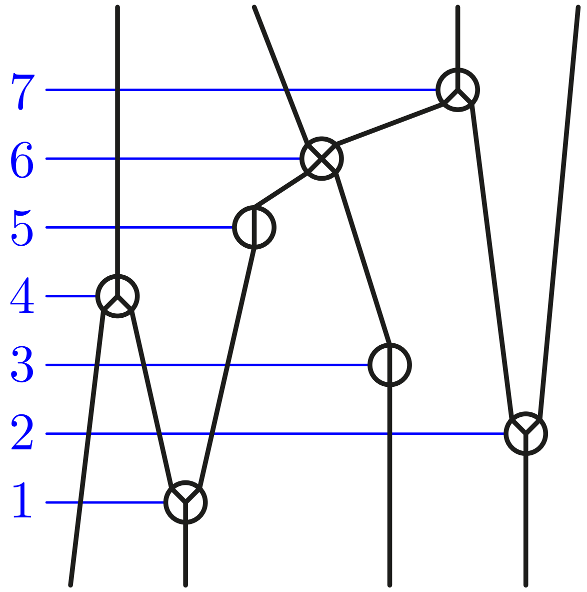
permutree $=$ directed (bottom to top) and labeled (bijectively by $[n]$) tree such that

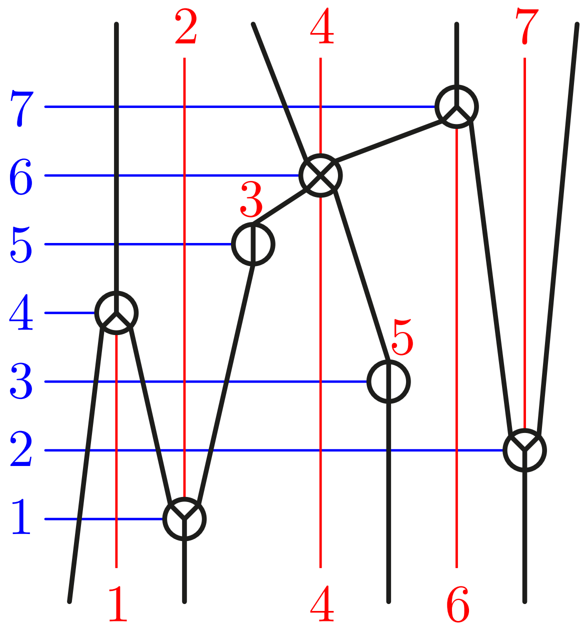


increasing tree $=$ directed and labeled tree such that labels increase along arcs

leveled permutree $=$ directed tree with a permutree labeling and an increasing labeling



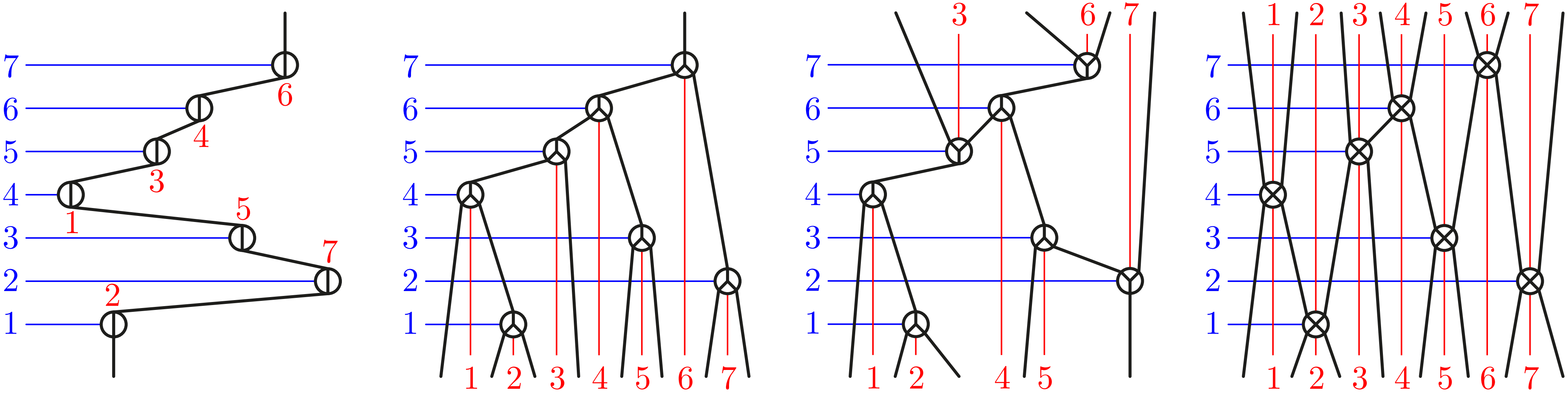




SPECIAL PERMUTREES

Examples.

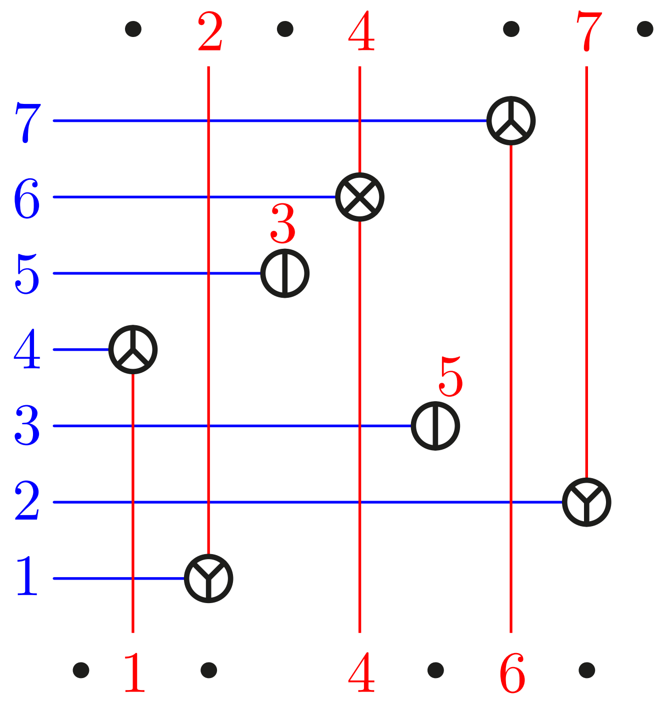
decoration		permutrees
⦶^{n}	$\longleftrightarrow$	permutations of $[n]$
☮^{n}	$\longleftrightarrow$	standard binary search trees
$\{$☮, Ⓨ$\}^n$	$\longleftrightarrow$	Cambrian trees
$\otimes^n$	$\longleftrightarrow$	binary sequences



PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}6}$

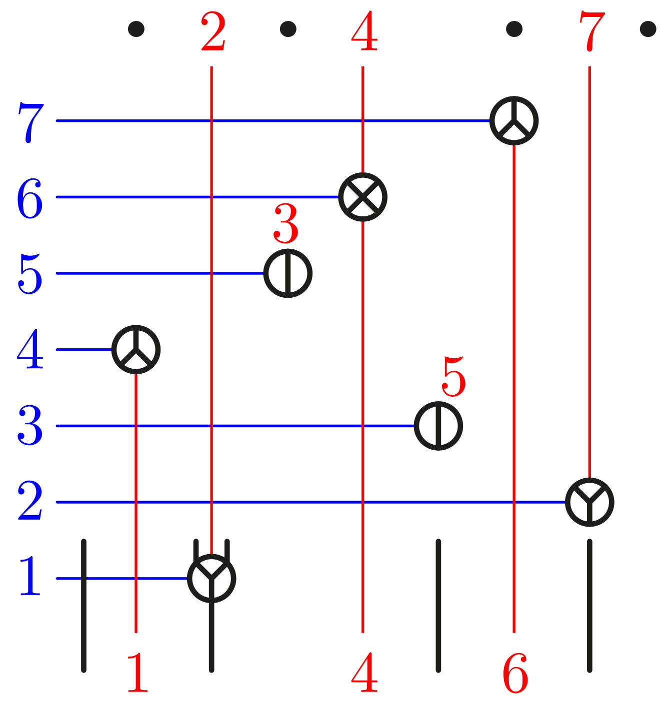


Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13^{+})
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation ⟼ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$



Reading, *Cambrian lattices* ('06)
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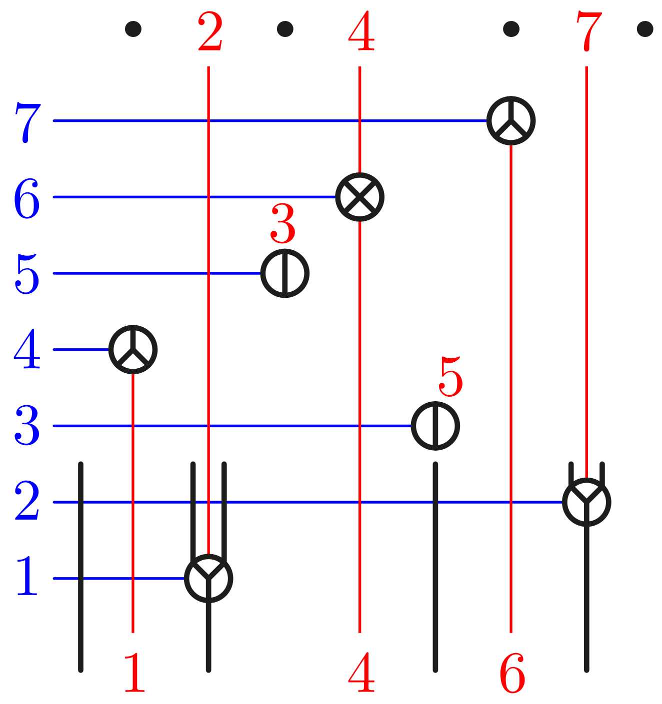
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE CORRESPONDENCE

permutree correspondence $=$ decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{27}5\underline{1}3\underline{\overline{4}6}$



Reading, *Cambrian lattices* ('06)
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P.-Pons, *Permutrees* ('17)

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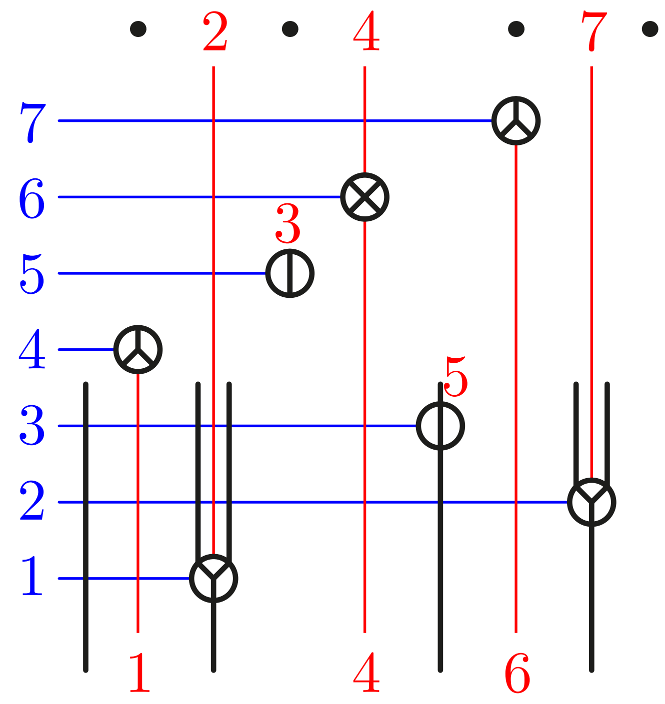

Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13^{+})
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE CORRESPONDENCE

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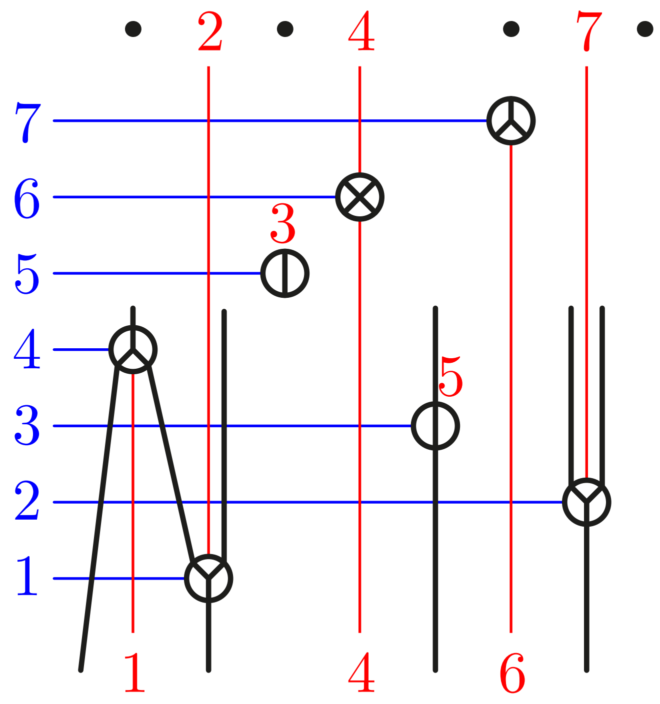
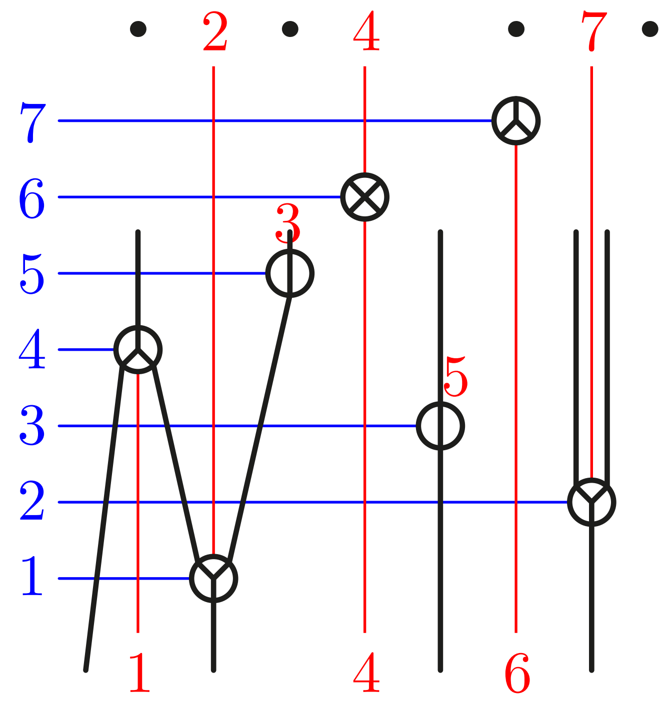


Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13^+)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$



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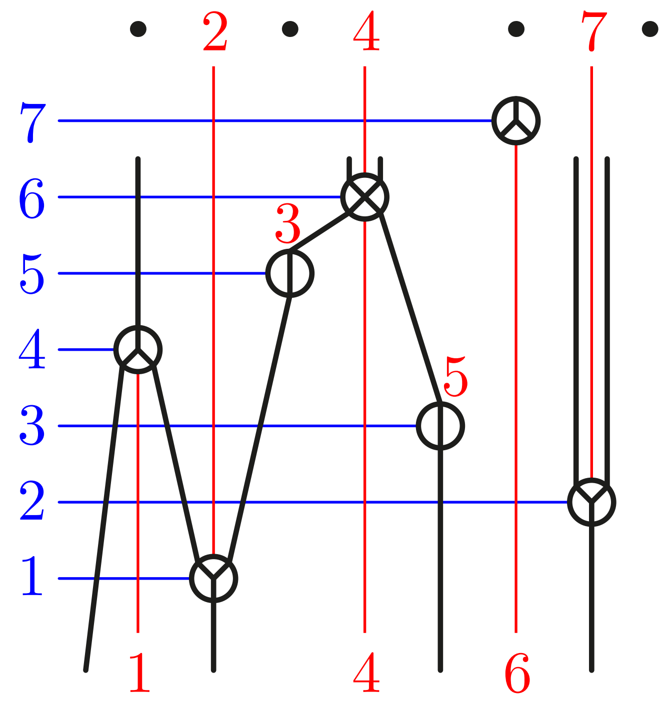
Lange-P., *Associahedra via spines* ('13^+)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation ⟼ leveled permutree

Exm: decorated permutation $\overline{27}5\underline{1}3\overline{\underline{4}}6$

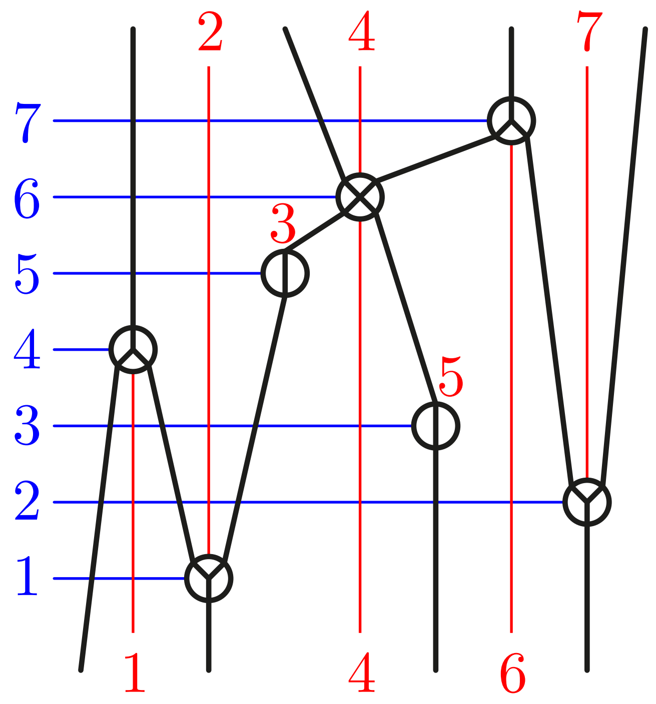


Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13^+)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}51\underline{1}3\overline{\underline{4}}6$

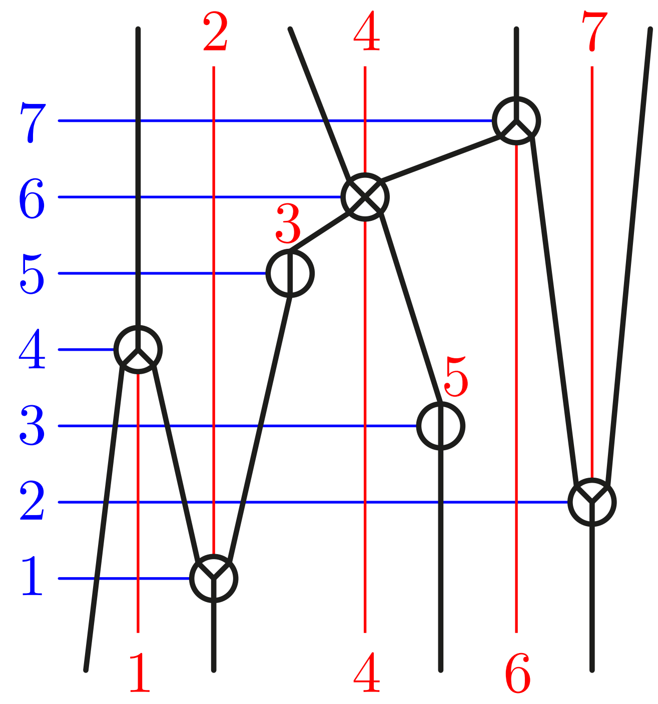


Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13^+)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation ⟼ leveled permutree

Exm: decorated permutation $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$



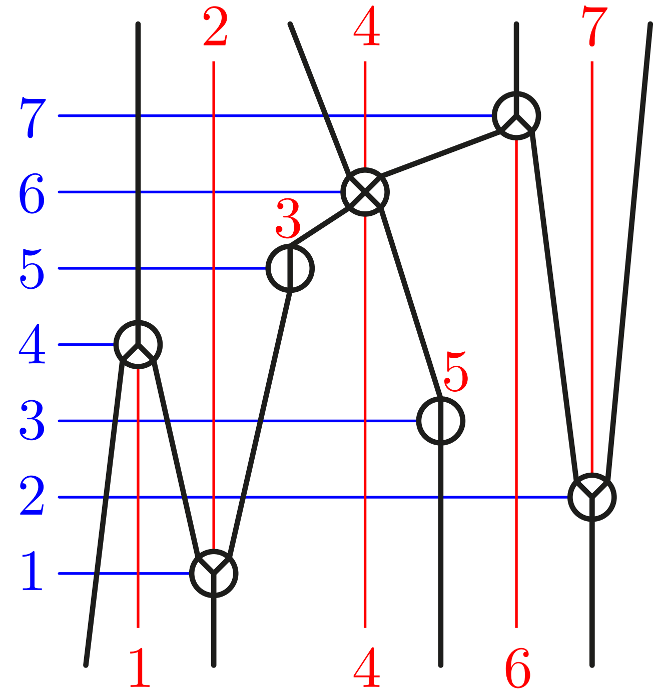
Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13^+)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

PROP. bijection decorated permutation ⟷ leveled permutree.

PERMUTREE CORRESPONDENCE

permutree correspondence = decorated permutation $\longmapsto$ leveled permutree

Exm: decorated permutation $\overline{27}5\underline{1}3\underline{\overline{4}6}$



Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13^+)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

$\mathbf{P}(\tau)$ = $\mathbf{P}$-symbol of τ = permutree produced by permutree correspondence

$\mathbf{Q}(\tau)$ = $\mathbf{Q}$-symbol of τ = increasing tree produced by permutree correspondence

(analogy to Robinson-Schensted algorithm)

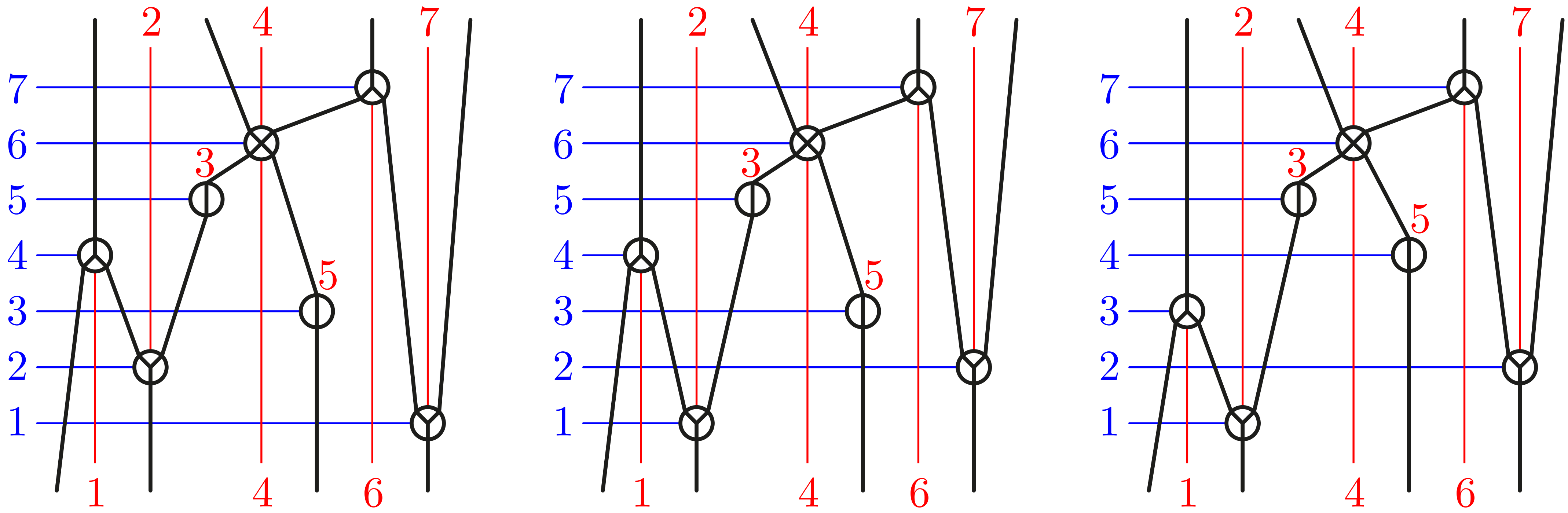
PERMUTREE CONGRUENCE

δ-permutree congruence = transitive closure of the rewriting rules

$$UacVbW \equiv_\delta UcaVbW \quad \text{if } a < b < c \text{ and } \delta_b \in \{☮, \otimes\}$$
$$UbVacW \equiv_\delta UbVcaW \quad \text{if } a < b < c \text{ and } \delta_b \in \{Ⓨ, \otimes\}$$

where a, b, c are elements of $[n]$ while U, V, W are words on $[n]$

PROP. $\tau \equiv_\delta \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$.



$$\overline{7}\overline{2}5\underline{1}3\overline{\underline{4}}\underline{6} \quad \equiv \quad \overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6} \quad \equiv \quad \overline{2}\overline{7}\underline{1}53\overline{\underline{4}}\underline{6}$$

PERMUTREE CONGRUENCE

δ-permutree congruence = transitive closure of the rewriting rules

$UacVbW \equiv_\delta UcaVbW$ if $a < b < c$ and $\delta_b \in$ {☮, ⊗}

$UbVacW \equiv_\delta UbVcaW$ if $a < b < c$ and $\delta_b \in$ {Ⓨ, ⊗}

where a, b, c are elements of $[n]$ while U, V, W are words on $[n]$

PROP. $\tau \equiv_\delta \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$.

PROP. The permutree congruence class labeled by permutree T is given by

$\left\{\tau \in \mathfrak{S}^\delta \;\middle|\; \mathbf{P}(\tau) = \mathrm{T}\right\} = \{$linear extensions of $\mathrm{T}\}$.

PROP. The permutree classes are intervals of the weak order.

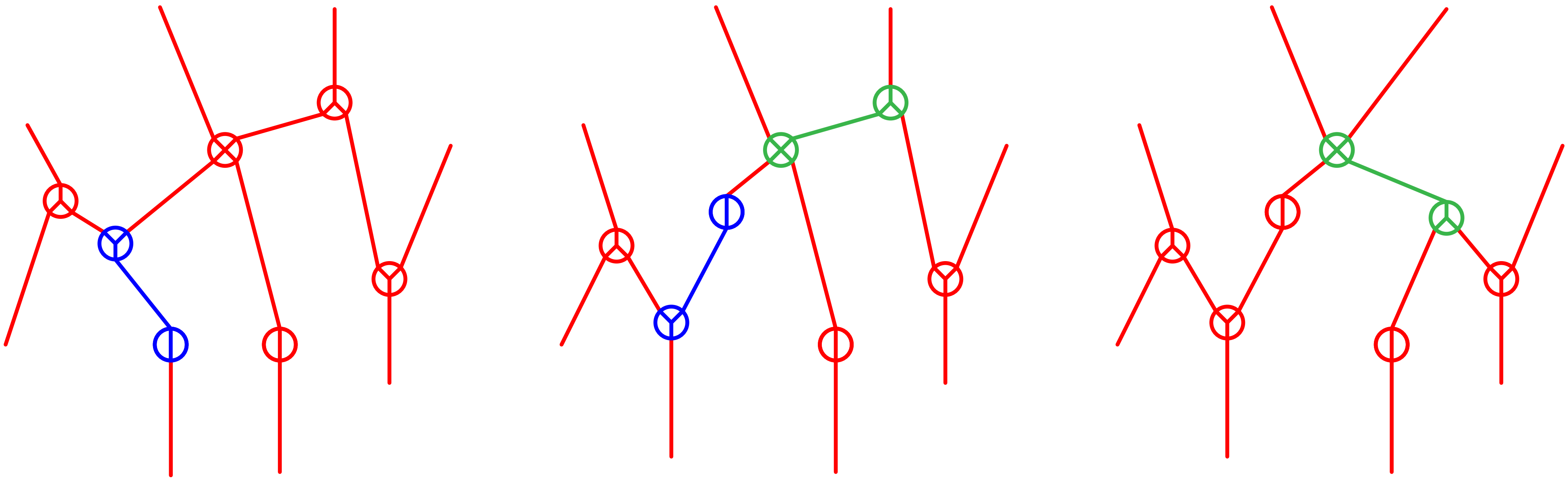
Minimums avoid $b - ca$ with $\delta_b \in$ {Ⓨ, ⊗} and $ca - b$ with $\delta_b \in$ {☮, ⊗}.

Maximums avoid $b - ac$ with $\delta_b \in$ {Ⓨ, ⊗} and $ac - b$ with $\delta_b \in$ {☮, ⊗}.

Reading, *Cambrian lattices* ('06)
P.-Pons, *Permutrees* ('17)

ROTATIONS AND PERMUTREE LATTICES

Rotation operation preserves permutrees:

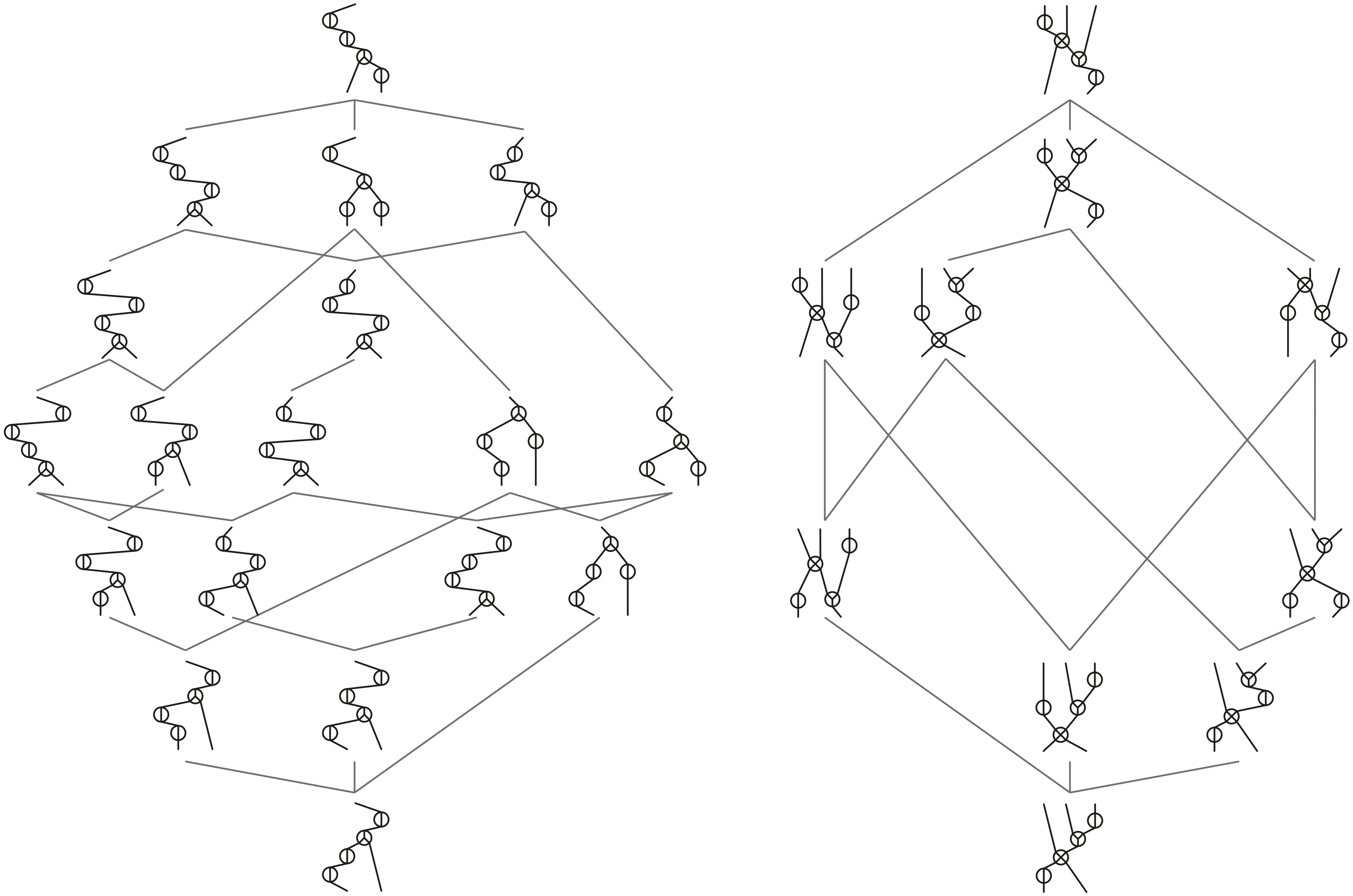


increasing rotation $=$ rotation of edge $i \to j$ where $i < j$

PROP. The transitive closure of the increasing rotation graph is the permutree lattice.
$\mathbf{P}$ defines a lattice homomorphism from the weak order to the permutree lattice.

Reading, *Cambrian lattices* ('06)
P.-Pons, *Permutrees* ('17)

ROTATIONS AND CAMBRIAN LATTICES

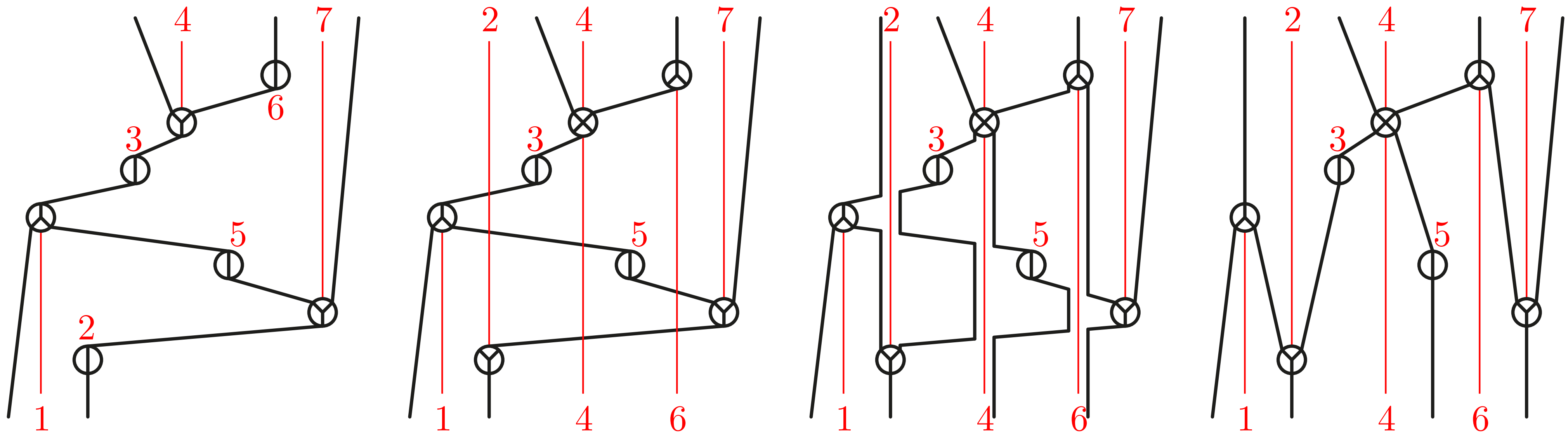


Reading, *Cambrian lattices* ('06)
P.-Pons, *Permutrees* ('17)

DECORATION REFINEMENTS

δ refines δ' when $\delta_i \preccurlyeq \delta'_i$ for all $i \in [n]$ for the order ⦶ $\preccurlyeq$ Ⓨ, ☮ $\preccurlyeq$ ⊗

PROP. When δ refines δ', the δ-permutree congruence classes refine the δ-permutree congruence classes: $\sigma \equiv_{\delta} \tau \Longrightarrow \sigma \equiv_{\delta'} \tau$.
It defines a surjection $\Psi_{\delta}^{\delta'}$ from the δ-permutrees to the δ'-permutrees.



PERMUTREEHEDRA

Loday, *Realization of the Stasheff polytope* ('04)
Hohlweg-Lange, Realizations of the associahedron and cyclohedron ('07)
Lange-P., *Using spines to revisit a construction of the associahedron* ('15)
Chatel-P., *Cambrian Hopf Algebras* ('17)
P.-Pons, *Permutrees* ('17)

PERMUTREE FAN

For a permutree T, define

$$\begin{aligned} \mathrm{C}^{\diamond}(\mathrm{T}) &:= \{\mathbf{x} \in \mathbb{H} \mid x_i \le x_j \text{ for any } i \to j \text{ in } \mathrm{T}\} \\ &= \mathbb{1} + \mathrm{cone}\left\{ \sum_{j \in J} |I| \mathbf{e}_j - \sum_{i \in I} |J| \mathbf{e}_i \;\middle|\; \text{for all edge cuts } (I \parallel J) \text{ in } \mathrm{T}) \right\} \end{aligned}$$

THM. For any $\delta \in \{⦶, ☮, Ⓨ, \otimes\}^n$, the collection of cones $\{\mathrm{C}^{\diamond}(\mathrm{T}) \mid \mathrm{T}\ \delta\text{-permutree}\}$ together with all their faces define a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$.

P.-Pons, *Permutrees* ('17)

Examples.

decoration		permutrees
$⦶^n$	$\longleftrightarrow$	braid fan
$☮^n$	$\longleftrightarrow$	binary tree fan
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	Cambrian fan
$\otimes^n$	$\longleftrightarrow$	fan of the arrangement $\{x_i = x_{i+1} \mid i \in [n-1]\}$

PERMUTREEHEDRA

THM. The permutree fan $\mathcal{F}(\delta)$ is the normal fan of the permutreehedron $\mathbb{PT}(\delta)$, defined equivalently as

(i) the convex hull of the points

$$\mathbf{a}(\mathrm{T})_i = \begin{cases} d+1 & \text{if } \delta_i = ⦶, \\ d+1+\underline{\ell r} & \text{if } \delta_i = ☮, \\ d+1-\overline{\ell r} & \text{if } \delta_i = Ⓨ, \\ d+1+\underline{\ell r}-\overline{\ell r} & \text{if } \delta_i = \otimes, \end{cases}$$

for all δ-permutrees T,

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

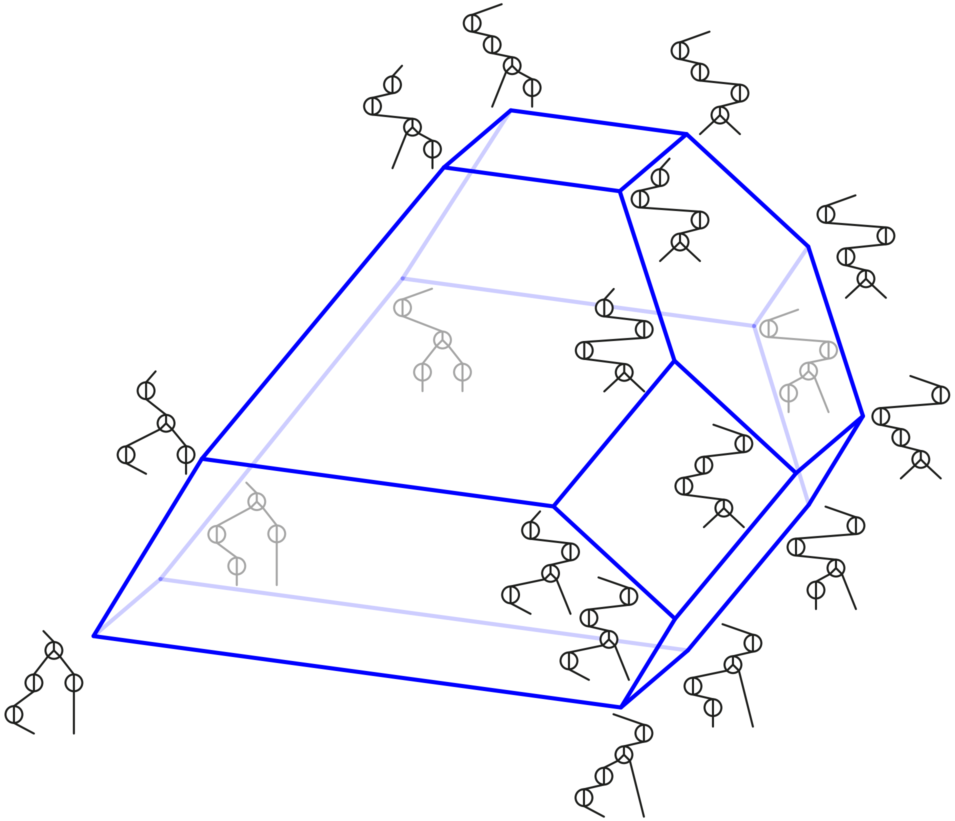
$$\mathbf{H}^{\geq}(I) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \right\}$$

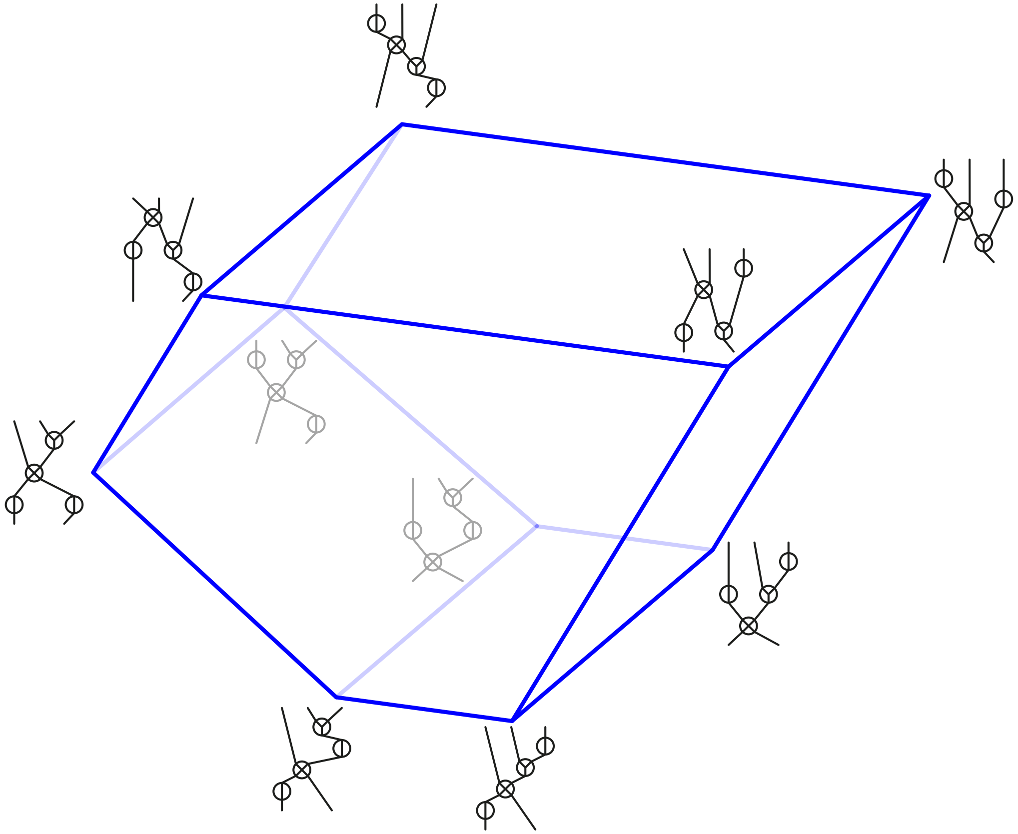
for all edge cuts $(I \parallel J)$ of all δ-permutrees.

P.-Pons, *Permutrees* ('17)

PERMUTREEHEDRA

THM. The permutree fan $\mathcal{F}(\delta)$ is the normal fan of the permutreehedron $\mathbb{PT}(\delta)$.

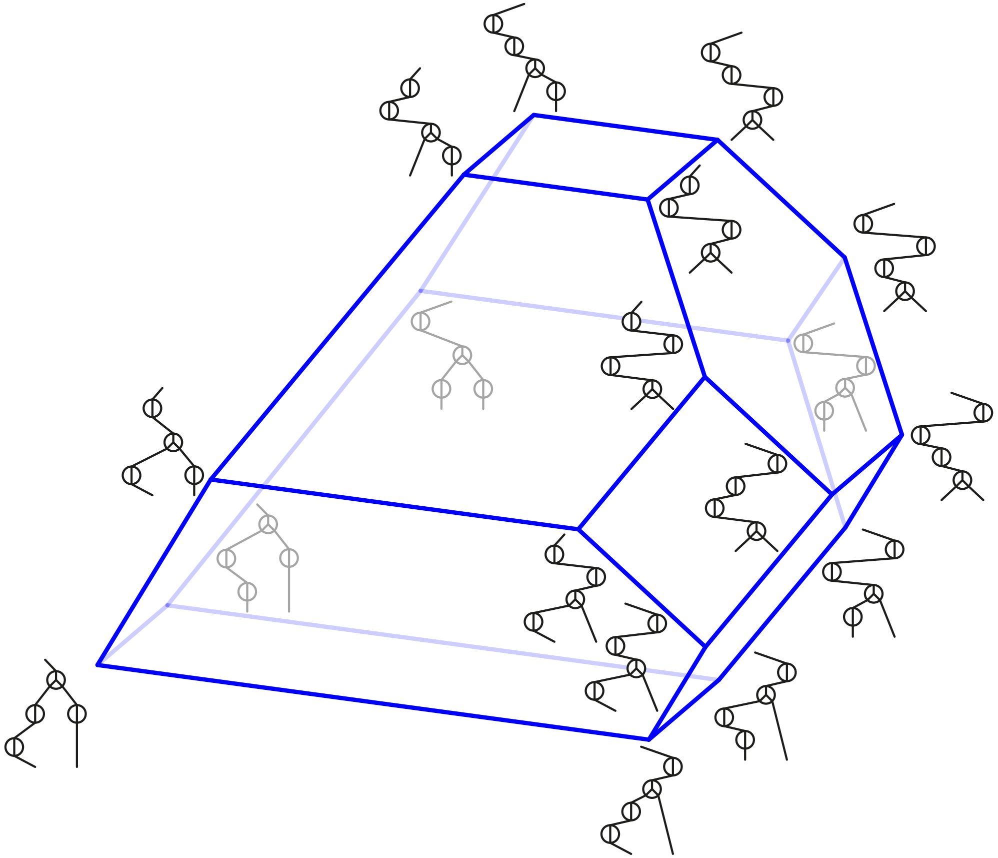


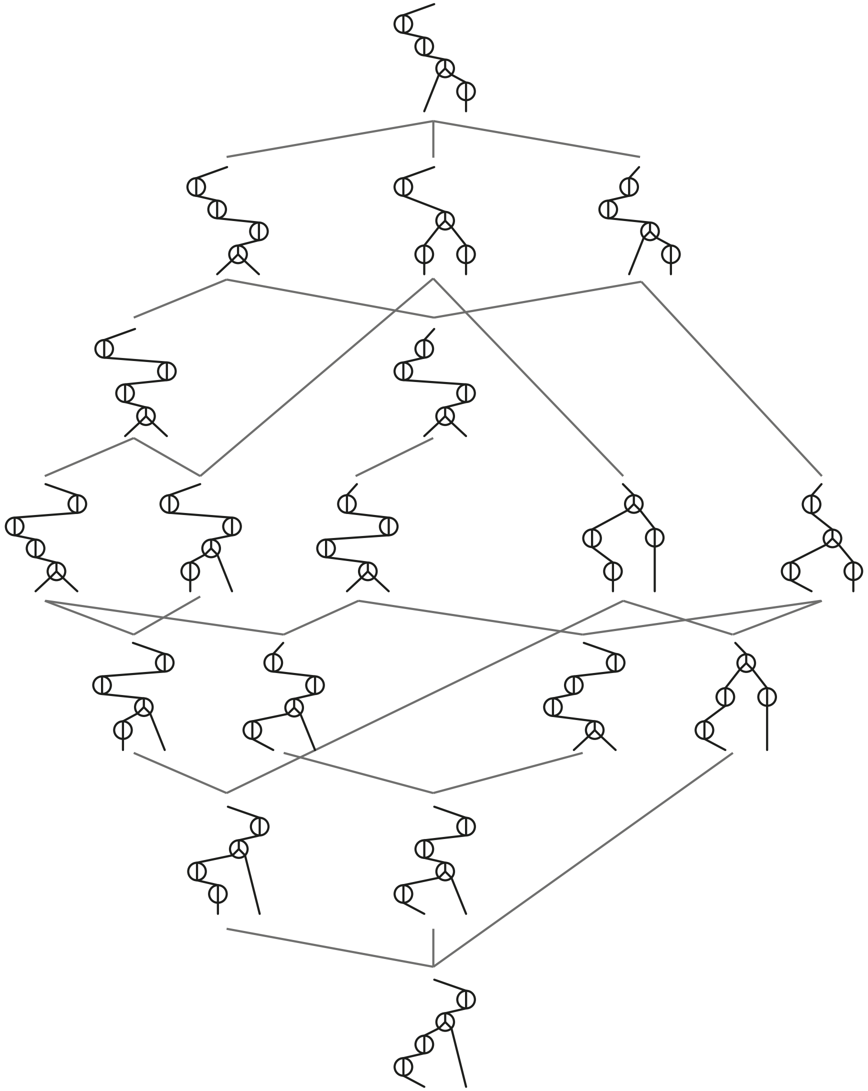


PERMUTREEHEDRA AND PERMUTREE LATTICES

PROP. $U \coloneqq (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = \sum_{i \in [n]} (n+1-2i)\, \mathbf{e}_i$

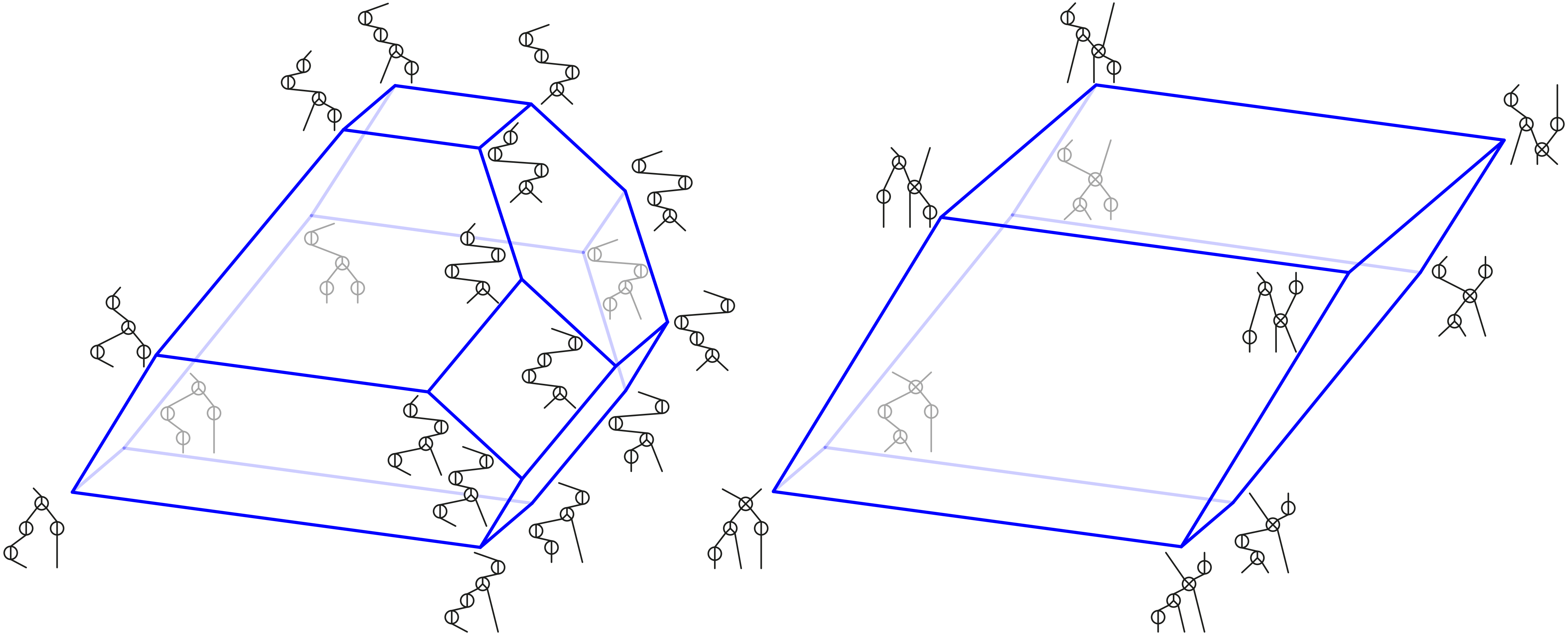
graph of $\mathbb{PT}(\delta)$ oriented by U = Hasse diagram of the δ-permutree lattice.



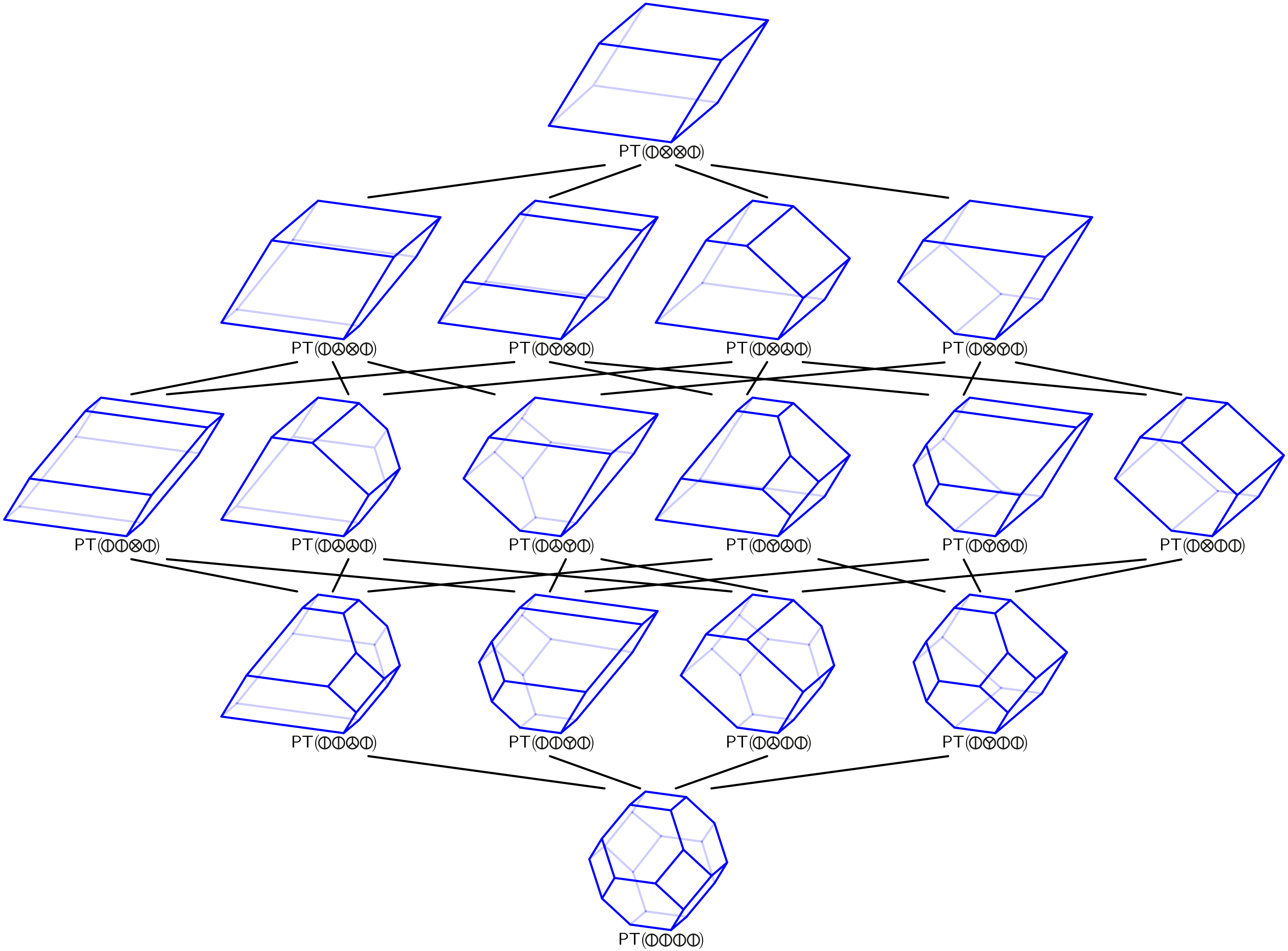


MATRIOCHKA PERMUTREEHEDRA

PROP. refinement $\delta \preccurlyeq \delta'$ $\implies$ inclusion $\mathbb{PT}(\delta) \subset \mathbb{PT}(\delta')$.



MATRIOCHKA PERMUTREEHEDRA



MATRIOCHKA PERMUTREEHEDRA

PERMUTREE ALGEBRA

Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)
Hivert-Novelli-Thibon, *The algebra of binary search trees* ('05)
Chatel-P., *Cambrian Hopf Algebras* ('17)
P.-Pons, *Permutrees* ('17)

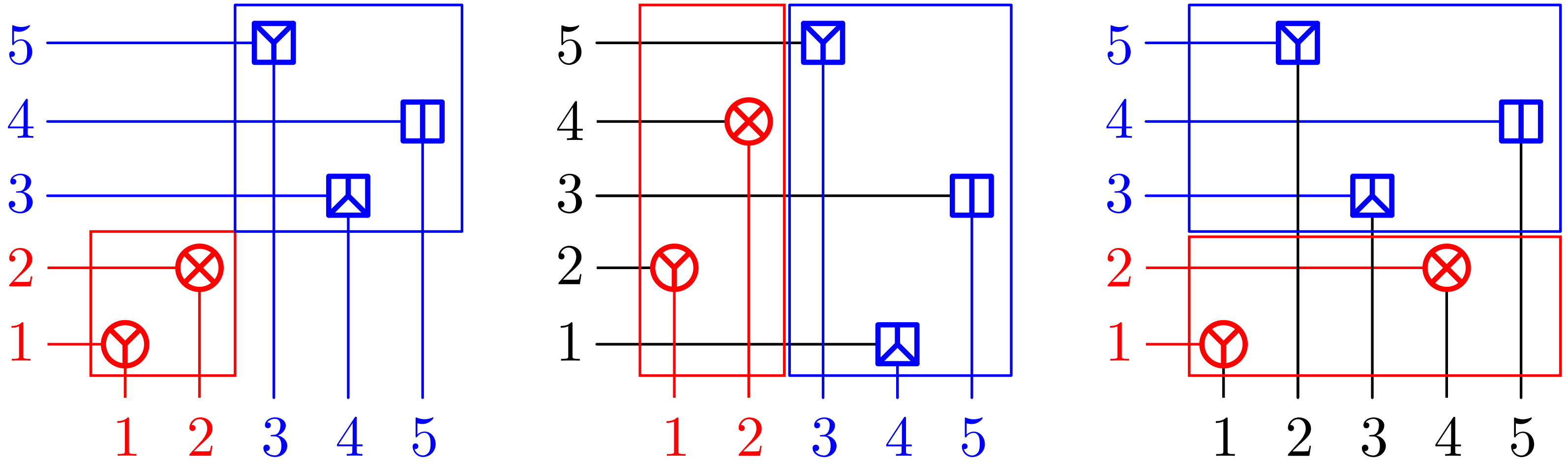
DECORATED VERSION

For decorated permutations:

- decorations are attached to values in the shuffle
- decorations are attached to positions in the convolution

Exm: $\overline{1}\overline{\underline{2}} \shuffle \underline{2}3\overline{1} = \{\overline{1}\overline{\underline{2}}\underline{4}5\overline{3}, \overline{1}\underline{4}\overline{\underline{2}}5\overline{3}, \overline{1}\underline{4}5\overline{\underline{2}}\overline{3}, \overline{1}\underline{4}5\overline{3}\overline{\underline{2}}, \underline{4}\overline{1}\overline{\underline{2}}5\overline{3}, \underline{4}\overline{1}5\overline{\underline{2}}\overline{3}, \underline{4}\overline{1}5\overline{3}\overline{\underline{2}}, \underline{4}5\overline{1}\overline{\underline{2}}\overline{3}, \underline{4}5\overline{1}\overline{3}\overline{\underline{2}}, \underline{4}5\overline{3}\overline{1}\overline{\underline{2}}\}$

$\overline{1}\overline{\underline{2}} \star \underline{2}3\overline{1} = \{\overline{1}\overline{\underline{2}}\underline{4}5\overline{3}, \overline{1}\overline{\underline{3}}\underline{4}5\overline{2}, \overline{1}\overline{\underline{4}}\underline{3}5\overline{2}, \overline{1}\overline{\underline{5}}\underline{3}4\overline{2}, \overline{2}\overline{\underline{3}}\underline{4}5\overline{1}, \overline{2}\overline{\underline{4}}\underline{3}5\overline{1}, \overline{2}\overline{\underline{5}}\underline{3}4\overline{1}, \overline{3}\overline{\underline{4}}\underline{2}5\overline{1}, \overline{3}\overline{\underline{5}}\underline{2}4\overline{1}, \overline{4}\overline{\underline{5}}\underline{2}3\overline{1}\}$



concatenation shuffle convolution

$\mathbf{k}\mathfrak{S}_{\{⦶,☮,Ⓨ,\otimes\}}$ = Hopf algebra with basis $(\mathbb{F}_\tau)_{\tau\in\mathfrak{S}_{\{⦶,☮,Ⓨ,\otimes\}}}$ and where

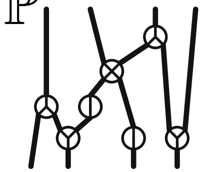
$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma\in\tau \shuffle \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle\mathbb{F}_\sigma = \sum_{\sigma\in\tau\star\tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

PERMUTREE ALGEBRA AS SUBALGEBRA

Permutree algebra = vector subspace $\mathbf{k}\mathfrak{PT}$ of $\mathbf{k}\mathfrak{S}_{\{⦶,☮,Ⓨ,\otimes\}}$ generated by

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S}_{\{⦶,☮,Ⓨ,\otimes\}} \\ \mathbf{P}(\tau)=\mathrm{T}}} \mathbb{F}_{\tau} = \sum_{\tau \in \mathcal{L}(\mathrm{T})} \mathbb{F}_{\tau},$$

for all permutrees T.

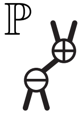
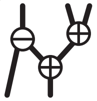
Exm: $\mathbb{P}$  $= \mathbb{F}_{\overline{2}\underline{1}35\overline{\underline{4}}\overline{7}\underline{6}} + \mathbb{F}_{\overline{2}\underline{1}35\overline{7}\overline{\underline{4}}\underline{6}} + \mathbb{F}_{\overline{2}\underline{1}3\overline{7}5\overline{\underline{4}}\underline{6}} + \cdots + \mathbb{F}_{\overline{7}5\overline{2}3\underline{1}\overline{\underline{4}}\underline{6}} + \mathbb{F}_{\overline{7}5\overline{2}3\overline{\underline{4}}\underline{1}\underline{6}} + \mathbb{F}_{\overline{7}5\overline{2}3\overline{\underline{4}}\underline{6}\underline{1}}$

THEO. $\mathbf{k}\mathfrak{PT}$ is a subalgebra of $\mathbf{k}\mathfrak{S}_{\{⦶,☮,Ⓨ,\otimes\}}$.

Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)
Hivert-Novelli-Thibon, *The algebra of binary search trees* ('05)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

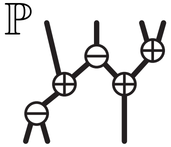
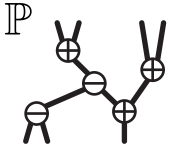
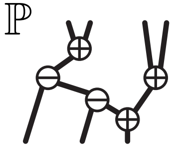
GAME: Explain the product and coproduct directly on the permutrees...

PRODUCT IN PERMUTREE ALGEBRA

$\mathbb{P}$  $\cdot\ \mathbb{P}$ 

$$= \mathbb{F}_{\underline{1}\overline{2}} \cdot \big(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{23}\underline{1}}\big)$$

$$= \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{24}\underline{3}\overline{5}} + \mathbb{F}_{\underline{1}\overline{245}\underline{3}} + \mathbb{F}_{\underline{1}\overline{42}\underline{3}\overline{5}} \\ + \mathbb{F}_{\underline{1}\overline{425}\underline{3}} + \mathbb{F}_{\underline{1}\overline{452}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\overline{25}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{52}\underline{3}} + \mathbb{F}_{\overline{45}\underline{1}\overline{2}\underline{3}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{25}} + \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{52}} \\ + \mathbb{F}_{\underline{1}\overline{45}\underline{3}\overline{2}} + \mathbb{F}_{\overline{4}\underline{13}\overline{25}} \\ + \mathbb{F}_{\overline{4}\underline{13}\overline{52}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\underline{3}\overline{2}} \\ + \mathbb{F}_{\overline{45}\underline{13}\overline{2}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\overline{4}\underline{31}\overline{25}} + \mathbb{F}_{\overline{4}\underline{31}\overline{52}} \\ + \mathbb{F}_{\overline{4}\underline{3}\overline{5}\underline{1}\overline{2}} + \mathbb{F}_{\overline{45}\underline{31}\overline{2}} \end{pmatrix}$$

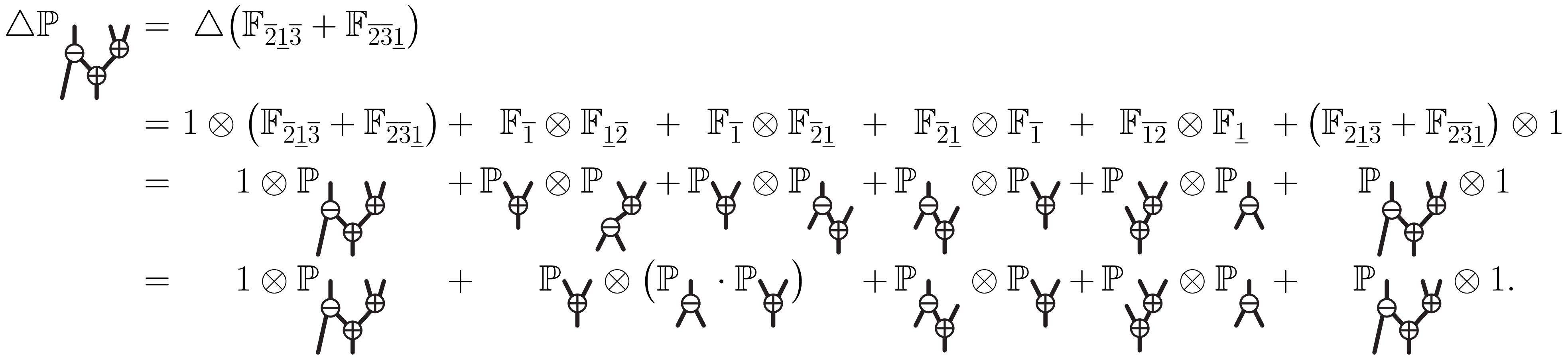
$= \mathbb{P}$  $+\ \mathbb{P}$  $+\ \mathbb{P}$ 

PROP. For any permutrees T and T',

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}$$

where S runs over the interval $\big[\mathrm{T} \nearrow \bar{\mathrm{T}}'\,,\ \mathrm{T} \nwarrow \bar{\mathrm{T}}'\big]$ in the $\delta(\mathrm{T})\delta(\mathrm{T}')$-permutree lattice.

COPRODUCT IN PERMUTREE ALGEBRA



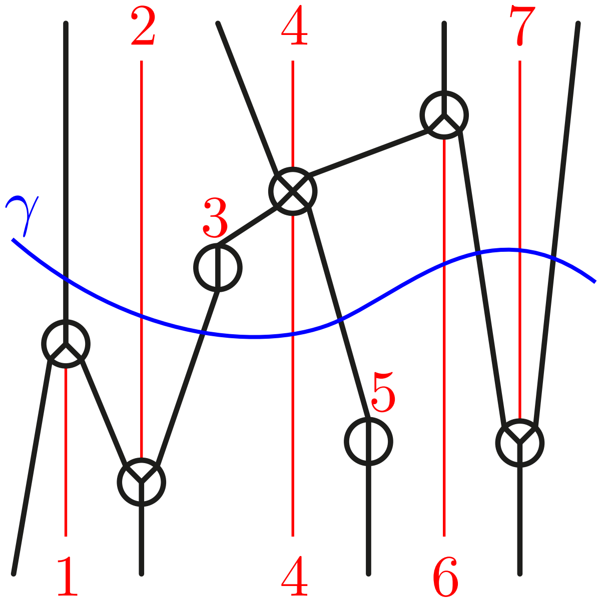
$$\triangle\mathbb{P} = \triangle\left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right)$$

$$= 1 \otimes \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\underline{1}\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\bar{2}\underline{1}} + \mathbb{F}_{\bar{2}\underline{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{\bar{1}\bar{2}} \otimes \mathbb{F}_{\underline{1}} + \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) \otimes 1$$

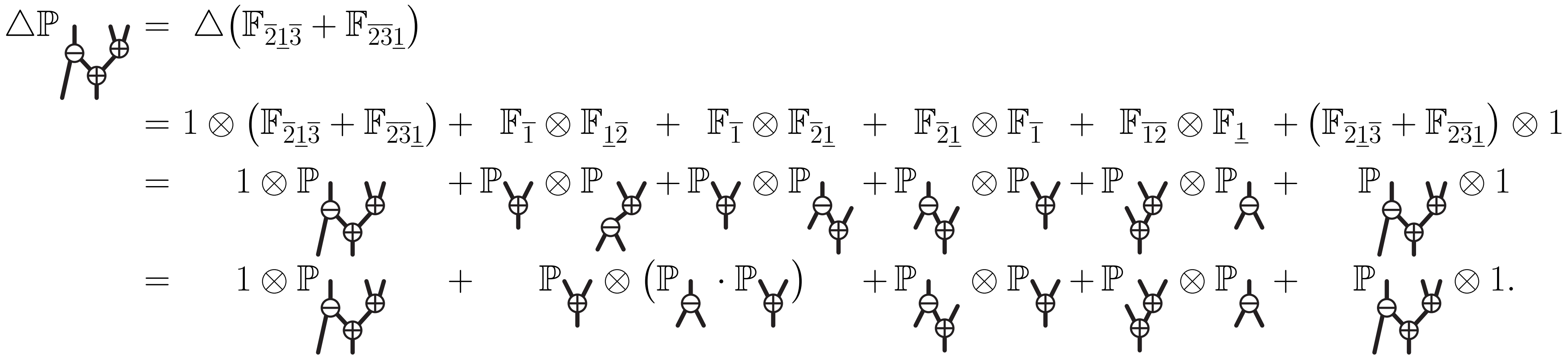
PROP. For any permutree S,

$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma} \Big(\prod_{\mathrm{T} \in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}} \Big) \otimes \Big(\prod_{\mathrm{T}' \in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'} \Big)$$

where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the forests above and below γ respectively.



COPRODUCT IN PERMUTREE ALGEBRA



$$\triangle\mathbb{P} = \triangle\big(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\big)$$

$$= 1 \otimes \big(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\big) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\underline{1}\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\bar{2}\underline{1}} + \mathbb{F}_{\bar{2}\underline{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{\bar{1}\bar{2}} \otimes \mathbb{F}_{\underline{1}} + \big(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\big) \otimes 1$$

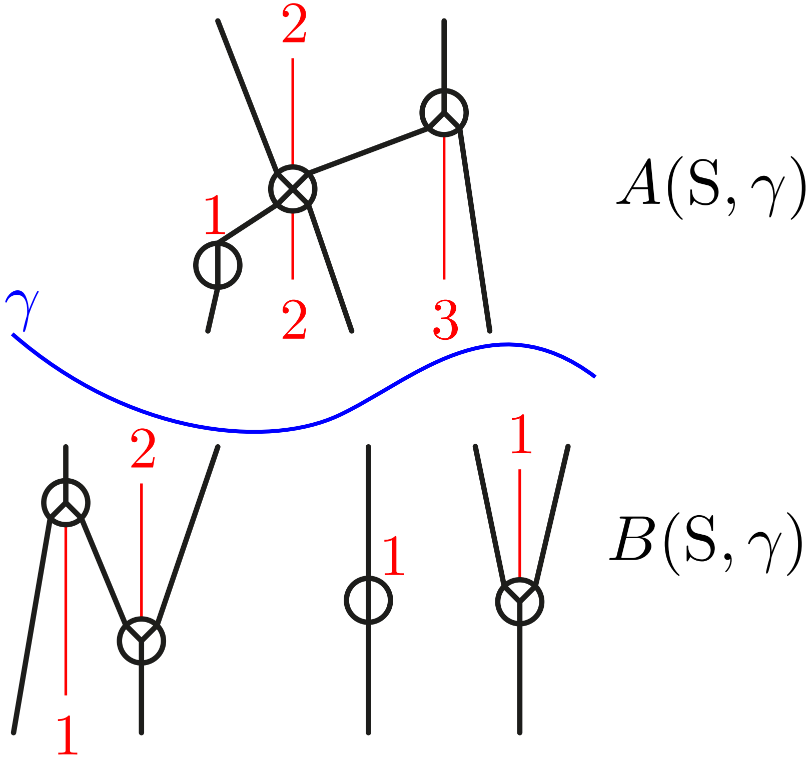
$$= 1 \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1$$

$$= 1 \otimes \mathbb{P} + \mathbb{P} \otimes (\mathbb{P} \cdot \mathbb{P}) + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes \mathbb{P} + \mathbb{P} \otimes 1.$$

PROP. For any permutree S,

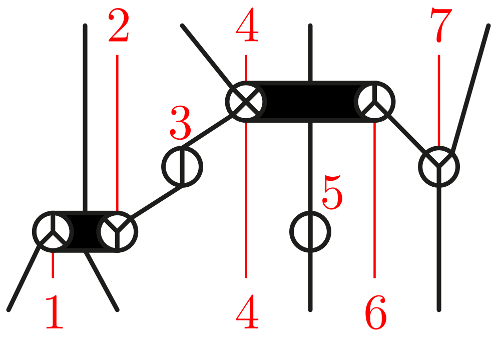
$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma}\Bigg(\prod_{\mathrm{T}\in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}}\Bigg) \otimes \Bigg(\prod_{\mathrm{T}'\in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'}\Bigg)$$

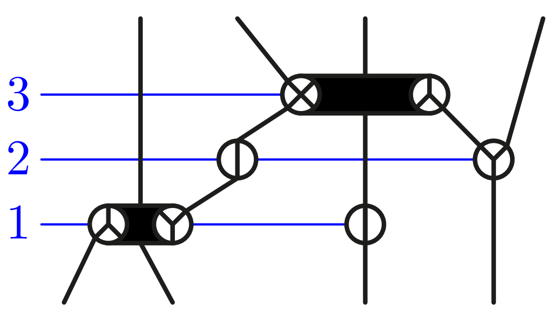
where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the forests above and below γ respectively.

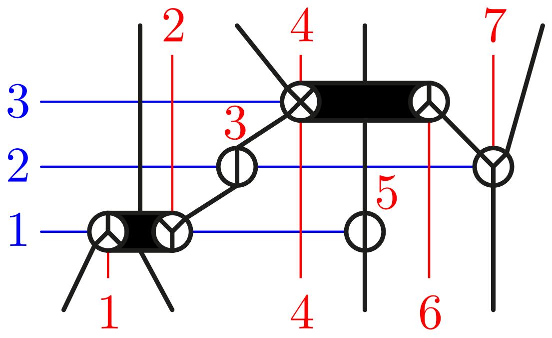


EXTENSIONS

- Schröder permutrees







EXTENSIONS

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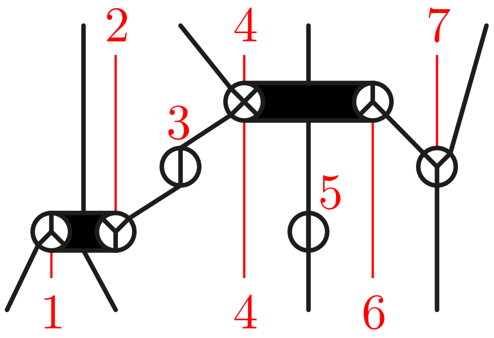
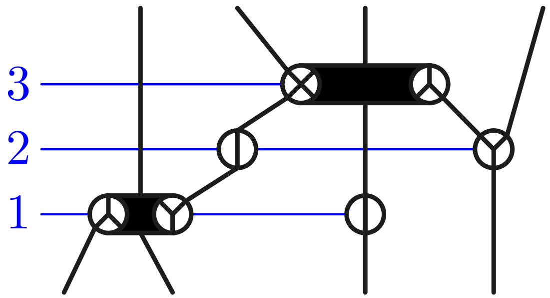
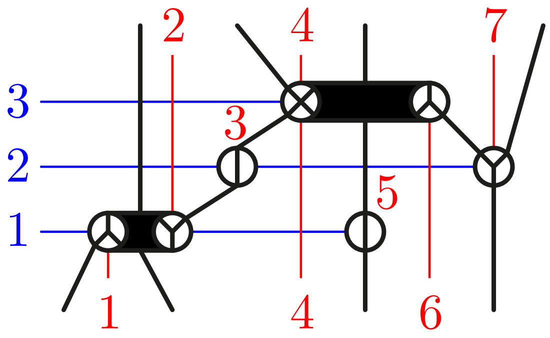
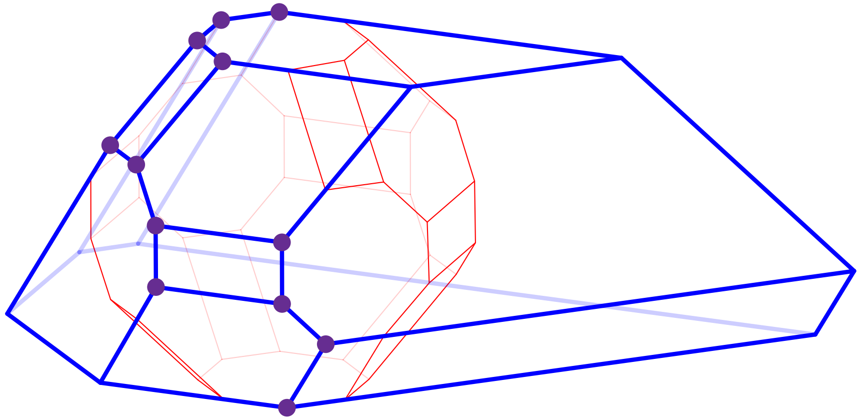
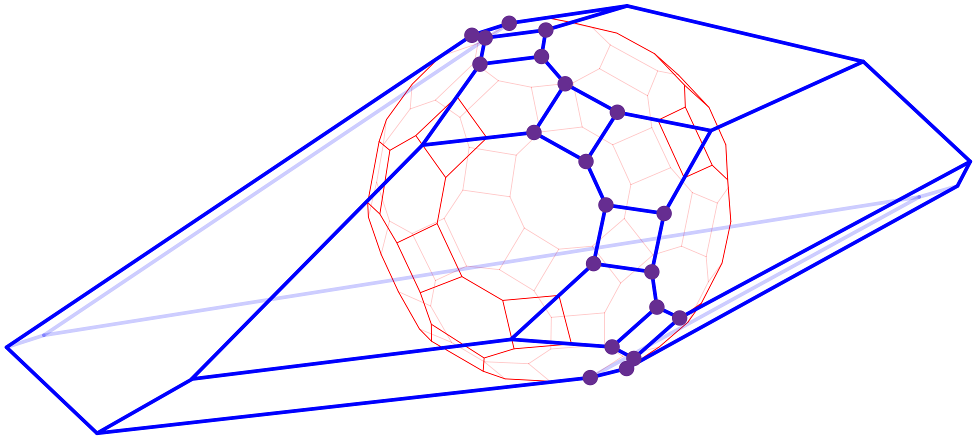






- arbitrary finite Coxeter groups
  somewhere between the W-permutahedron and the W-associahedron





Reading, *Cambrian lattices* ('06)

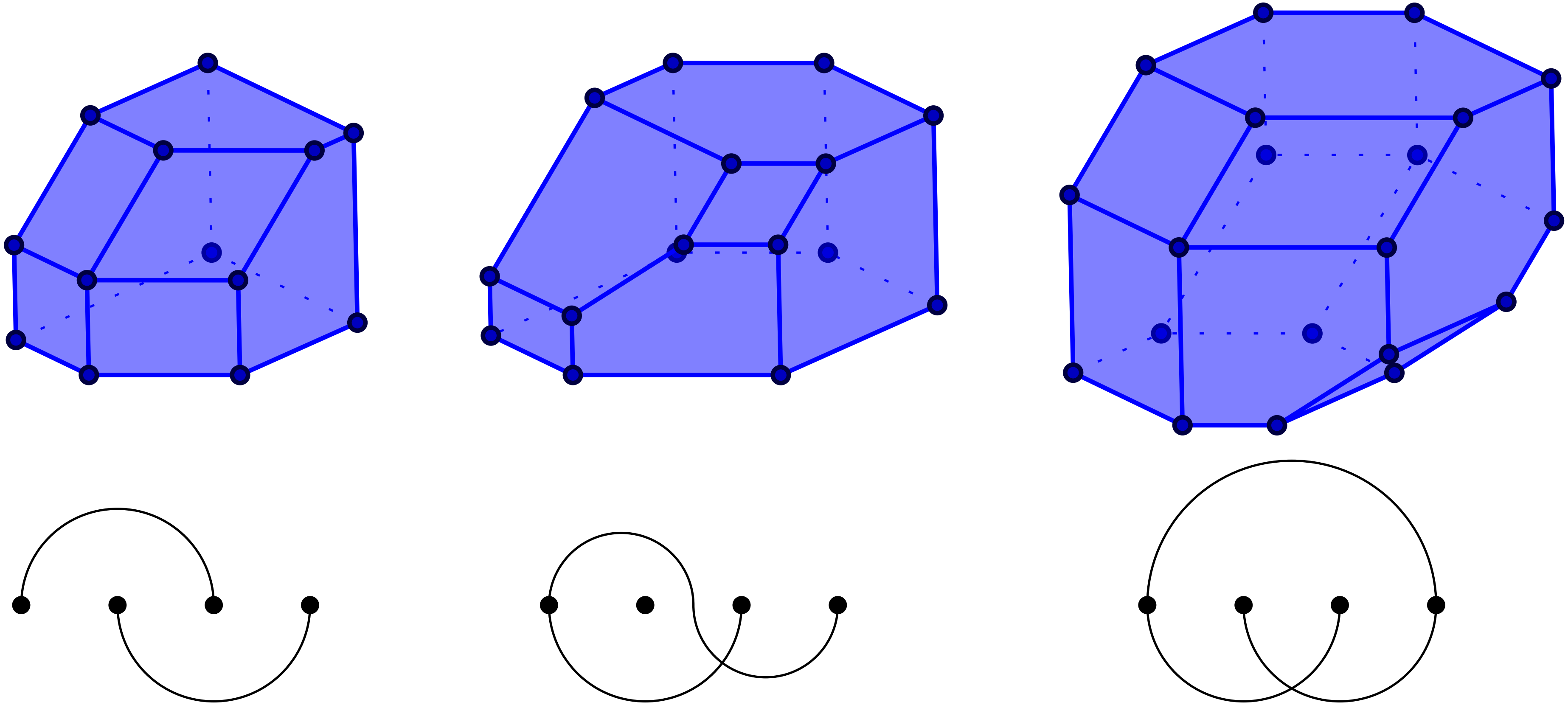
Reading-Speyer, *Cambrian fans* ('09)

Hohlweg-Lange-Thomas, *Permutahedra and generalized associahedra* ('11)

P.-Stump, *Brick polytopes of spherical subword complexes & gen. assoc.* ('15)

Hohlweg-P.-Stella, *Polytopal realizations of finite type* $\mathbf{g}$*-vector fans* ('17$^+$)

II. QUOTIENTOPES



P.-Santos, *Quotientopes* ('17$^+$)

P., *Hopf algebras on decorated noncrossing arc diagrams* ('17$^+$)

LATTICE SETUP

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Noncrossing arc diagrams and canonical join representations* ('15)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)

CANONICAL JOIN REPRESENTATIONS

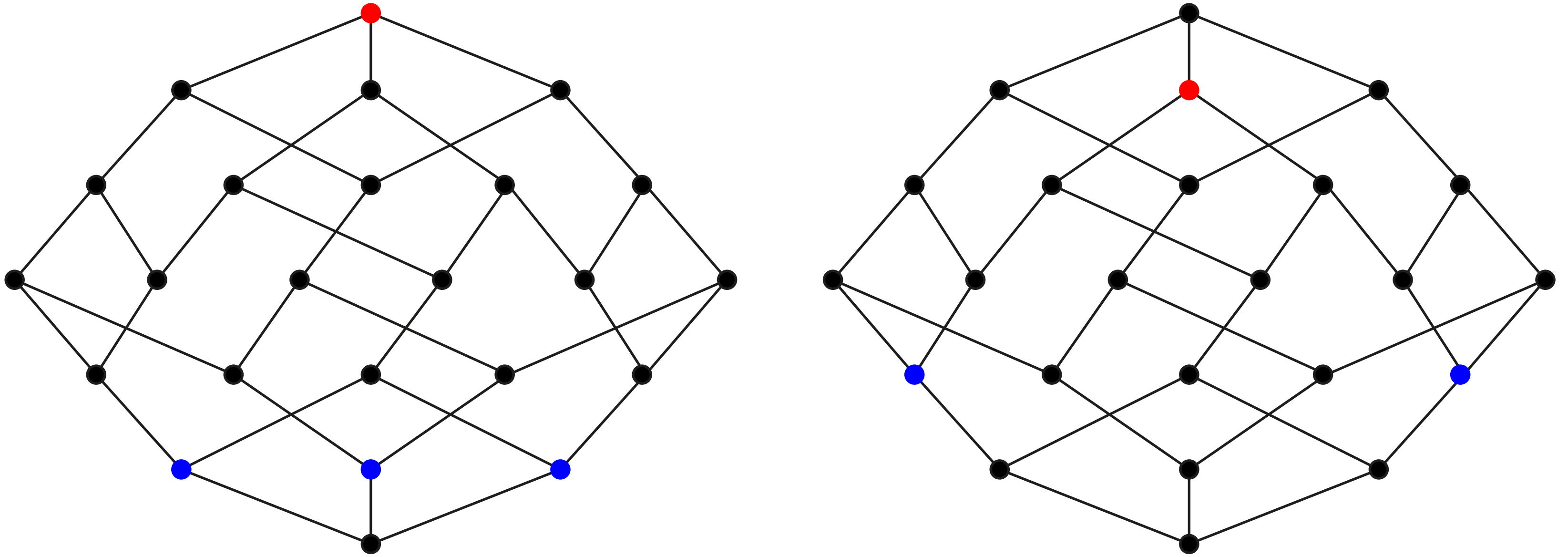
lattice = poset $(L, \leq)$ with a meet $\wedge$ and a join $\vee$

join representation of $x \in L$ = subset $J \subseteq L$ such that $x = \bigvee J$.

$x = \bigvee J$ irredundant if $\nexists\, J' \subsetneq J$ with $x = \bigvee J'$

JR are ordered by containement of order ideals: $J \leq J' \iff \forall\, y \in J,\ \exists\, y' \in J',\ y \leq y'$

canonical join representation of x = minimal irred. join representation of x (if it exists)



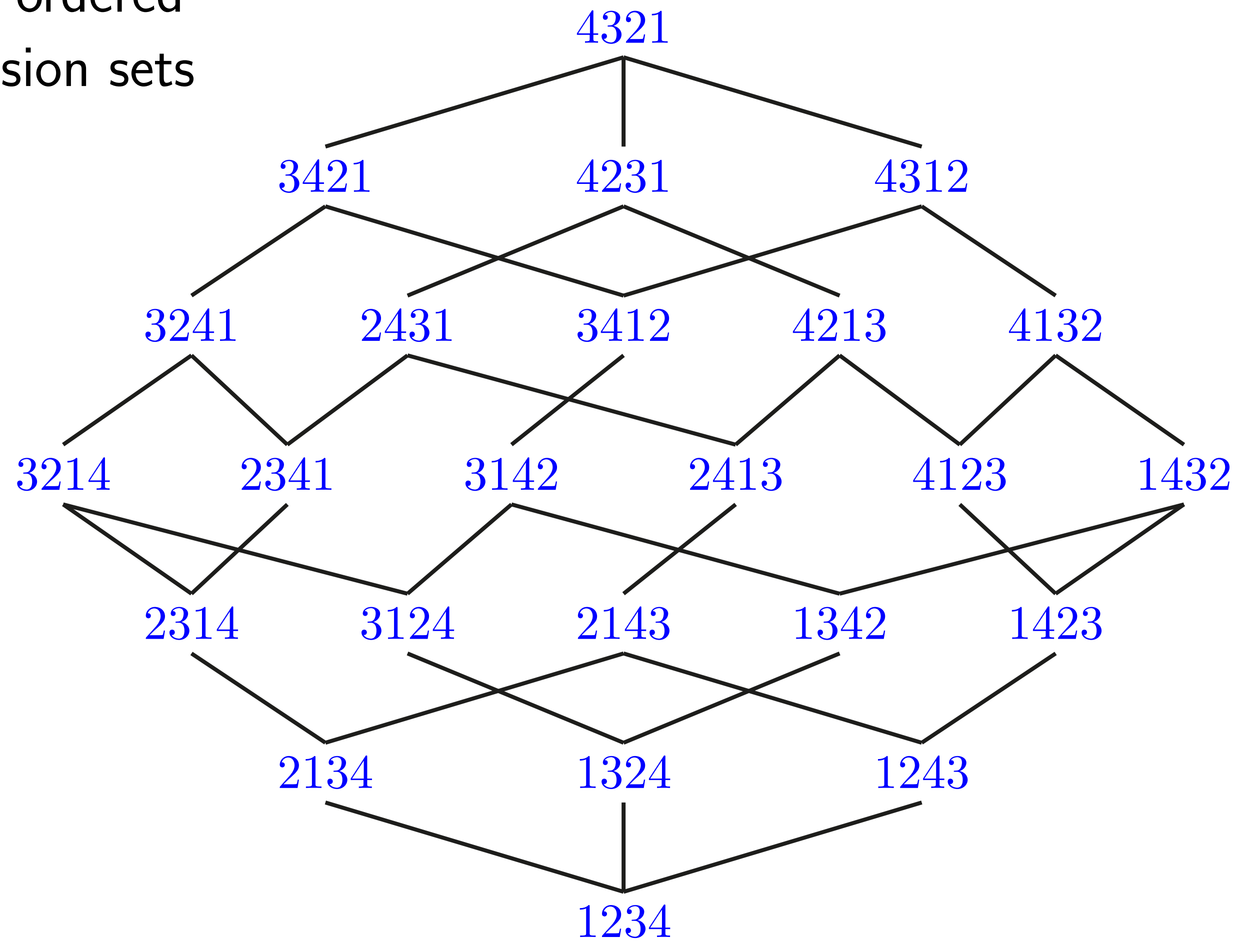
$\implies$ "lowest way to write x as a join"

CANONICAL JOIN REPRESENTATIONS IN THE WEAK ORDER

σ permutation

inversions of σ = pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$

weak order = permutations of $\mathfrak{S}_n$ ordered
by inclusion of inversion sets



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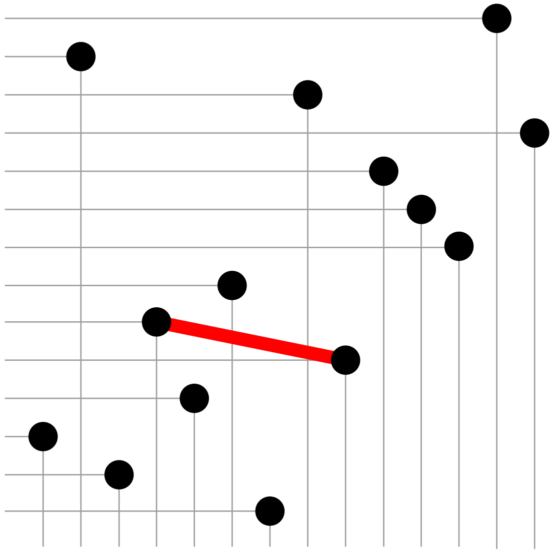
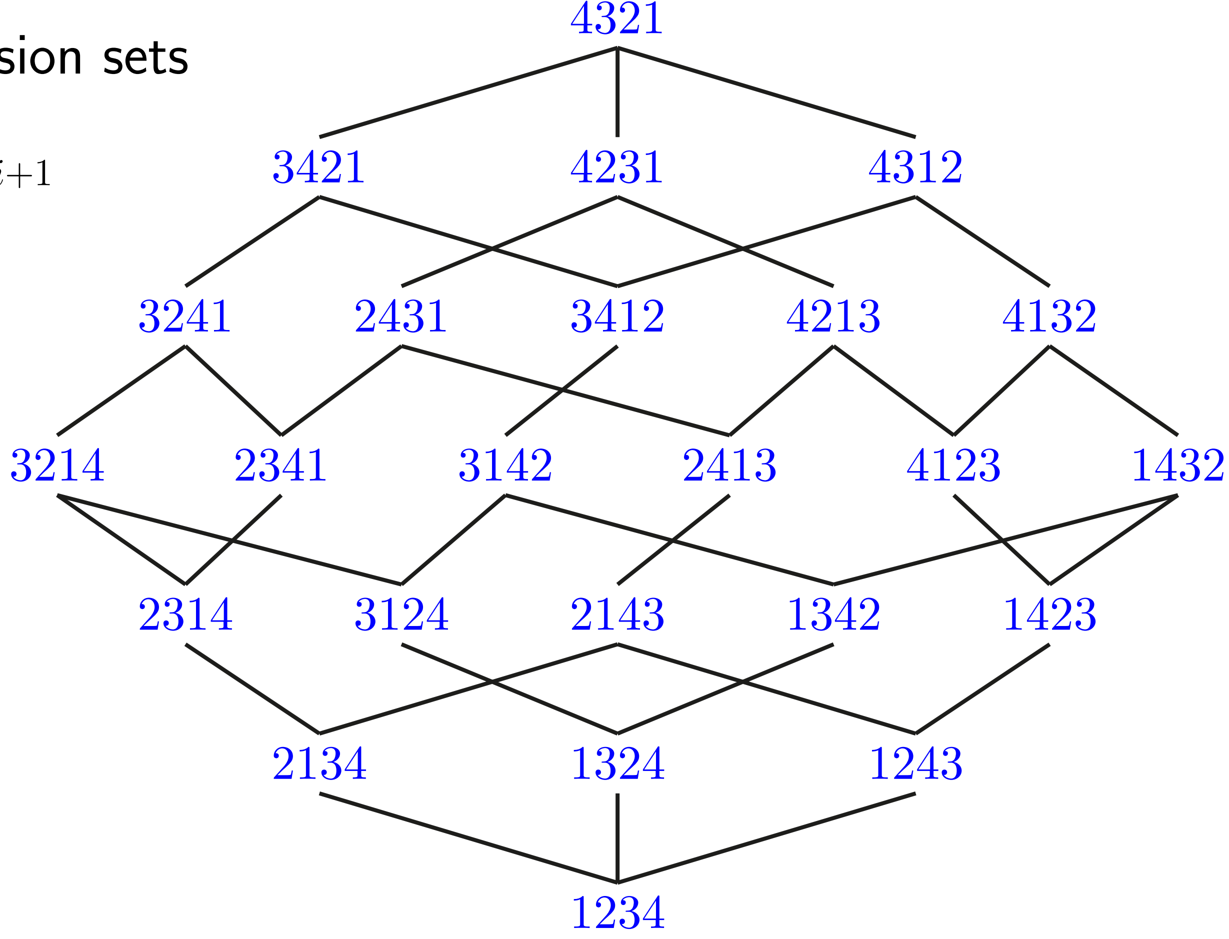
descent of σ = i such that $\sigma_i > \sigma_{i+1}$

$\sigma =$





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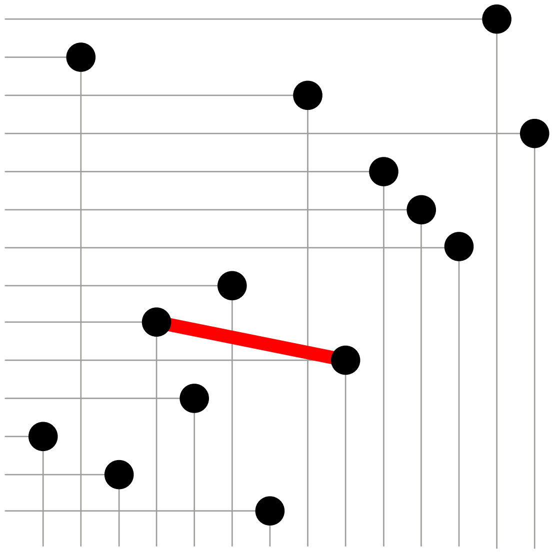
by inclusion of inversion sets

descent of σ $=$ i such that $\sigma_i > \sigma_{i+1}$

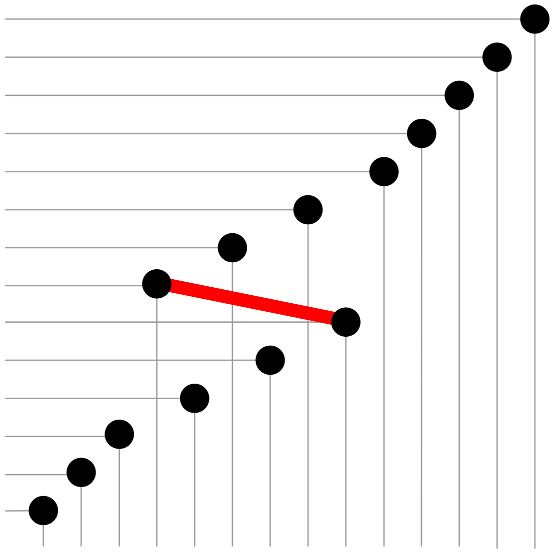
join-irreducible $\lambda(\sigma, i)$

$\sigma =$



$\longrightarrow$

$\lambda(\sigma, i) =$



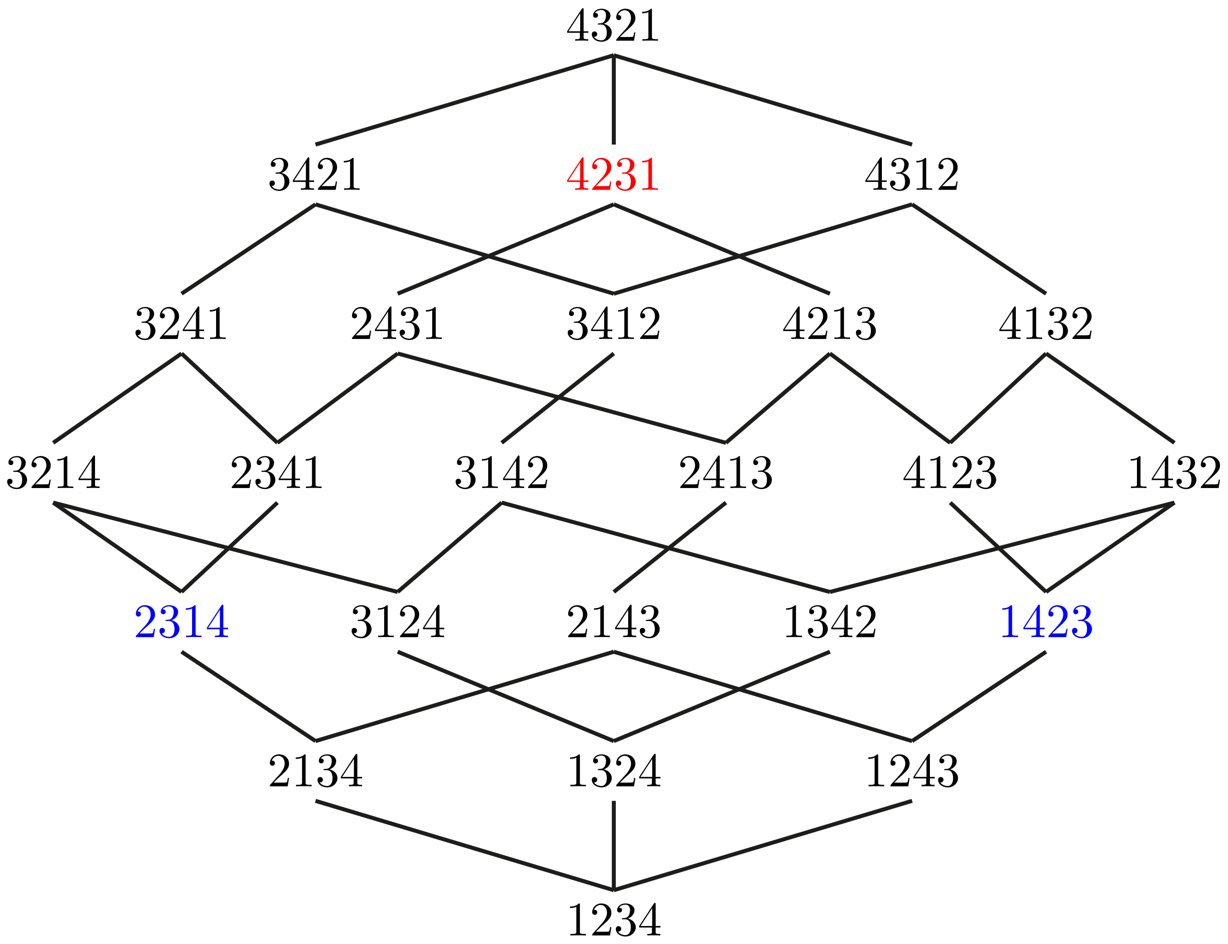
THM. Canonical join representation of $\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)$.

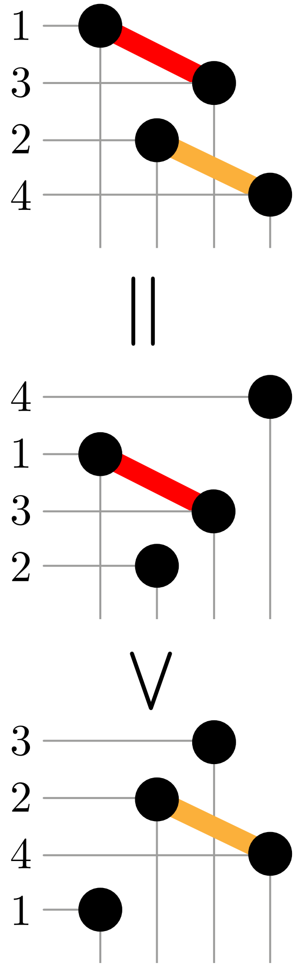
Reading, *Noncrossing arc diagrams and canonical join representations* ('15)

CANONICAL JOIN REPRESENTATIONS IN THE WEAK ORDER

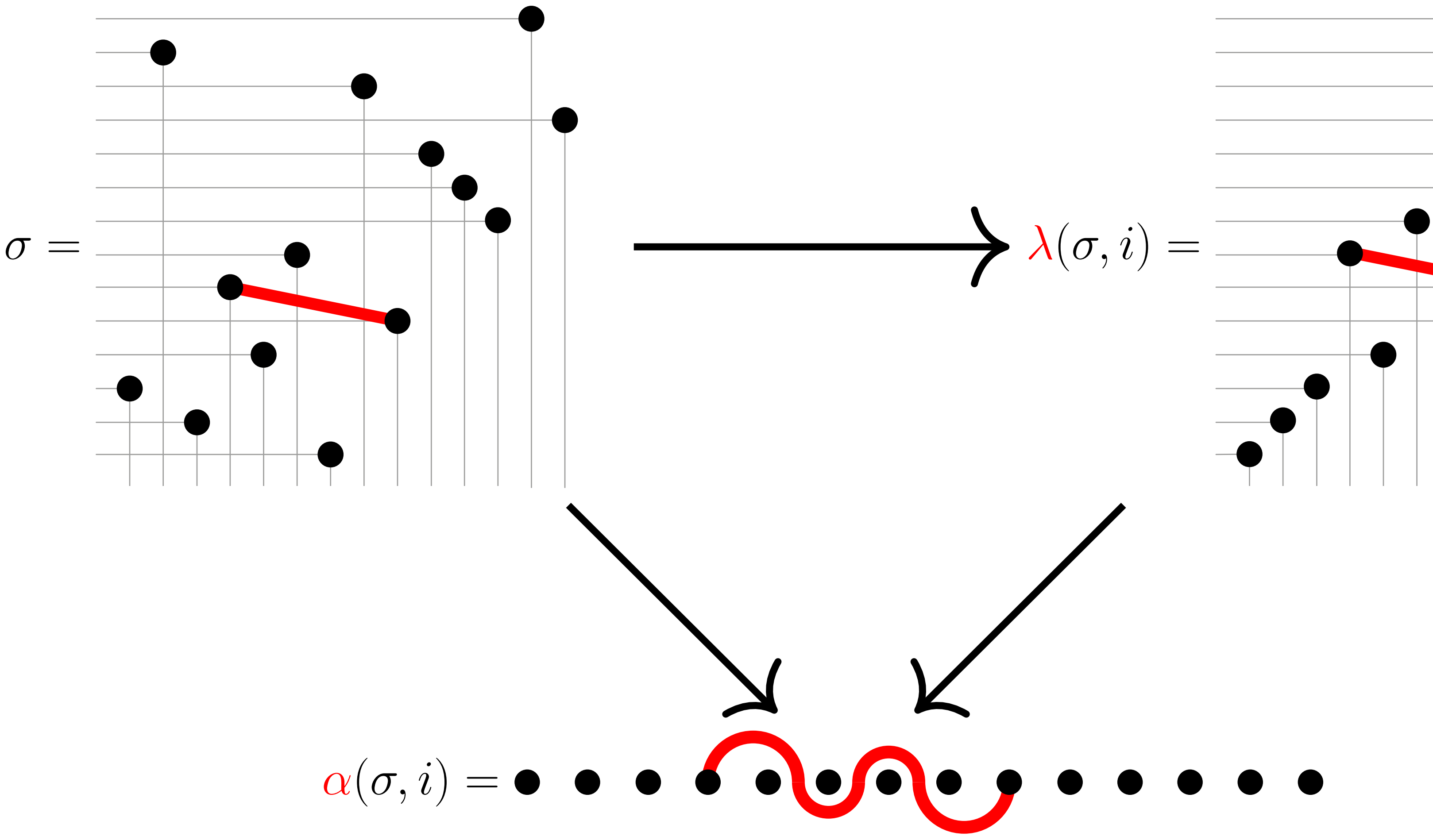
THM. Canonical join representation of $\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)$.

Reading, *Noncrossing arc diagrams and canonical join representations* ('15)





ARCS



$\sigma =$

$\lambda(\sigma, i) =$

$\alpha(\sigma, i) =$

$\underline{\text{arc}} = (a, b, n, S)$ with $1 \leq a < b \leq n$ and $S \subseteq \,]a, b[$

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

$\sigma = 2537146$

draw the table of points (σ_i, i)
draw all arcs (σ_i, i) — $(\sigma_{i+1}, i+1)$ with
descents in red and ascent in green

project down the red arcs and up the green arcs
allowing arcs to bend but not to cross or pass points

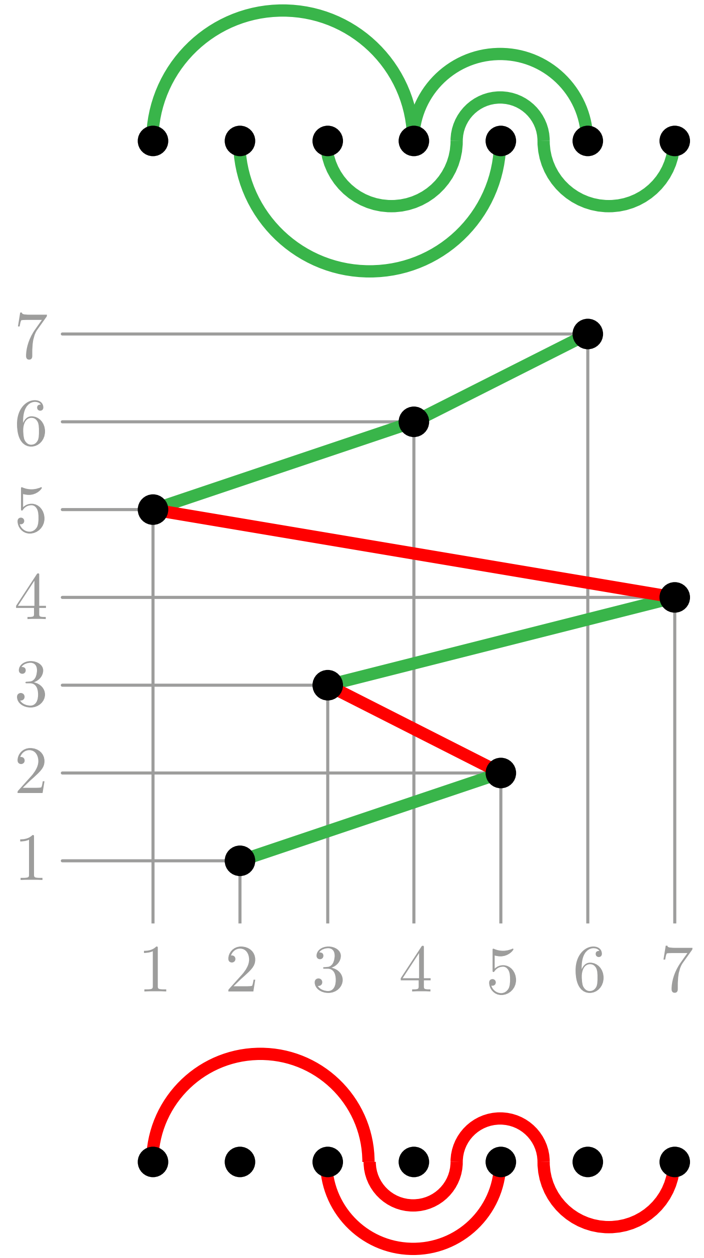
$\delta(\sigma)$ = projected red arcs
$\delta(\sigma)$ = projected green arcs

noncrossing arc diagrams = set $\mathcal{D}$ of arcs st. $\forall \, \alpha, \beta \in \mathcal{D}$:

- left$(\alpha) \neq$ left(β) and right$(\alpha) \neq$ right(β),
- α and β are not crossing.

THM. $\sigma \to \delta(\sigma)$ and $\sigma \to \delta(\sigma)$ are bijections from permutations to noncrossing arc diagrams.

Reading, *Noncrossing arc diagrams and can. join representations* ('15)



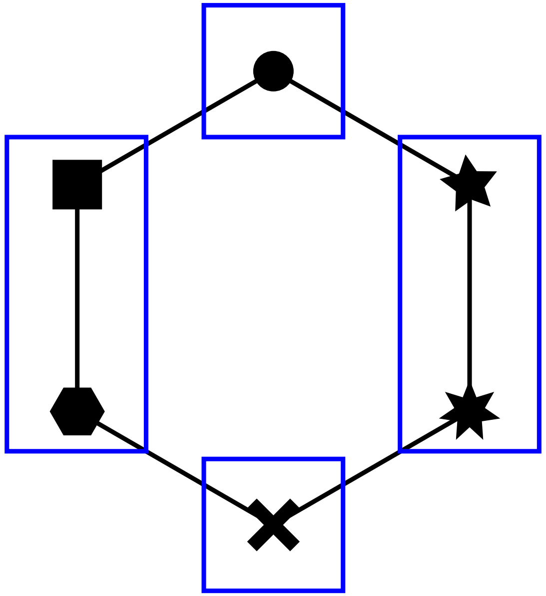
LATTICE CONGRUENCES

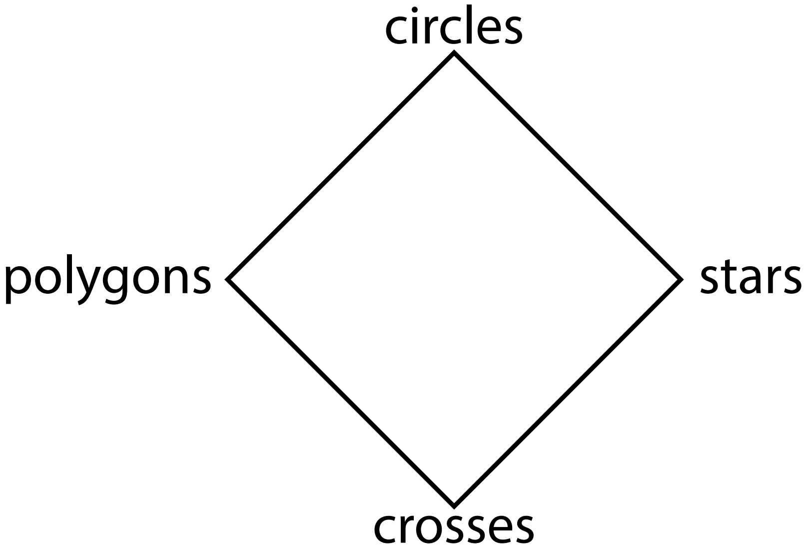
lattice congruence = equiv. rel. $\equiv$ on L which respects meets and joins

$$x \equiv x' \quad \text{and} \quad y \equiv y' \quad \Longrightarrow \quad x \wedge y \equiv x' \wedge y' \quad \text{and} \quad x \vee y \equiv x' \vee y'$$

lattice quotient of $L/\equiv$ = lattice on equiv. classes of L under $\equiv$ where

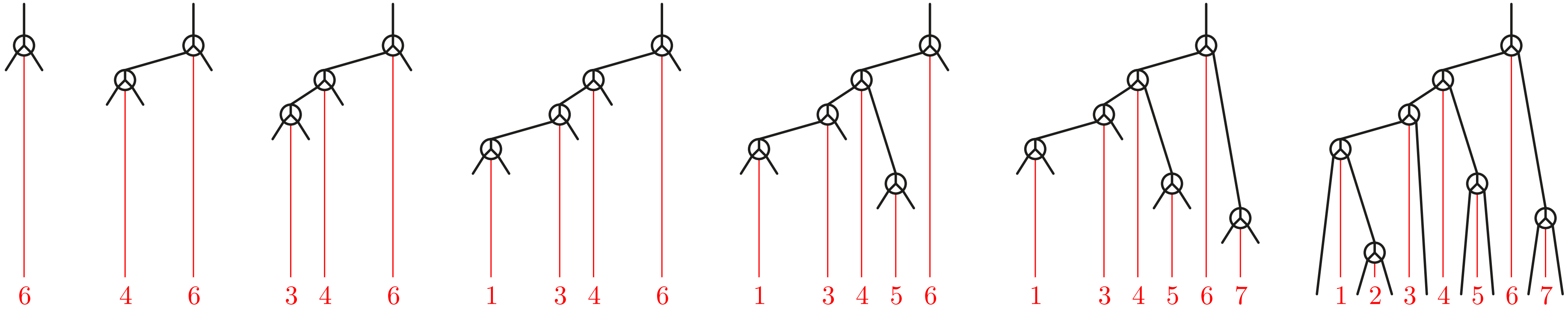
- $X \le Y \iff \exists x \in X,\ y \in Y,\quad x \le y$
- $X \wedge Y$ = equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ = equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$





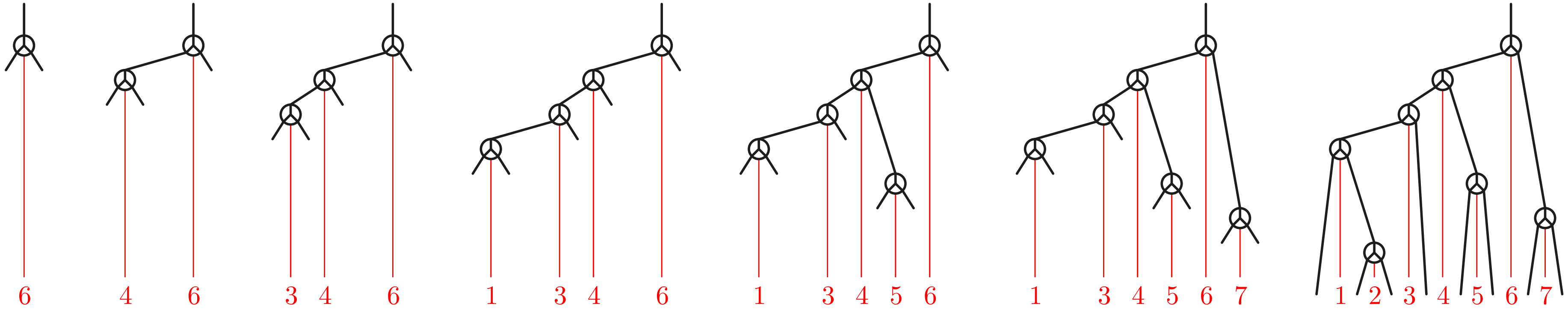
EXM: TAMARI LATTICE AS LATTICE QUOTIENT OF WEAK ORDER

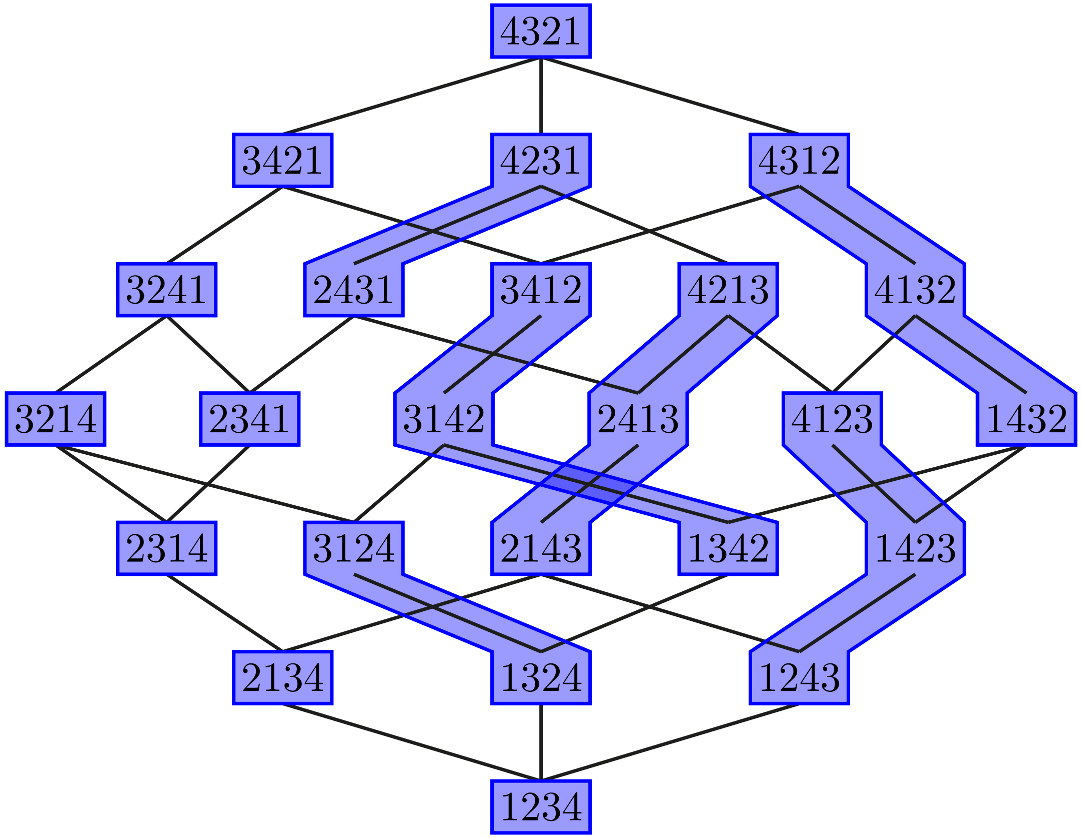
binary search tree insertion of 2751346

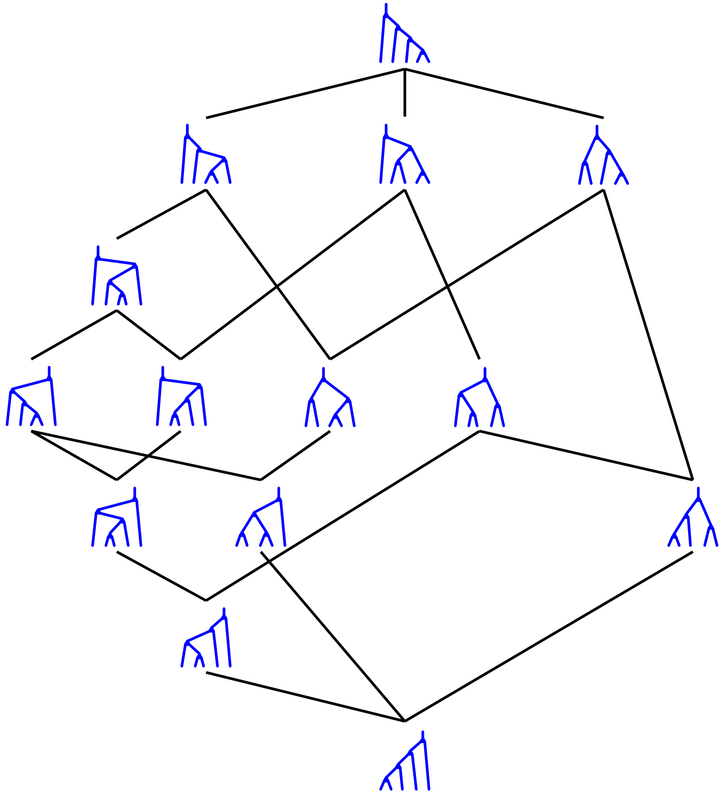


EXM: TAMARI LATTICE AS LATTICE QUOTIENT OF WEAK ORDER

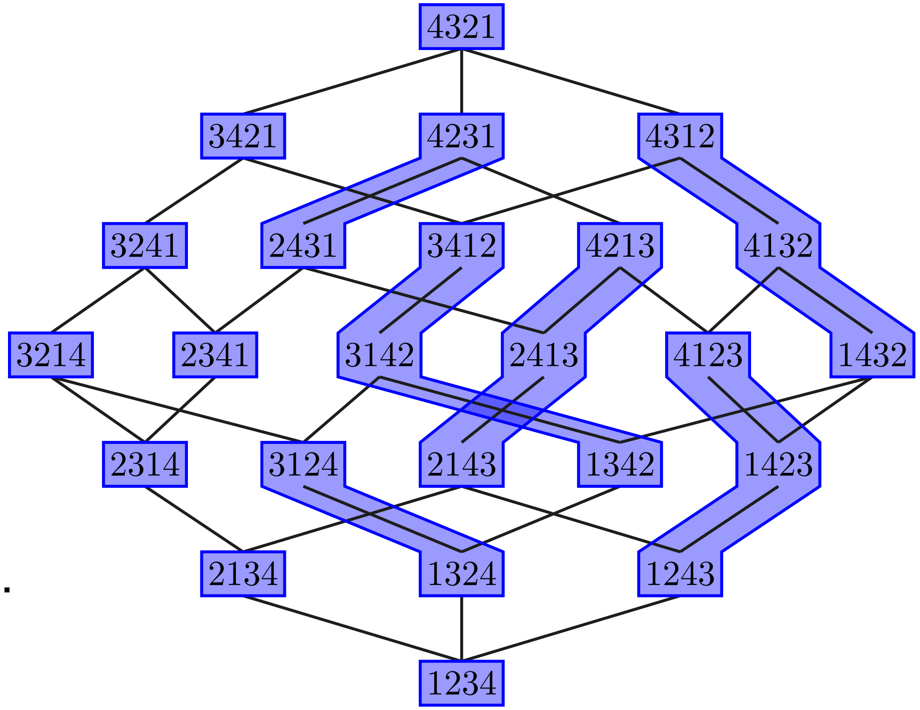
binary search tree insertion of 2751346







LATTICE QUOTIENTS AND CANONICAL JOIN REPRESENTATIONS



$\equiv$ lattice congruence on L, then

- each class X is an interval $[\pi_\downarrow(X), \pi^\uparrow(X)]$
- $L/\equiv$ is isomorphic to $\pi_\downarrow(L)$ (as poset)
- canonical join representations in $L/\equiv$ are canonical join representations in L that do not involve join irreducibles x with $\pi_\downarrow(x) \neq x$.

THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$

Let $\mathcal{I}_\equiv$ = arcs corresponding to join irreducibles σ with $\pi_\downarrow(\sigma) = \sigma$

- $\pi_\downarrow(\sigma) = \sigma \iff \sigma$ has no descent i st. $\alpha(\sigma, i) \notin \mathcal{I}_\equiv$.
- the map $\mathfrak{S}_n/\equiv \longrightarrow$ {nc arc diagrams in $\mathcal{I}_\equiv$} is a bijection.

  $X \longmapsto \delta(\pi_\downarrow(X))$
- $\equiv$ is the transitive closure of the rewriting rule $\sigma \to \sigma \cdot (i \;\; i+1)$ where i descent of σ such that $\alpha(\sigma, i) \notin \mathcal{I}_\equiv$.

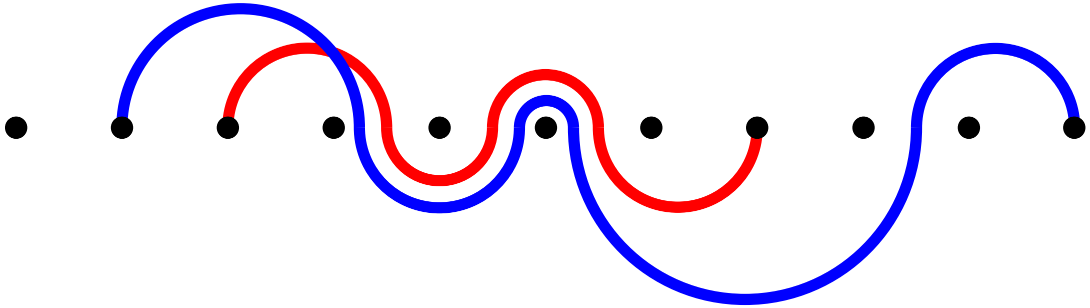
Reading, *Noncrossing arc diagrams and can. join representations* ('15)

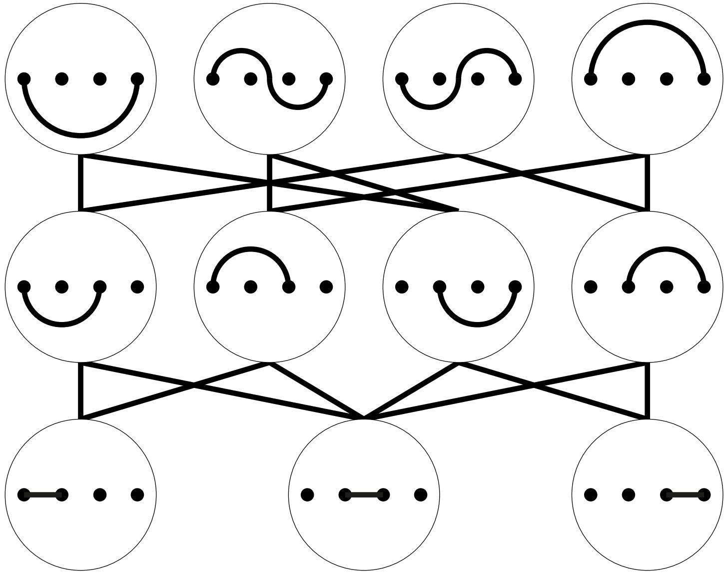
FORCING AND ARC IDEALS

THM. $\mathcal{I}_{\equiv}$ = arcs corresponding to join irreducibles σ with $\pi_{\downarrow}(\sigma) = \sigma$.
Bijection $\mathfrak{S}_n/\!\equiv \;\longleftrightarrow\;$ {nc arc diagrams in $\mathcal{I}_{\equiv}$}.

What sets of arcs can be $\mathcal{I}_{\equiv}$?

(a, d, n, S) <u>forces</u> (b, c, n, T) when $a \le b < c \le d$ and $T = S \cap \,]b, c[$



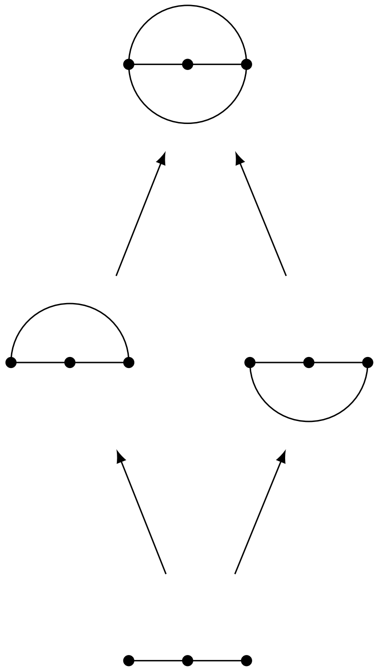


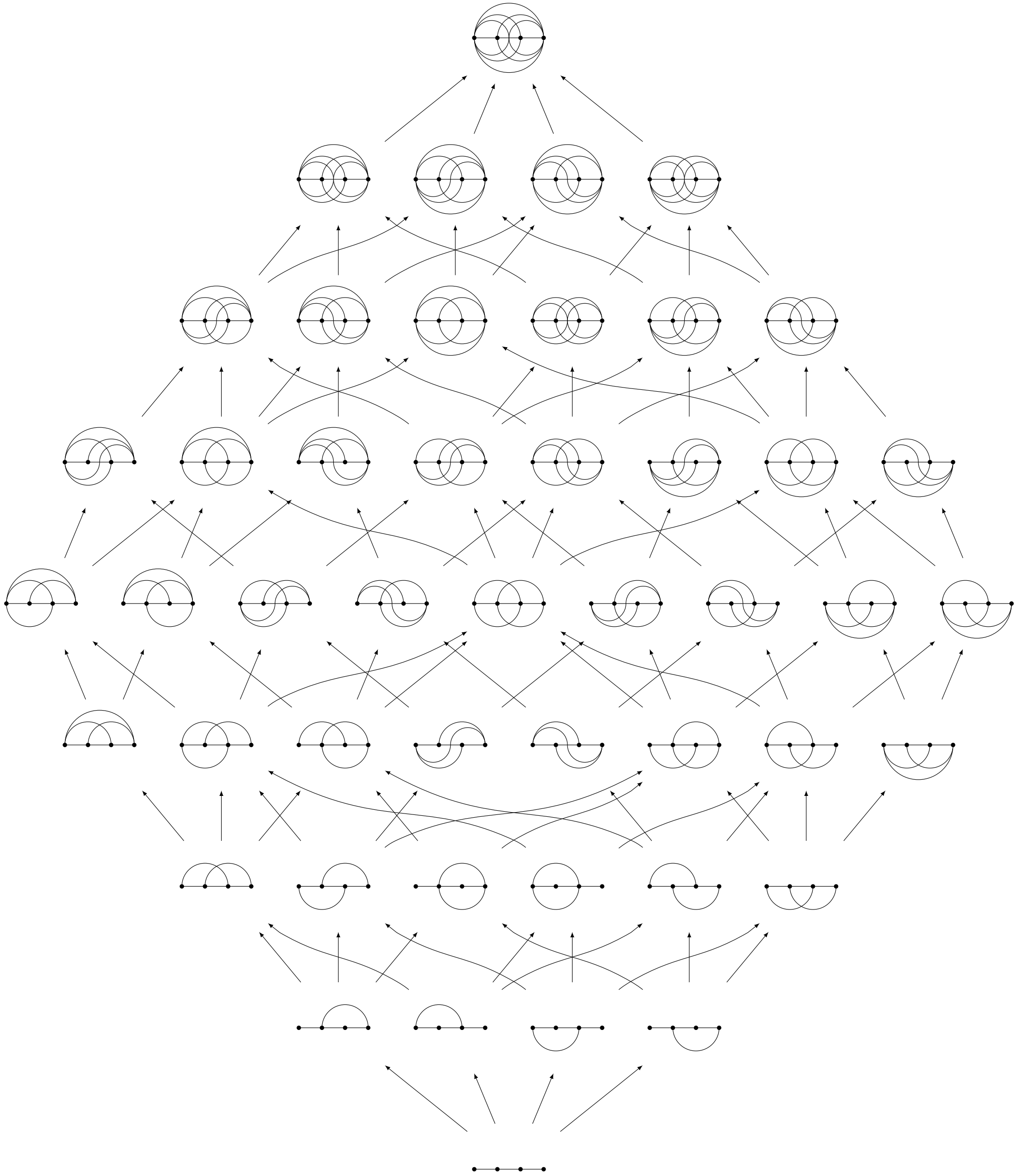
THM. $\mathcal{I}$ set of arcs. $\exists$ lattice cong. $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{I} = \mathcal{I}_{\equiv} \iff \mathcal{I}$ closed by forcing.

Reading, *Noncrossing arc diagrams and can. join representations* ('15)

ARC IDEALS

arc ideal = ideal of the forcing poset on arcs = subsets of arcs closed by forcing

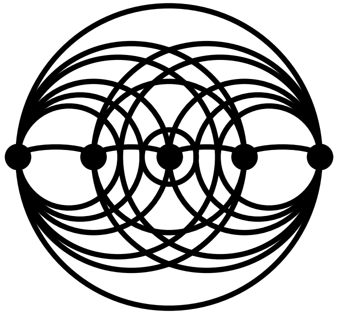




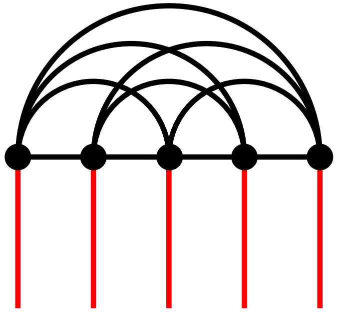
BOUNDED CROSSINGS ARC IDEALS

arc ideal $=$ ideal of the forcing poset on arcs $=$ subsets of arcs closed by forcing

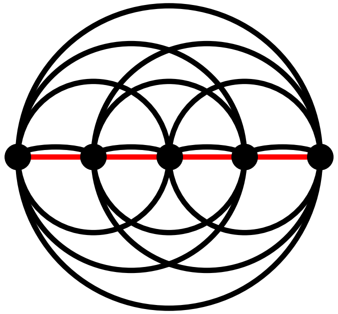
fix $k \geq 0$ and some red walls above, below and in between the points
allow arcs that cross at most k walls



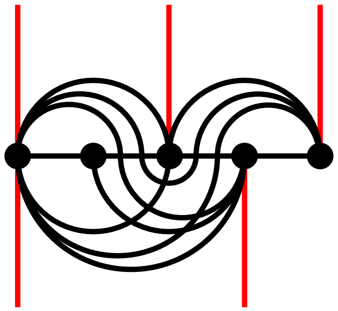
weak order



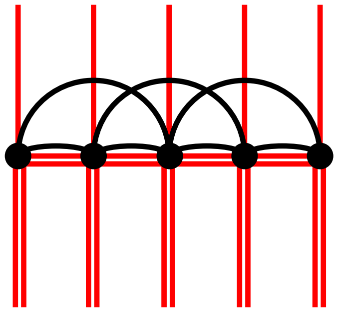
Tamari lattice



diagonal rectangulations



Cambrian lattices



k-sashes lattices

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS AGAIN

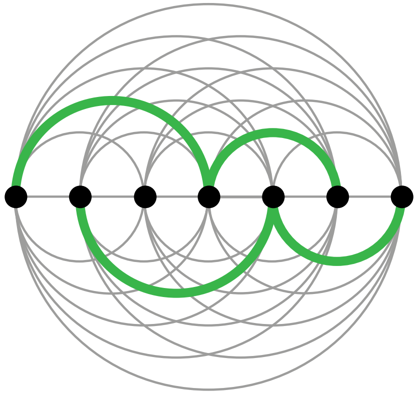
$\mathcal{I}_{\equiv}$ closed by forcing

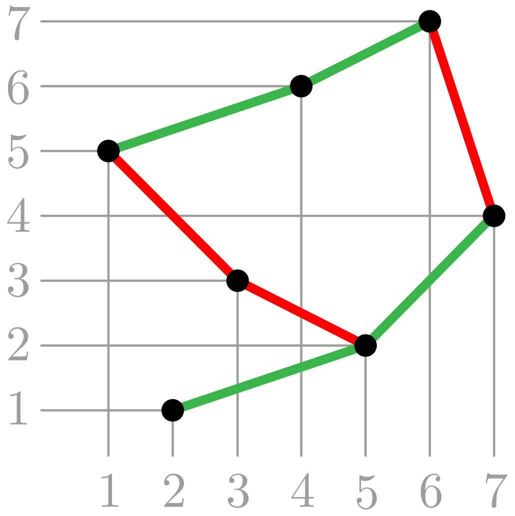
$$\begin{array}{llcr} \text{bijecton} & \mathfrak{S}_n/{\equiv} & \longrightarrow & \{\text{nc arc diagrams in } \mathcal{I}_{\equiv}\} \\ & X & \longmapsto & \delta(\pi_{\downarrow}(X)) \end{array}$$

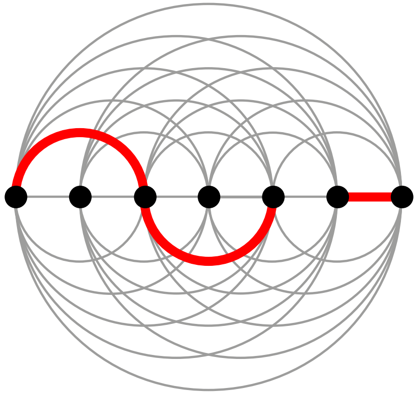
FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS AGAIN

$\mathcal{I}_{\equiv}$ closed by forcing

$$\begin{array}{rccc} \text{surjection} & \mathfrak{S}_n & \longrightarrow & \{\text{nc arc diagrams in } \mathcal{I}_{\equiv}\} \\ & \sigma & \longmapsto & \delta(\pi_{\downarrow}(\sigma)) \end{array}$$







FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS AGAIN

$\mathcal{I}_\equiv$ closed by forcing

surjection $\quad \begin{array}{ccc} \mathfrak{S}_n & \longrightarrow & \{\text{nc arc diagrams in } \mathcal{I}_\equiv\} \\ \sigma & \longmapsto & \delta(\pi_\downarrow(\sigma)) \end{array}$

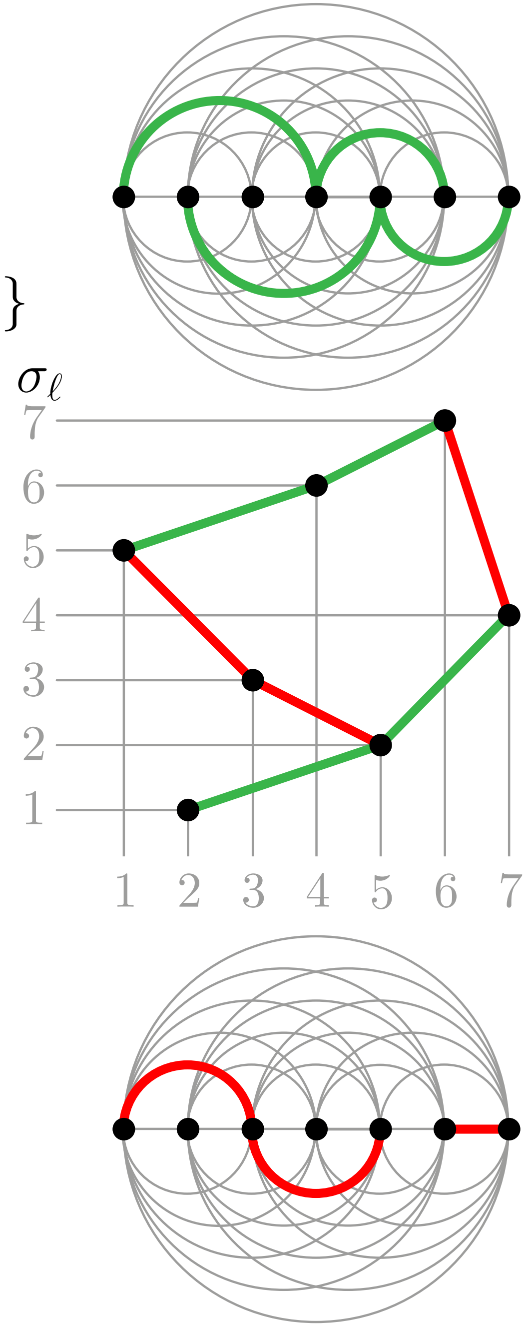
$\boxbslash(\sigma) = \{(i,j) \mid 1 \le i < j \le n,\ \sigma_i > \sigma_j \text{ and } \sigma(]i,j[) \cap]\sigma_j, \sigma_i[= \varnothing\}$

ordered by $(i,j) \prec (k,\ell) \iff i \le k < \ell \le j$ and $\sigma_k \ge \sigma_j > \sigma_i \ge \sigma_\ell$

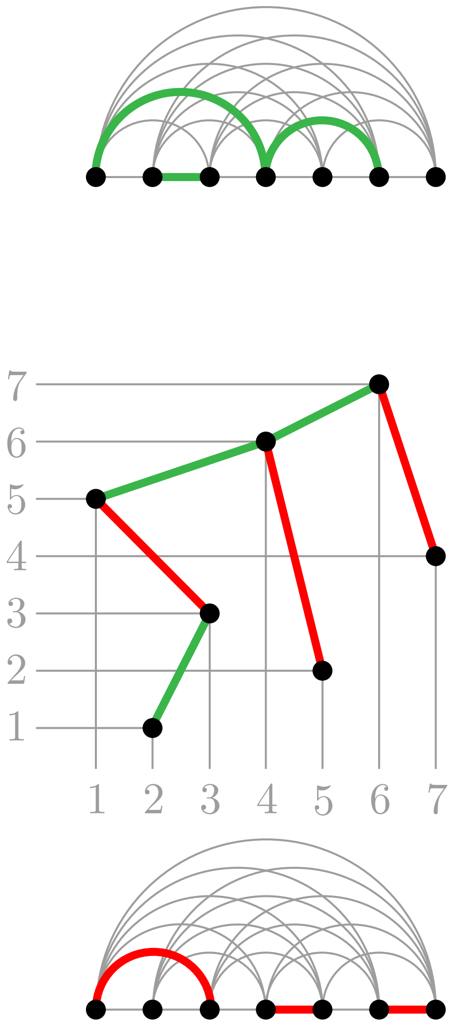
$\alpha(i,j,\sigma) = (\sigma_j, \sigma_i, n, \{\sigma_k \mid j < k \text{ and } \sigma_k \in]\sigma_j, \sigma_i[\})$

$\boxslash_{\mathcal{I}_\equiv}(\sigma) = \prec$-maximal elem. in $\{(i,j) \in \boxbslash(\sigma) \mid \alpha(i,j,\sigma) \in \mathcal{I}_\equiv\}$

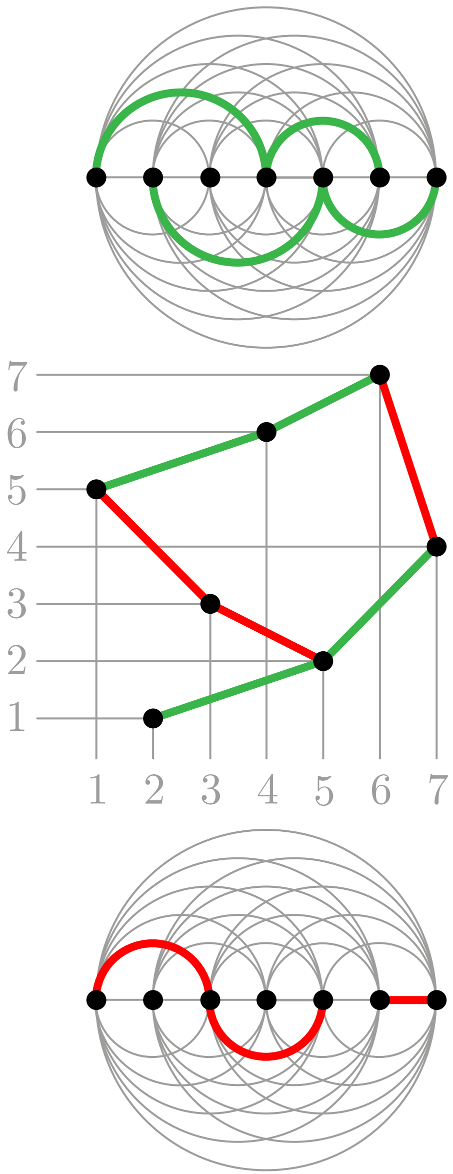
PROP. $\delta(\pi_\downarrow(\sigma)) = \{\alpha(i,j,\sigma) \mid (i,j) \in \boxslash_{\mathcal{I}_\equiv}(\sigma)\}$.



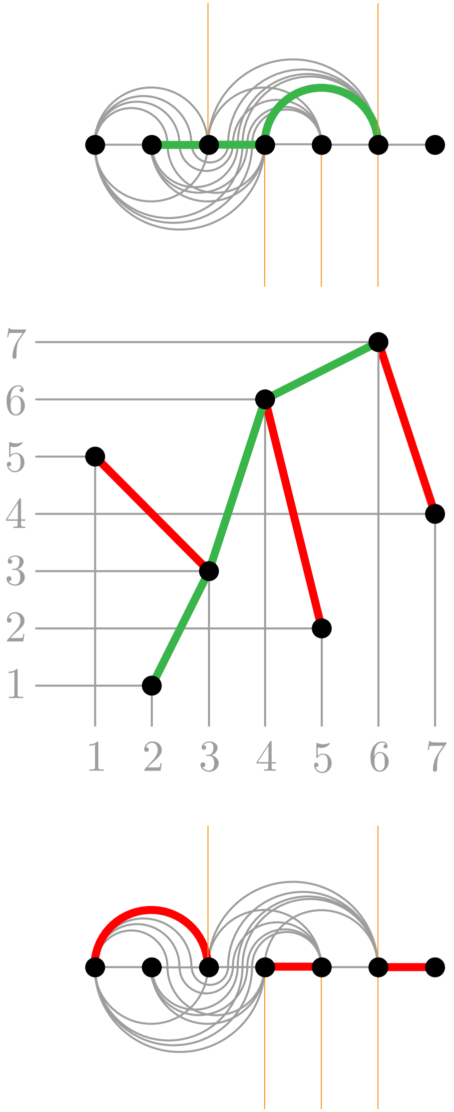
FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS AGAIN



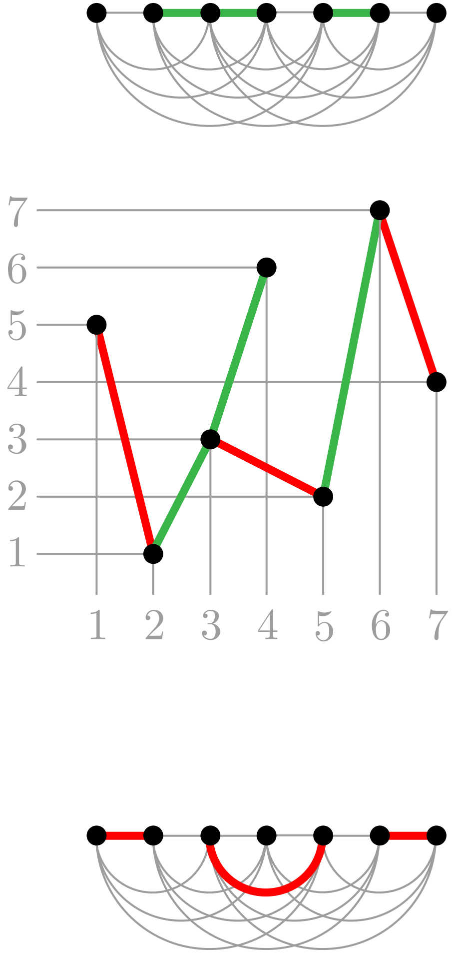
binary trees



diagonal quadrangulations



permutrees



k-sashes

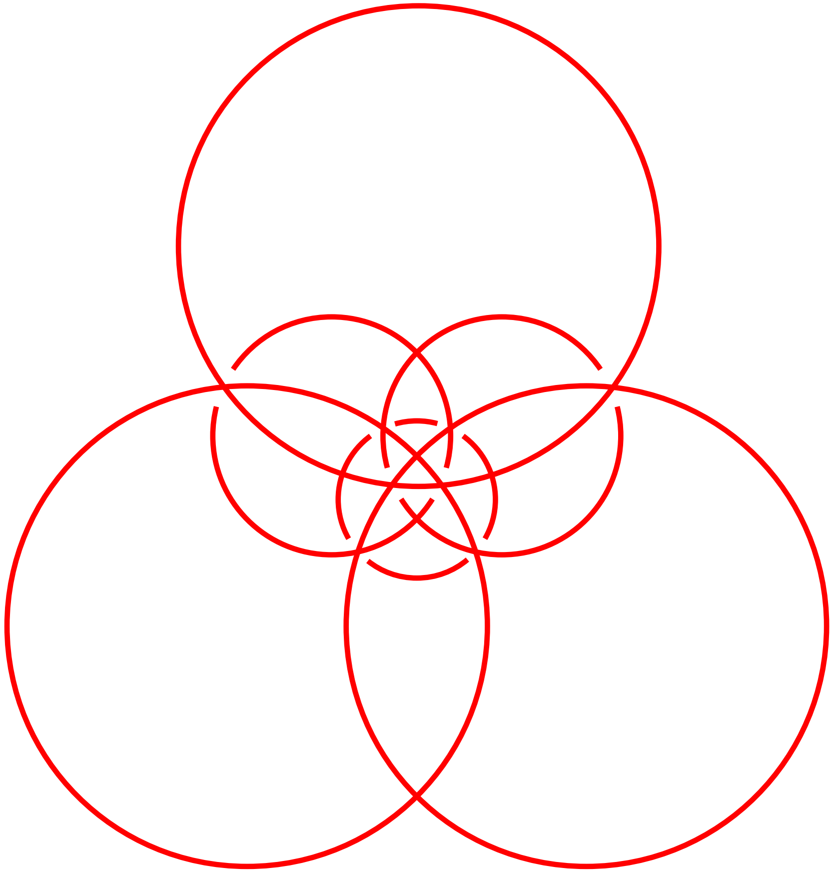
QUOTIENTOPES

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
P.-Santos, *Quotientopes* ('17^+)

SHARDS AND QUOTIENT FAN

arcs decompose the hyperplanes $\{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}$ of the braid arrangement into shards

<u>shard</u> $\Sigma(i,j,n,S) := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; x_i = x_j \text{ and } \left[\begin{array}{l} x_i \le x_k \text{ for all } k \in S \text{ while} \\ x_i \ge x_k \text{ for all } k \in \,]i,j[\, \smallsetminus S \end{array} \right. \right\}$



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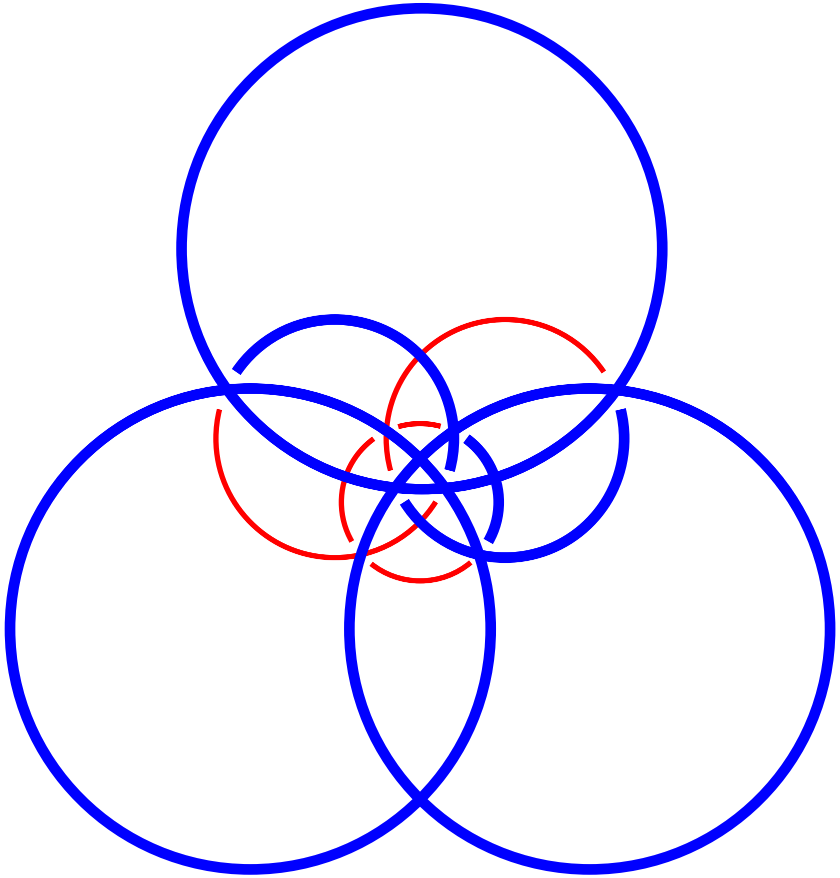


THM. $\equiv$ lattice congruence on $\mathfrak{S}_n$ with arcs $\mathcal{I}_\equiv$
The collection of cones defined equiv. as

- the cones obtained by glueing the Coxeter regions of the permutations in the same congruence class of $\equiv$
- the complements of the union of the shards $\Sigma(\alpha)$ for all arcs $\alpha \in \mathcal{I}_\equiv$

forms a fan $\mathcal{F}_\equiv$ of $\mathbb{R}^n$ whose dual graph realizes the lattice quotient $\mathfrak{S}_n/\equiv$.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

QUOTIENTOPE

fix a forcing dominant function f : arcs $\to \mathbb{R}_{>0}$ ie. st. $f(\alpha) > \sum_{\alpha' \succ \alpha} f(\alpha')$ for any arc α.

for an arc $\alpha = (i, j, n, S)$ and a subset $\varnothing \neq R \subsetneq [n]$ define the contribution

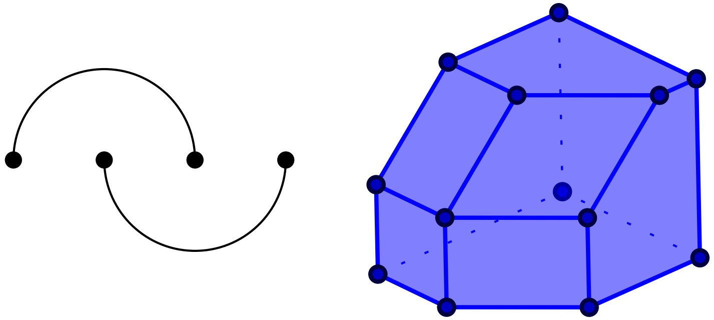
$$\gamma(\alpha, R) \coloneqq \begin{cases} 1 & \text{if } |R \cap \{i, j\}| = 1 \text{ and } S = R \cap \,]i, j[, \\ 0 & \text{otherwise} \end{cases}$$

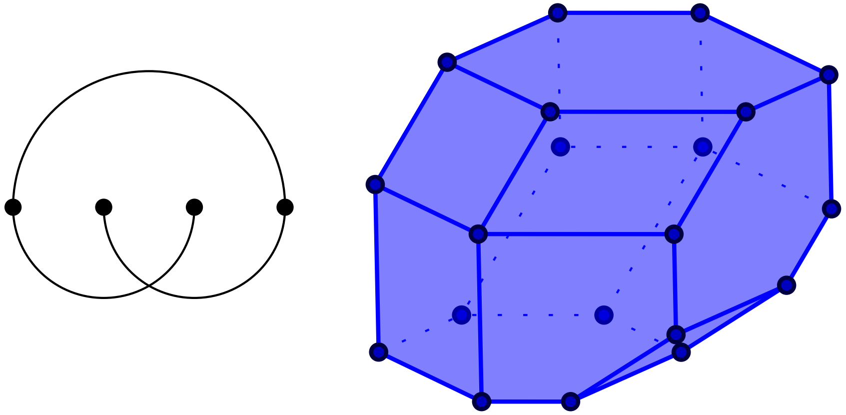
define height function h for $\varnothing \neq R \subsetneq [n]$ by $h^f_{\equiv}(R) \coloneqq \sum_{\alpha \in \mathcal{I}_{\equiv}} f(\alpha) \, \gamma(\alpha, R)$.

THM. For any lattice congruence $\equiv$ on $\mathfrak{S}_n$ and any forcing dominant function f : arcs $\to \mathbb{R}_{>0}$, the quotient fan $\mathcal{F}_{\equiv}$ is the normal fan of the polytope

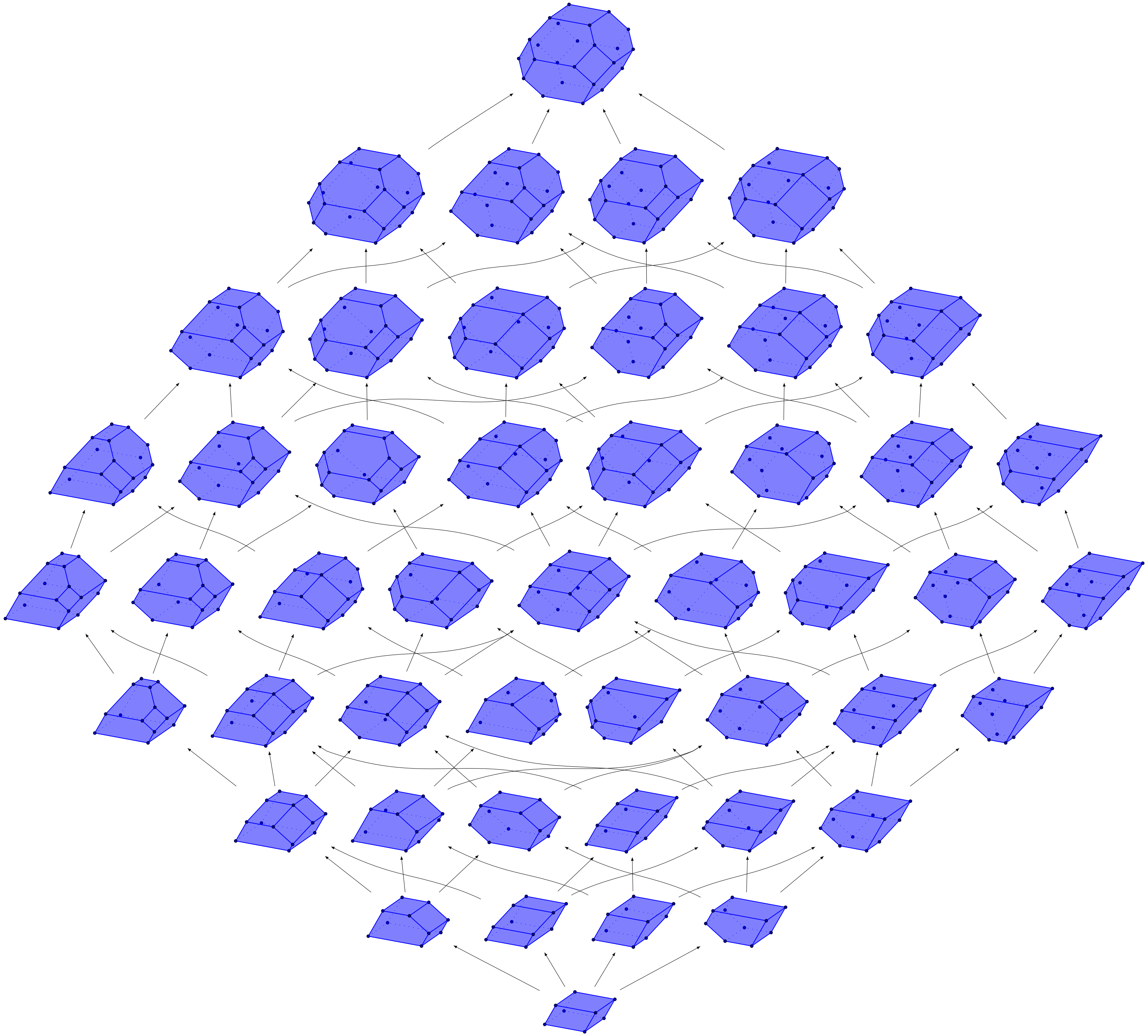
$$P^f_{\equiv} \coloneqq \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \langle \, \mathbf{r}(R) \mid \mathbf{x} \, \rangle \leq h^f_{\equiv}(R) \text{ for all } \varnothing \neq R \subsetneq [n] \right\}.$$

P.-Santos, *Quotientopes* ('17$^+$)

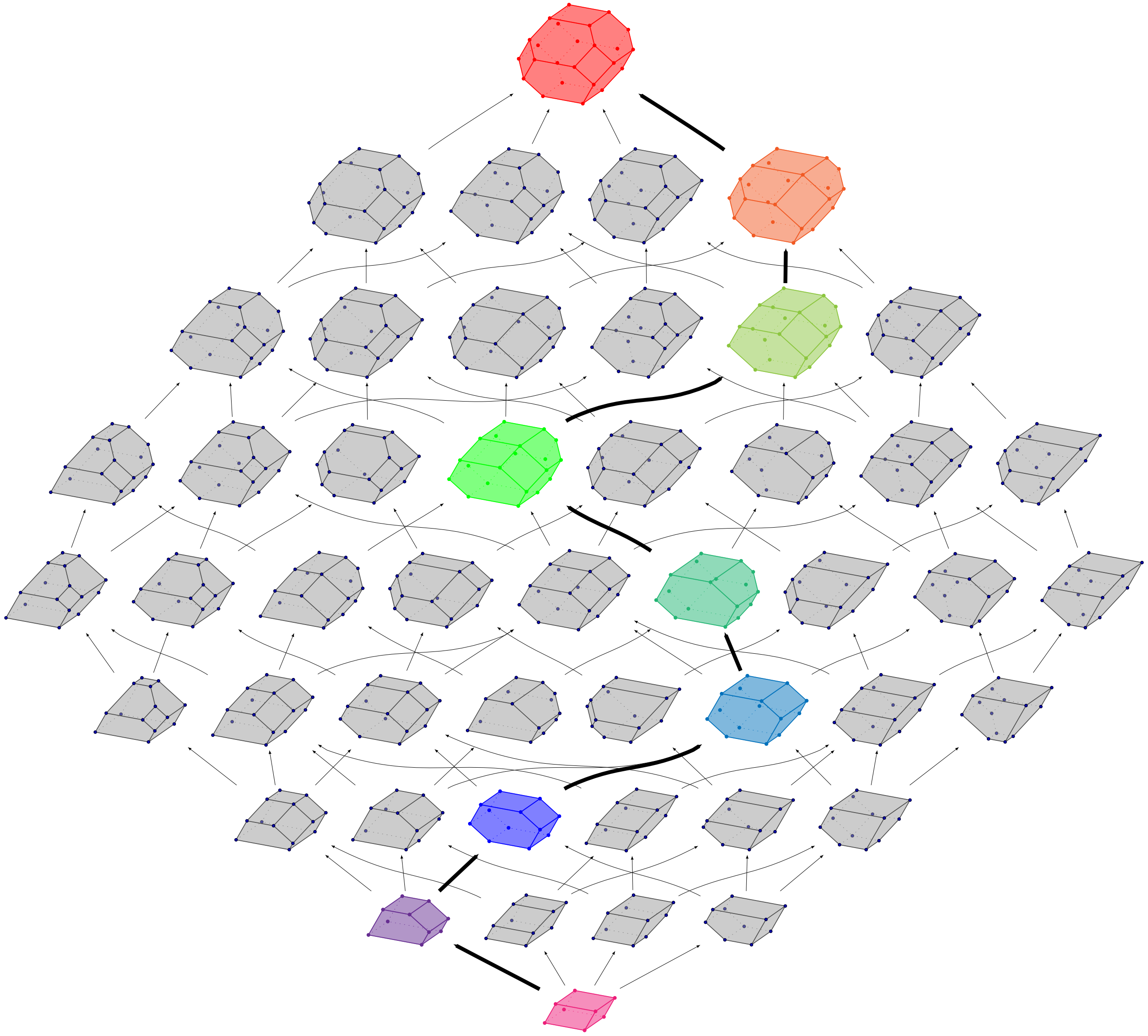




QUOTIENTOPE LATTICE



QUOTIENTOPE LATTICE



POLYWOOD

INSIDAHEDRA / OUTSIDAHEDRA

outsidahedra

permutrees

insidahedra

quotientopes

TOWARDS QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS

$\mathcal{H}$ hyperplane arrangement in $\mathbb{R}^n$
B distinguished region of $\mathbb{R}^n \smallsetminus \mathcal{H}$
inversion set of a region $C =$ set of hyperplanes of $\mathcal{H}$ that separate B and C
poset of regions $\operatorname{Pos}(\mathcal{H}, B) =$ regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ ordered by inclusion of inversion sets

THM. The poset of regions $\operatorname{Pos}(\mathcal{H}, B)$

- is never a lattice when B is not a simple region,
- is always a lattice when $\mathcal{H}$ is a simplicial arrangement.

Björner-Edelman-Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)

THM. If $\operatorname{Pos}(\mathcal{H}, B)$ is a lattice, and $\equiv$ is a lattice congruence of $\operatorname{Pos}(\mathcal{H}, B)$, the cones obtained by glueing together the regions of $\mathbb{R}^n \smallsetminus \mathcal{H}$ in the same congruence class form a complete fan.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

Is the quotient fan polytopal?

HOPF ALGEBRAS ON ARC DIAGRAMS

P., *Hopf algebras on decorated noncrossing arc diagrams* ('17^+)

DECORATED PERMUTATION

decoration set = a graded set $\mathfrak{X} := \bigsqcup_{n\geq 0} \mathfrak{X}_n$ endowed with

- a concatenation concat : $\mathfrak{X}_m \times \mathfrak{X}_n \longrightarrow \mathfrak{X}_{m+n}$
- a selection select : $\mathfrak{X}_m \times \binom{[m]}{k} \longrightarrow \mathfrak{X}_k$

such that

(i) concat$(\mathcal{X}, \text{concat}(\mathcal{Y}, \mathcal{Z})) = \text{concat}(\text{concat}(\mathcal{X}, \mathcal{Y}), \mathcal{Z})$

(ii) select$(\text{select}(\mathcal{X}, R), S) = \text{select}(\mathcal{X}, \{r_s \mid s \in S\})$

(iii) concat$(\text{select}(\mathcal{X}, R), \text{select}(\mathcal{Y}, S)) = \text{select}(\text{concat}(\mathcal{X}, \mathcal{Y}), R \cup S^{\to m})$
where $S^{\to m} := \{s + m \mid s \in S\}$.

Exm:

- $\mathcal{A}^* =$ words on an alphabet $\mathcal{A}$, with concatenation and subwords
- labeled graphs, with shifted union and induced subgraphs
- ...

DECORATED PERMUTATION

decoration set = a graded set $\mathfrak{X} := \bigsqcup_{n \ge 0} \mathfrak{X}_n$ endowed with

- a concatenation concat : $\mathfrak{X}_m \times \mathfrak{X}_n \longrightarrow \mathfrak{X}_{m+n}$
- a selection select : $\mathfrak{X}_m \times \binom{[m]}{k} \longrightarrow \mathfrak{X}_k$

such that

(i) $\mathrm{concat}(\mathcal{X}, \mathrm{concat}(\mathcal{Y}, \mathcal{Z})) = \mathrm{concat}(\mathrm{concat}(\mathcal{X}, \mathcal{Y}), \mathcal{Z})$

(ii) $\mathrm{select}(\mathrm{select}(\mathcal{X}, R), S) = \mathrm{select}(\mathcal{X}, \{r_s \mid s \in S\})$

(iii) $\mathrm{concat}(\mathrm{select}(\mathcal{X}, R), \mathrm{select}(\mathcal{Y}, S)) = \mathrm{select}(\mathrm{concat}(\mathcal{X}, \mathcal{Y}), R \cup S^{\rightarrow m})$
where $S^{\rightarrow m} := \{s + m \mid s \in S\}$.

$\mathfrak{X}$-decorated permutation = pair $(\sigma, \mathcal{X})$ with $\sigma \in \mathfrak{S}_n$ and $\mathcal{X} \in \mathfrak{X}_n$.

standardization $\mathrm{std}((\rho, \mathcal{Z}), R) := \big(\mathrm{stdp}(\rho, R), \mathrm{select}(\mathcal{Z}, \rho^{-1}(R))\big)$

THM. The product $\cdot$ and coproduct $\triangle$ defined by

$$\mathbb{F}_{(\sigma,\mathcal{X})} \cdot \mathbb{F}_{(\tau,\mathcal{Y})} := \sum_{\rho \in \sigma \,\bar{\sqcup}\, \tau} \mathbb{F}_{(\rho, \mathrm{concat}(\mathcal{X},\mathcal{Y}))} \quad \text{and} \quad \triangle \mathbb{F}_{(\rho,\mathcal{Z})} := \sum_{k=0}^{p} \mathbb{F}_{\mathrm{std}((\rho,\mathcal{Z}),[k])} \otimes \mathbb{F}_{\mathrm{std}((\rho,\mathcal{Z}),[p] \smallsetminus [k])}$$

endow the vector space of decorated permutations with a graded Hopf algebra structure.

P., *Hopf algebras on decorated noncrossing arc diagrams* ('17$^+$)

DECORATED NONCROSSING ARC DIAGRAMS

a graded function $\Psi : \mathfrak{X} = \bigsqcup_{n \ge 0} \mathfrak{X}_n \longrightarrow \mathfrak{I} = \bigsqcup_{n \ge 0} \mathfrak{I}_n$ is conservative if

(i) $\Psi(\mathcal{X})^{+n}$ and $\Psi(\mathcal{Y})^{\to m}$ are both subsets of $\Psi(\text{concat}(\mathcal{X}, \mathcal{Y}))$

(ii) $(r_a, r_b, p, S) \in \Psi(\mathcal{Z})$ implies $(a, b, q, \{c \mid r_c \in S\}) \in \Psi(\text{select}(\mathcal{Z}, R))$

$\mathcal{I}$ collection of arcs closed by forcing

surjection $\begin{array}{rccc} \eta_{\mathcal{I}} : & \mathfrak{S}_n & \longrightarrow & \{\text{nc arc diagrams in } \mathcal{I}\} \\ & \sigma & \longmapsto & \eta_{\mathcal{I}}(\sigma) = \delta(\pi_{\downarrow}(\sigma)) \end{array}$

$\mathfrak{X}$-decorated noncrossing arc diagram $= (\mathcal{D}, \mathcal{X})$ where $\mathcal{D}$ is a non crossing arc diagram contained in $\Psi(\mathcal{X})$

THM. For a decorated noncrossing arc diagram $(\mathcal{D}, \mathcal{X})$, define

$$\mathbb{P}_{(\mathcal{D},\mathcal{X})} := \sum \mathbb{F}_{(\sigma,\mathcal{X})},$$

where σ ranges over the permutations such that $\eta_{\Psi(\mathcal{X})}(\sigma) = \mathcal{D}$. The graded vector subspace $\mathbf{k}\mathfrak{D} := \bigoplus_{n \ge 0} \mathbf{k}\mathfrak{D}_n$ of $\mathbf{k}\mathfrak{P}$ generated by the elements $\mathbb{P}_{(\mathcal{D},\mathcal{X})}$, for all $\mathfrak{X}$-decorated noncrossing arc diagrams $(\mathcal{D}, \mathcal{X})$, is a Hopf subalgebra of $\mathbf{k}\mathfrak{P}$.

P., *Hopf algebras on decorated noncrossing arc diagrams* ('17^{+})

APPLICATIONS

fix $k \geq 0$

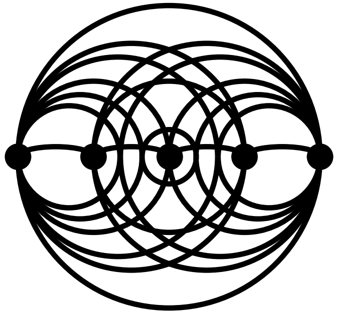
$\mathfrak{X}$ = words on $\mathbb{N}^4$ (each letter a is made of 4 numbers $\ell_a \overset{u_a}{\underset{d_a}{+}} r_a$)

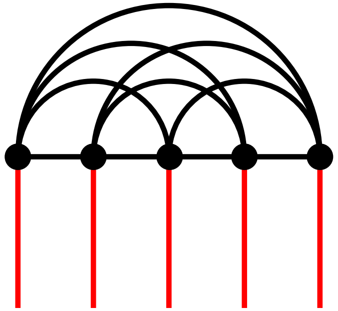
with concat$(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_m b_1 \cdots b_n$

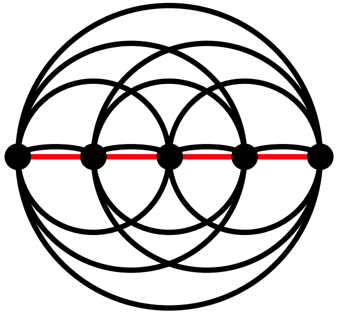
select$(c_1 \cdots c_p, R) = \bar{c}_{r_1} \cdots \bar{c}_{r_q}$ where $\bar{c}_{r_i} = \min\limits_{r_{i-1} < k \leq r_i} \ell_{c_k} \overset{u_{c_{r_i}}}{\underset{d_{c_{r_i}}}{+}} \min\limits_{r_i \leq k < r_{i+1}} r_{c_k}$

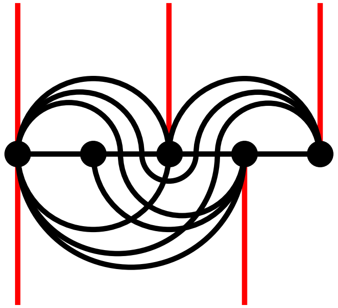
$$\Psi(a_1 \cdots a_m) = \text{draw} \left| \begin{array}{ll} u_{a_i} & \\ d_{a_i} & \text{red walls} \\ \min(r_{a_i}, \ell_{a_{i+1}}) & \end{array} \right| \begin{array}{l} \text{above } i \\ \text{below } i \\ \text{between } i \text{ and } i+1 \end{array}$$

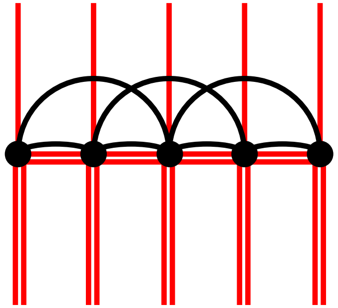
allow arcs that cross at most k walls


weak order


Tamari lattice


diagonal rectangulations


Cambrian lattices

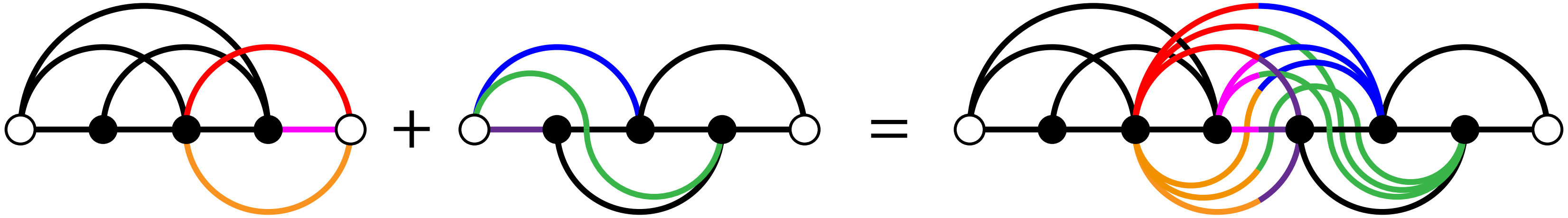

k-sashes lattices

APPLICATIONS

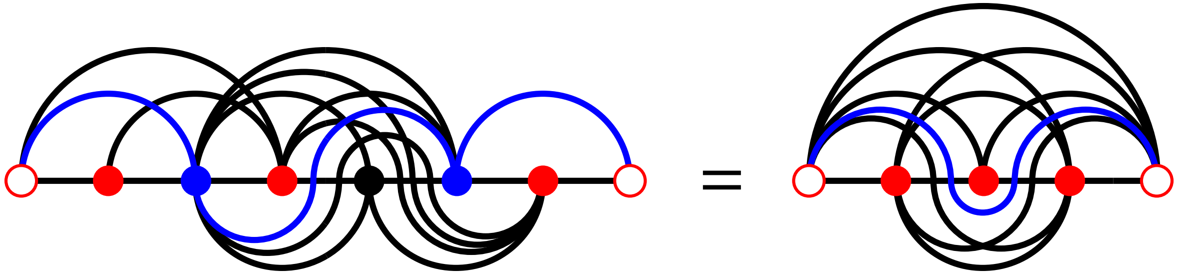
extended arc = arc allowed to start at 0 or end at $n+1$

$\mathfrak{X}$ = extended arc ideals with

concatenation:



selection:

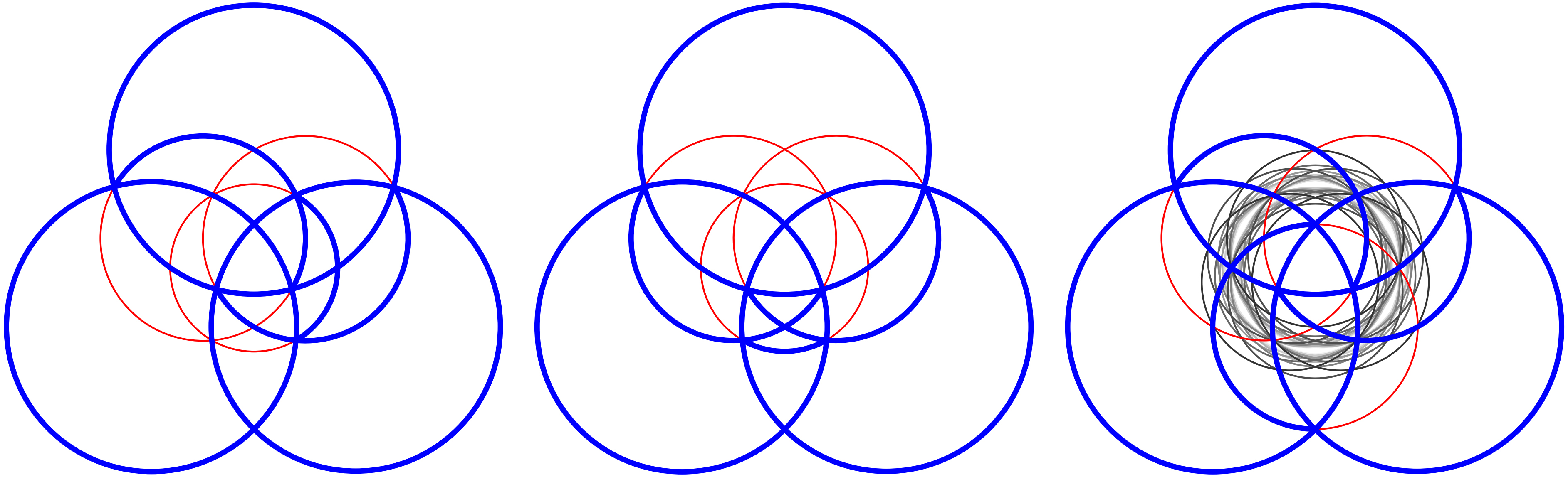


$\Psi(\mathcal{X})$ = strict arcs in $\mathcal{X}$

$\Longrightarrow$ Hopf algebra on all arc ideals containing the permutree algebra

P., *Hopf algebras on decorated noncrossing arc diagrams* ('17^{+})

III. THE UNIVERSAL ASSOCIAHEDRON AND ITS PROJECTIONS



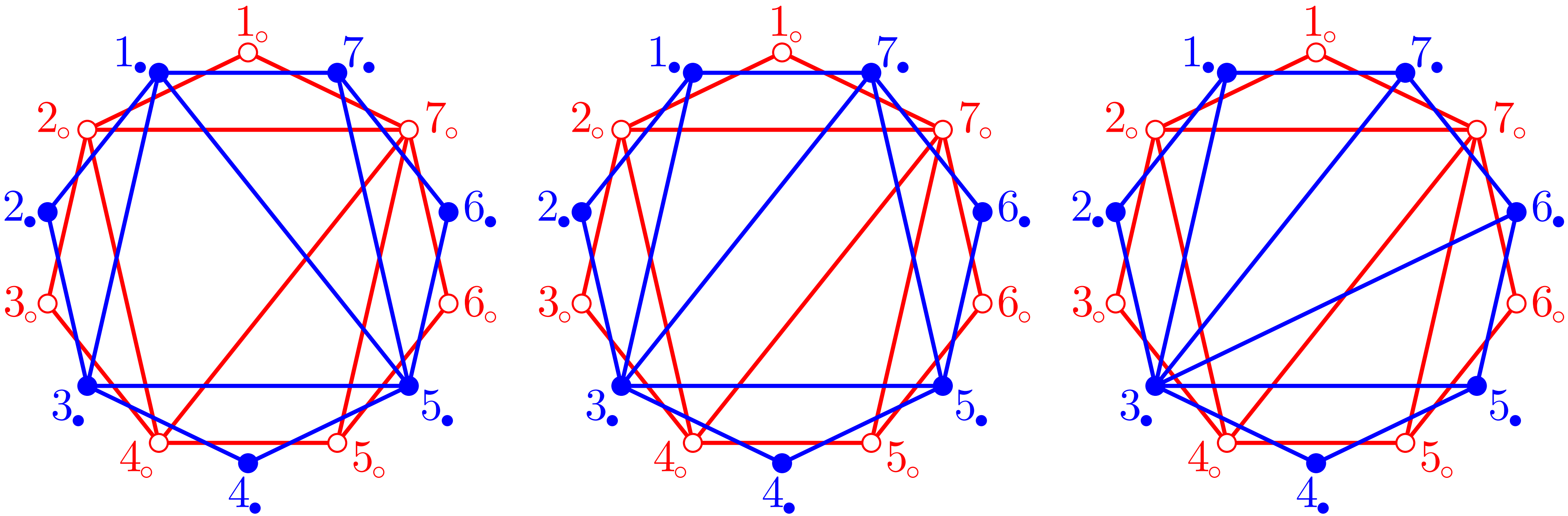
Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)
Manneville-P., *Geometric realizations of the accordion complex* ('17^+)

G- AND C-VECTORS

TWO POLYGONS

Consider simultaneously two n-gons:

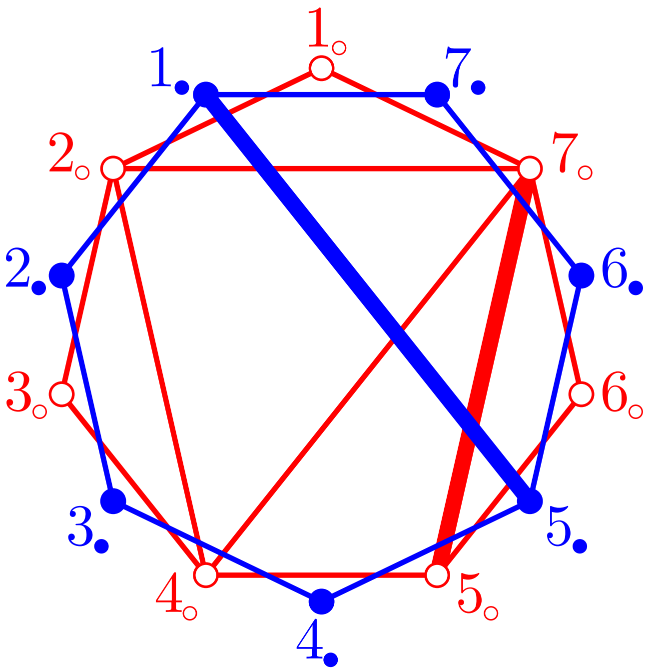
- the red polygon supports a reference triangulation,
- the blue polygon is the ground set.

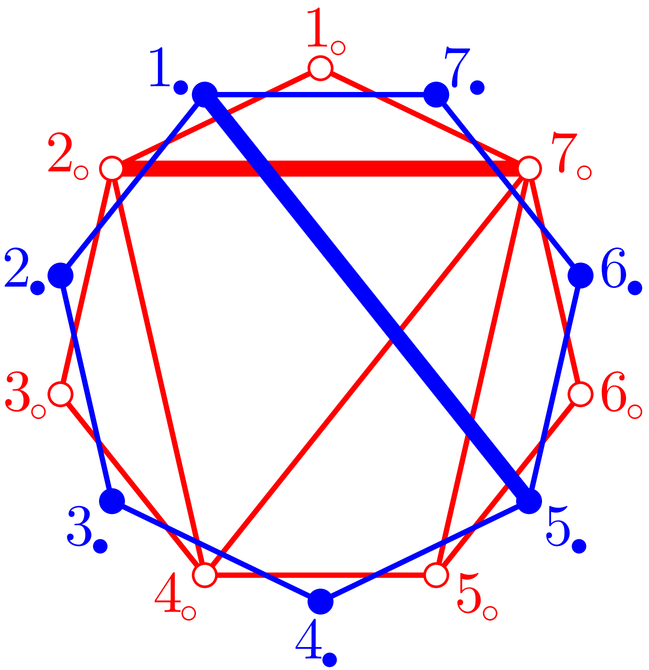


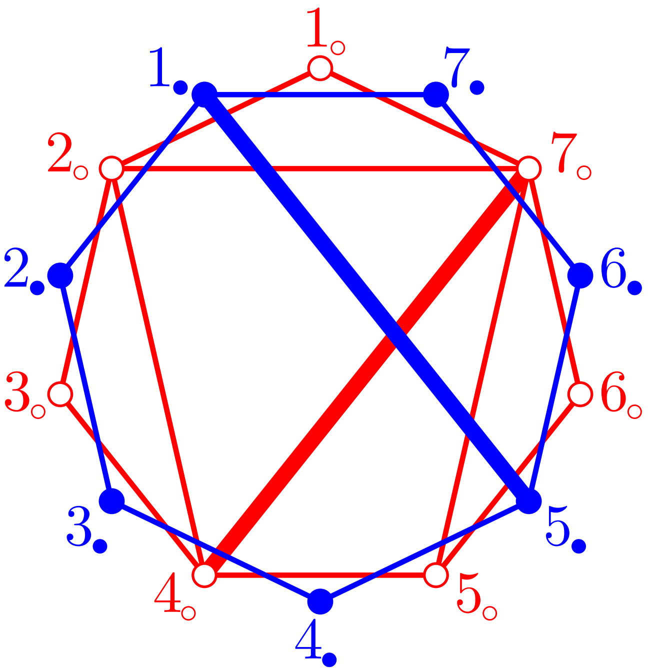
G-VECTORS

For $\mathrm{T}_\circ$ red triangulation, $\delta_\circ \in \mathrm{T}_\circ$ and $\delta_\bullet$ a blue diagonal, let

$$\varepsilon_\circ(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet) = \begin{cases} 1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as a Z} \\ -1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in \mathrm{T}_\circ \text{ as an } \reflectbox{Z} \\ 0 & \text{otherwise} \end{cases}$$







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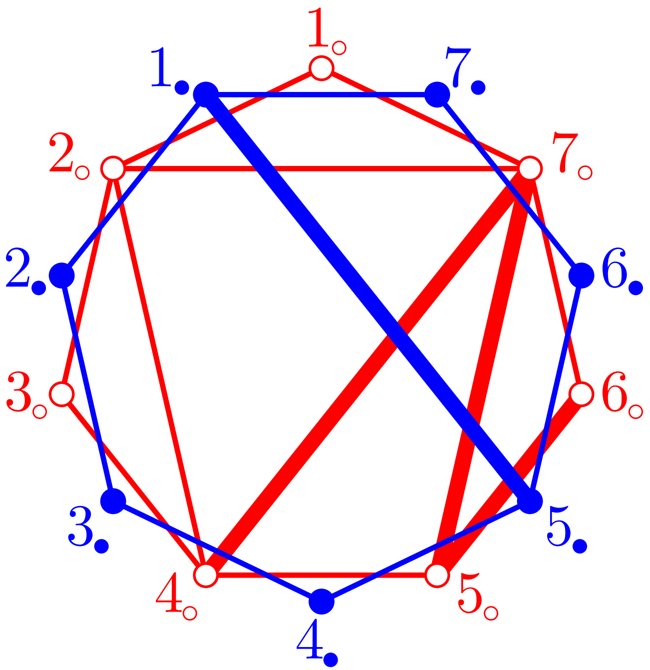
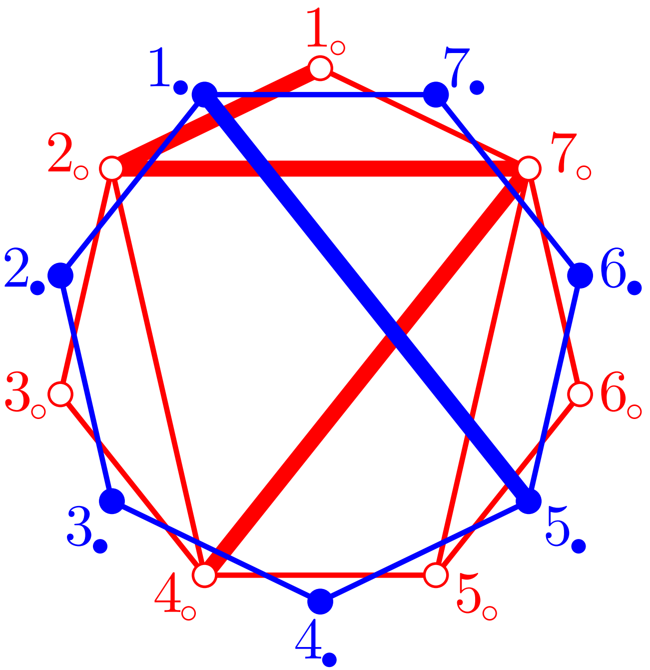
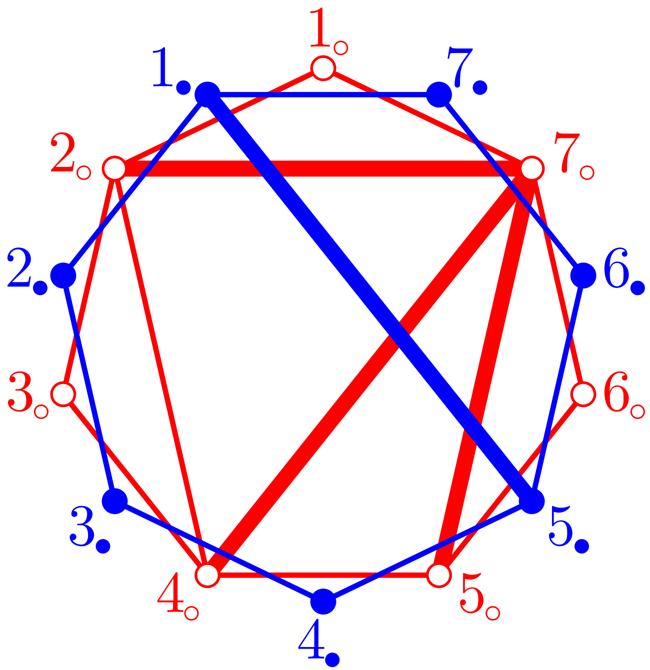


$\mathsf{Z} = 1$



$\Sigma = -1$



$\mathsf{V} = 0$

G-VECTORS

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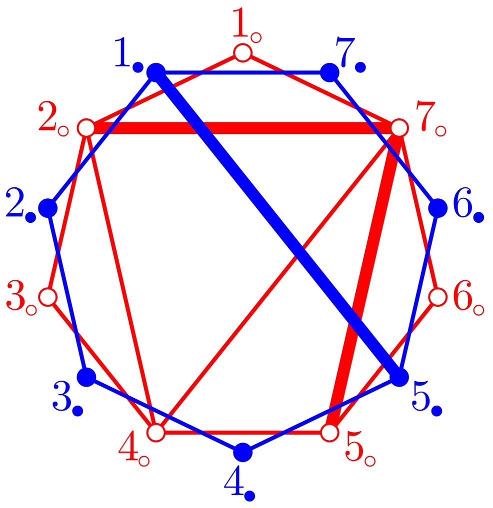
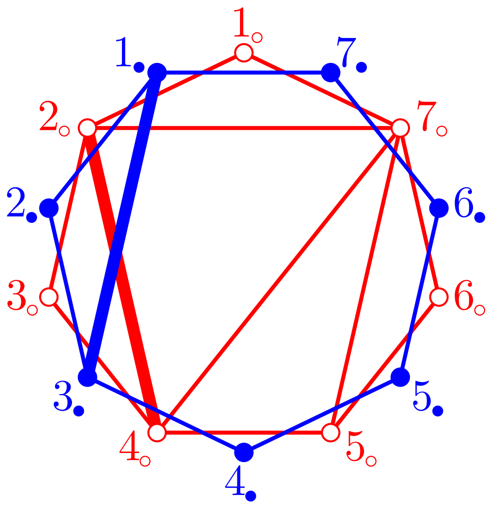
$\mathbf{g}(\mathrm{T}_\circ, \delta_\bullet)$ = g-vector of $\delta_\bullet$ with respect to $\mathrm{T}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet\big)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$

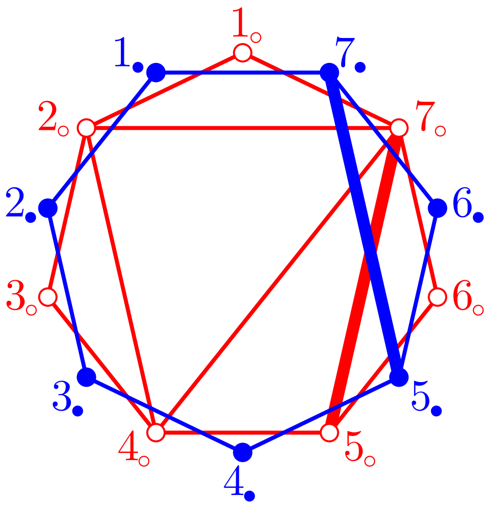
= alternating ± 1 along the zigzag crossed by $\delta_\bullet$ in $\mathrm{T}_\circ$



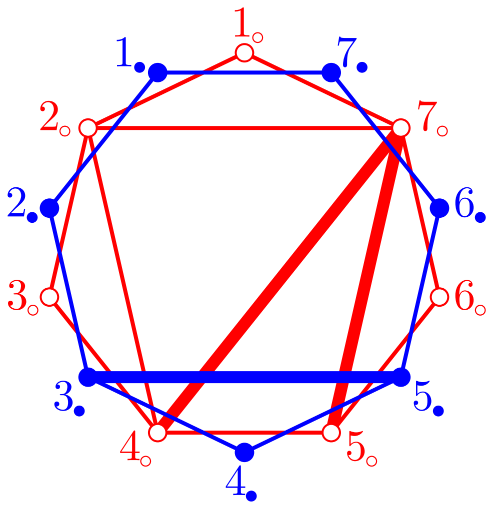
$\mathbf{g}(\mathrm{T}_\circ, (1_\bullet, 5_\bullet)) =$
$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{2_\circ 7_\circ}$



$\mathbf{g}(\mathrm{T}_\circ, (1_\bullet, 3_\bullet)) =$
$-\mathbf{e}_{2_\circ 4_\circ}$



$\mathbf{g}(\mathrm{T}_\circ, (5_\bullet, 7_\bullet)) =$
$\mathbf{e}_{5_\circ 7_\circ}$



$\mathbf{g}(\mathrm{T}_\circ, (3_\bullet, 5_\bullet)) =$
$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{4_\circ 7_\circ}$

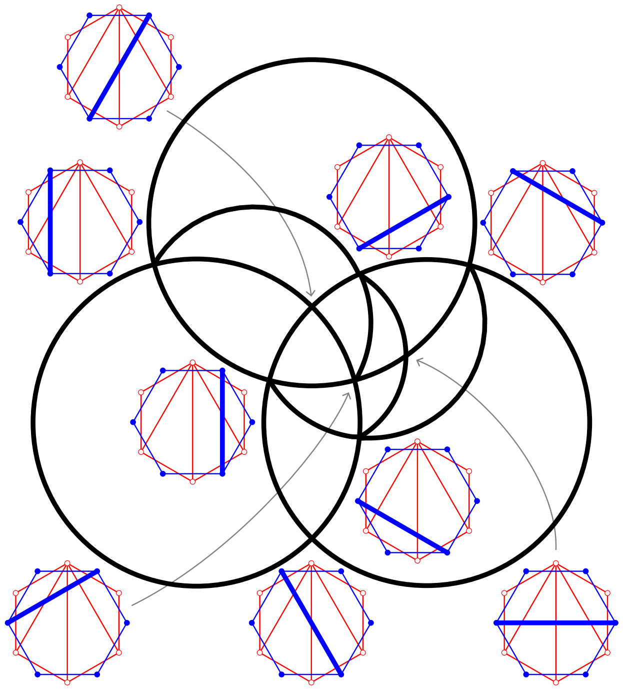
G-VECTOR FAN

$\mathbf{g}(\mathrm{T}_\circ, \delta_\bullet)$ = g-vector of $\delta_\bullet$ with respect to $\mathrm{T}_\circ = \Big[\varepsilon_\circ(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$

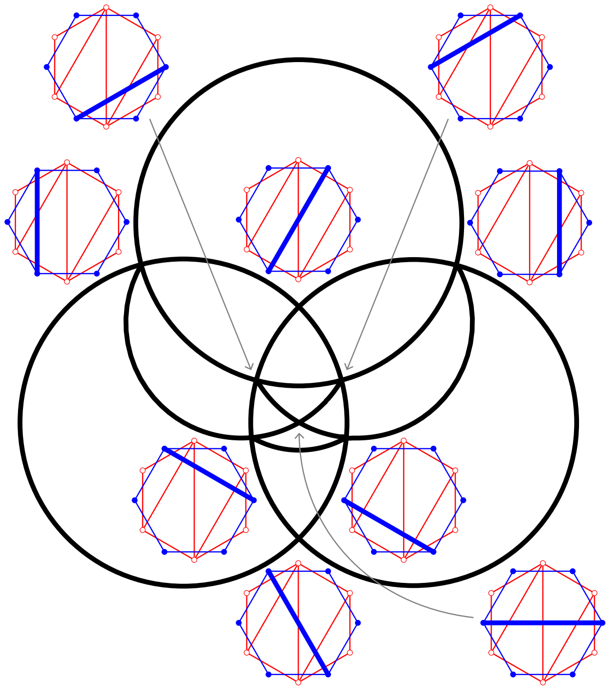
THM. For any red triangulation $\mathrm{T}_\circ$, the collection of cones

$$\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ) := \big\{ \mathbb{R}_{\geq 0}\mathbf{g}(\mathrm{T}_\circ, \mathrm{D}_\bullet) \;\big|\; \mathrm{D}_\bullet \text{ any blue dissection} \big\}$$

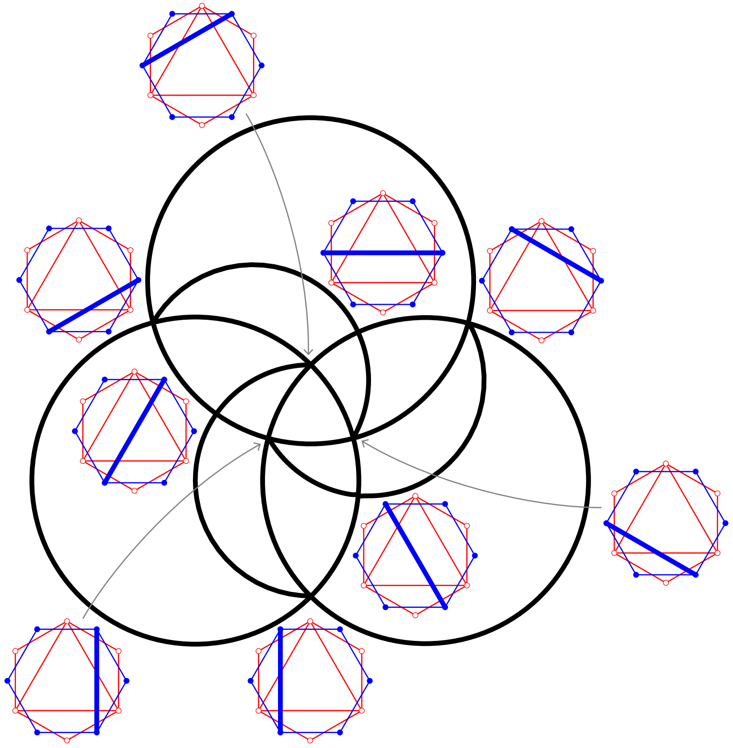
forms a complete simplicial fan, called g-vector fan of $\mathrm{T}_\circ$.



Loday



Hohlweg-Lange / Reading



Hohlweg-P.-Stella

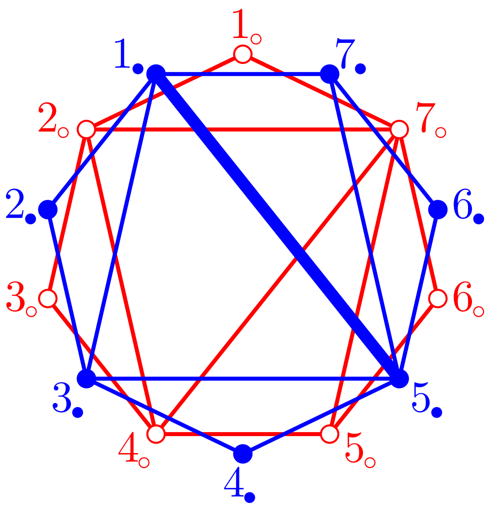
C-VECTORS

For $\mathrm{T}_\circ$ red triangulation and $\mathrm{T}_\bullet$ blue triangulation
and two diagonals $\delta_\circ \in \mathrm{T}_\circ$ and $\delta_\bullet \in \mathrm{T}_\bullet$, let

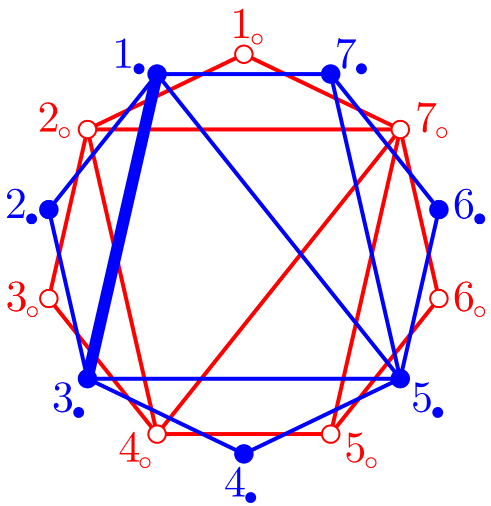
$$\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) = \begin{cases} 1 & \text{if } \delta_\circ \text{ slaloms on } \delta_\bullet \in \mathrm{T}_\bullet \text{ as a } \Sigma \\ -1 & \text{if } \delta_\circ \text{ slaloms on } \delta_\bullet \in \mathrm{T}_\bullet \text{ as an } Z \\ 0 & \text{otherwise} \end{cases}$$

$\mathbf{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) =$ <u>c-vector</u> of $\delta_\bullet$ in $\mathrm{T}_\bullet$ with respect to $\mathrm{T}_\circ$ $= \Big[\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$

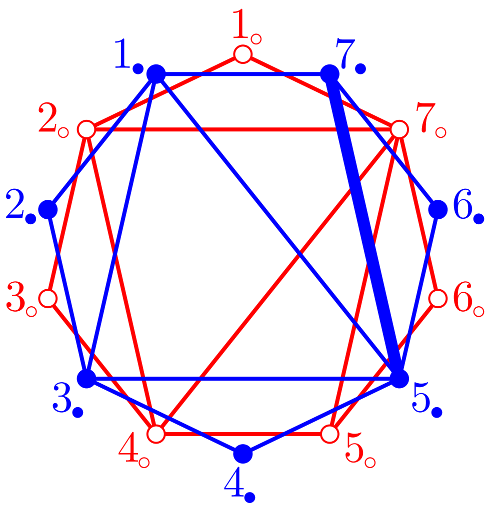
$= \pm$ charac. vector of diagonals of $\mathrm{T}_\circ$ crossed by opposite neighbors of $\delta_\bullet$



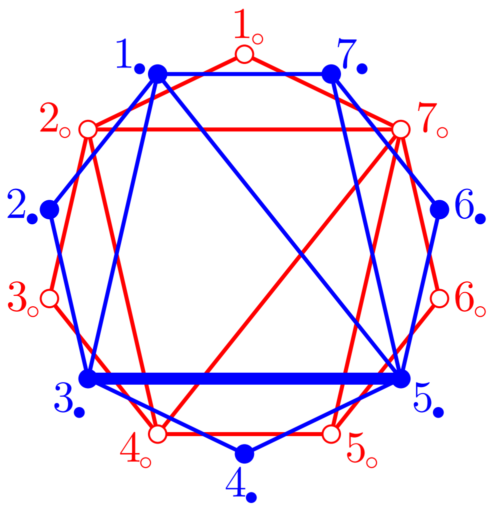
$\mathbf{c}(\mathrm{T}_\circ, (1_\bullet, 5_\bullet) \in \mathrm{T}_\bullet) =$
$-\mathbf{e}_{2_\circ 7_\circ}$



$\mathbf{c}(\mathrm{T}_\circ, (1_\bullet, 3_\bullet) \in \mathrm{T}_\bullet) =$
$-\mathbf{e}_{2_\circ 4_\circ}$



$\mathbf{c}(\mathrm{T}_\circ, (5_\bullet, 7_\bullet) \in \mathrm{T}_\bullet) =$
$\mathbf{e}_{2_\circ 7_\circ} + \mathbf{e}_{4_\circ 7_\circ} + \mathbf{e}_{5_\circ 7_\circ}$



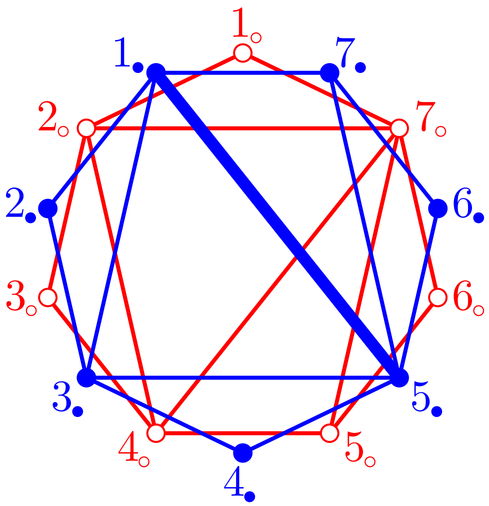
$\mathbf{c}(\mathrm{T}_\circ, (5_\bullet, 7_\bullet) \in \mathrm{T}_\bullet) =$
$-\mathbf{e}_{4_\circ 7_\circ}$

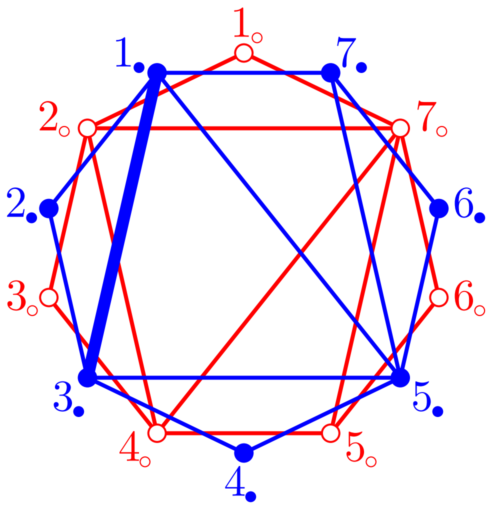
G- AND C-VECTORS

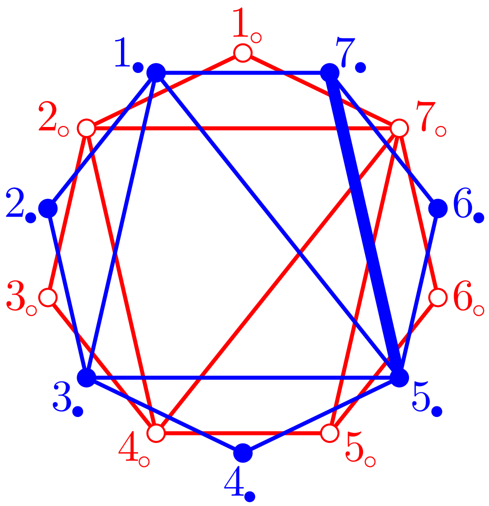
For $\mathrm{T}_\circ$ red triangulation and $\mathrm{T}_\bullet$ blue triangulation

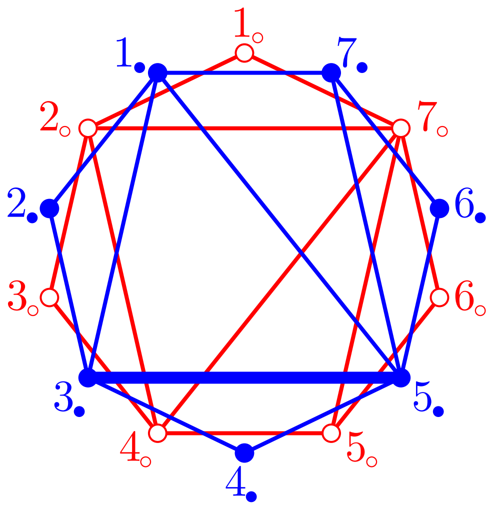
$$\mathbf{g}(\mathrm{T}_\circ, \delta_\bullet) = \underline{\text{g-vector}} \text{ of } \delta_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\circ\big(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet\big) \Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$

$$\mathbf{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \underline{\text{c-vector}} \text{ of } \delta_\bullet \text{ in } \mathrm{T}_\bullet \text{ with respect to } \mathrm{T}_\circ = \Big[\varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$$









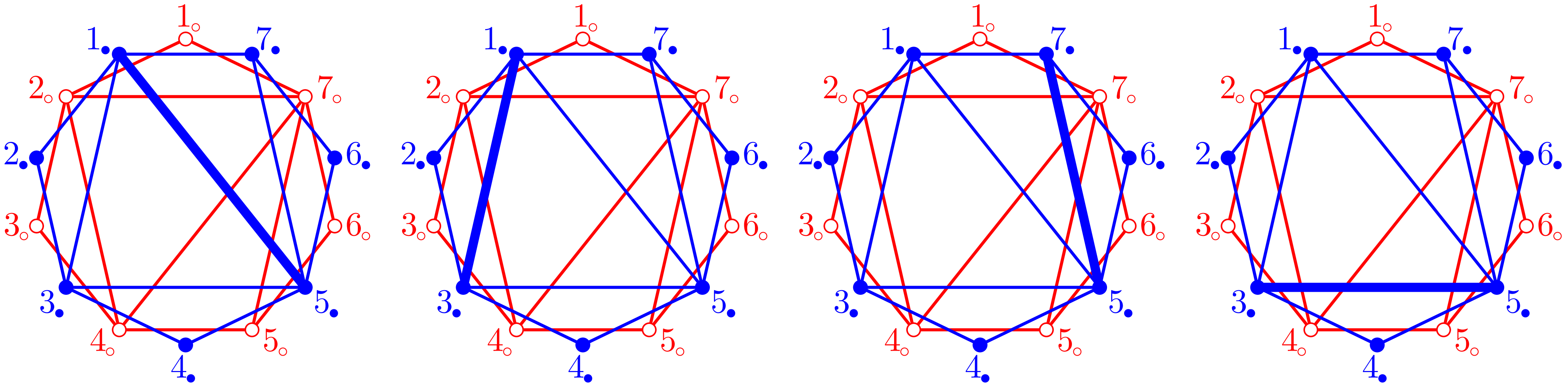
g	$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{5_\circ 7_\circ}$	$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{4_\circ 7_\circ}$
c	$-\mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{2_\circ 7_\circ} + \mathbf{e}_{4_\circ 7_\circ} + \mathbf{e}_{5_\circ 7_\circ}$	$-\mathbf{e}_{4_\circ 7_\circ}$

G- AND C-VECTORS

For $\mathrm{T}_\circ$ red triangulation and $\mathrm{T}_\bullet$ blue triangulation

$\mathbf{g}(\mathrm{T}_\circ, \delta_\bullet)$ = $\underline{\mathbf{g}\text{-vector}}$ of $\delta_\bullet$ with respect to $\mathrm{T}_\circ$ $= \Big[\varepsilon_\circ(\delta_\circ \in \mathrm{T}_\circ, \delta_\bullet)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$

$\mathbf{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet)$ = $\underline{\mathbf{c}\text{-vector}}$ of $\delta_\bullet$ in $\mathrm{T}_\bullet$ with respect to $\mathrm{T}_\circ$ $= \Big[\varepsilon_\bullet(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet)\Big]_{\delta_\circ \in \mathrm{T}_\circ} \in \mathbb{R}^{\mathrm{T}_\circ}$



$\mathbf{g}$	$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{5_\circ 7_\circ}$	$\mathbf{e}_{5_\circ 7_\circ} - \mathbf{e}_{4_\circ 7_\circ}$
$\mathbf{c}$	$-\mathbf{e}_{2_\circ 7_\circ}$	$-\mathbf{e}_{2_\circ 4_\circ}$	$\mathbf{e}_{2_\circ 7_\circ} + \mathbf{e}_{4_\circ 7_\circ} + \mathbf{e}_{5_\circ 7_\circ}$	$-\mathbf{e}_{4_\circ 7_\circ}$

PROP. The g-vectors $\mathbf{g}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$ and the c-vectors $\mathbf{c}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$ form dual bases.

PROP. Duality: $\mathbf{g}(\mathrm{T}_\circ, \mathrm{T}_\bullet) = -\mathbf{c}(\mathrm{T}_\bullet, \mathrm{T}_\circ)^t$ and $\mathbf{c}(\mathrm{T}_\circ, \mathrm{T}_\bullet) = -\mathbf{g}(\mathrm{T}_\bullet, \mathrm{T}_\circ)^t$

ASSOCIAHEDRA FOR g-VECTOR FANS

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17$^+$)

$\mathrm{T}_\circ$-ZONOTOPE

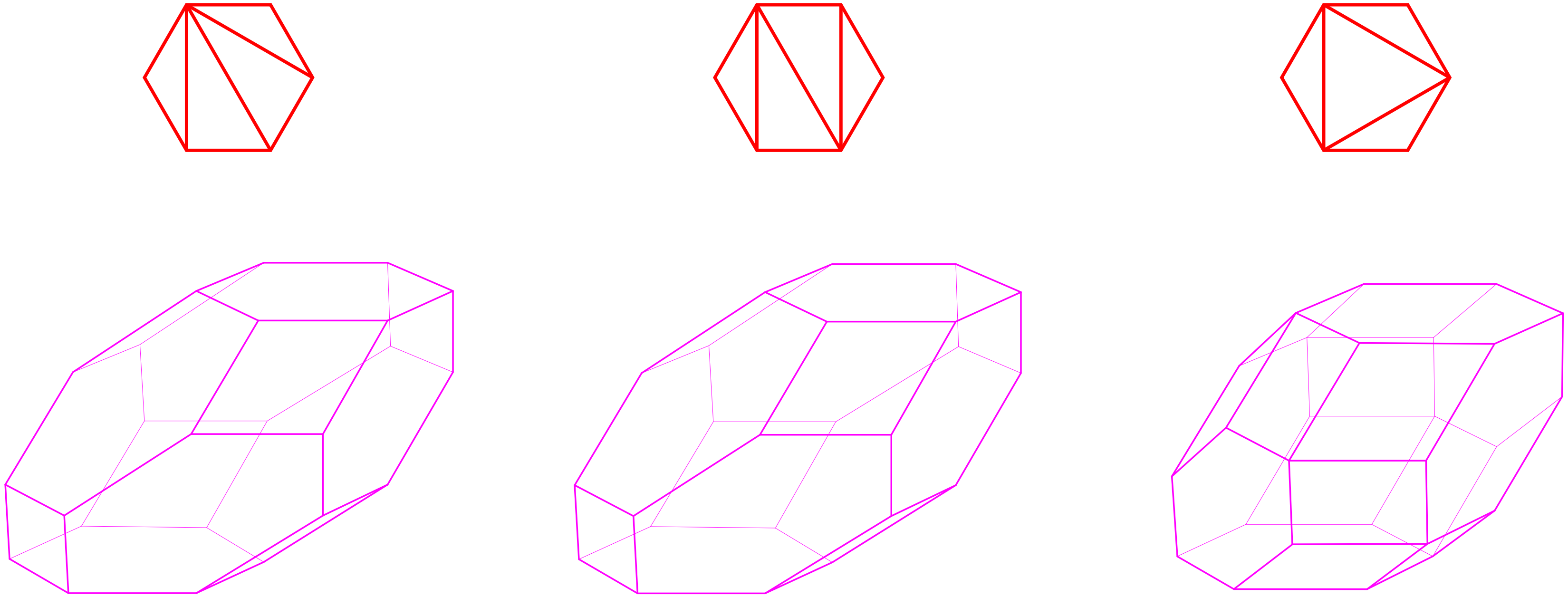
$\mathrm{T}_\circ$-zonotope $=$ Zono$(\mathrm{T}_\circ)$ $=$ Minkowski sum of all $\mathbf{c}$-vectors $\mathbf{C}(\mathrm{T}_\circ) = \bigcup_{\mathrm{T}_\bullet} \mathbf{c}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$

$$\text{Zono}(\mathrm{T}_\circ) = \sum_{\mathbf{c} \in \mathbf{C}(\mathrm{T}_\circ)} \mathbf{c}.$$

PROP. For any diagonal $\gamma_\bullet$, Zono$(\mathrm{T}_\circ)$ has a facet defined by the inequality

$$\langle\, \mathbf{g}(\mathrm{T}_\circ, \gamma_\bullet) \mid \mathbf{x} \,\rangle \le \omega(\gamma_\bullet)$$

where $\omega(\gamma_\bullet)$ $=$ number of red diagonals that cross $\gamma_\bullet$.



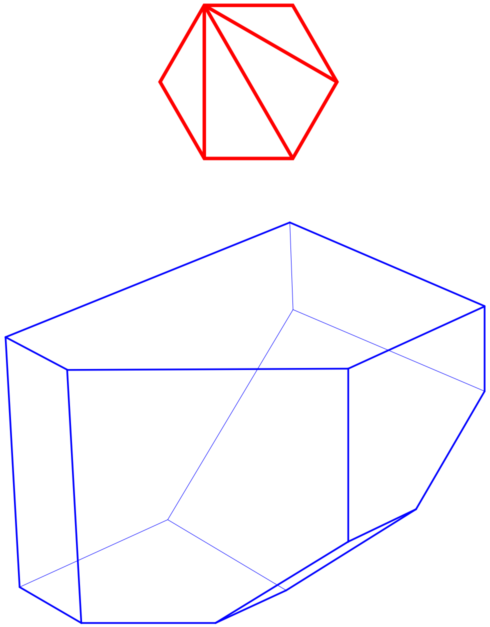
$\mathrm{T}_\circ$-ASSOCIAHEDRON

Define $$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) := \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \mathbf{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet)$$

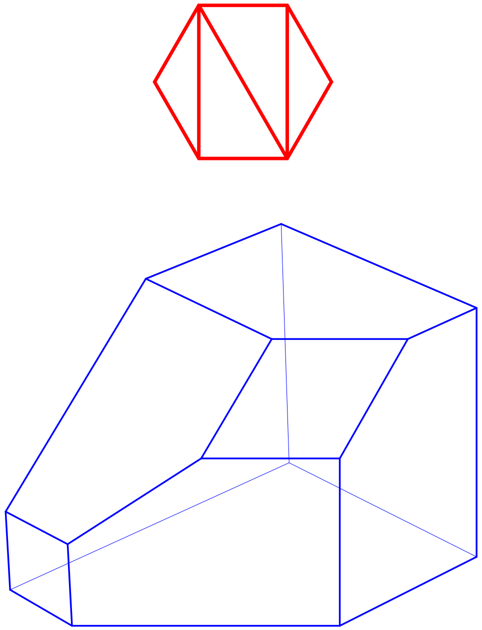
THM. For any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\begin{aligned} \mathsf{Asso}(\mathrm{T}_\circ) &= \operatorname{conv}\left\{ \mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \mid \mathrm{T}_\bullet \text{ blue triangulation} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{T}_\circ} \;\middle|\; \langle\, \mathbf{g}(\mathrm{T}_\circ, \delta_\bullet) \mid \mathbf{x} \,\rangle \le \omega(\delta_\bullet) \text{ for any blue diagonal } \delta_\bullet \right\}. \end{aligned}$$

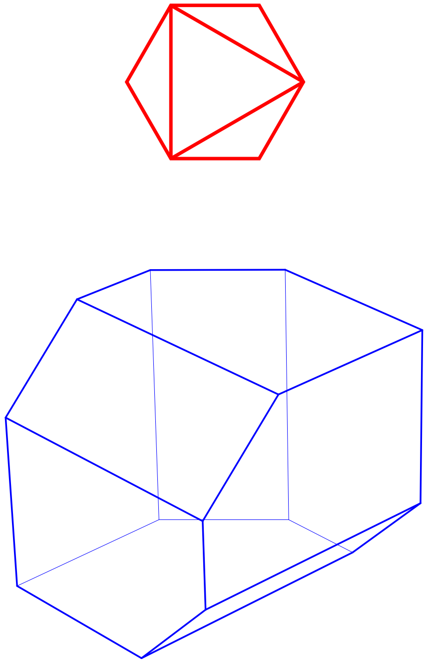
Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17$^+$)



Loday



Hohlweg-Lange



Hohlweg-P.-Stella

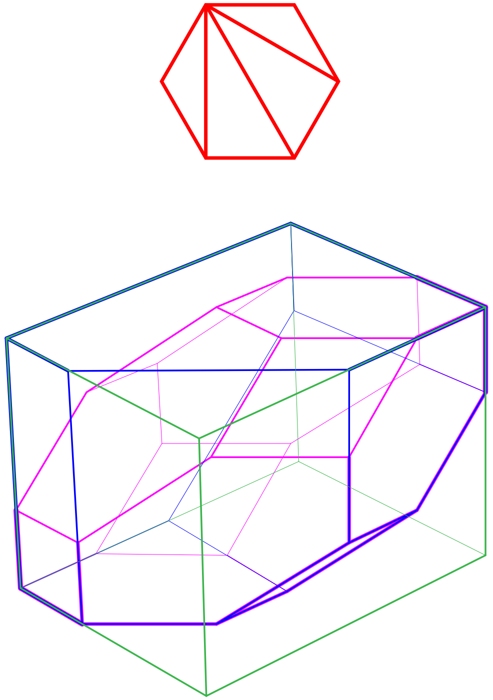
$\mathrm{T}_\circ$-ASSOCIAHEDRON

Define $$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \coloneqq \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \mathbf{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet)$$

THM. For any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\begin{aligned} \text{Asso}(\mathrm{T}_\circ) &= \operatorname{conv}\left\{\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \mid \mathrm{T}_\bullet \text{ blue triangulation}\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^{\mathrm{T}_\circ} \;\middle|\; \langle\, \mathbf{g}(\mathrm{T}_\circ, \delta_\bullet) \mid \mathbf{x} \,\rangle \le \omega(\delta_\bullet) \text{ for any blue diagonal } \delta_\bullet\right\}. \end{aligned}$$

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)



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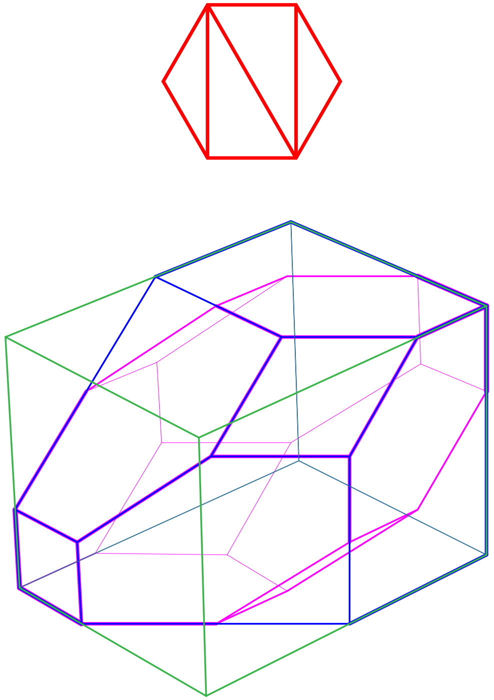
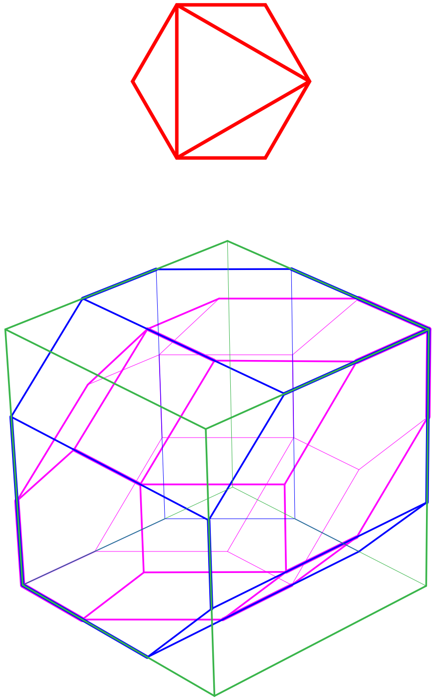


Hohlweg-Lange



Hohlweg-P.-Stella

UNIVERSAL ASSOCIAHEDRON

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)

UNIVERSAL ASSOCIAHEDRON

THM. For any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\mathsf{Asso}(\mathrm{T}_\circ) = \operatorname{conv}\left\{\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \mid \mathrm{T}_\bullet \text{ blue triangulation}\right\}$$

where

$$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) := \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \mathbf{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \sum_{\delta_\circ \in \mathrm{T}_\circ} \left(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big)\right) \mathbf{e}_{\delta_\circ} \; \in \mathbb{R}^{\mathrm{T}_\circ}.$$

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17$^+$)

$\Longrightarrow$ the $\delta_\circ$-coordinate of $\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet)$ does not really depends on $\mathrm{T}_\circ$

UNIVERSAL ASSOCIAHEDRON

THM. For any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of

$$\mathsf{Asso}(\mathrm{T}_\circ) = \operatorname{conv}\left\{\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) \;\middle|\; \mathrm{T}_\bullet \text{ blue triangulation}\right\}$$

where

$$\mathbf{p}(\mathrm{T}_\circ, \mathrm{T}_\bullet) := \sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \mathbf{c}(\mathrm{T}_\circ, \delta_\bullet \in \mathrm{T}_\bullet) = \sum_{\delta_\circ \in \mathrm{T}_\circ} \Bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Bigg) \, \mathbf{e}_{\delta_\circ} \;\in\; \mathbb{R}^{\mathrm{T}_\circ}.$$

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Bigg) \, \mathbf{e}_{\delta_\circ} \;\in\; \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathsf{Asso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)

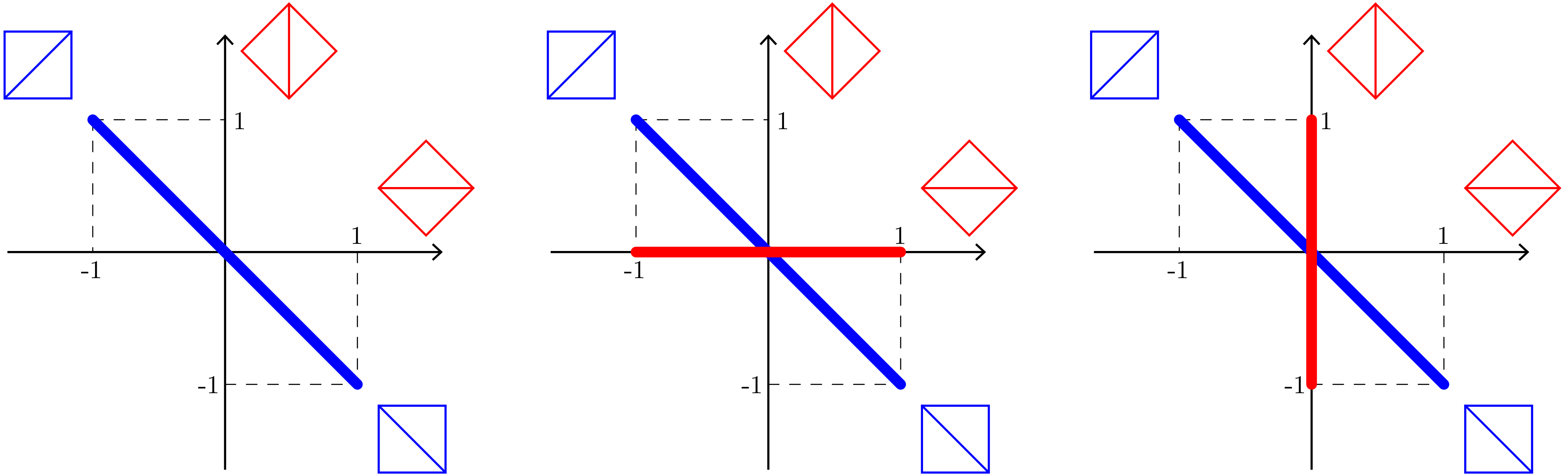
UNIVERSAL ASSOCIAHEDRON

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Big(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet) \Big) \, \mathbf{e}_{\delta_\circ} \; \in \; \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathrm{Asso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^{+})



UNIVERSAL ASSOCIAHEDRON

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Bigg) \mathbf{e}_{\delta_\circ} \in \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathrm{Asso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathrm{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17$^+$)

n	dimension of ambient space	dimension	# vertices	# facets	# vertices / facet	# facets / vertex
1	2	1	2	2	1	1
2	5	4	5	5	4	4
3	9	8	14	60	$9 \le \cdot \le 10$	$30 \le \cdot \le 42$
4	14	13	42	8960	$14 \le \cdot \le 28$	$3463 \le \cdot \le 4244$

UNIVERSAL ASSOCIAHEDRON

THM. Let $\mathrm{X}_\circ$ be the set of all internal red diagonals.
Define the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\mathrm{T}_\bullet) := \sum_{\delta_\circ \in \mathrm{X}_\circ} \Bigg(\sum_{\delta_\bullet \in \mathrm{T}_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet\big(\delta_\circ, \delta_\bullet \in \mathrm{T}_\bullet\big) \Bigg) \mathbf{e}_{\delta_\circ} \in \mathbb{R}^{\mathrm{X}_\circ}$$

over all blue triangulations $\mathrm{T}_\bullet$.
Then for any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection $\mathsf{Asso}(\mathrm{T}_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)

THM. The origin is the vertex barycenter of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)

CORO. For any red triangulation $\mathrm{T}_\circ$, the origin is the vertex barycenter of the $\mathrm{T}_\circ$-associahedron $\mathsf{Asso}(\mathrm{T}_\circ)$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17^+)

SECTIONS AND PROJECTIONS

Manneville-P., *Geometric realizations of the accordion complex* ('17^+)

SECTIONS AND PROJECTIONS

THM. For any red triangulation $\mathrm{T}_\circ$, the g-vector fan $\mathcal{F}^{\mathbf{g}}(\mathrm{T}_\circ)$ is the normal fan of the projection Asso$(\mathrm{T}_\circ)$ of the universal associahedron Asso$_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{T}_\circ}$.

What happens if we project on other coordinate planes?
No clue in general, but...

For a red dissection $\mathrm{D}_\circ$, define

$$\text{Asso}(\mathrm{D}_\circ) = \text{projection of Asso}_{\mathrm{un}}(n) \text{ on the coordinate plane } \mathbb{R}^{\mathrm{D}_\circ}$$

Since normal fan of projections are sections of normal fans,

$$\begin{aligned}\text{normal fan of Asso}(\mathrm{D}_\circ) &= \text{section of the normal fan of Asso}_{\mathrm{un}}(n) \text{ by the plane } \mathbb{R}^{\mathrm{D}_\circ}\\ &= \text{subfan of the normal fan of Asso}_{\mathrm{un}}(n) \text{ induced by the rays in } \mathbb{R}^{\mathrm{D}_\circ}\\ &= \text{subfan of the normal fan of Asso}(\mathrm{T}_\circ) \text{ induced by the rays in } \mathbb{R}^{\mathrm{D}_\circ}\end{aligned}$$

for a triangulation $\mathrm{T}_\circ$ containing $\mathrm{D}_\circ$

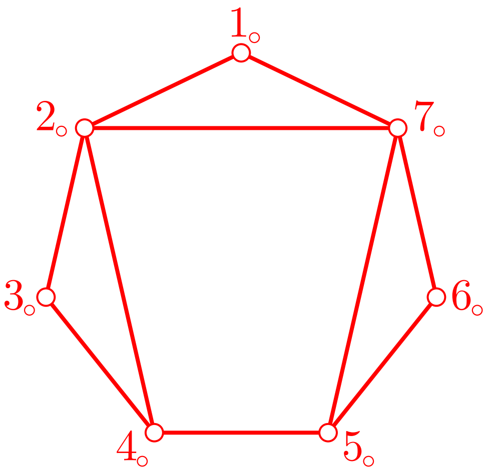
ACCORDION COMPLEX

LEM. For a red dissection $\mathrm{D}_\circ$ contained in a red triangulation $\mathrm{T}_\circ$, and a blue diagonal $\delta_\bullet$, $\mathbf{g}(\mathrm{T}_\circ, \delta_\bullet) \in \mathbb{R}^{\mathrm{D}_\circ} \iff \delta_\bullet$ never crosses a cell of $\mathrm{D}_\circ$ through two non-consecutive edges

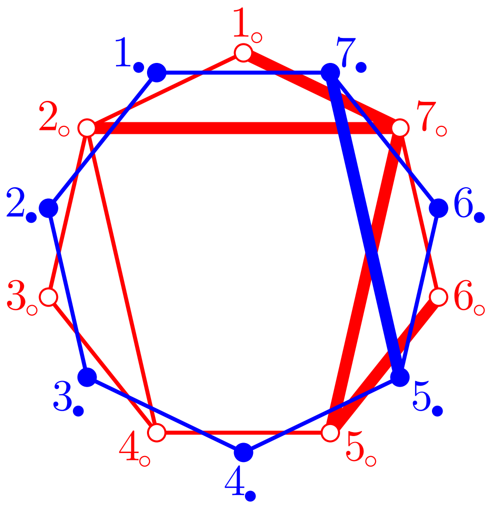
$\mathrm{D}_\circ$-accordion diagonal $=$ diagonal of the blue solid polygon that crosses an accordion of $\mathrm{D}_\circ$

$\mathrm{D}_\circ$-accordion dissection $=$ set of non-crossing $\mathrm{D}_\circ$-accordion diagonals

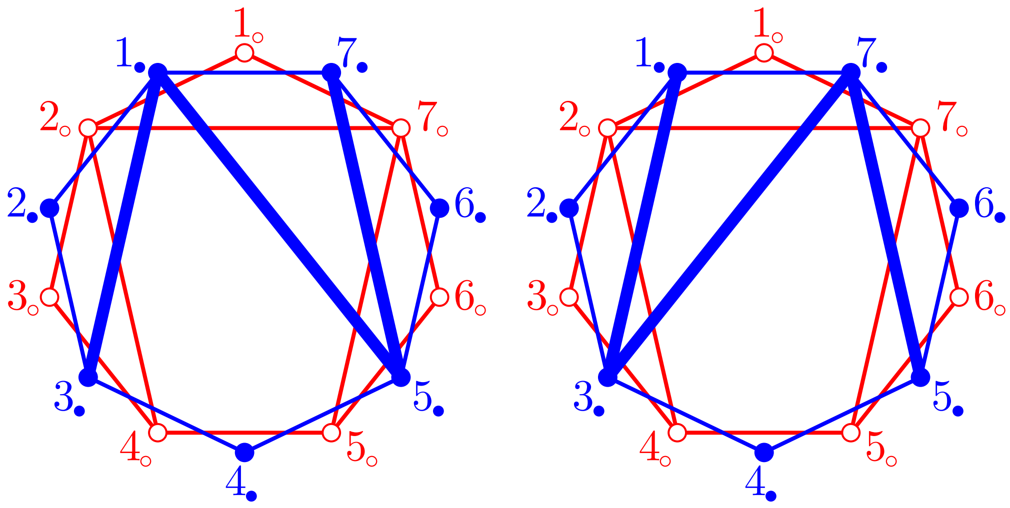
$\mathrm{D}_\circ$-accordion complex $=$ simplicial complex of $\mathrm{D}_\circ$-accordion dissections



dissection $\mathrm{D}_\circ$



$\mathrm{D}_\circ$-accordion diagonal

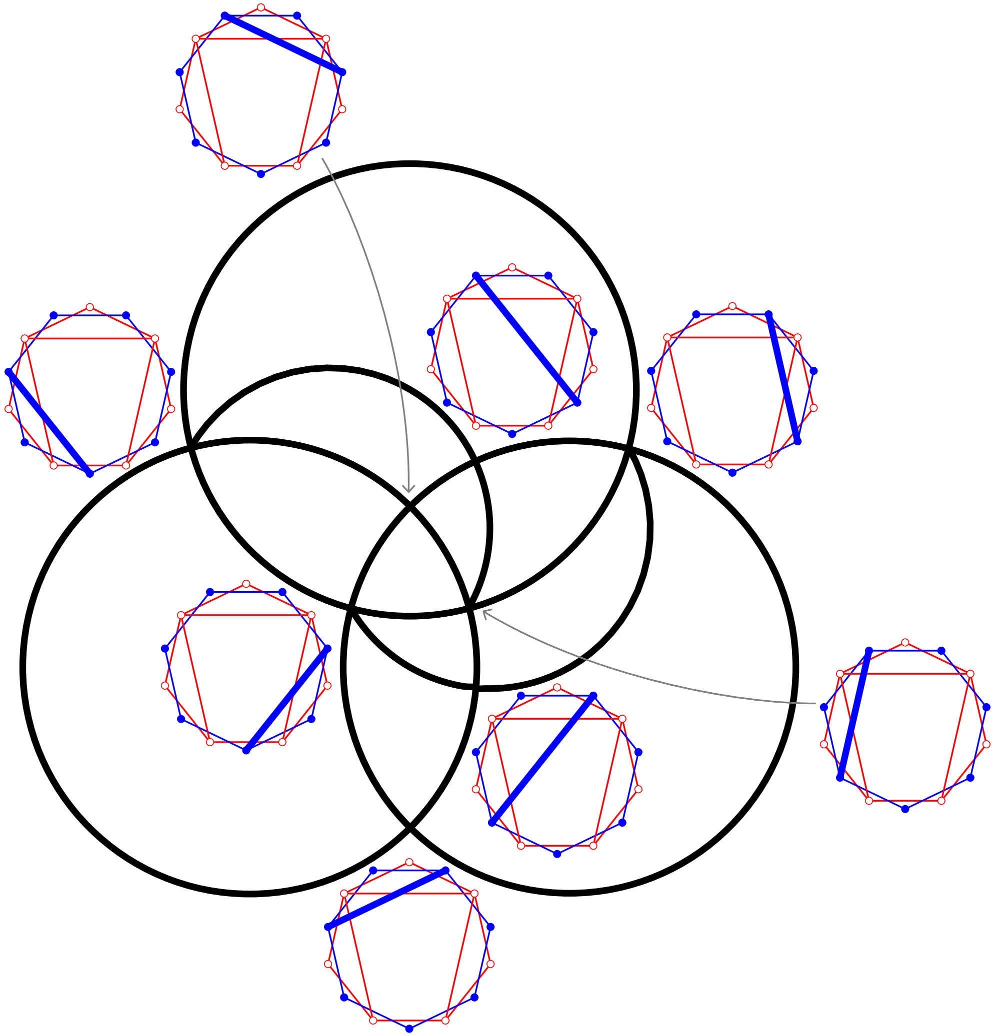


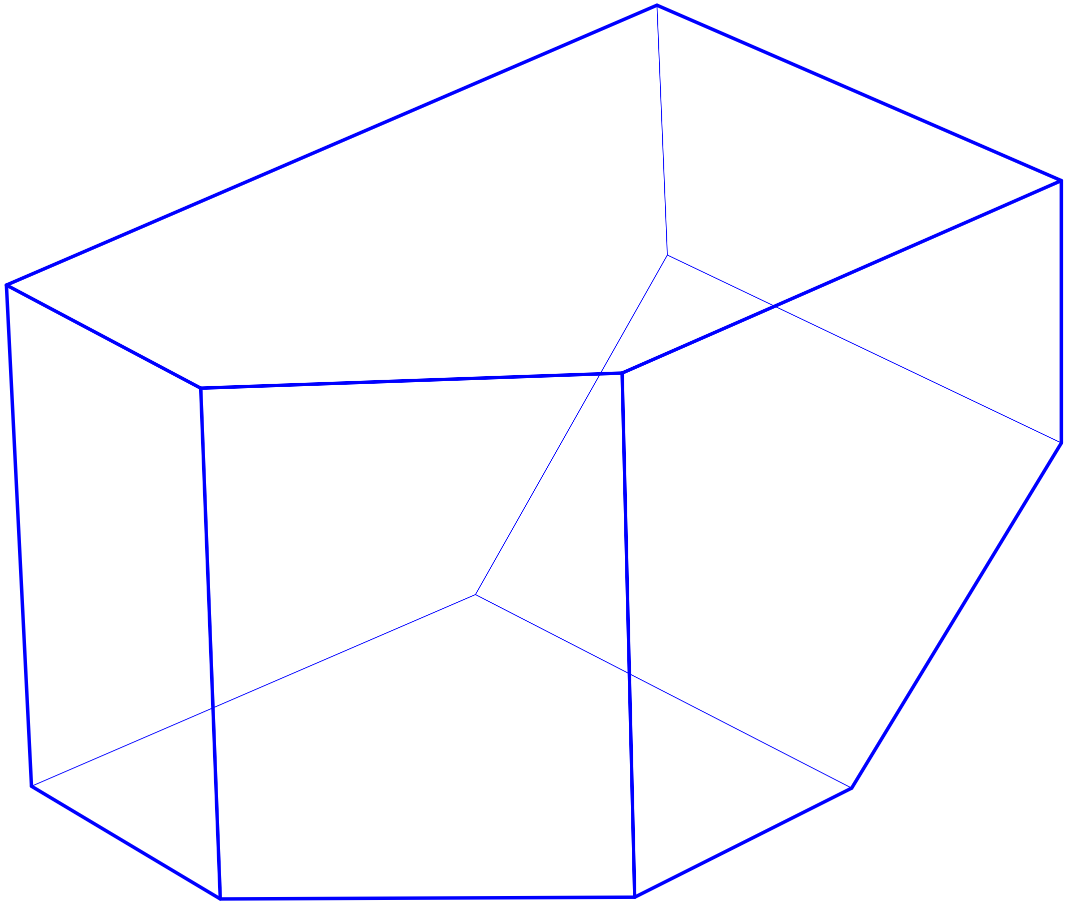
two maximal $\mathrm{D}_\circ$-accordion dissections

ACCORDIOHEDRON

THM. For any red dissection $\mathrm{D}_\circ$, the projection $\mathsf{Asso}(\mathrm{D}_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(n)$ on the coordinate plane $\mathbb{R}^{\mathrm{D}_\circ}$ realizes the $\mathrm{D}_\circ$-accordion complex.

Manneville-P., *Geometric realizations of the accordion complex* ('17^+)

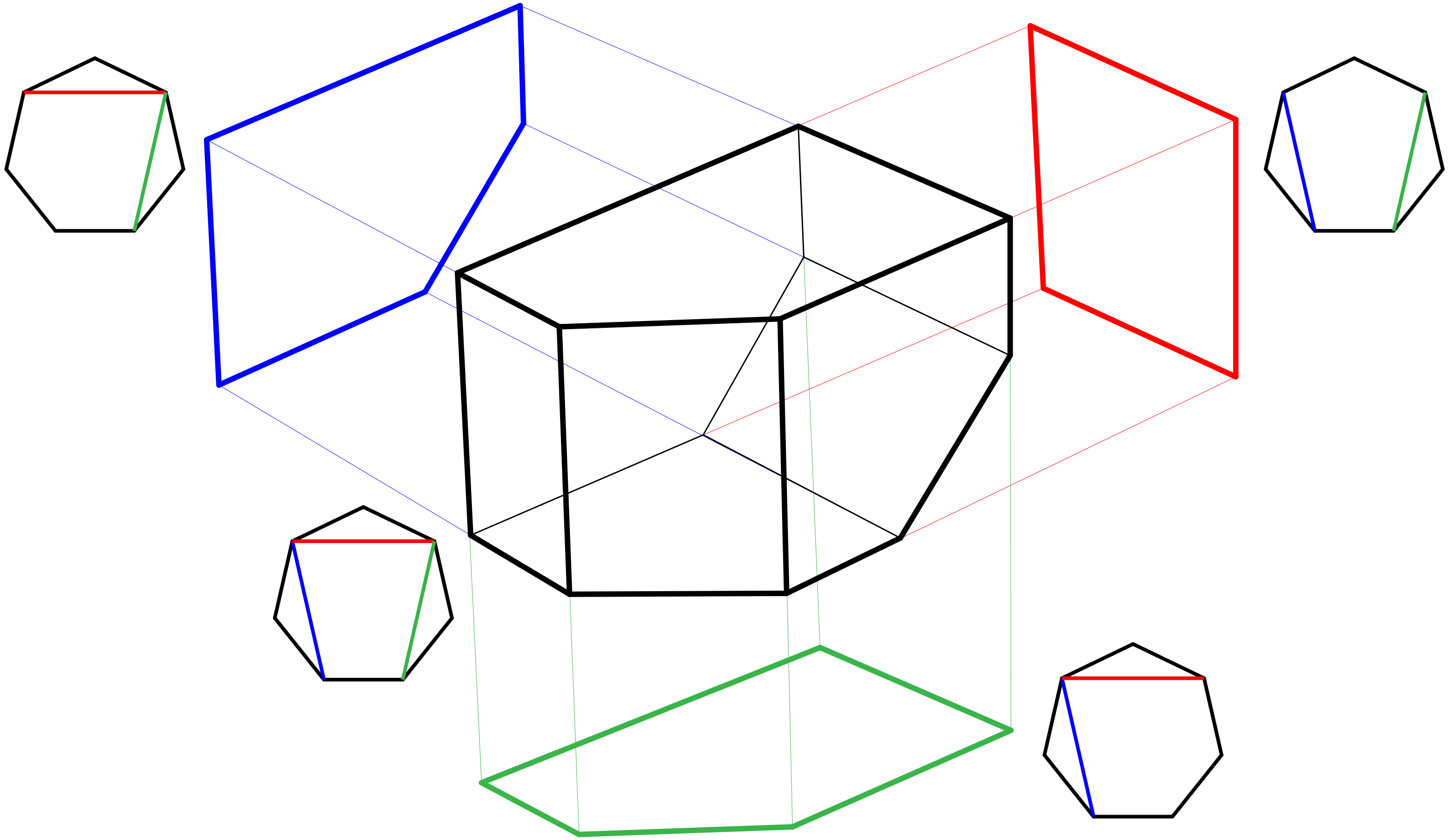




PROJECTIONS OF PROJECTIONS

PROP. If $\mathrm{D}_\circ \subseteq \mathrm{D}'_\circ$, then

- $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ is the section of $\mathcal{F}^{\mathbf{g}}(\mathrm{D}'_\circ)$ with the coordinate plane $\langle \mathbf{e}_{\delta_\circ} \mid \delta_\circ \in \mathrm{D}_\circ \rangle$,
- therefore, $\mathcal{F}^{\mathbf{g}}(\mathrm{D}_\circ)$ is also realized by the projection of $\mathrm{Asso}(\mathrm{D}_\circ)$ on $\langle \mathbf{e}_{\delta_\circ} \mid \delta_\circ \in \mathrm{D}_\circ \rangle$.



EXTENSIONS TO CLUSTER ALGEBRAS

Fomin-Zelevinsky, *Cluster Algebras I, II, III, IV* ('02 – '07)

CLUSTER ALGEBRAS

cluster algebra $=$ commutative ring generated by distinguished cluster variables grouped into overlapping clusters

clusters computed by a mutation process :

cluster seed $=$ algebraic data $\{x_1, \dots, x_n\}$, combinatorical data B (matrix or quiver)
cluster mutation $= \big(\{x_1, \dots, x_k, \dots, x_n\}, B\big) \xleftrightarrow{\mu_k} \big(\{x_1, \dots, x'_k, \dots, x_n\}, \mu_k(B)\big)$

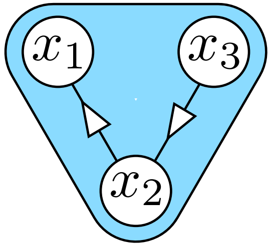
$$x_k \cdot x'_k = \prod_{\{i \mid b_{ik}>0\}} x_i^{b_{ik}} + \prod_{\{i \mid b_{ik}<0\}} x_i^{-b_{ik}}$$

$$\big(\mu_k(B)\big)_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + |b_{ik}| \cdot b_{kj} & \text{if } k \notin \{i, j\} \text{ and } b_{ik} \cdot b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

cluster complex $=$ simplicial complex w/ vertices $=$ cluster variables & facets $=$ clusters

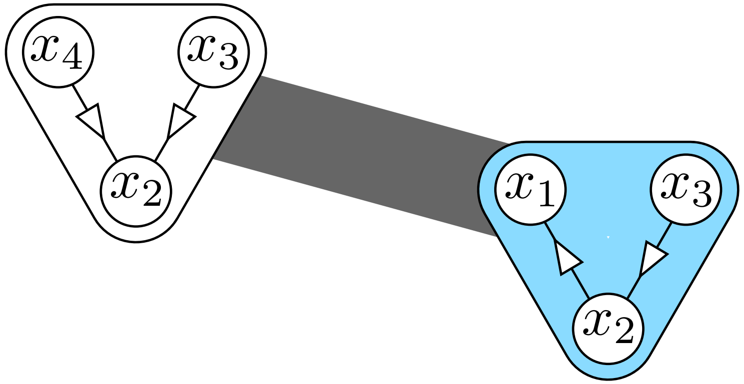
Fomin-Zelevinsky, *Cluster Algebras I, II, III, IV* ('02 – '07)

CLUSTER MUTATION



CLUSTER MUTATION

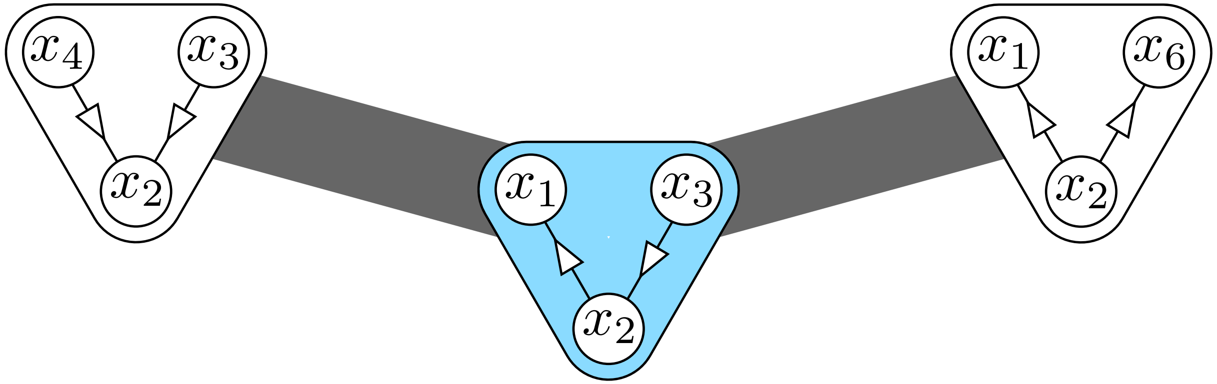
$$x_4 = \frac{1 + x_2}{x_1}$$



CLUSTER MUTATION

$$x_4 = \frac{1 + x_2}{x_1}$$

$$x_6 = \frac{1 + x_2}{x_3}$$



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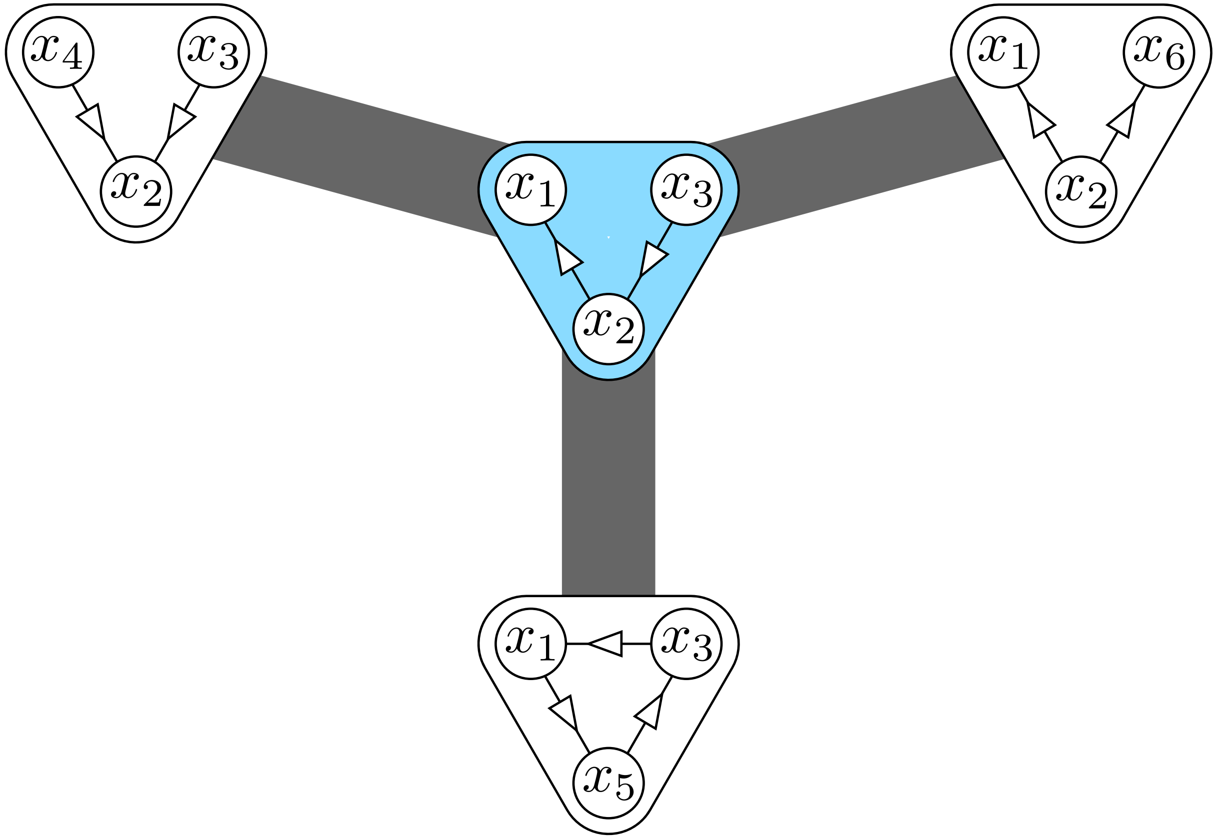
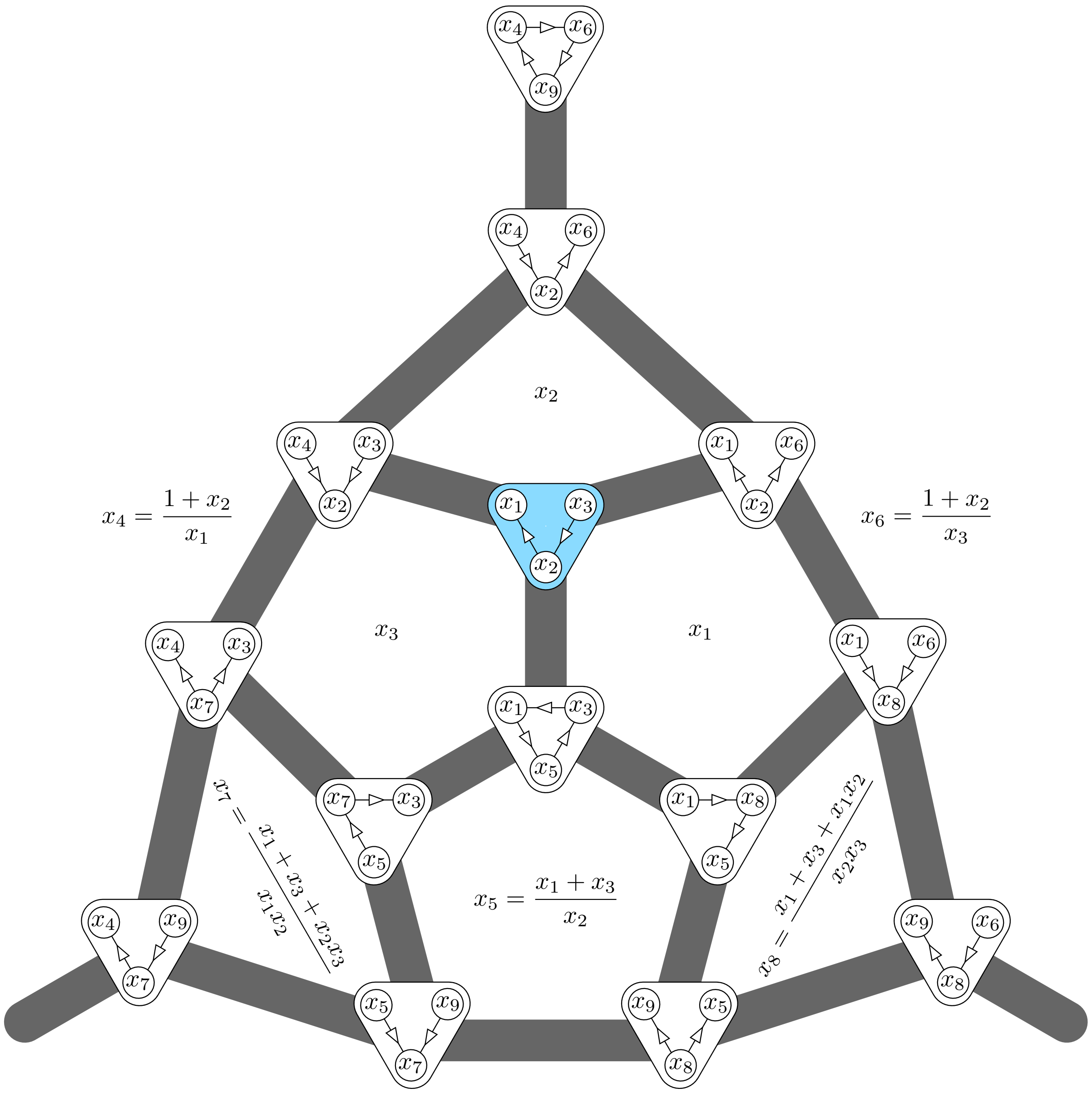


$$x_5 = \frac{x_1 + x_3}{x_2}$$

CLUSTER MUTATION GRAPH



$$x_4 = \frac{1+x_2}{x_1}$$

$$x_6 = \frac{1+x_2}{x_3}$$

$$x_7 = \frac{x_1+x_3+x_2x_3}{x_1x_2}$$

$$x_5 = \frac{x_1+x_3}{x_2}$$

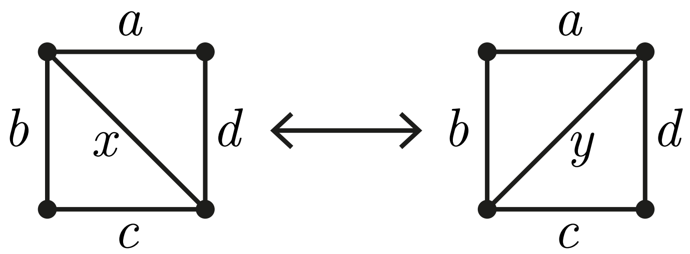
$$x_8 = \frac{x_1+x_3+x_1x_2}{x_2x_3}$$

$$x_9 = \frac{(x_1+x_3)(1+x_2)}{x_1x_2x_3}$$

CLUSTER ALGEBRA FROM TRIANGULATIONS

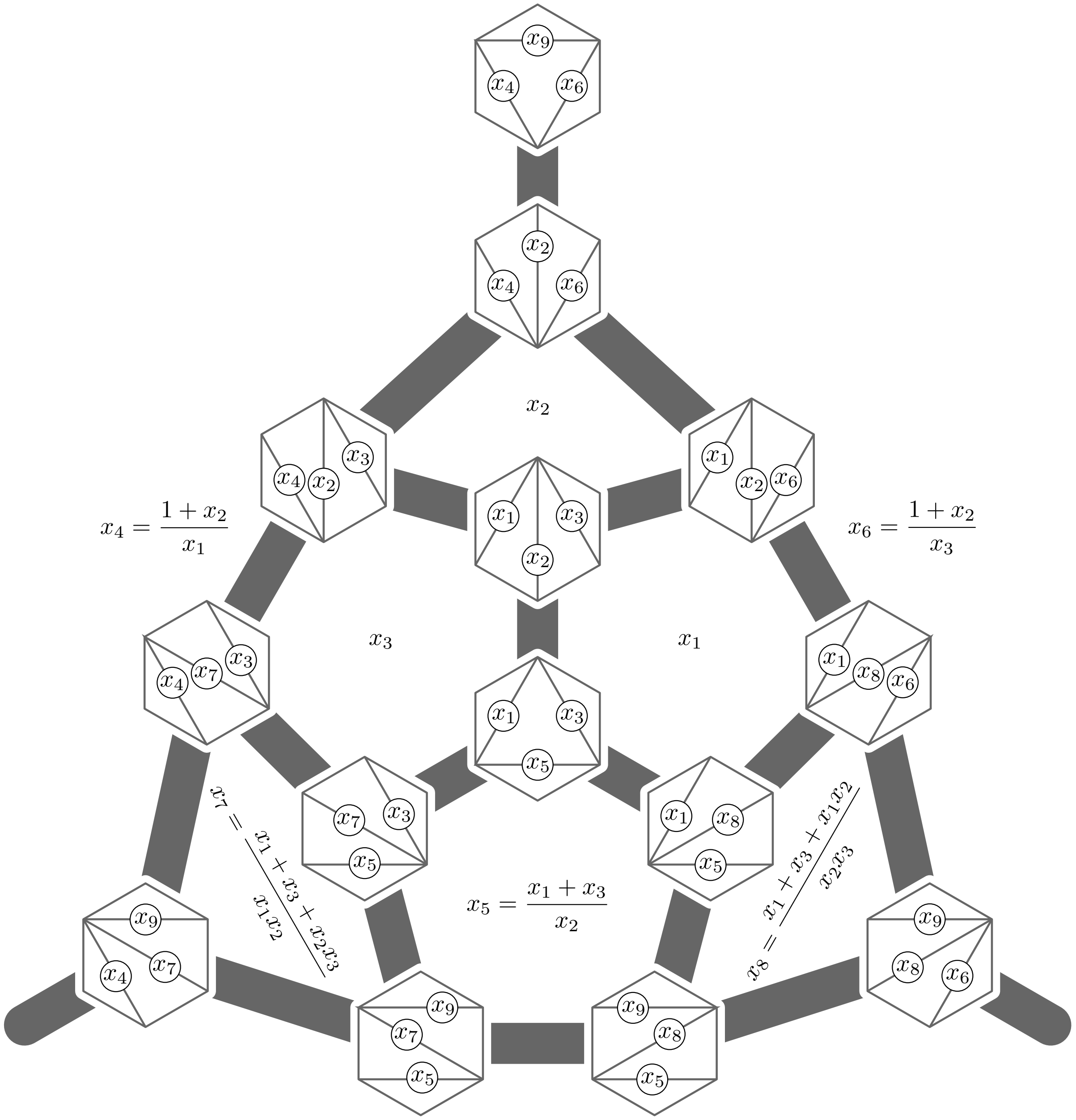
One constructs a cluster algebra from the triangulations of a polygon:

diagonals	$\longleftrightarrow$	cluster variables
triangulations	$\longleftrightarrow$	clusters
flip	$\longleftrightarrow$	mutation



$\longleftrightarrow$ $xy = ac + bd$

CLUSTER MUTATION GRAPH



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$$x_9 = \frac{(x_1+x_3)(1+x_2)}{x_1x_2x_3}$$

CLUSTER ALGEBRAS

THM. (Laurent phenomenon)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.

Fomin-Zelevinsky, *Cluster algebras I: Fundations* ('02)

THM. (Classification)
Finite type cluster algebras are classified by the Cartan-Killing classification for finite type crystallographic root systems.

Fomin-Zelevinsky, *Cluster algebras II: Finite type classification* ('03)

for a root system Φ, and an acyclic initial cluster $X = \{x_1, \dots, x_n\}$, there is a bijection

$$\begin{array}{ccc} \text{cluster variables of } \mathcal{A}_\Phi & \xleftrightarrow{\theta_X} & \Phi_{\geq -1} = \Phi^+ \cup -\Delta \\ y = \dfrac{F(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} & \xleftrightarrow{\theta_X} & \beta = d_1\alpha_1 + \cdots + d_n\alpha_n \\ \text{cluster of } \mathcal{A}_\Phi & \xleftrightarrow{\theta_X} & X\text{-cluster in } \Phi_{\geq -1} \\ \text{cluster complex of } \mathcal{A}_\Phi & \xleftrightarrow{\theta_X} & X\text{-cluster complex in } \Phi_{\geq -1} \end{array}$$

see a short introduction to finite Coxeter groups

COXETER UNIVERSAL ASSOCIAHEDRON

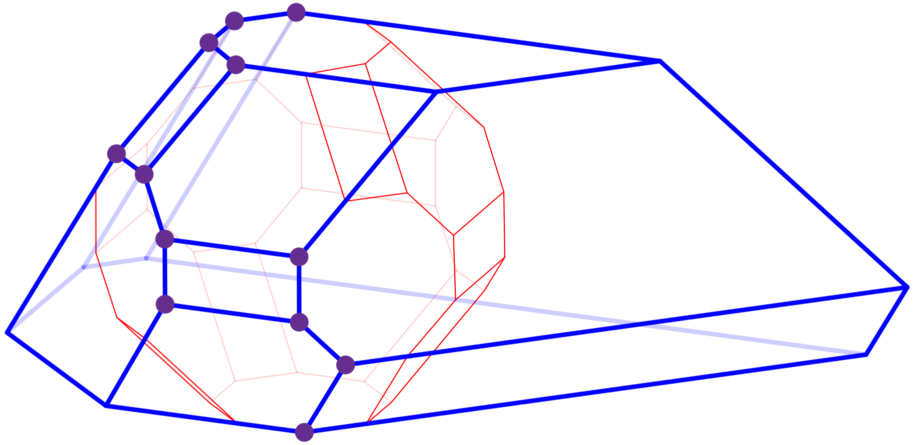
$\mathbf{g}$- and $\mathbf{c}$-vectors of cluster variables are defined using principal coefficients
universal $\mathbf{c}$-vectors are defined using universal coefficients

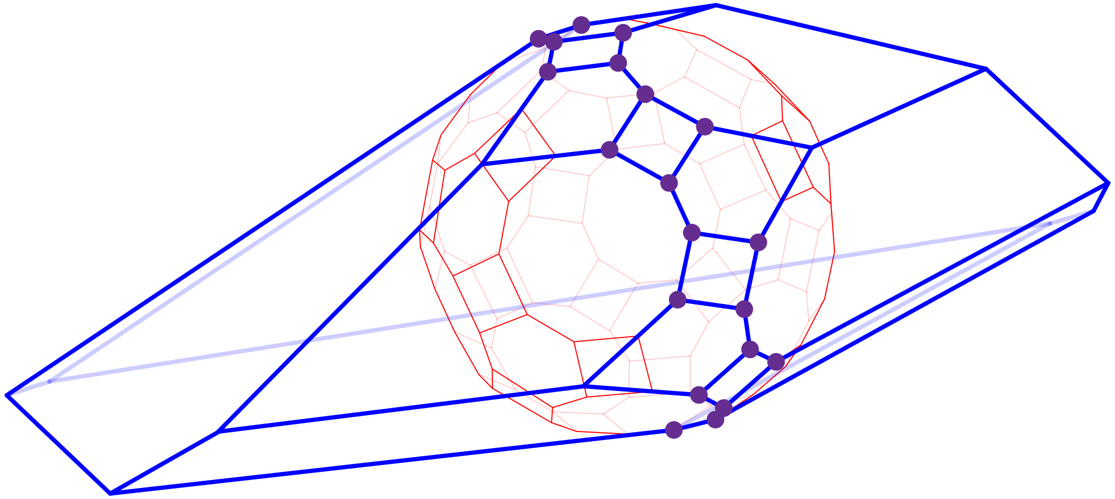
THM. Γ finite type Dynkin diagram and h : cluster vars $\to \mathbb{R}$ exchange submodular.
Define the universal Γ-associahedron $\mathsf{Asso}_{\mathrm{un}}(\Gamma)$ as the convex hull of the points

$$\mathbf{p}_{\mathrm{un}}(\Sigma) := \sum_{x \in \Sigma} h(x) \cdot \mathbf{c}_{\mathrm{un}}(x \in \Sigma)$$

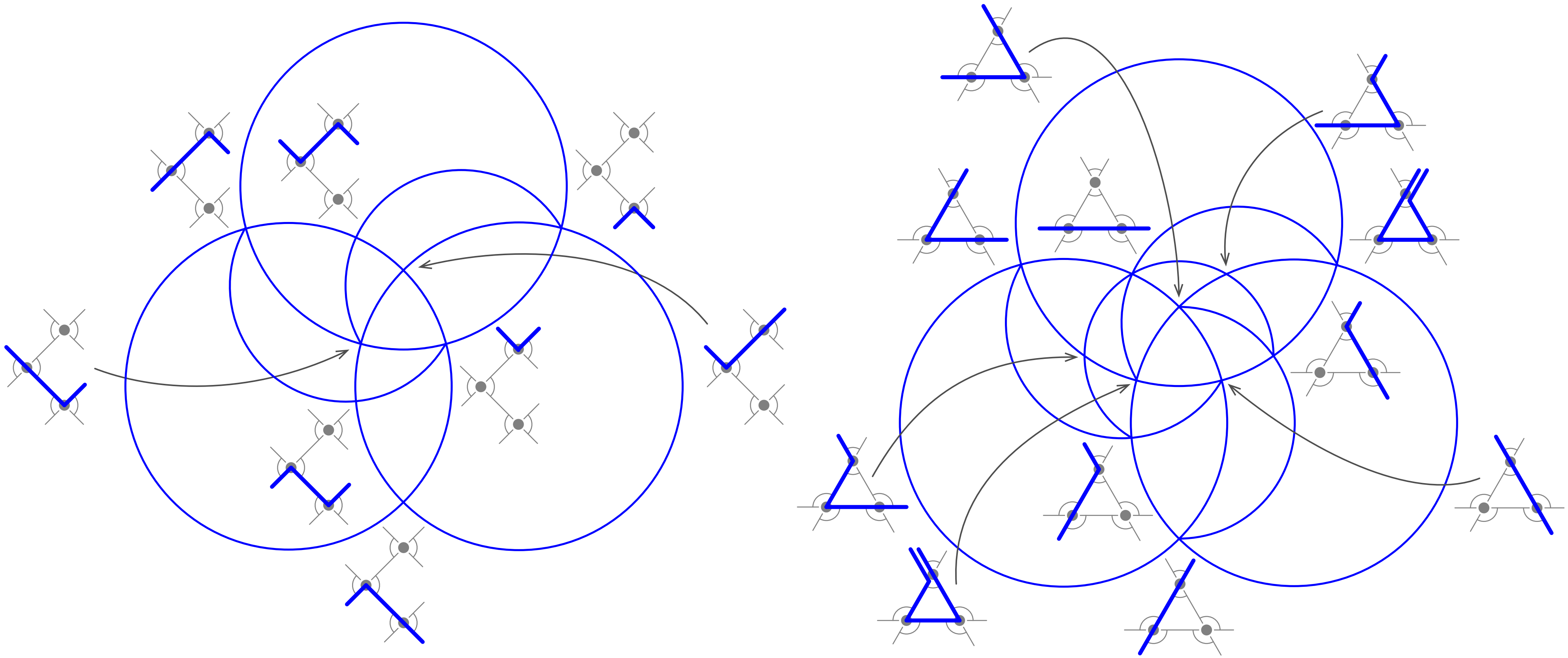
for all seeds Σ in the cluster algebra of type Γ.
Then for any initial seed $\Sigma_\circ$, the $\mathbf{g}$-vetor fan $\mathcal{F}^{\mathbf{g}}(\Sigma_\circ)$ is the normal fan of the projection $\mathsf{Asso}(\Sigma_\circ)$ of the universal associahedron $\mathsf{Asso}_{\mathrm{un}}(\Gamma)$ on the coordinate plane $\mathbb{R}^\Gamma$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17$^+$)





IV. NON-KISSING COMPLEXES AND GENTLE ASSOCIAHEDRA

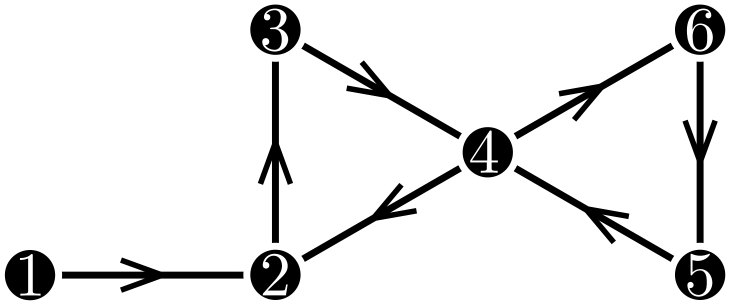


Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17$^+$)

NON-KISSING COMPLEX

Petersen-Pylyavskyy-Speyer, *A non-crossing standard monomial theory* ('10)
Santos-Stump-Welker, *Non-crossing sets and the Grassmann-assoc.* ('17)
McConville, *Lattice structures of grid Tamari orders* ('17)
Garver-McConville, *Enumerative properties of grid-associahedra* ('17^+)
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17^+)

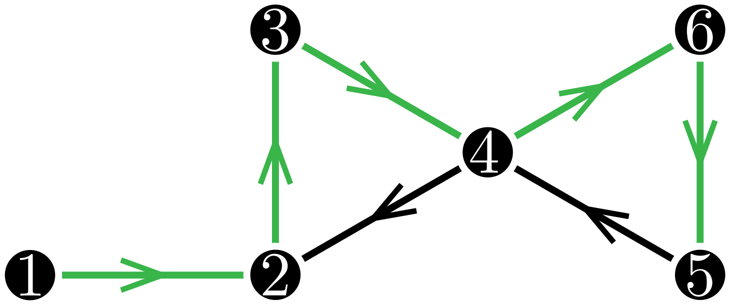
QUIVERS



quiver = oriented graph
(loops and multiple edges allowed)

$Q = (Q_0, Q_1, s, t)$
$Q_0 =$ vertices
$Q_1 =$ edges
$s, t : Q_1 \to Q_0$ source and target maps

QUIVERS



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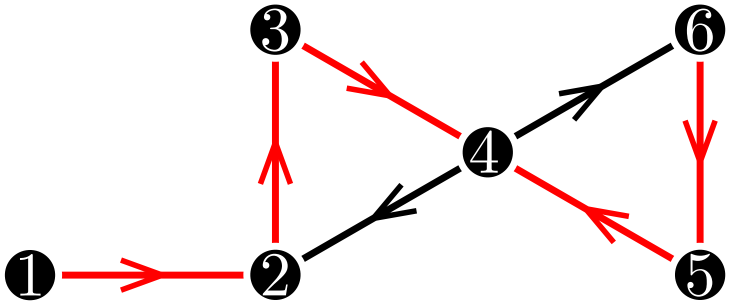
path $= \alpha_1 \dots \alpha_\ell$ with $\alpha_k \in Q_1$ and $t(\alpha_k) = s(\alpha_{k+1})$

path algebra $\mathbb{K}Q = \langle e_\pi \mid \pi$ path of $Q \rangle$ with concatenation product

$$e_{\alpha_1 \dots \alpha_\ell} \cdot e_{\beta_1 \dots \beta_k} = \begin{cases} e_{\alpha_1 \dots \alpha_\ell \beta_1 \dots \beta_k} & \text{if } t(\alpha_\ell) = s(\beta_1) \\ 0 & \text{otherwise} \end{cases}$$

QUIVERS



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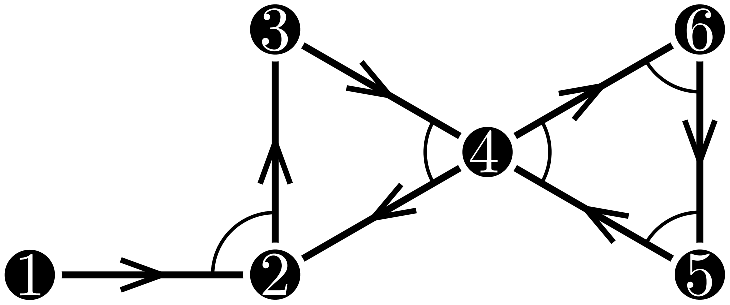

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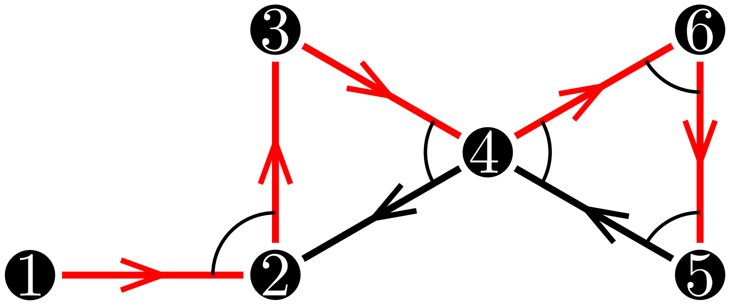
bound quiver = quiver with relations
$\bar{Q} = (Q, I)$ where I is an admissible ideal of $\mathbb{K}Q$.

Complicated way to say that we forbid certain paths

QUIVERS



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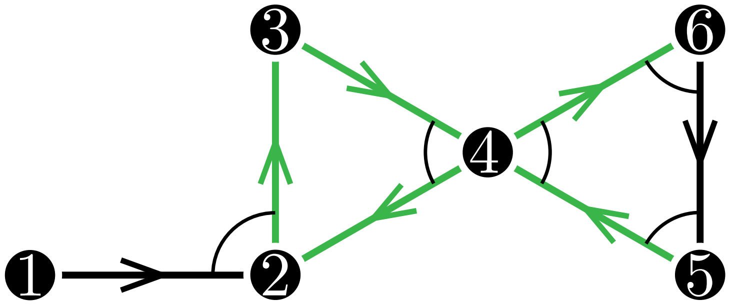

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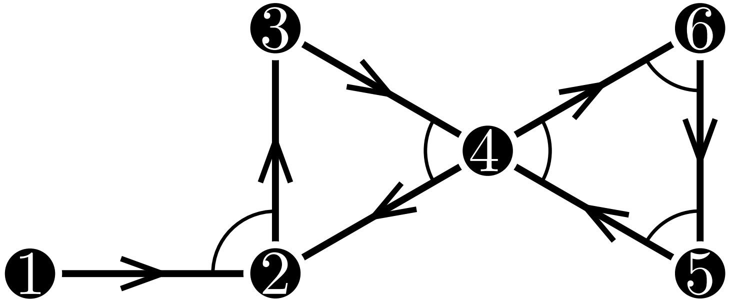
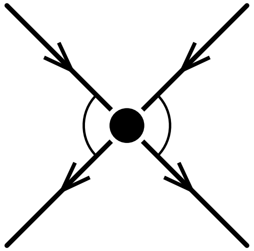


bound quiver $\bar{Q} = (Q, I)$

gentle quiver $=$

- forbidden paths all of length 2
- locally at each vertex, subgraph of



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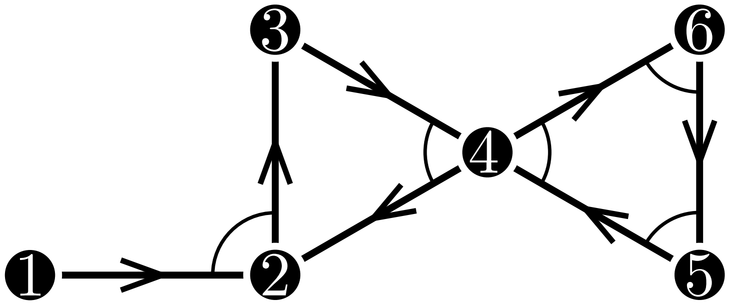

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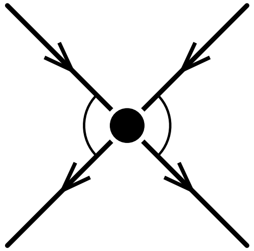
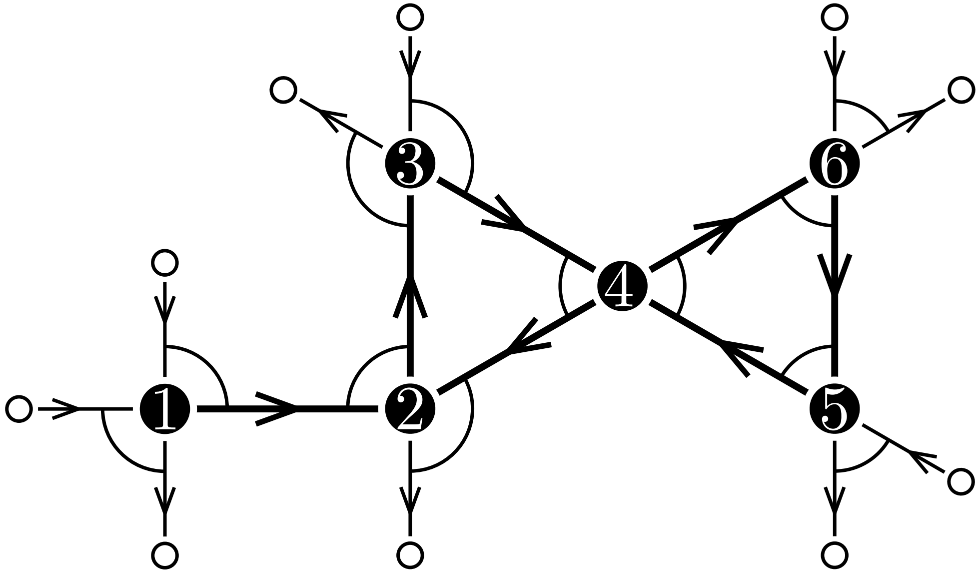
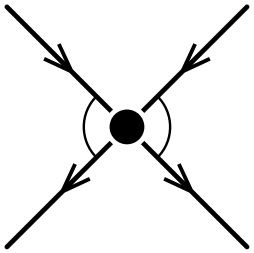
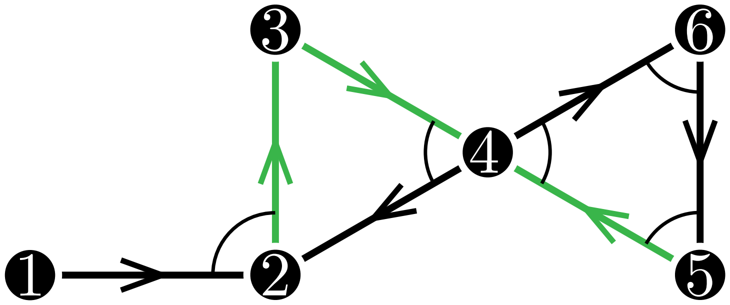




blossoming quiver $\bar{Q}^{❀}$ $=$ add blossoms to complete each vertex to



STRINGS AND WALKS



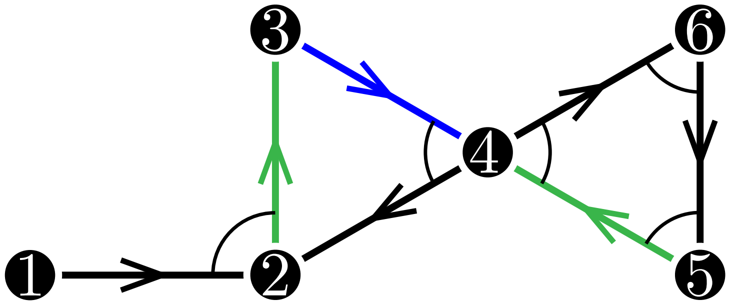
string $\sigma = \alpha_1^{\varepsilon_1} \dots \alpha_\ell^{\varepsilon_\ell}$

with $\alpha_k \in Q_1$,

$\varepsilon_k \in \{-1, 1\}$

and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$

STRINGS AND WALKS



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and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$

substrings of $\sigma = \left\{\alpha_i^{\varepsilon_i} \dots \alpha_j^{\varepsilon_j} \;\middle|\; 1 \le i \le j-1 \le k\right\}$

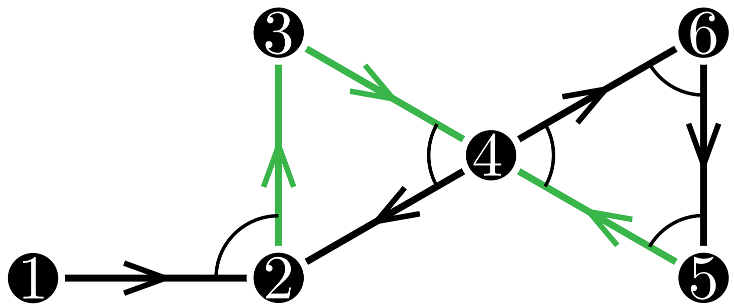
bottom substring of $\sigma =$ substring ρ of σ such that σ either ends
or has an outgoing arrow at each endpoint of ρ

$\Sigma_{\text{bot}}(\sigma) = \{$ bottom substrings of σ $\}$

top substring of $\sigma =$ substring ρ of σ such that σ either ends
or has an incoming arrow at each endpoint of ρ

$\Sigma_{\text{top}}(\sigma) = \{$ top substrings of σ $\}$

STRINGS AND WALKS



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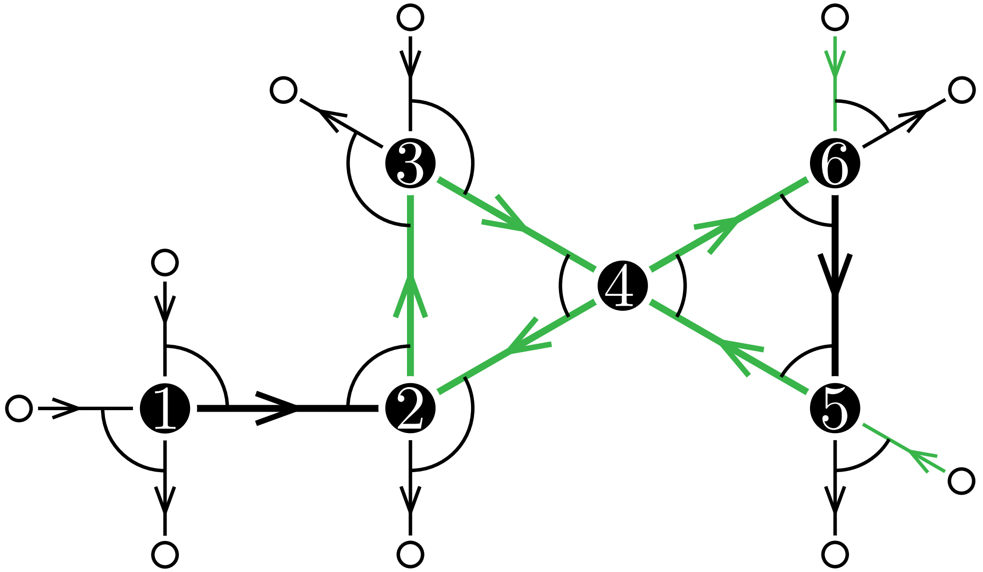
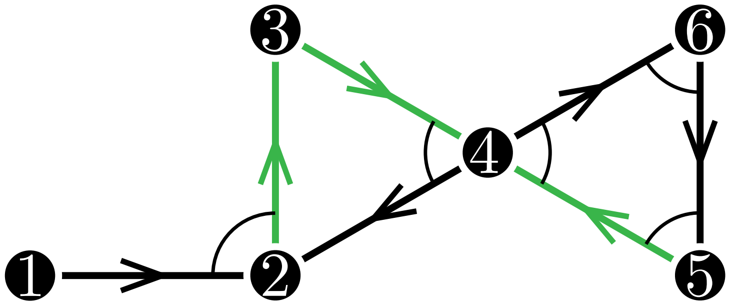
and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$



<u>walk</u> $\omega =$ maximal string in $Q^{❀}$
from blossoms to blossoms

STRINGS AND WALKS



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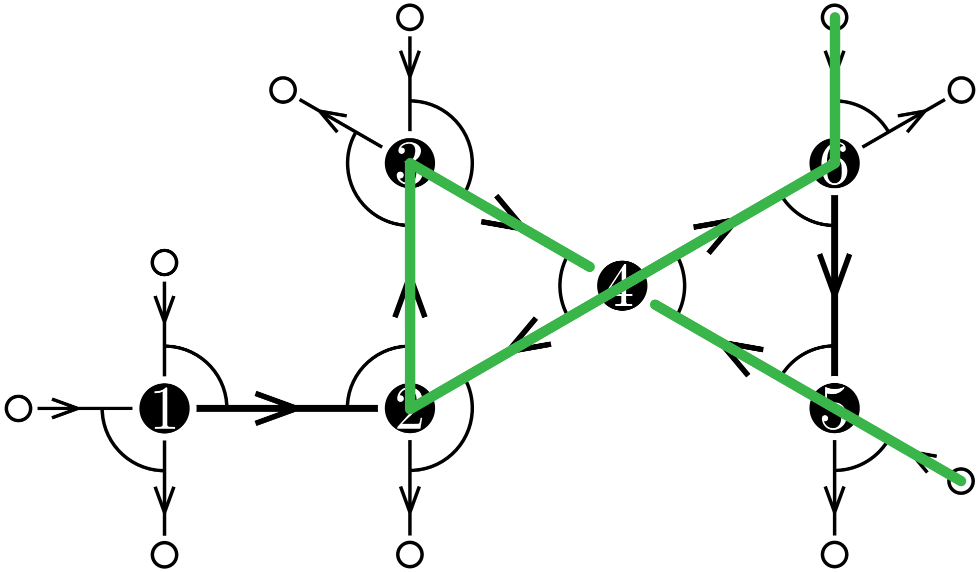

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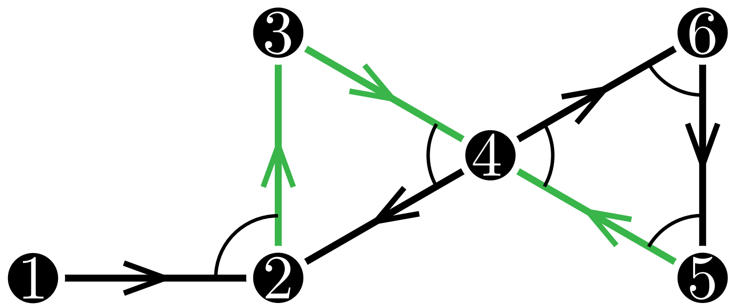
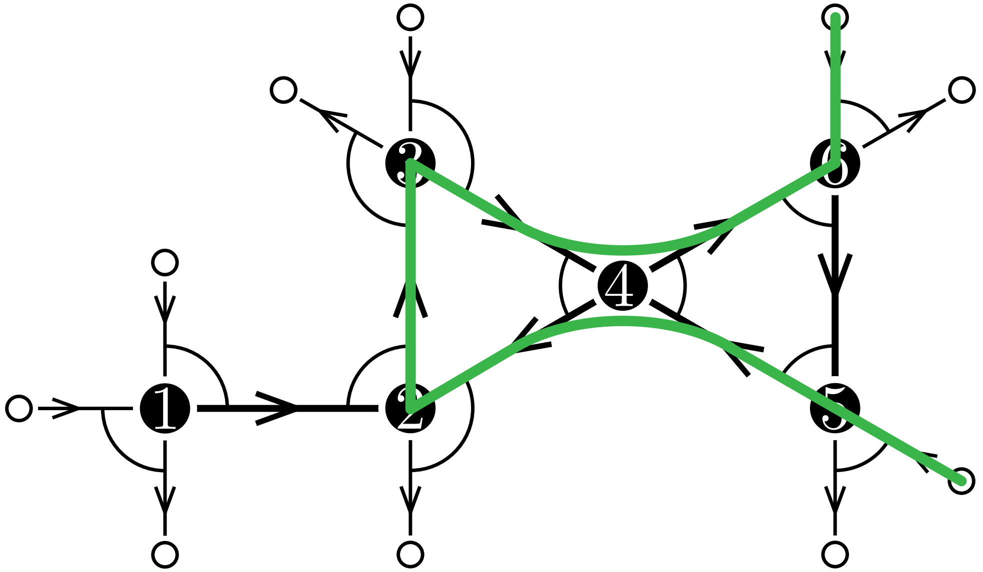
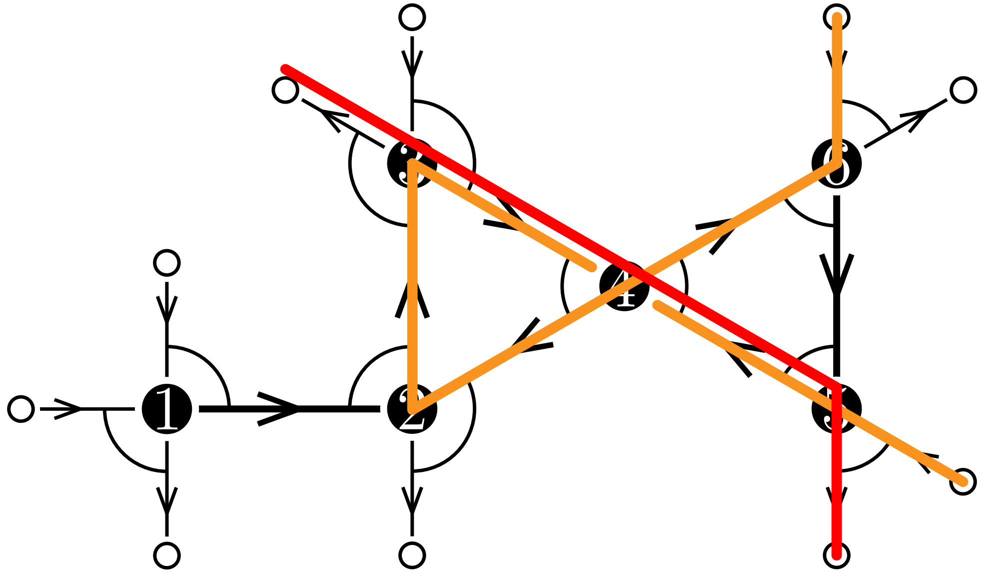


string $\sigma = \alpha_1^{\varepsilon_1} \dots \alpha_\ell^{\varepsilon_\ell}$
with $\alpha_k \in Q_1$,
$\varepsilon_k \in \{-1, 1\}$
and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$



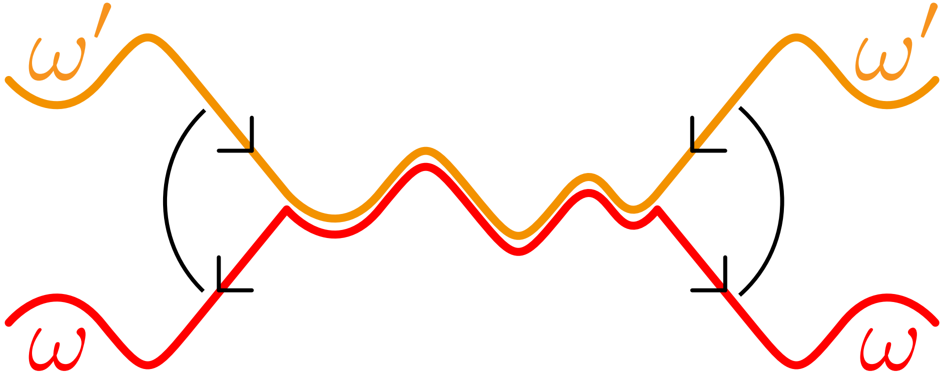
walk $\omega =$ maximal string in $Q^{❀}$
from blossoms to blossoms

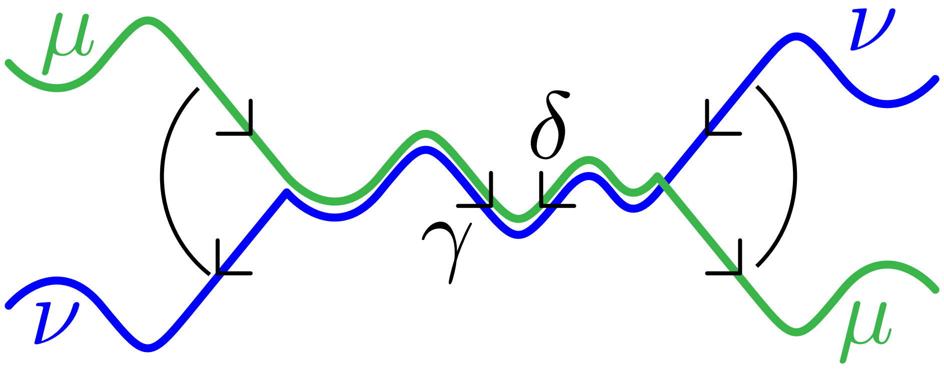
NON-KISSING COMPLEX



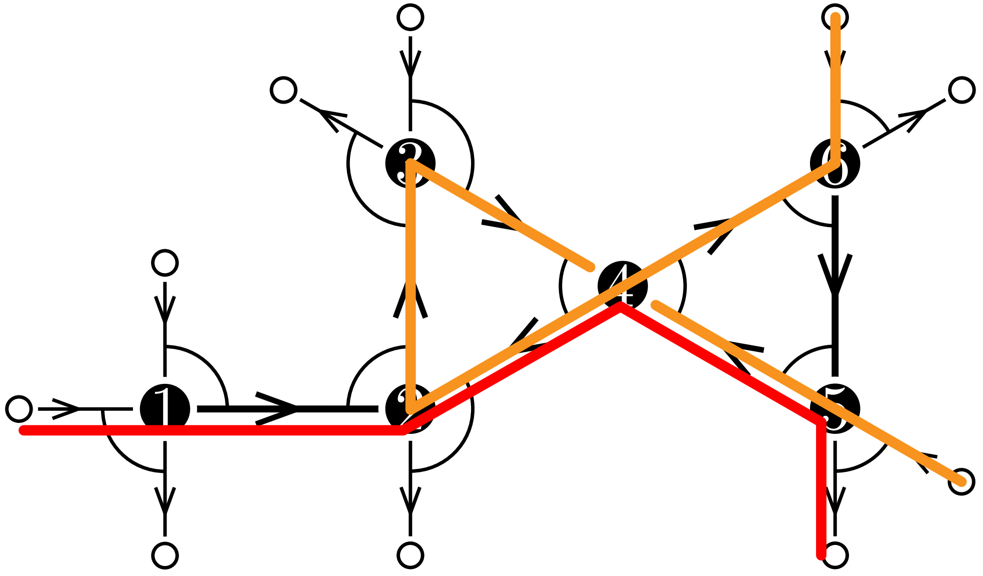
walk $\omega =$ maximal string in $Q^{\text{❀}}$
from blossoms to blossoms

ω kisses ω' if $\Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega') \neq \varnothing$





NON-KISSING COMPLEX



walk $\omega =$ maximal string in $Q^{✿}$
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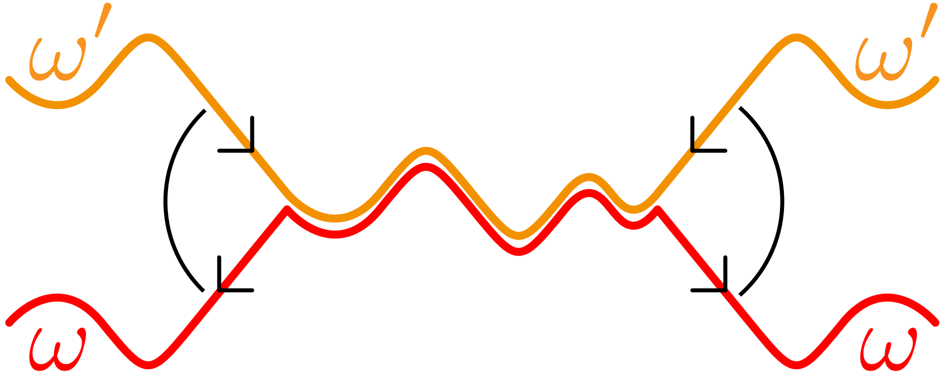
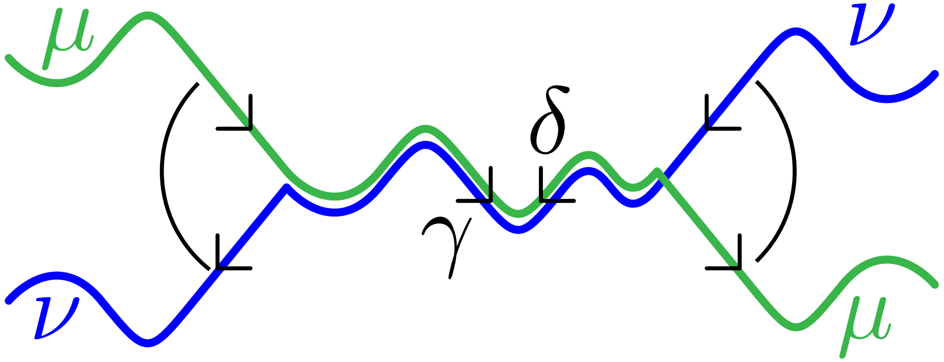
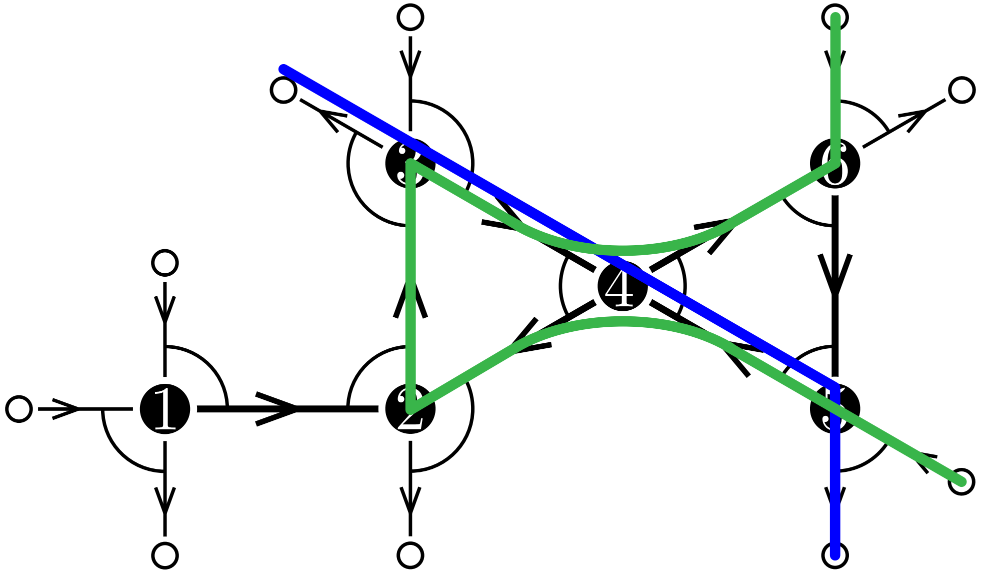




NON-KISSING COMPLEX



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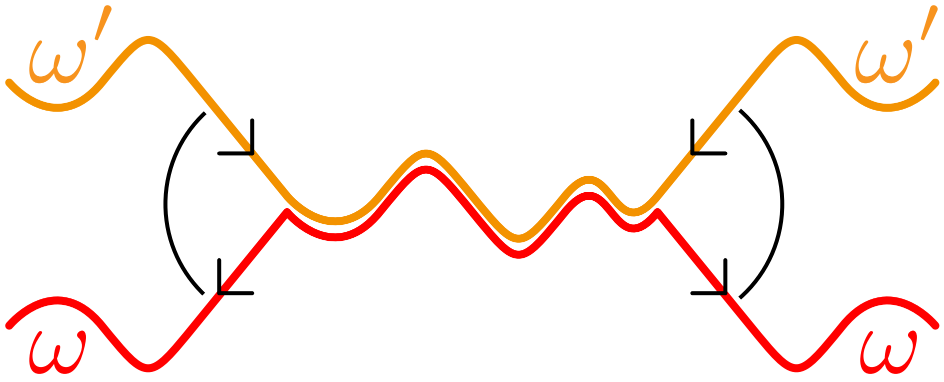

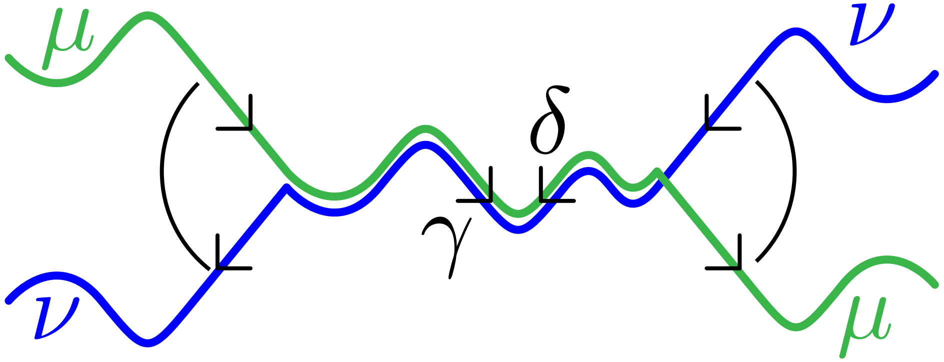

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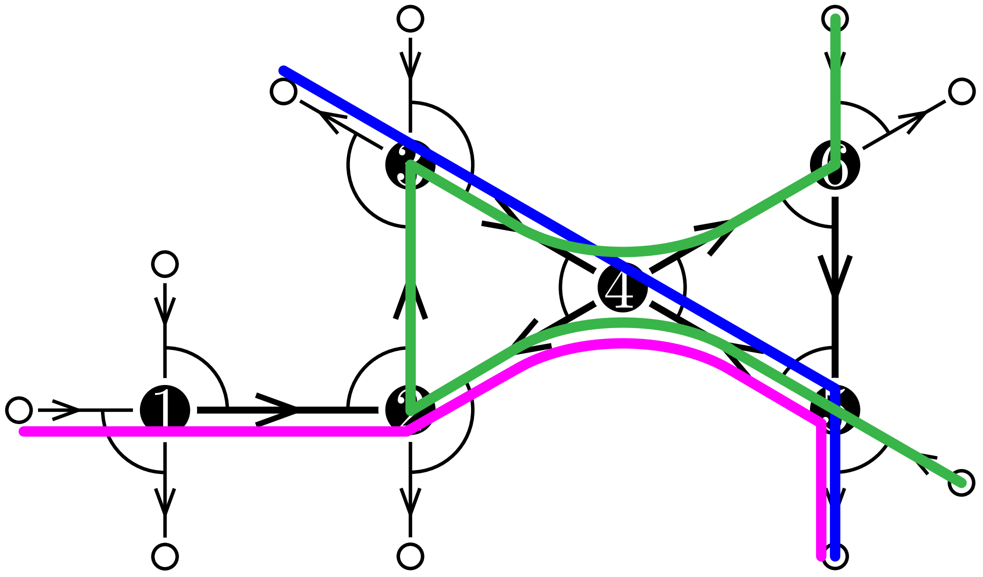

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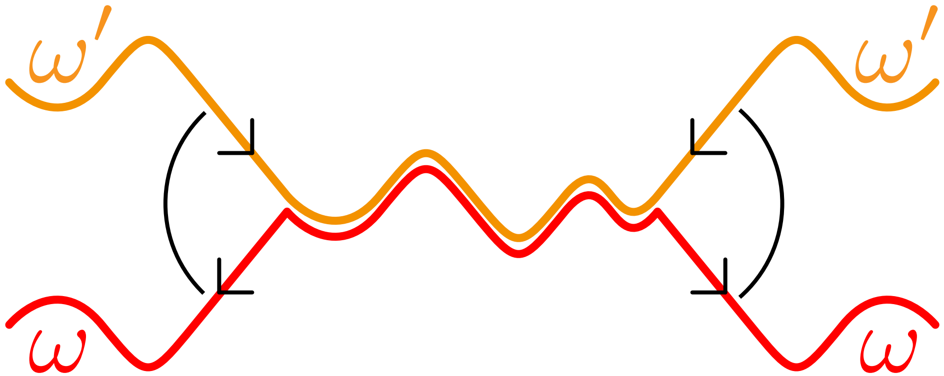

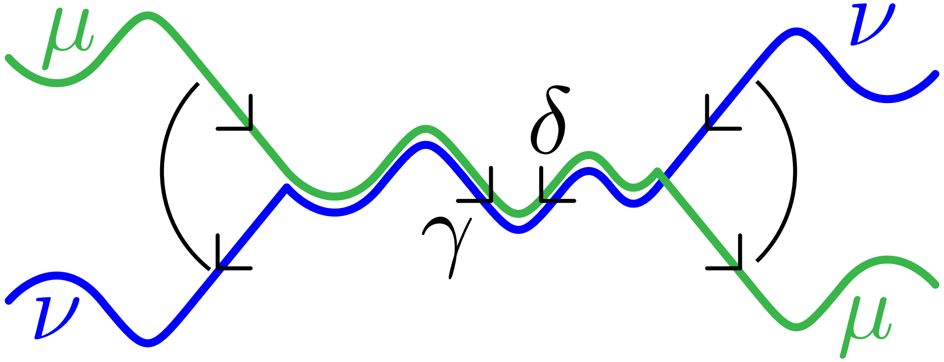

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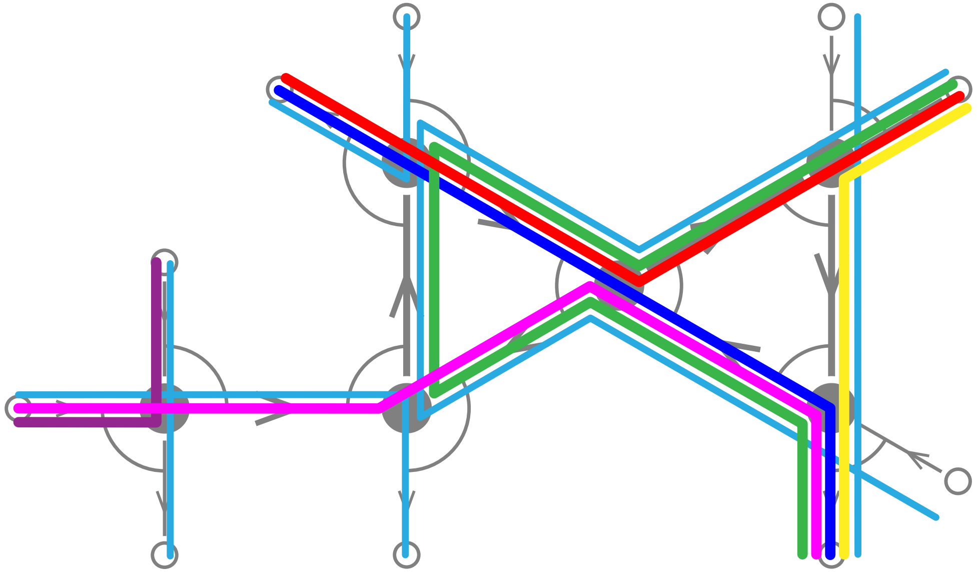

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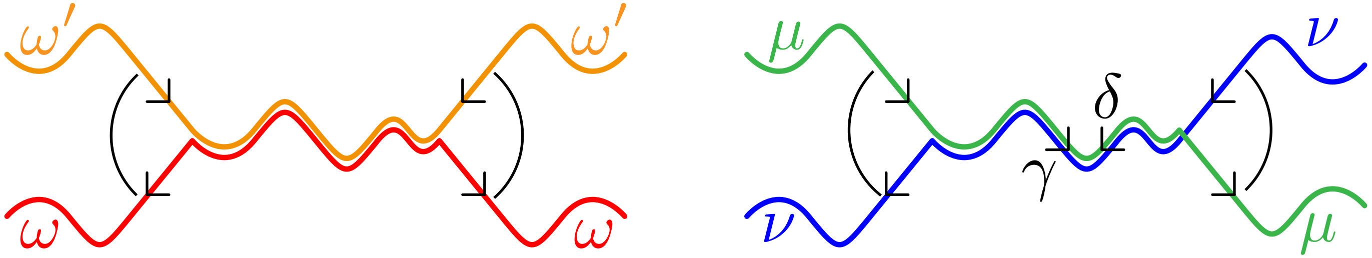
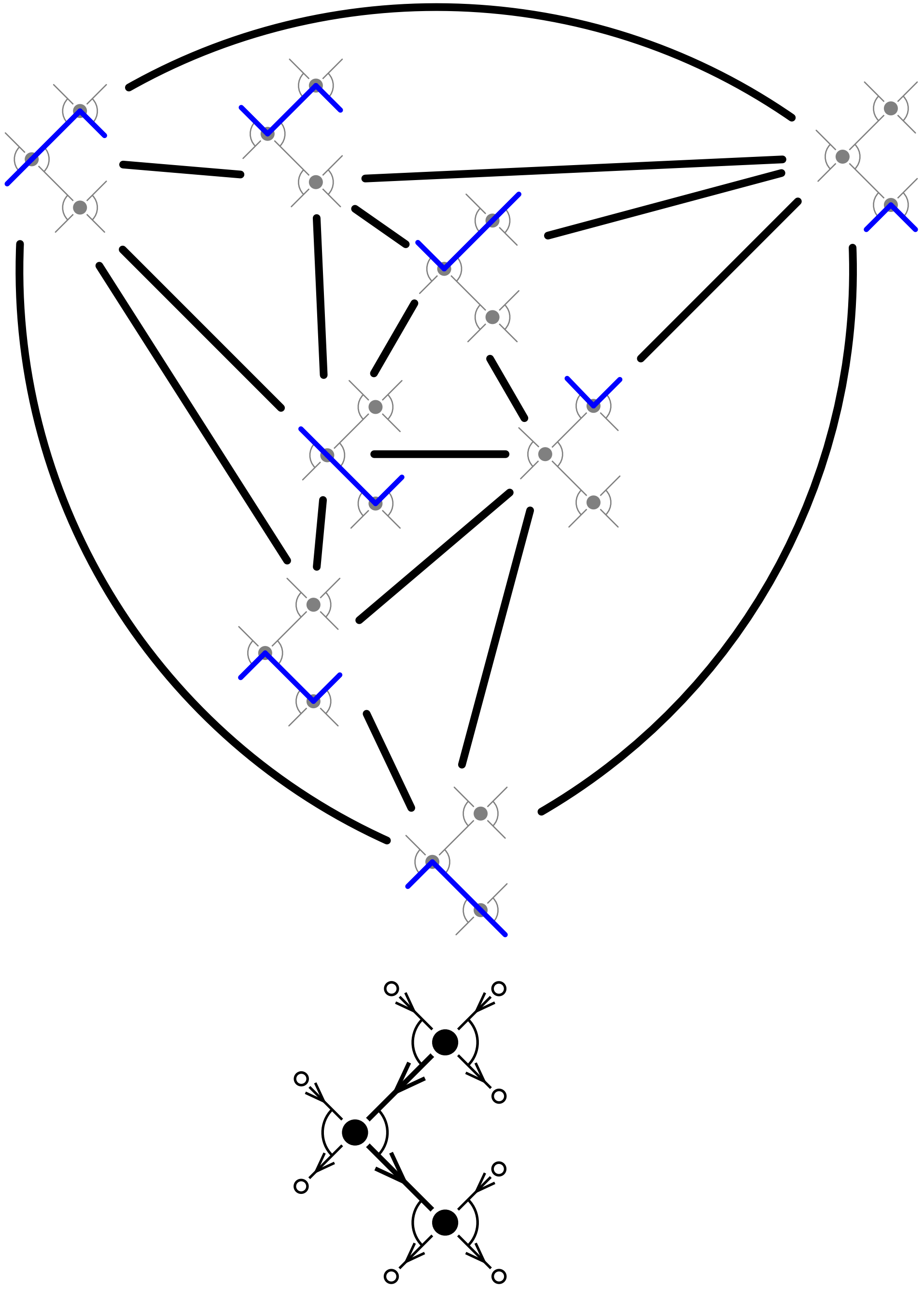
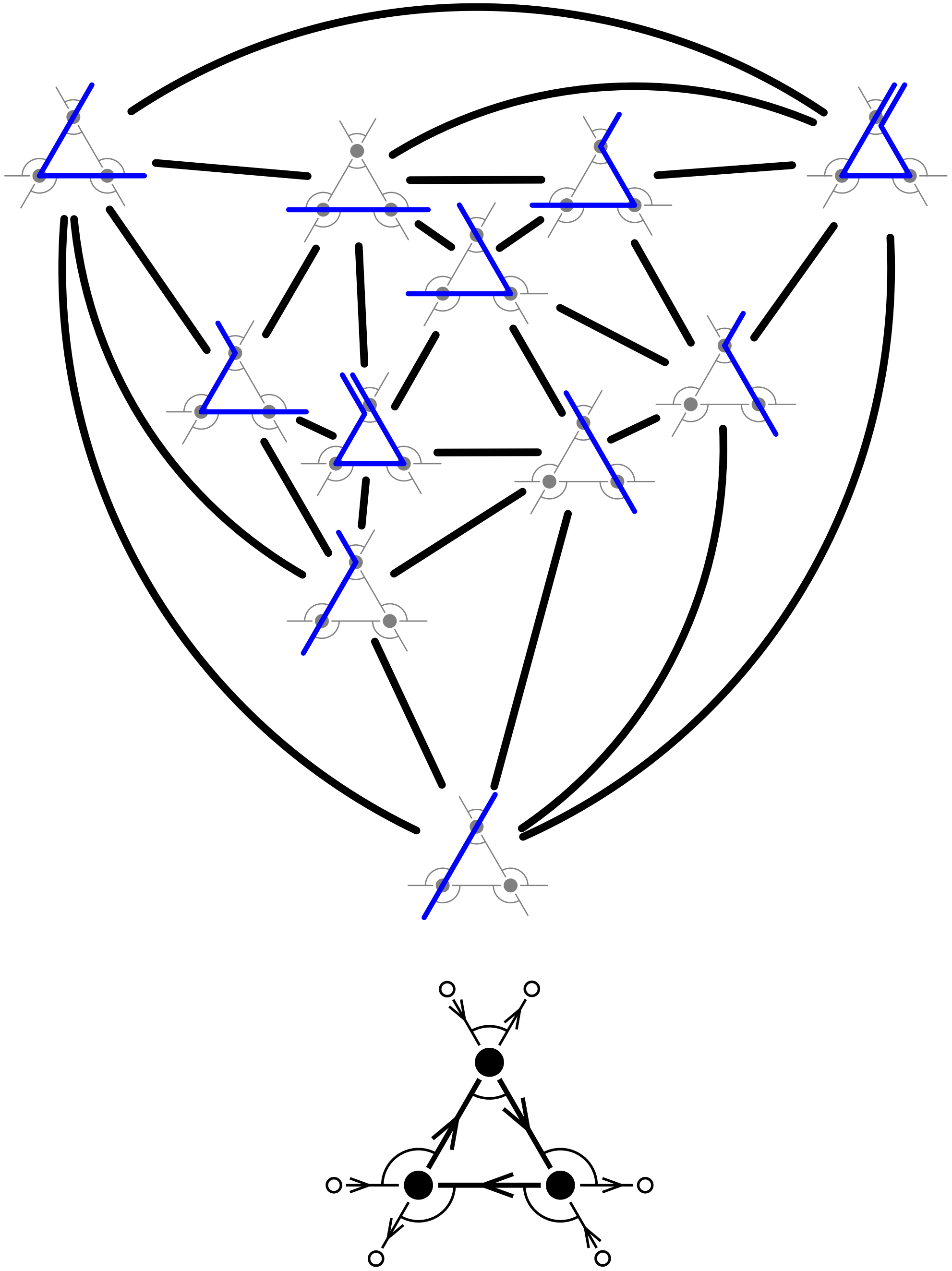


[reduced] non-kissing complex $\mathcal{K}_{\text{nk}}(\bar{Q})$ = simplicial complex with

- vertices = [bended] walks of $\bar{Q}$ (that are not self-kissing)
- faces = collections of pairwise non-kissing [bended] walks of $\bar{Q}$

REDUCED NON-KISSING COMPLEX

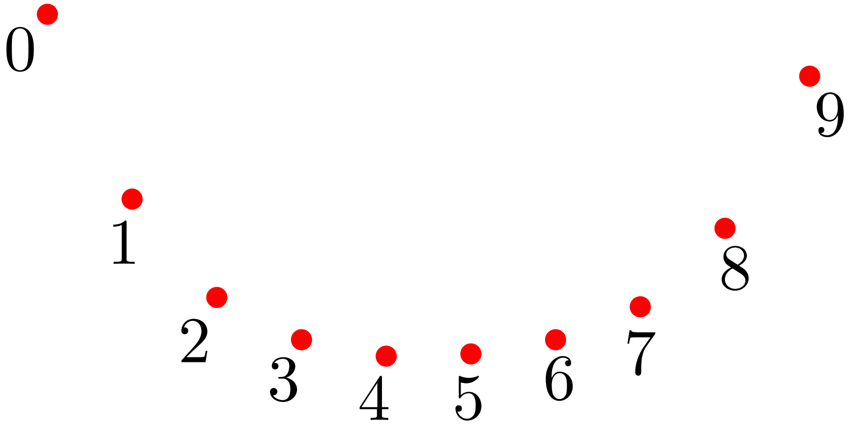


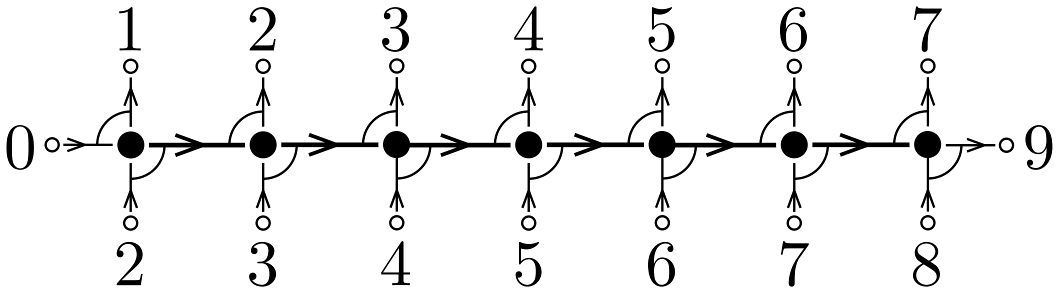


SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

[reduced] simplicial associahedron = simplicial complex with

- vertices = [internal] diagonals of an $(n+3)$-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the $(n+3)$-gon





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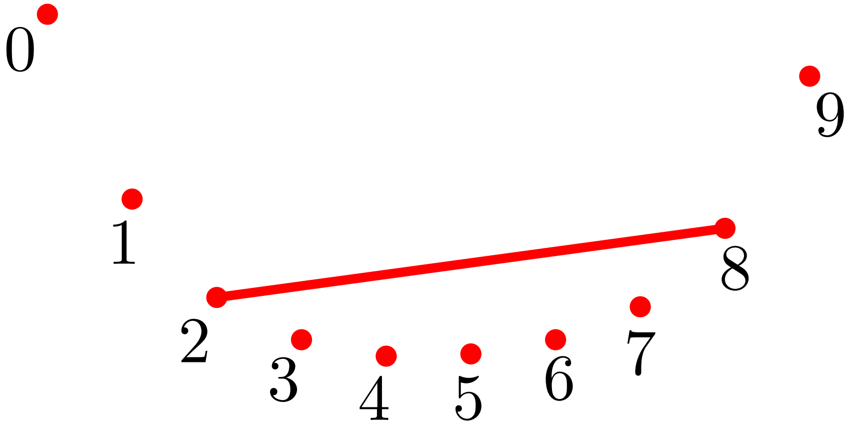

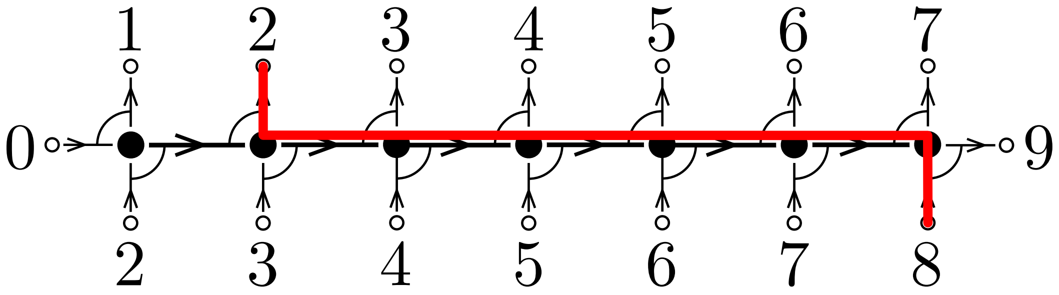

diagonal $\longleftrightarrow$ walk

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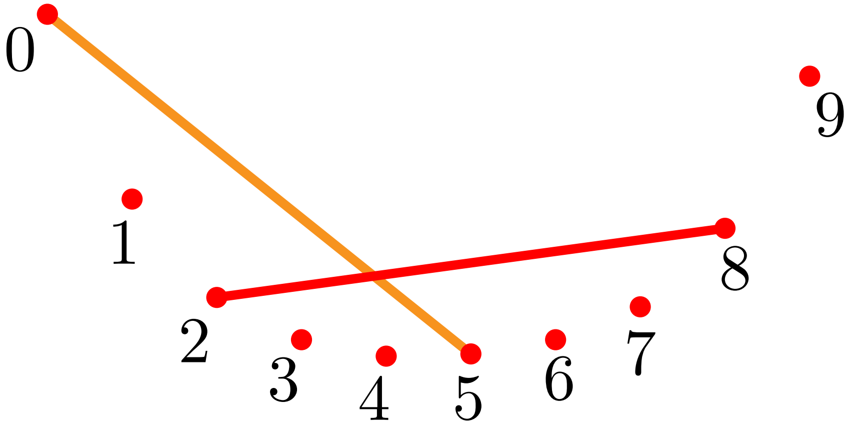

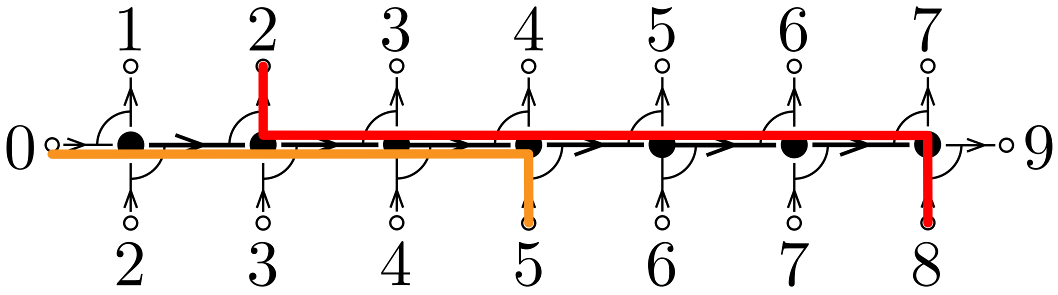

diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing

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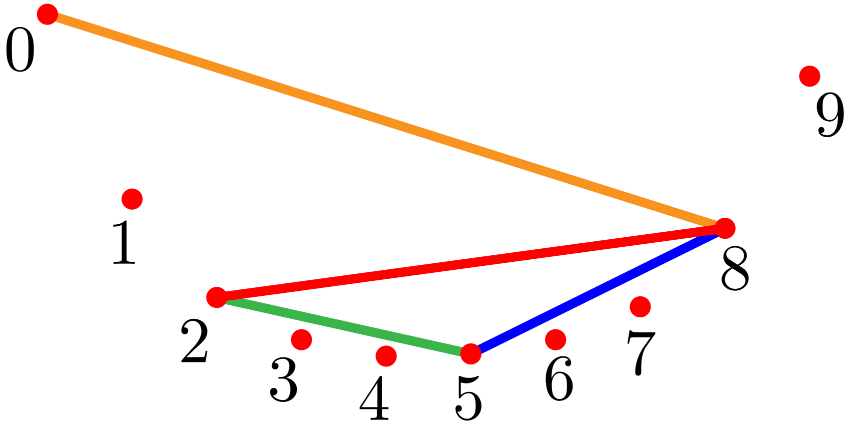

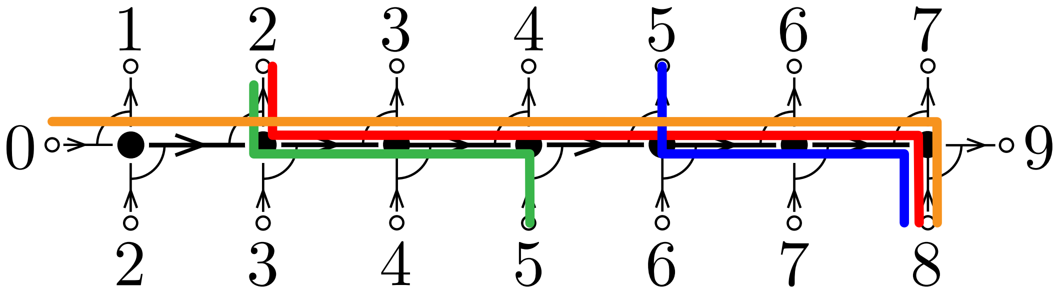

diagonal	⟷	walk
crossing	⟷	kissing
dissection	⟷	non-kissing face

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

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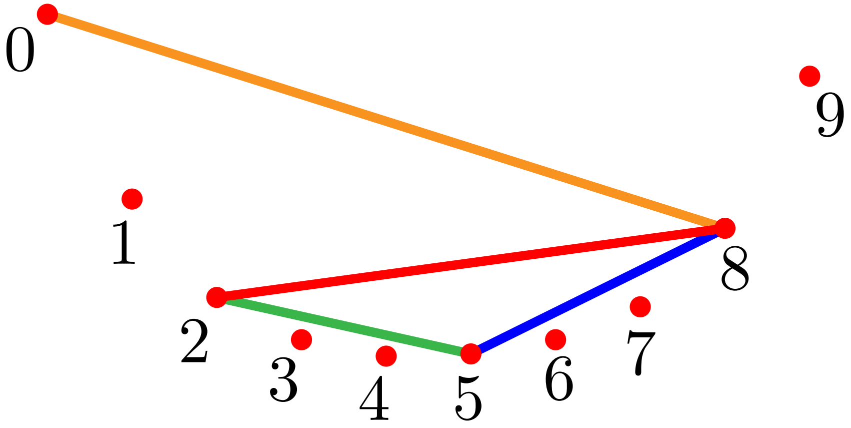

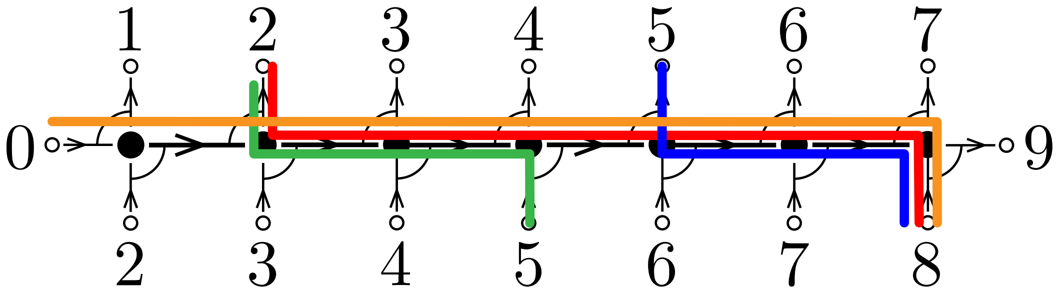

diagonal	$\longleftrightarrow$	walk
crossing	$\longleftrightarrow$	kissing
dissection	$\longleftrightarrow$	non-kissing face
simplicial associahedron	$\longleftrightarrow$	non-kissing complex

McConville, *Lattice structures of grid Tamari orders* ('17)

SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES

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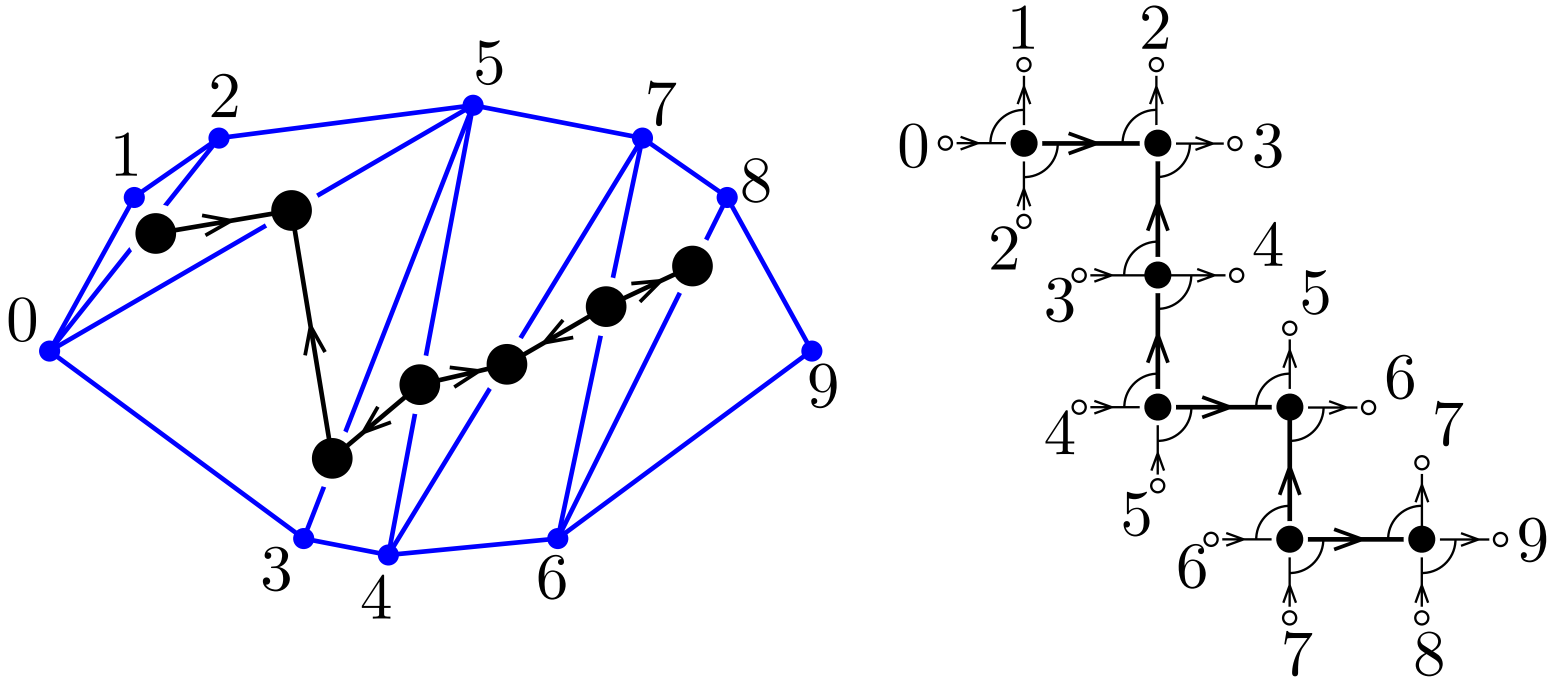
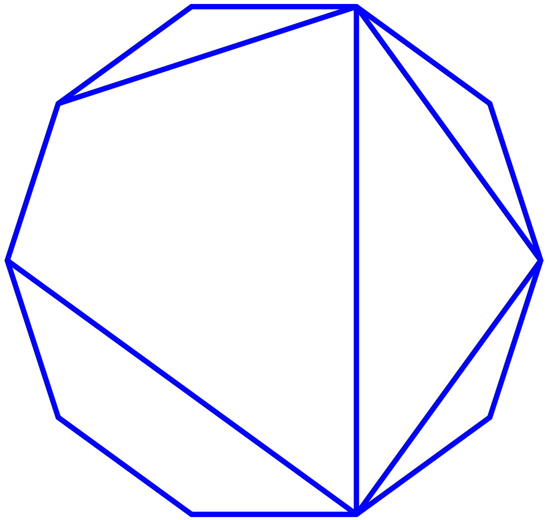
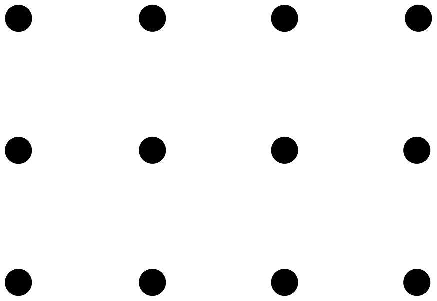


McConville, *Lattice structures of grid Tamari orders* ('17)

TWO FAMILIES OF NON-KISSING COMPLEXES

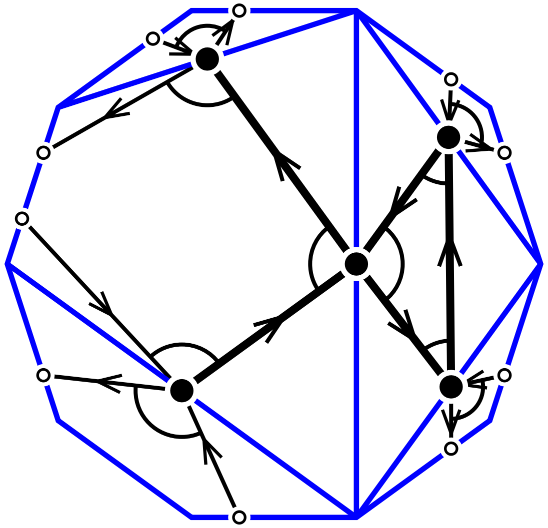


dissection

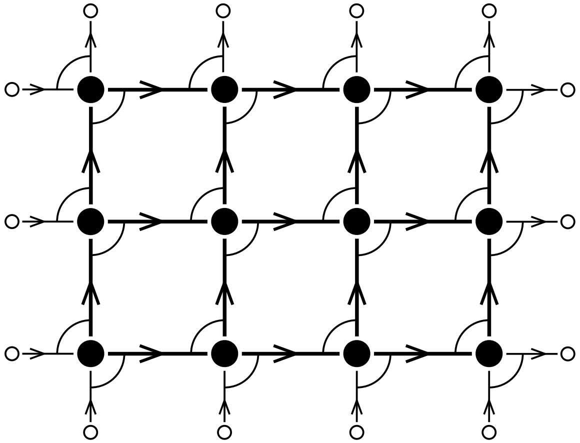


subset of $\mathbb{Z}^2$

TWO FAMILIES OF NON-KISSING COMPLEXES

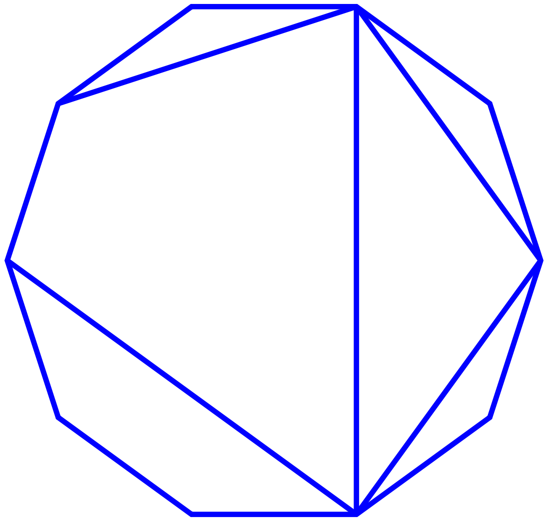


dissection

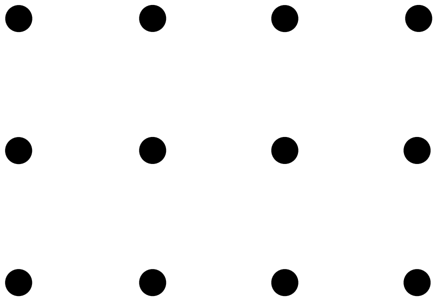


subset of $\mathbb{Z}^2$

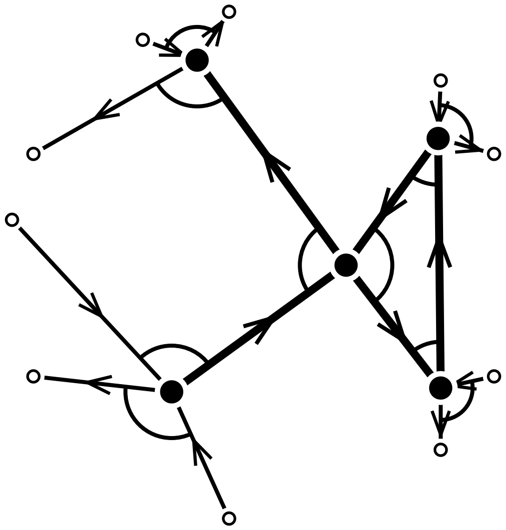
TWO FAMILIES OF NON-KISSING COMPLEXES



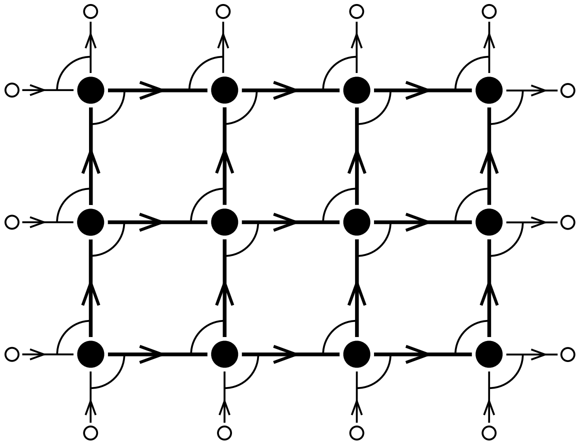
dissection



subset of $\mathbb{Z}^2$

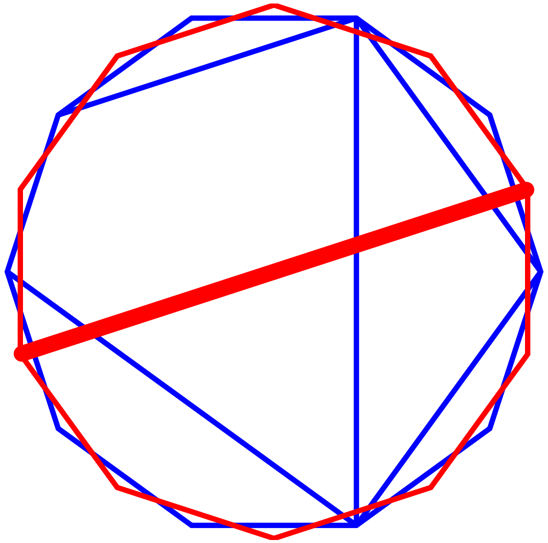


dissection quiver

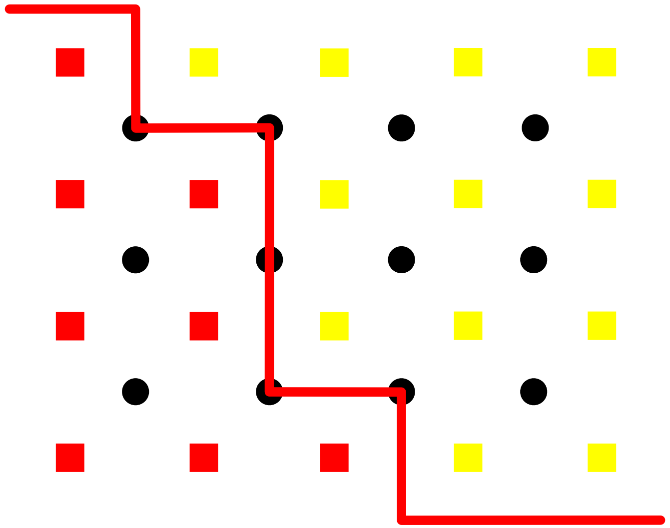


grid quiver

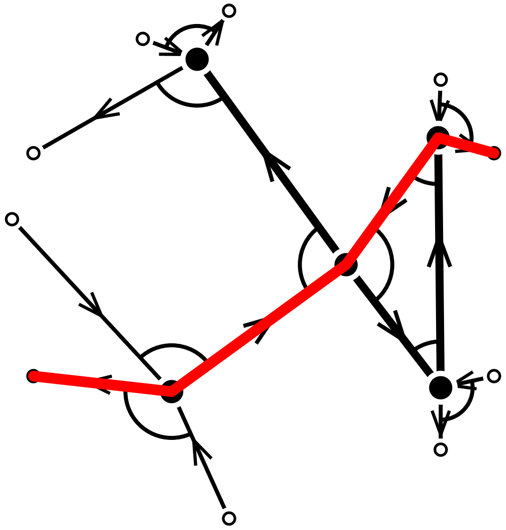
TWO FAMILIES OF NON-KISSING COMPLEXES



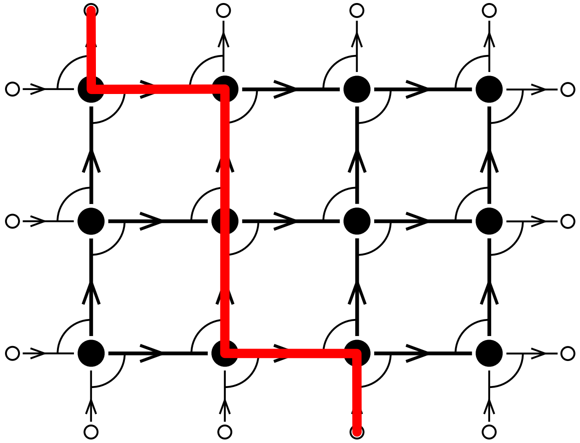
accordion



2457 subset of $[n+m]$

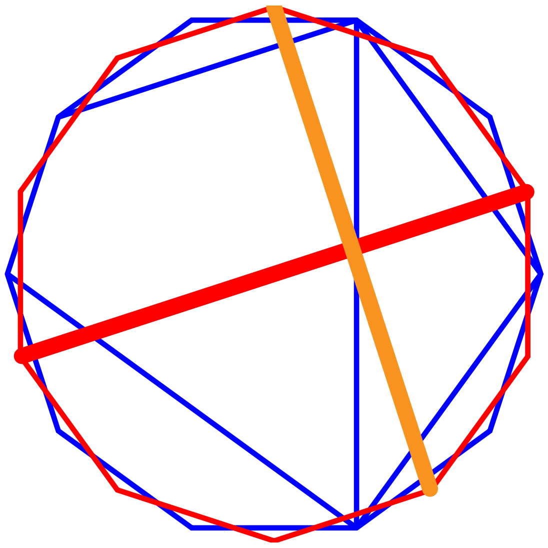


walk

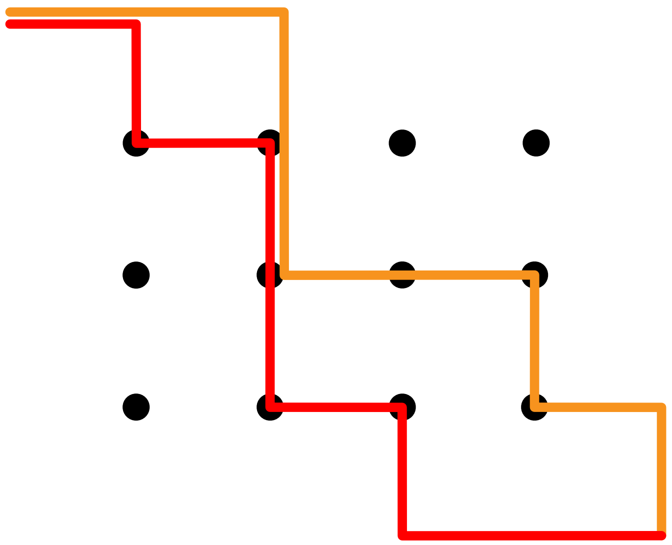


walk

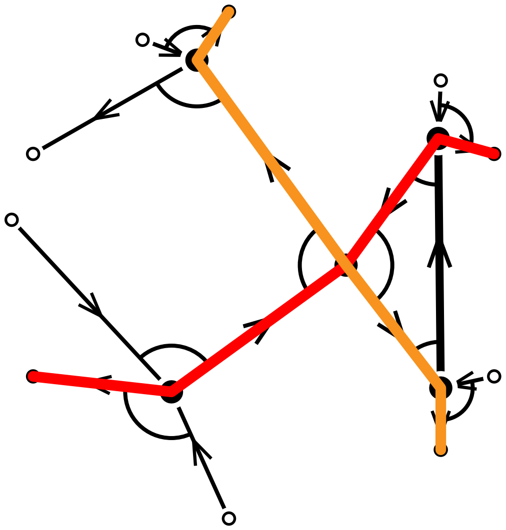
TWO FAMILIES OF NON-KISSING COMPLEXES



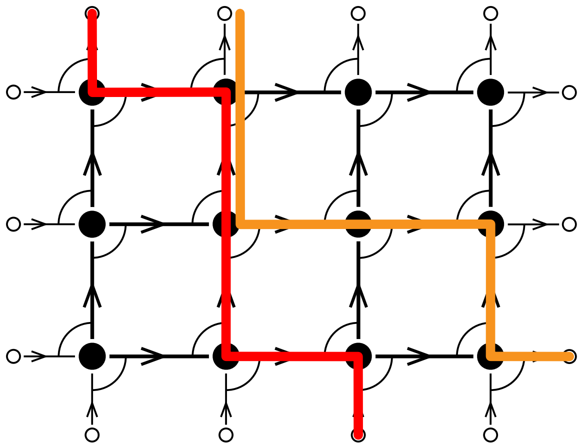
crossing accordions



crossing subsets of $[n+m]$

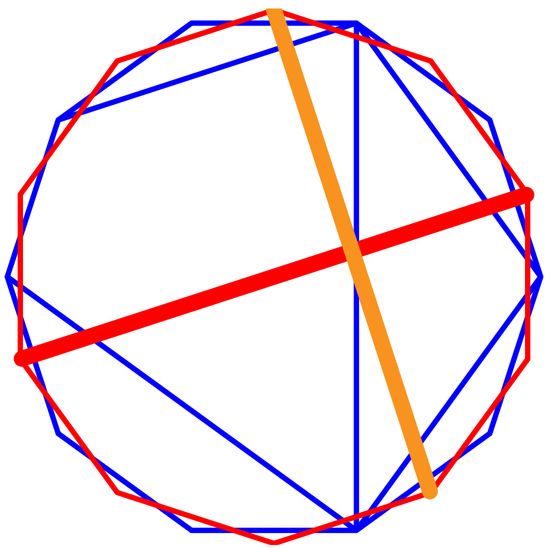


kissing walks

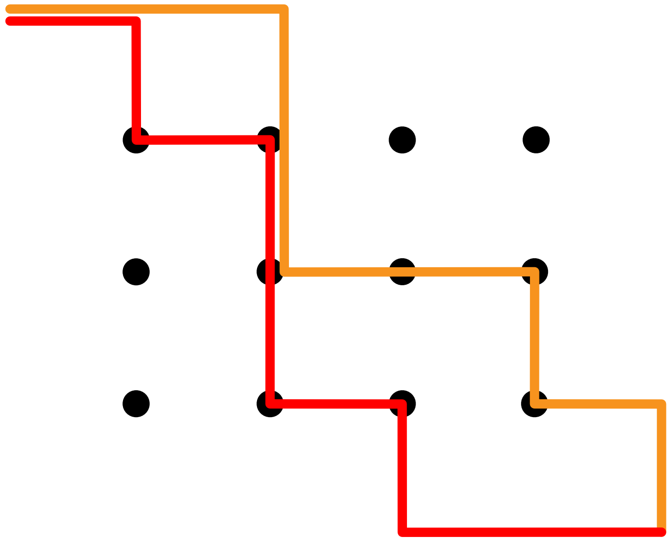


kissing walks

TWO FAMILIES OF NON-KISSING COMPLEXES



accordion complex

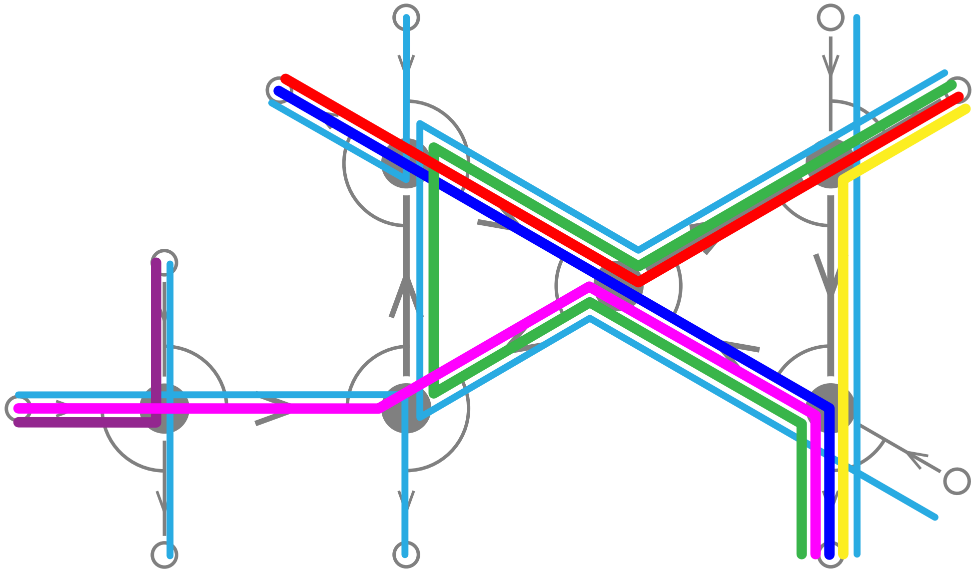


grid Tamari complex

Baryshnikov, *On Stokes sets* ('01)
Chapoton, *Stokes posets and serpent nests* ('16)
Garver-McConville, *Oriented flip graphs and non-crossing tree partitions* ('17)

Petersen-Pylyavskyy-Speyer, *A non-crossing standard monomial theory* ('10)
Santos-Stump-Welker, *Non-crossing sets and the Grassmann-assoc.* ('17)
McConville, *Lattice structures of grid Tamari orders* ('17)
Garver-McConville, *Enumerative properties of grid-associahedra* ('17^{+})

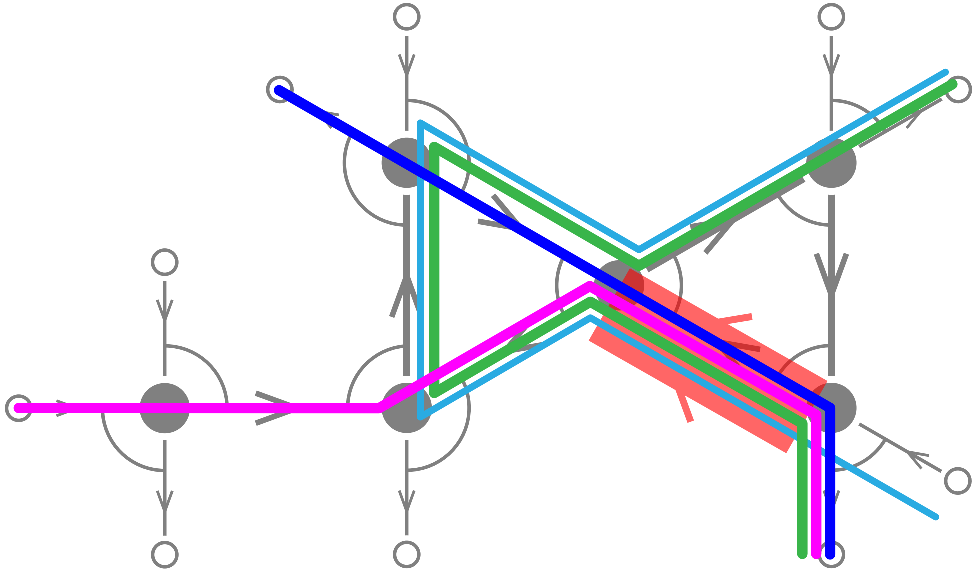
DISTINGUISHED WALKS, ARROWS AND STRINGS



F face of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$

$\alpha \in Q_1$

DISTINGUISHED WALKS, ARROWS AND STRINGS



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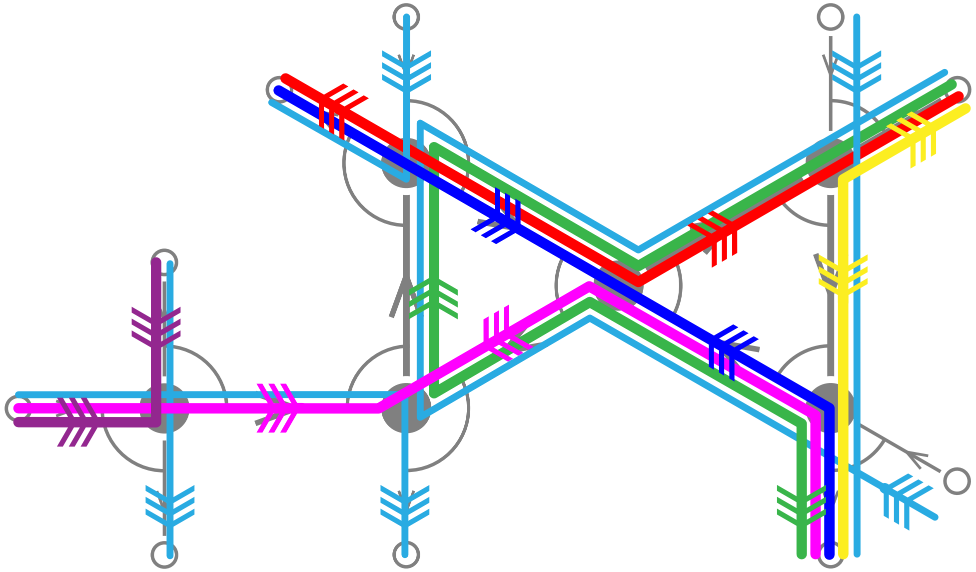
$F_\alpha = \{\omega \in F \mid \alpha \in \omega\}$

$\lambda \prec_\alpha \omega$ countercurrent order at α

DISTINGUISHED WALKS, ARROWS AND STRINGS



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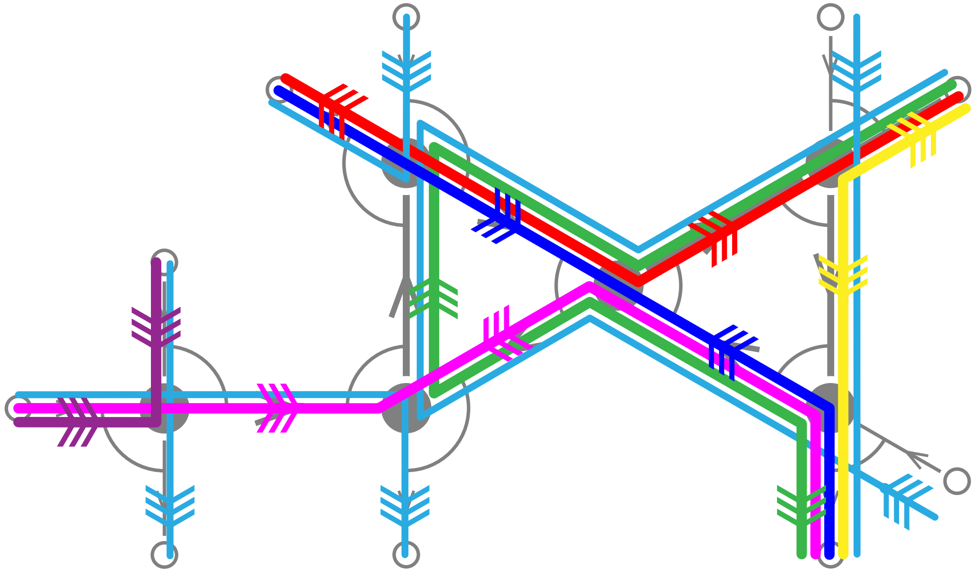
dw$(\alpha, F) = \max_{\prec_\alpha} F_\alpha$

da$(\omega, F) = \{\alpha \in Q_1 \mid \omega =$ dw$(\alpha, F)\}$

DISTINGUISHED WALKS, ARROWS AND STRINGS



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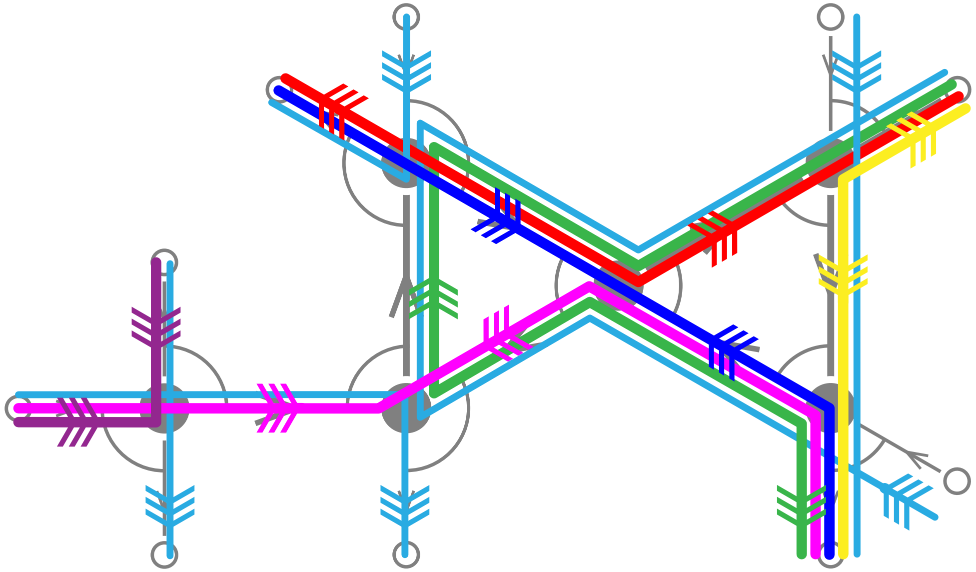
$\lambda \prec_\alpha \omega$ countercurrent order at α

$\mathsf{dw}(\alpha, F) = \max_{\prec_\alpha} F_\alpha$
$\mathsf{da}(\omega, F) = \{\alpha \in Q_1 \mid \omega = \mathsf{dw}(\alpha, F)\}$

PROP. For any facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$,

- each bended walk of F contains 2 distinguished arrows in F pointing opposite,
- each straight walk of F contains 1 distinguished arrows in F pointing as the walk.

DISTINGUISHED WALKS, ARROWS AND STRINGS



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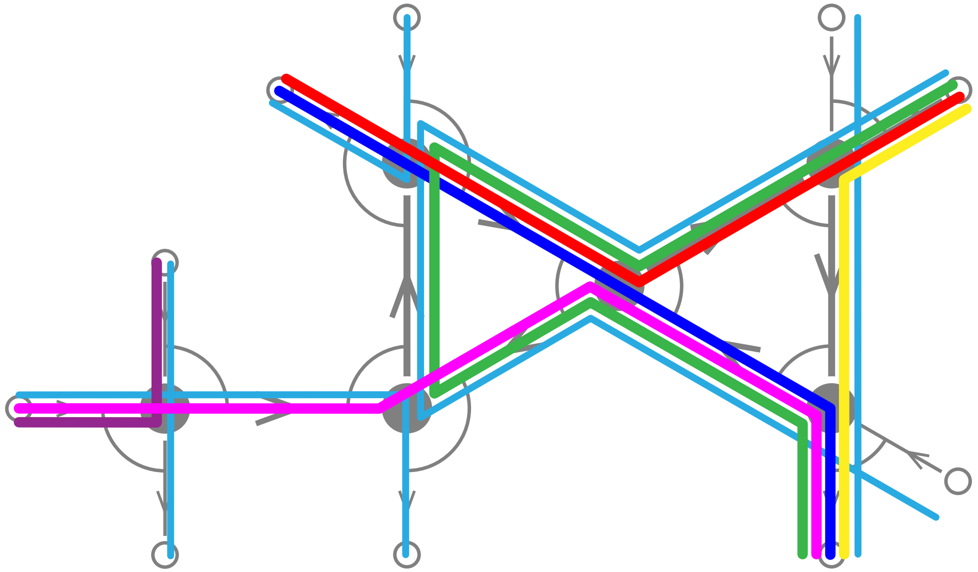
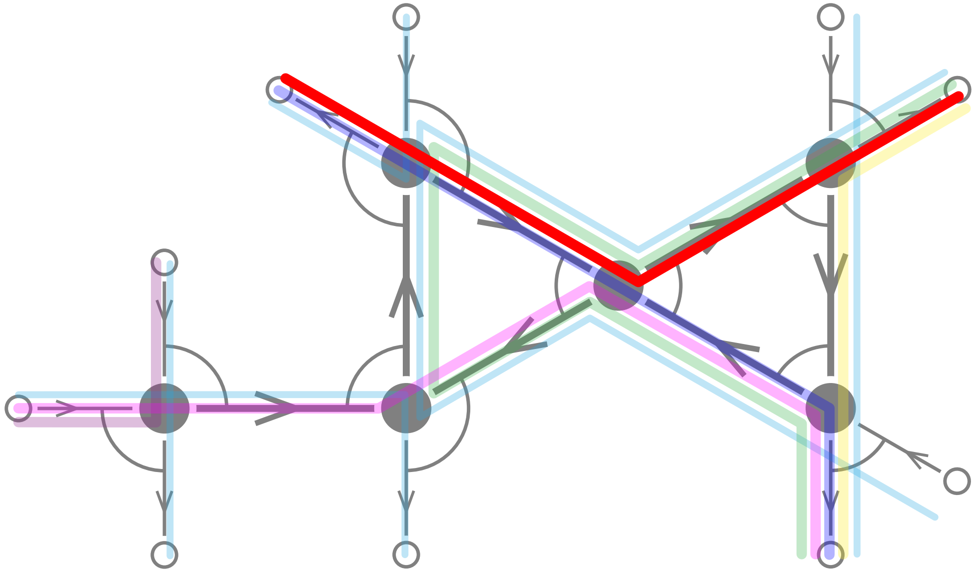
CORO. $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ is pure of dimension $|Q_0|$.

FLIPS

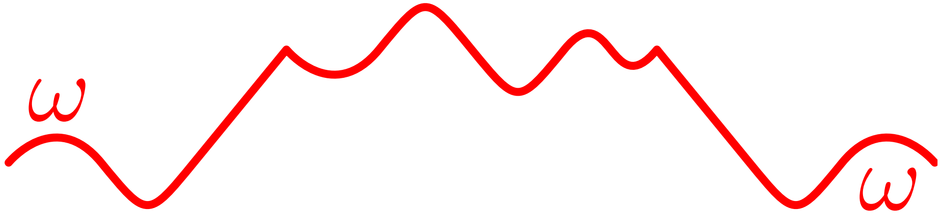


F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

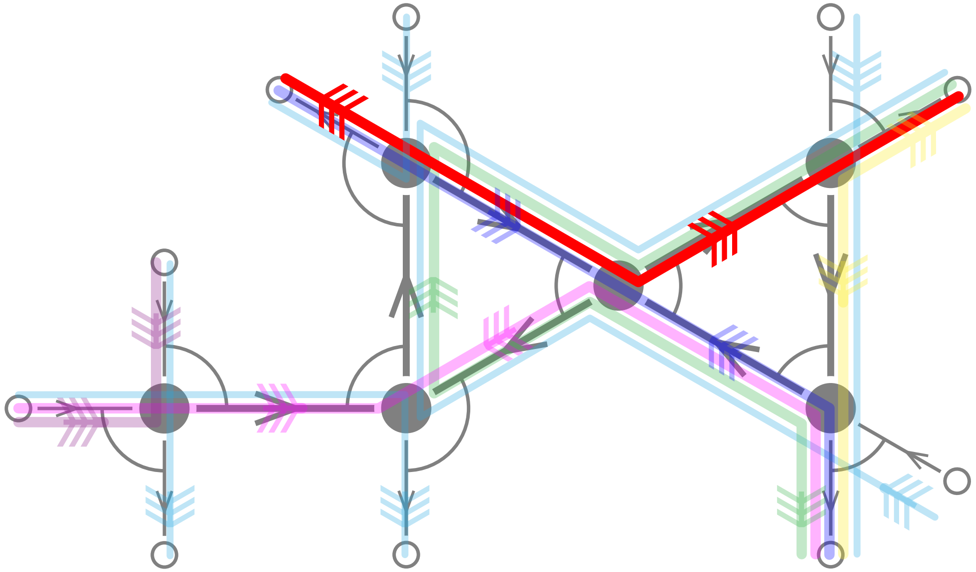
FLIPS



F facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$\omega \in F$ we want to “flip”



FLIPS



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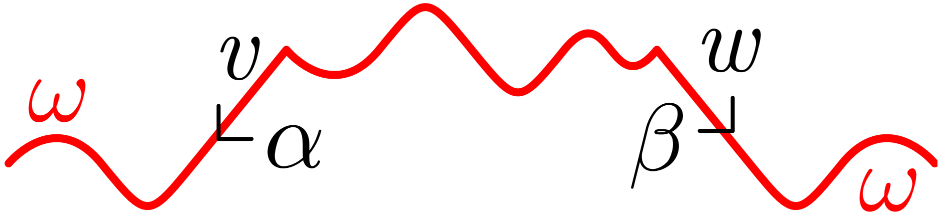
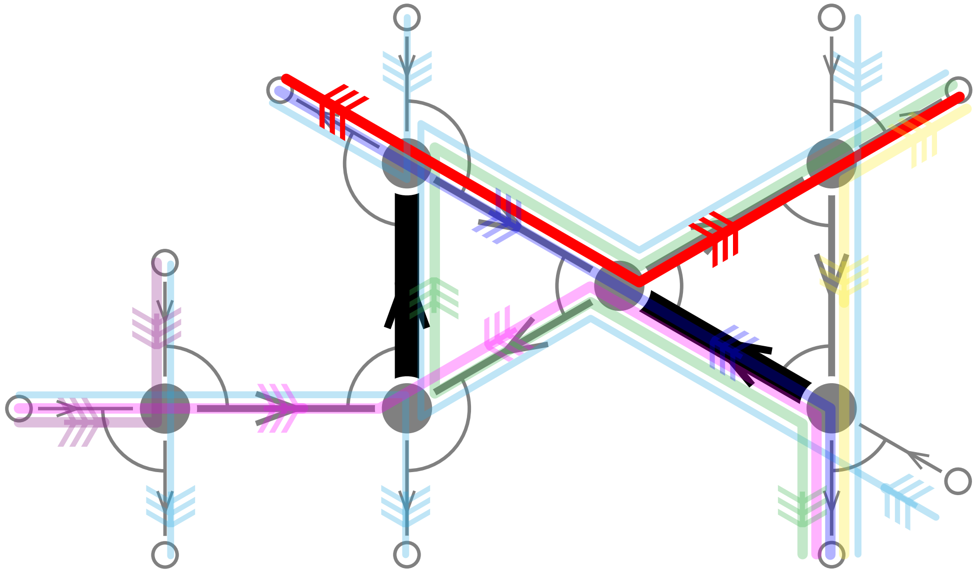


FLIPS



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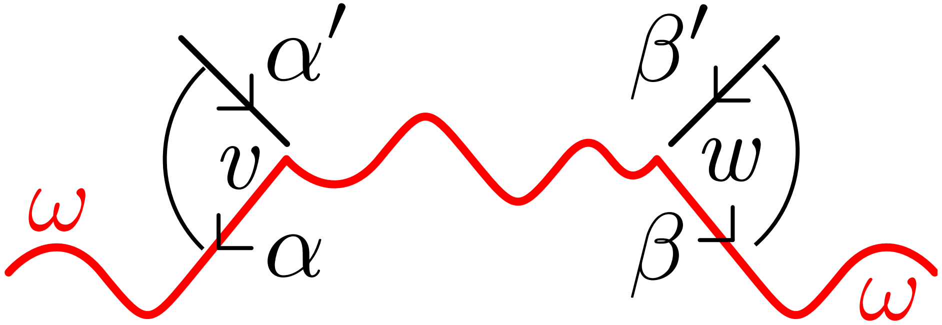
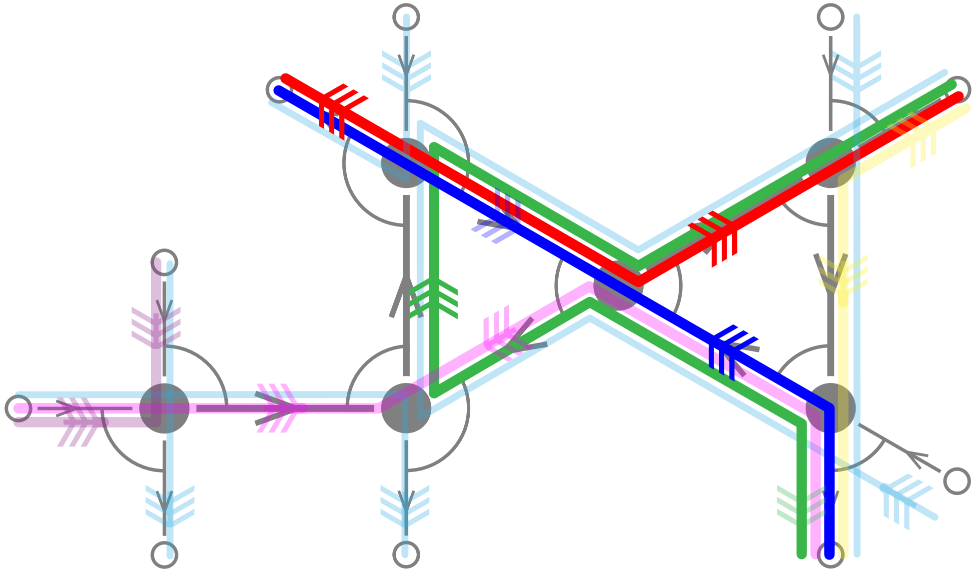
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$



FLIPS



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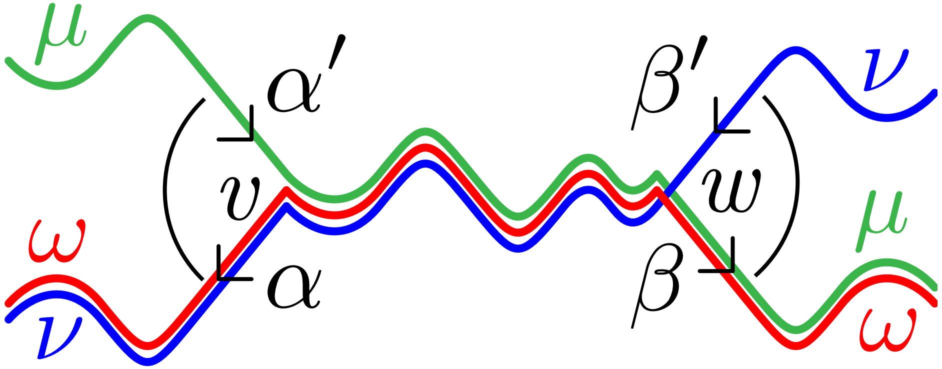
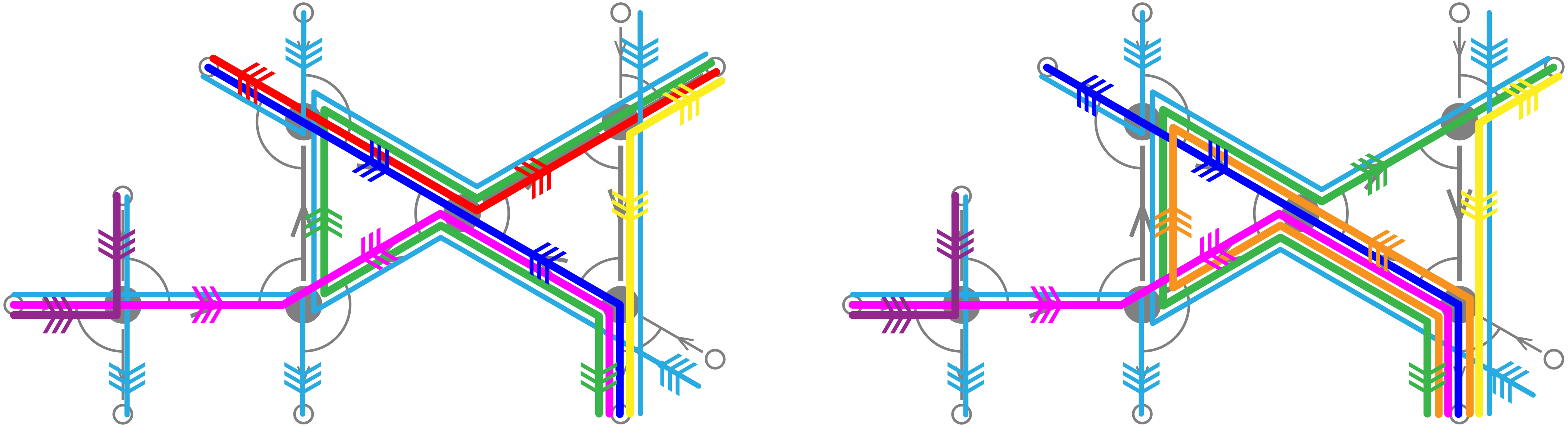
$\mu = \mathsf{dw}(\alpha, F)$ and $\nu = \mathsf{dw}(\beta, F)$



FLIPS



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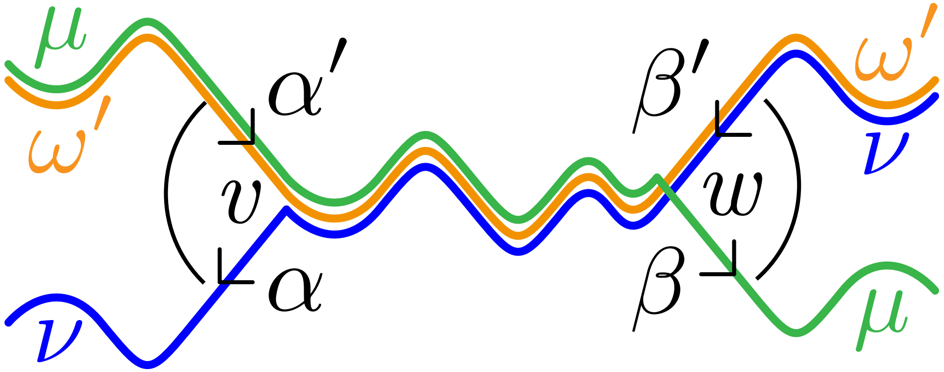
$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = \mathrm{dw}(\alpha, F)$ and $\nu = \mathrm{dw}(\beta, F)$

$\omega' = \mu[\cdot, v]\,\sigma\,\nu[w, \cdot]$



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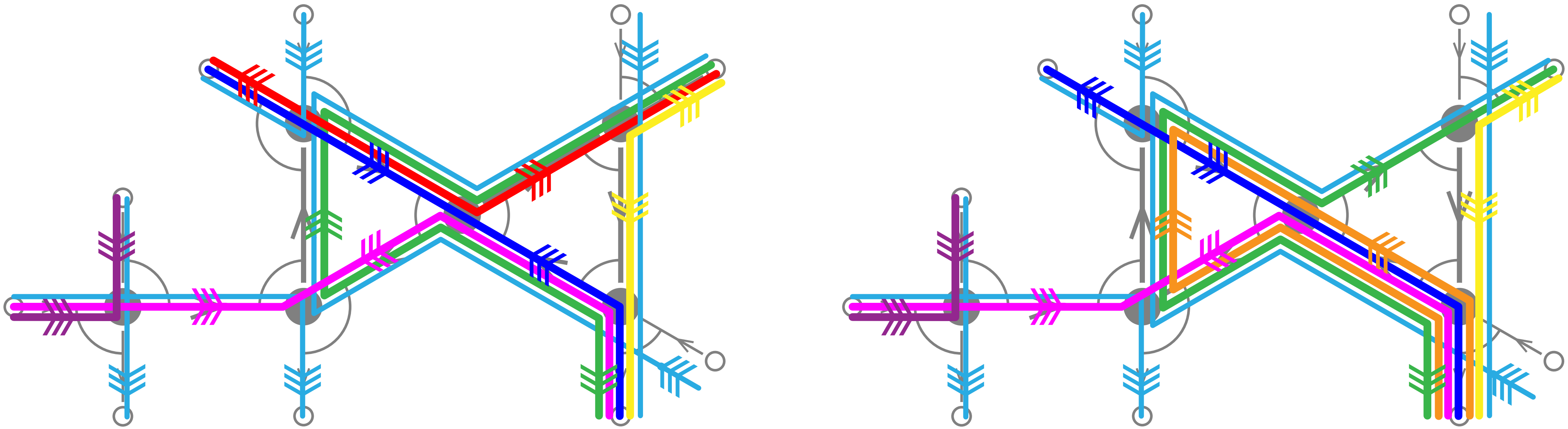
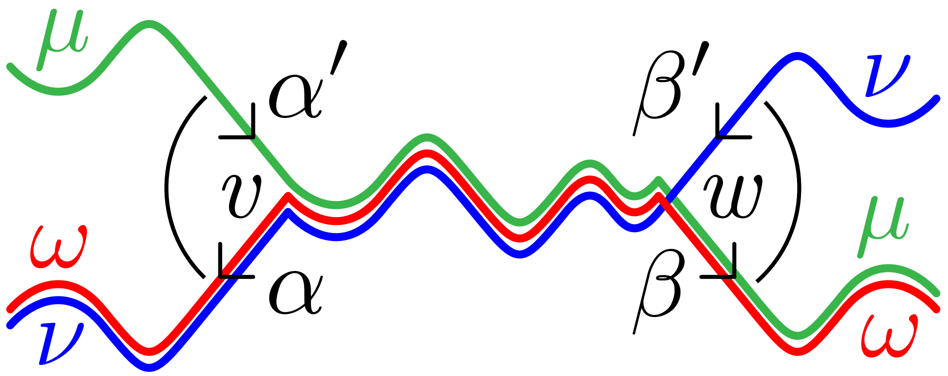
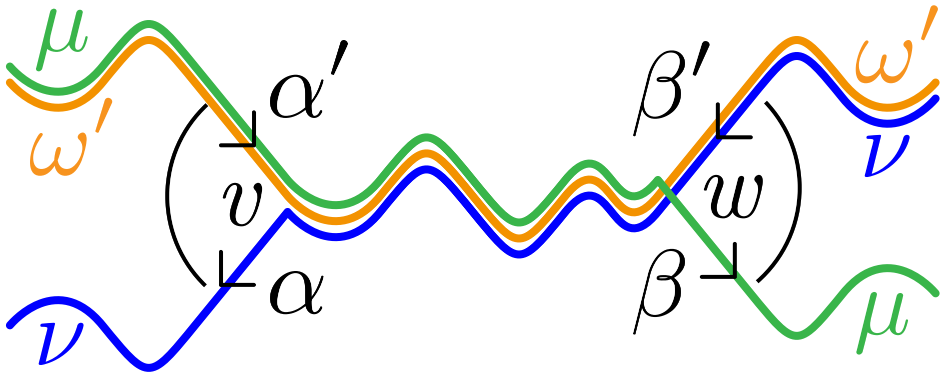
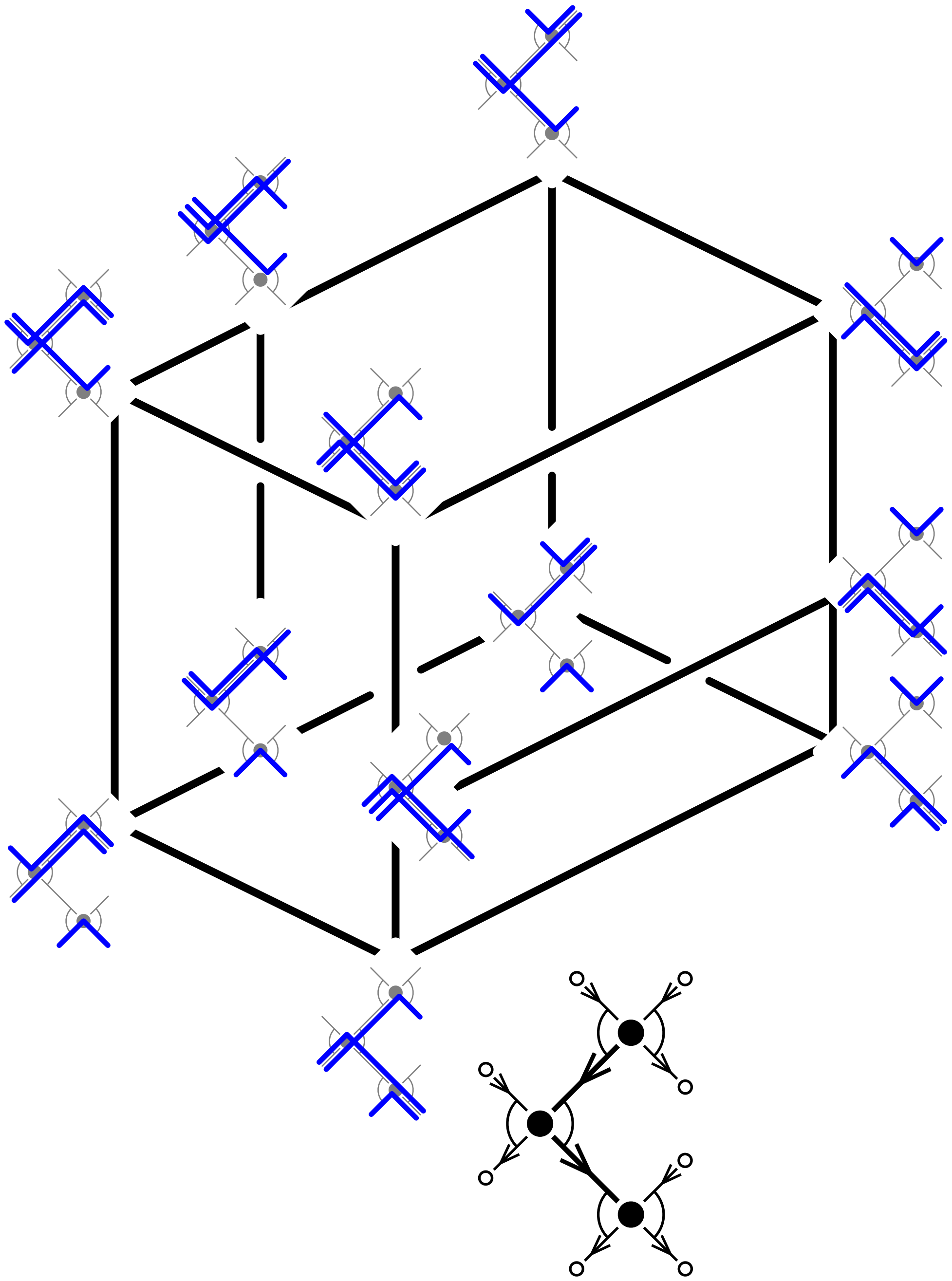


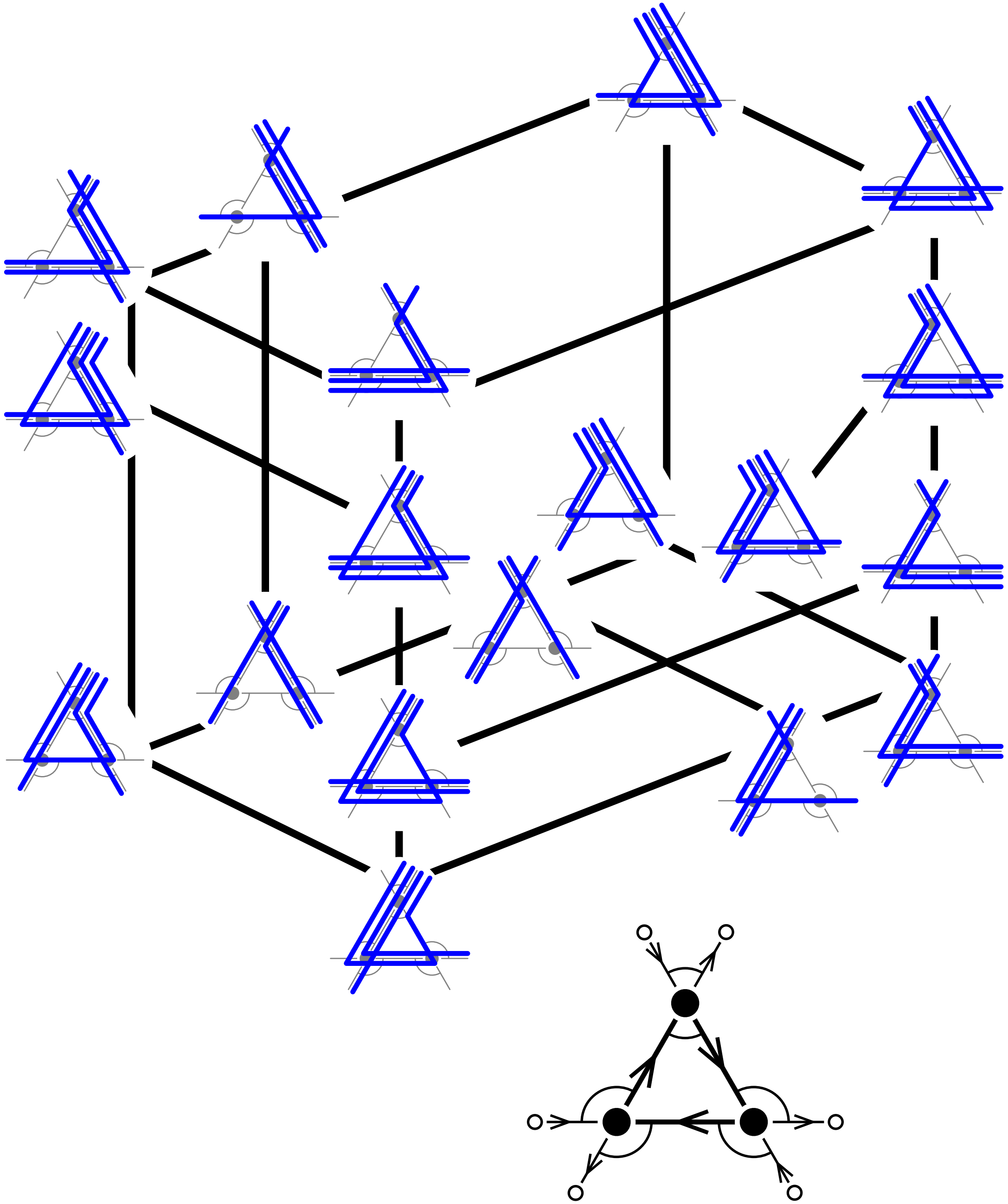
PROP. ω' kisses ω but no other walk of F. Moreover, ω' is the only such walk.





FLIP GRAPH





GENTLE ASSOCIAHEDRA

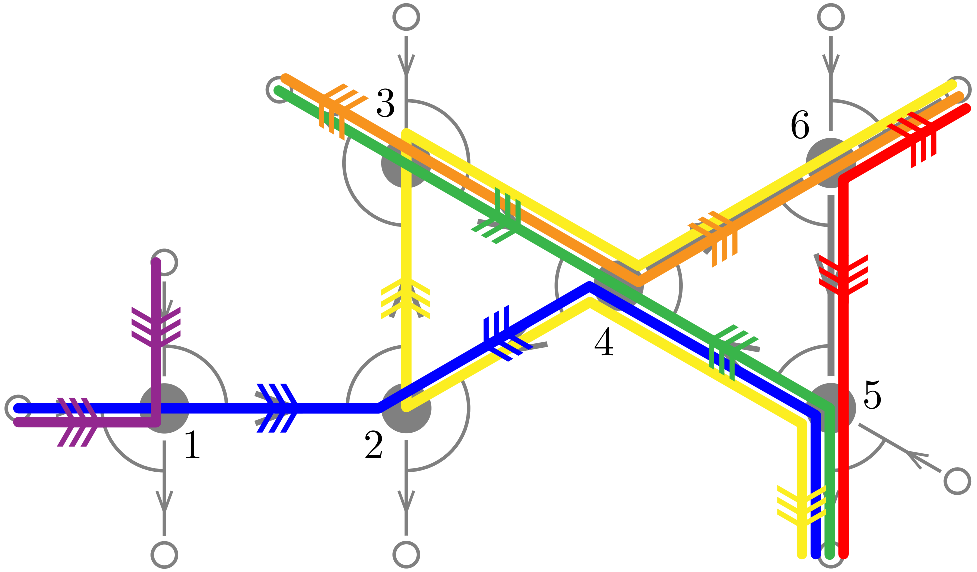
Manneville-P., *Geometric realizations of the accordion complex* ('17^+)
Hohlweg-P.-Stella, Polytopal realizations of finite type g-vector fans ('17^+)
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17^+)

G-VECTORS & C-VECTORS

multiplicity vector $\mathbf{m}_V$ of multiset $V = \{\{v_1, \dots, v_m\}\}$ of $Q_0 \quad = \quad \sum_{i \in [m]} \mathbf{e}_{v_i} \; \in \; \mathbb{R}^{Q_0}$

g-vector $\mathbf{g}(\omega)$ of a walk $\omega \quad = \quad \mathbf{m}_{\mathsf{peaks}(\omega)} - \mathbf{m}_{\mathsf{deeps}(\omega)}$

c-vector $\mathbf{c}(\omega \in F)$ of a walk ω in a non-kissing facet $F \quad = \quad \varepsilon(\omega, F)\, \mathbf{m}_{\mathsf{ds}(\omega, F)}$



$$\begin{array}{c|cccccc} & \text{🔴} & \text{🟠} & \text{🟡} & \text{🟢} & \text{🔵} & \text{🟣} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & -1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 1 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$\mathbf{g}F$

$$\begin{array}{c|cccccc} & \text{🔴} & \text{🟠} & \text{🟡} & \text{🟢} & \text{🔵} & \text{🟣} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 1 & 0 & -1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & -1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$\mathbf{c}F$

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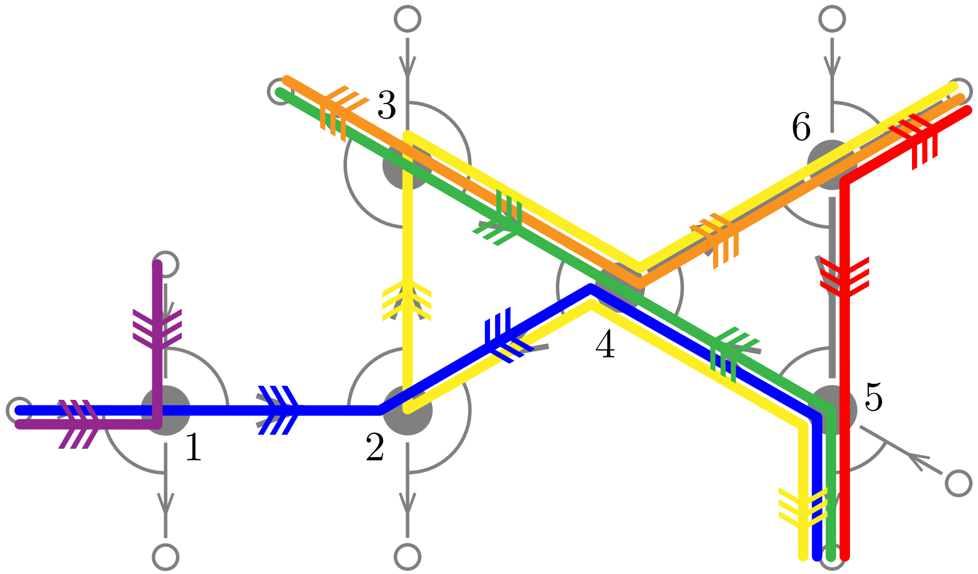

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\end{array} \\
\mathbf{c}F
\end{array}
$$

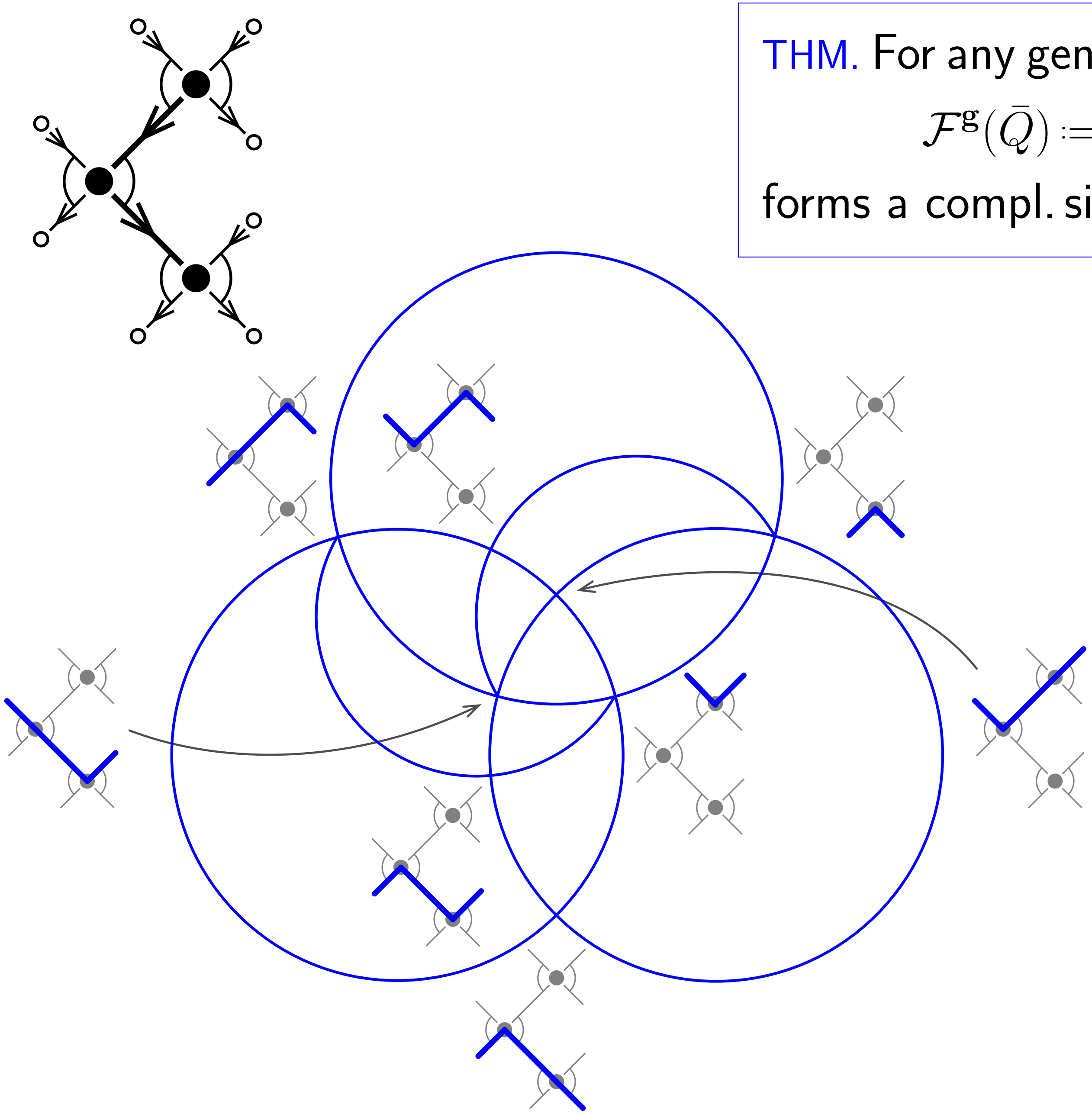
PROP. For any non-kissing facet F, the sets of vectors

$$\mathbf{g}(F) := \{\mathbf{g}(\omega) \mid \omega \in F\} \qquad \text{and} \qquad \mathbf{c}(F) := \{\mathbf{c}(\omega \in F) \mid \omega \in F\}$$

form dual bases.

Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17^+)

G-VECTOR FAN

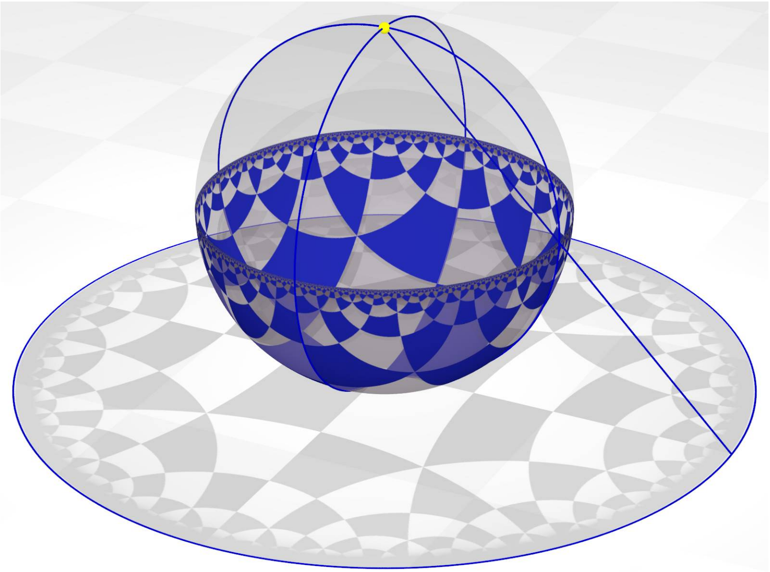


THM. For any gentle quiver $\bar{Q}$, the collection of cones

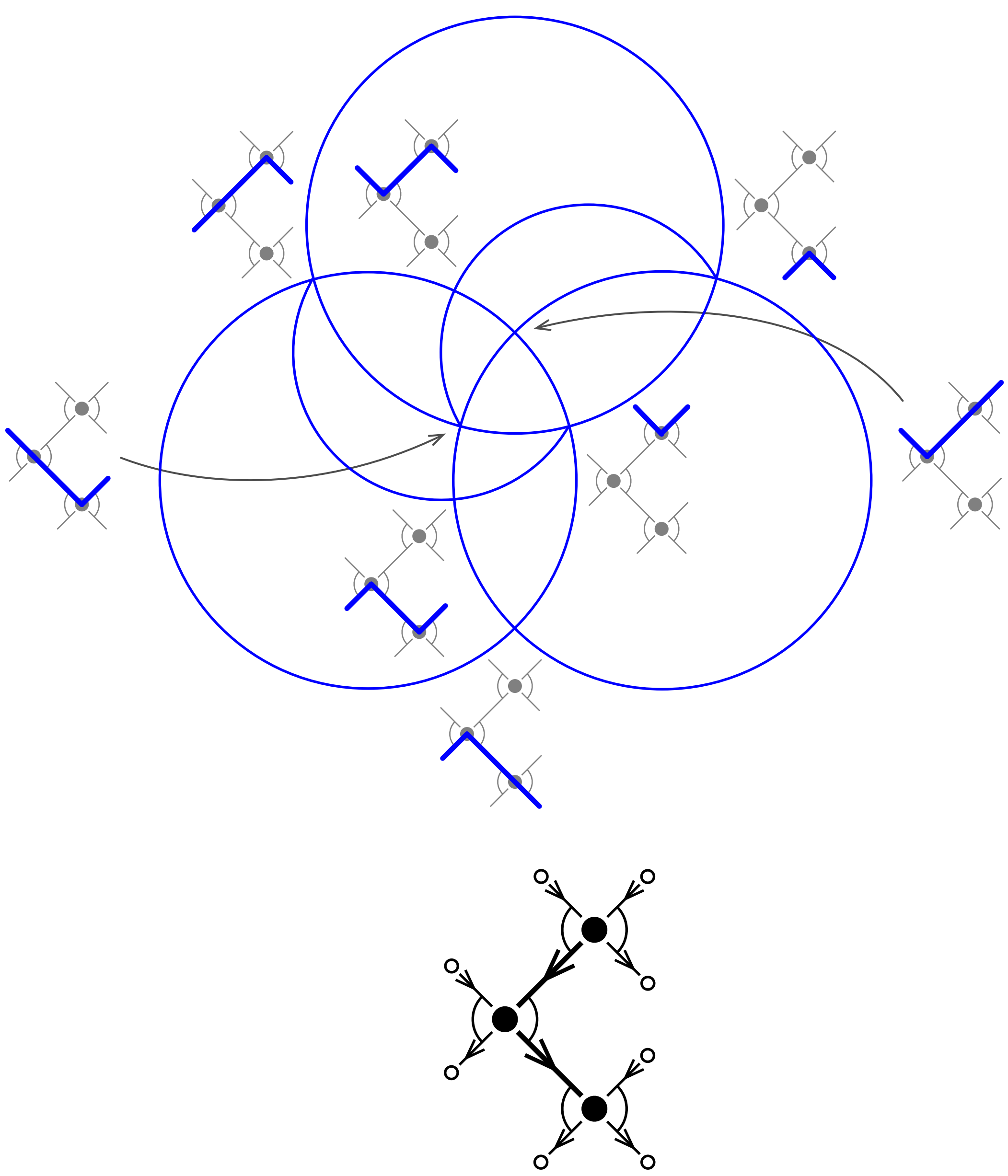
$$\mathcal{F}^{\mathbf{g}}(\bar{Q}) := \left\{ \mathbb{R}_{\geq 0}\mathbf{g}(F) \;\middle|\; F \in \mathcal{C}_{\mathrm{nk}}(\bar{Q}) \right\}$$

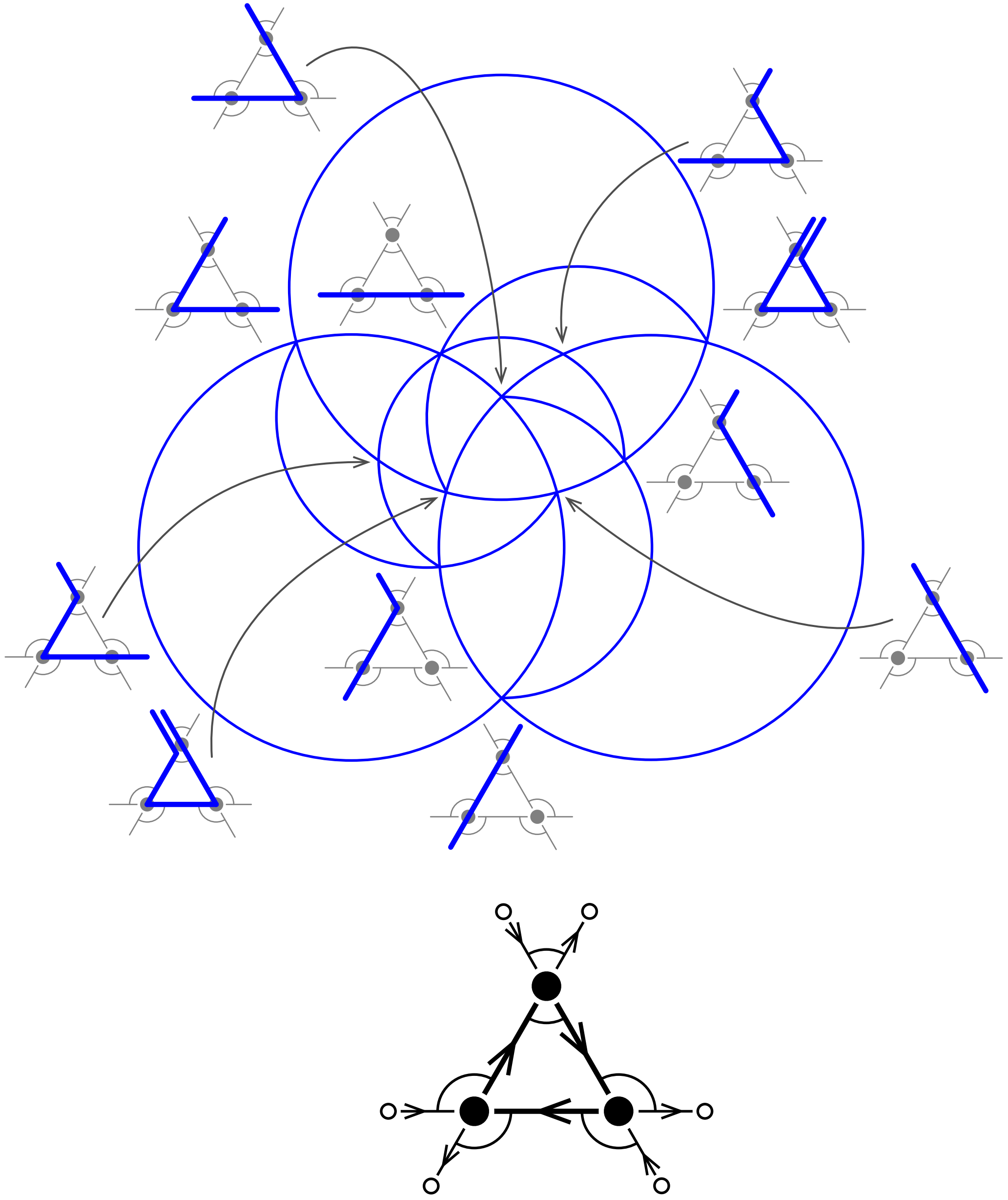
forms a compl. simpl. fan, called g-vector fan of $\bar{Q}$.

stereographic projection
from $(1,1,1)$



G-VECTOR FAN





NON-KISSING ASSOCIAHEDRON

kissing number $\kappa(\omega, \omega') \;=\;$ number of times ω kisses ω'

kissing number $\mathsf{kn}(\omega) \;=\; \sum_{\omega'} \kappa(\omega, \omega') + \kappa(\omega', \omega)$

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the two sets of $\mathbb{R}^{Q_0}$ given by

(i) the convex hull of the points

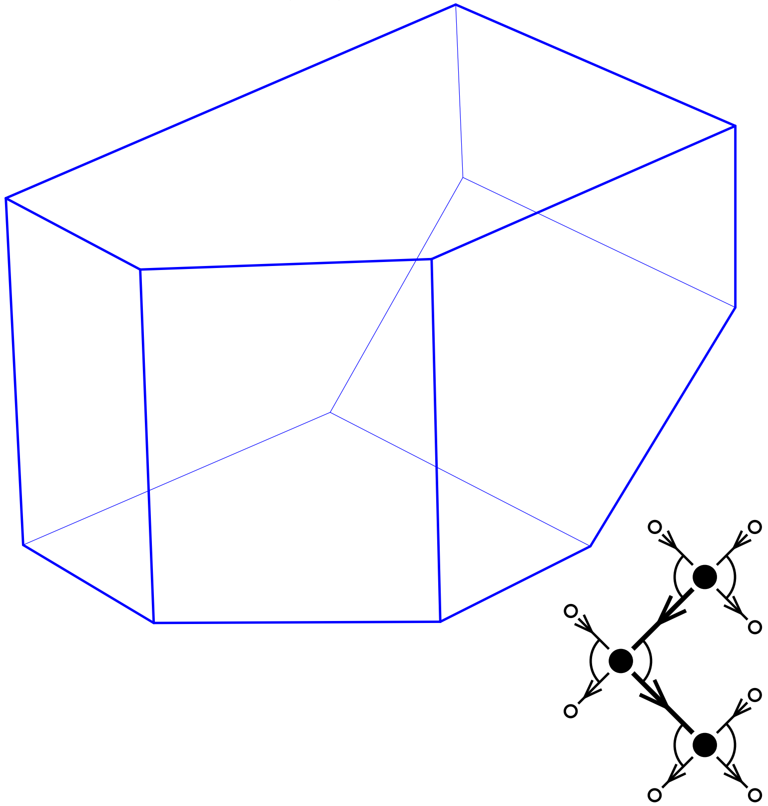
$$\mathbf{p}(F) := \sum_{\omega \in F} \mathsf{kn}(\omega)\, \mathbf{c}(\omega \in F),$$

for all non-kissing facets $F \in \mathcal{C}_{\mathrm{nk}}(\bar{Q})$,

(ii) the intersection of the halfspaces

$$\mathbf{H}^{\geq}(\omega) := \left\{ \mathbf{x} \in \mathbb{R}^{Q_0} \;\middle|\; \langle\, \mathbf{g}(\omega) \mid \mathbf{x} \,\rangle \leq \mathsf{kn}(\omega) \right\}.$$

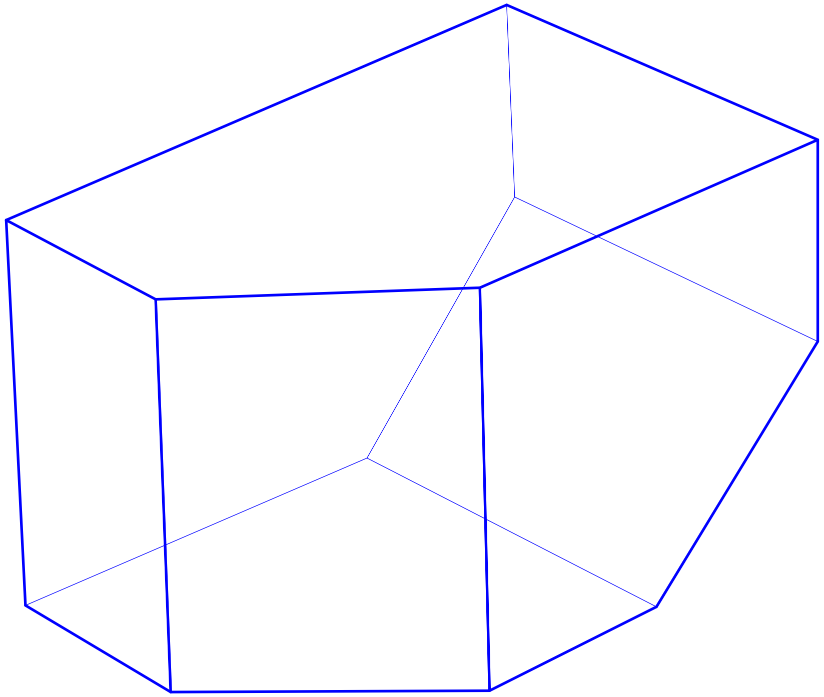
for all walks ω of $\bar{Q}$,

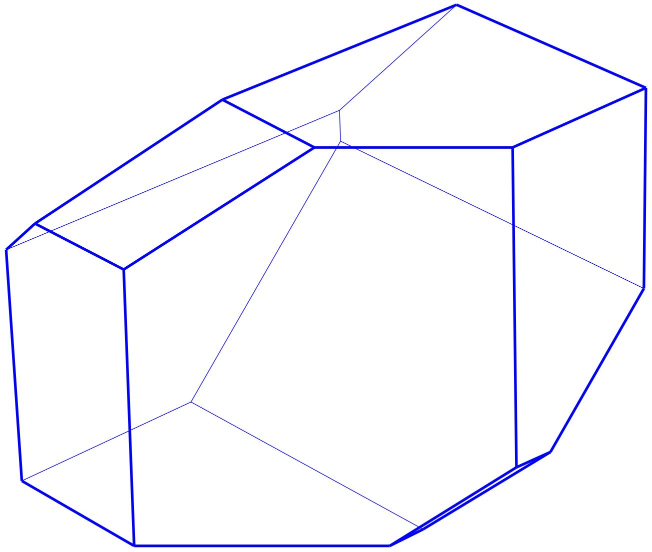


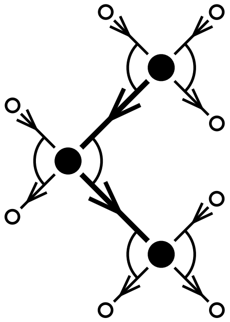
define the same polytope, whose normal fan is the g-vector fan $\mathcal{F}^{\mathbf{g}}$. We call it the $\bar{Q}$-associahedron and denote it by Asso.

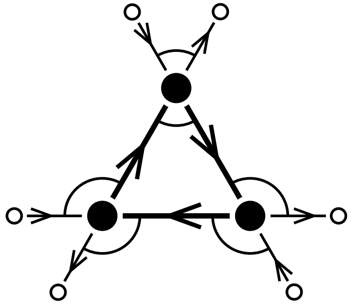
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17$^+$)

NON-KISSING ASSOCIAHEDRON

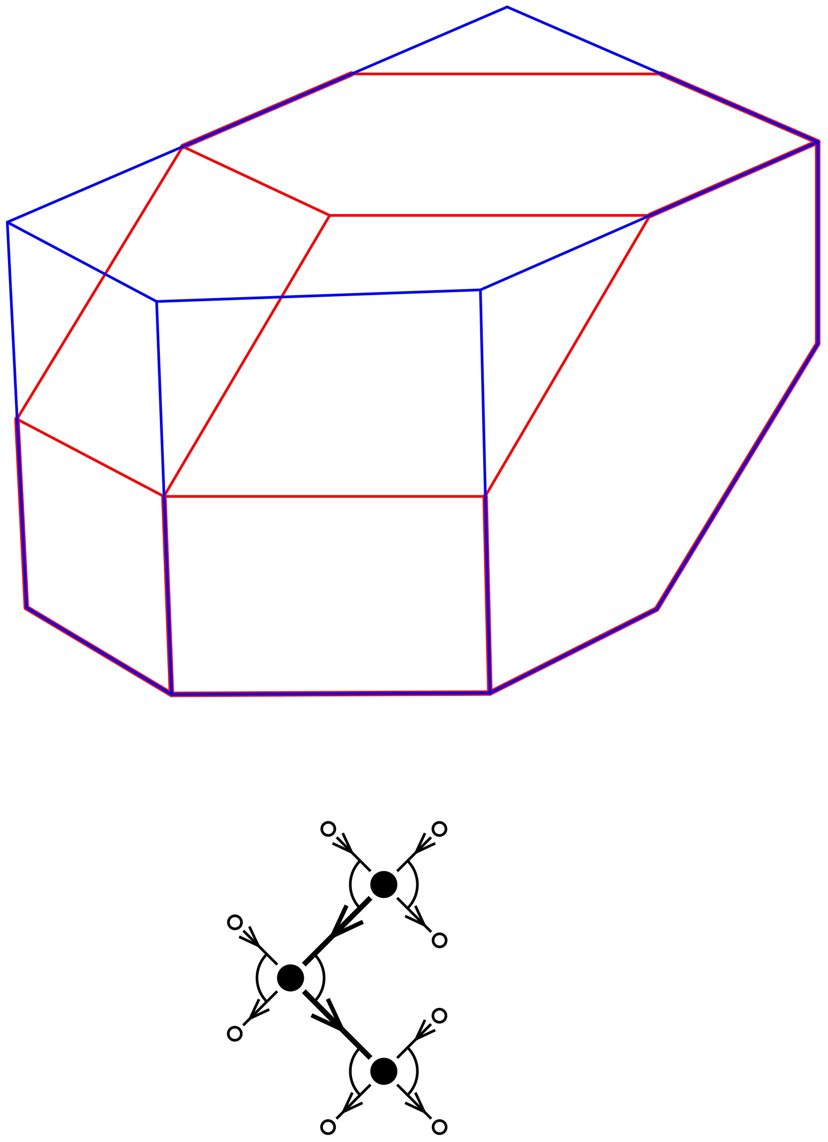


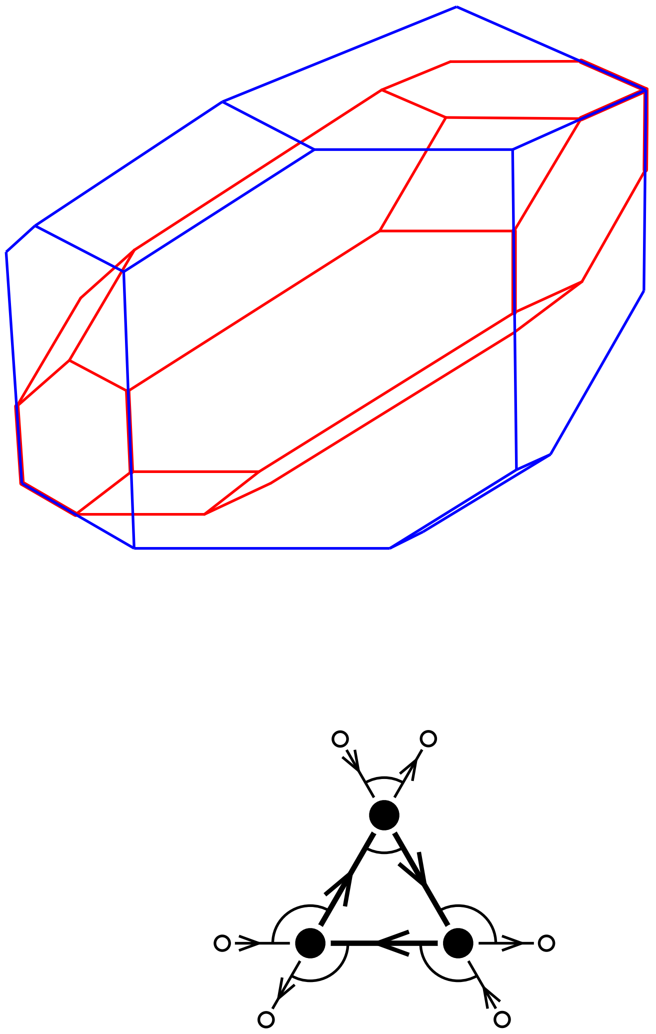






NON-KISSING ASSOCIAHEDRON VS ZONOTOPES



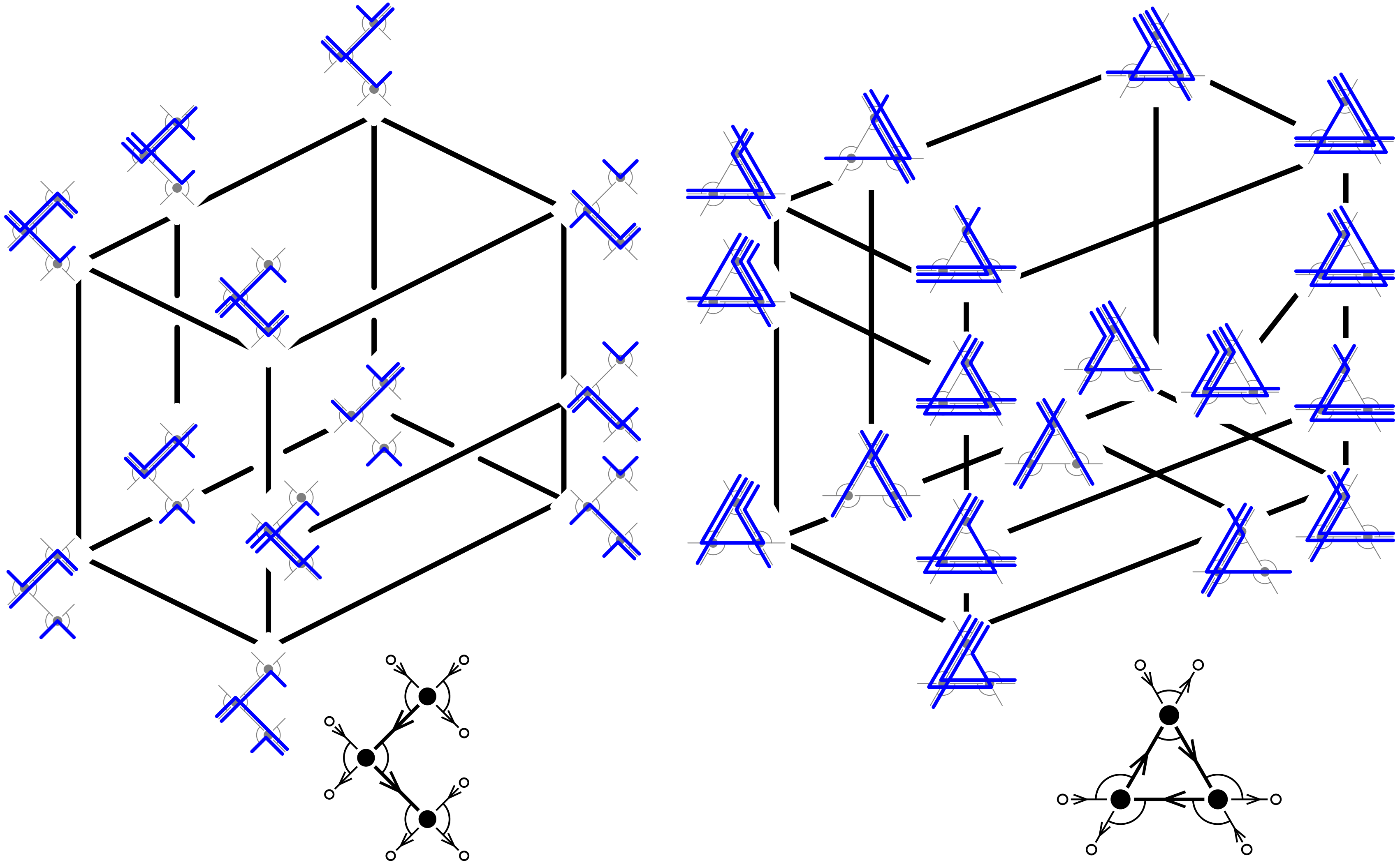


NON-KISSING LATTICE

McConville, *Lattice structures of grid Tamari orders* ('17)
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17^+)

NON-KISSING LATTICE

THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.



Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17^+)

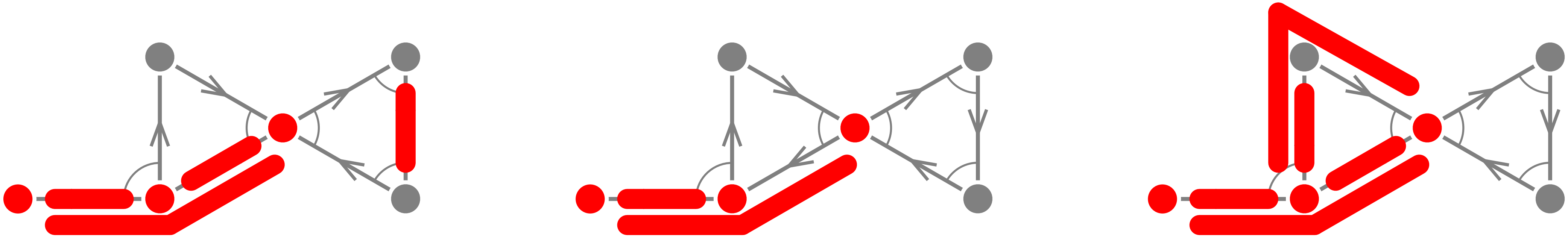
BICLOSED SETS OF SEGMENTS

σ, τ oriented strings

concatenation $\sigma \circ \tau = \left\{ \sigma\alpha\tau \;\middle|\; \alpha \in Q_1 \text{ and } \sigma\alpha\tau \text{ string of } \bar{Q} \right\}$

closure $S^{\mathrm{cl}} = \bigcup\limits_{\substack{\ell \in \mathbb{N} \\ \sigma_1, \dots, \sigma_\ell \in S}} \sigma_1 \circ \cdots \circ \sigma_\ell =$ all strings obtained by concatenation of some strings of S

closed $\iff S^{\mathrm{cl}} = S$ coclosed $\iff \bar{S}^{\mathrm{cl}} = \bar{S}$ biclosed $=$ closed and coclosed



> **THM.** For any gentle quiver $\bar{Q}$ such that $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ is finite, the inclusion poset on biclosed sets of strings of $\bar{Q}$ is a congruence-uniform lattice.

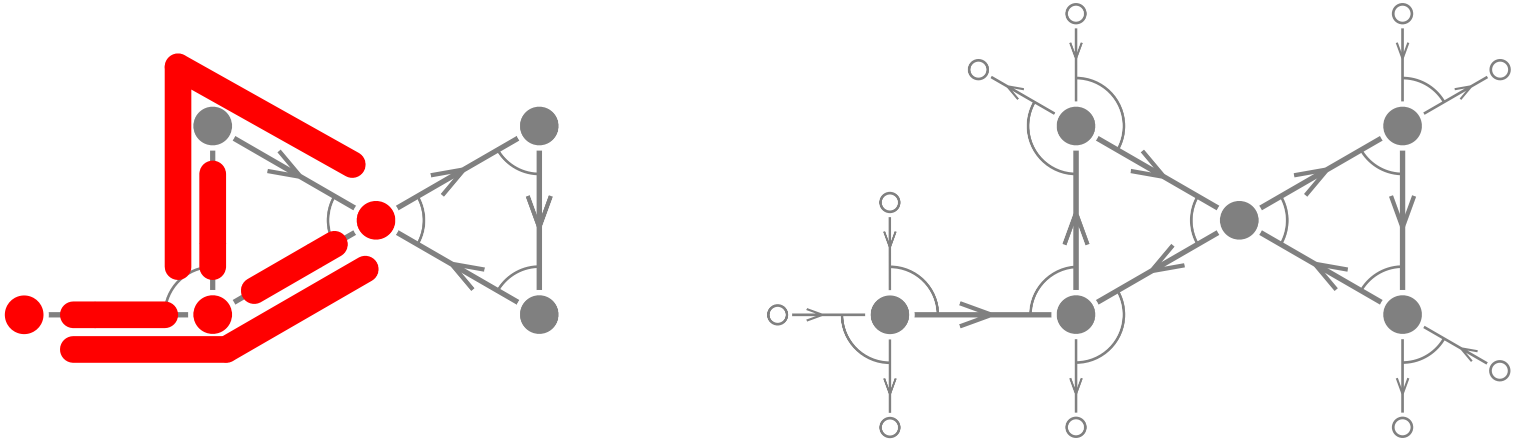
McConville, *Lattice structures of grid Tamari orders* ('17)

Garver-McConville, *Oriented flip graphs and non-crossing tree partitions* ('17^{+})

Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17^{+})

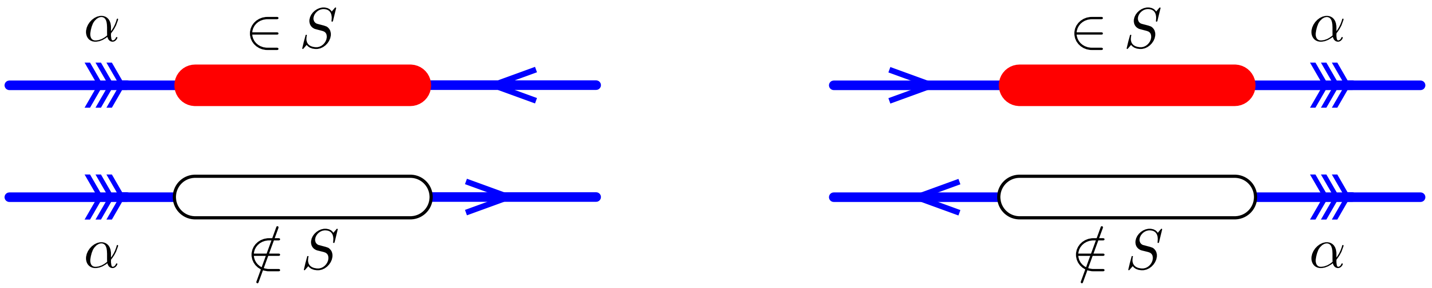
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



S biclosed, $\alpha \in Q_1$

$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

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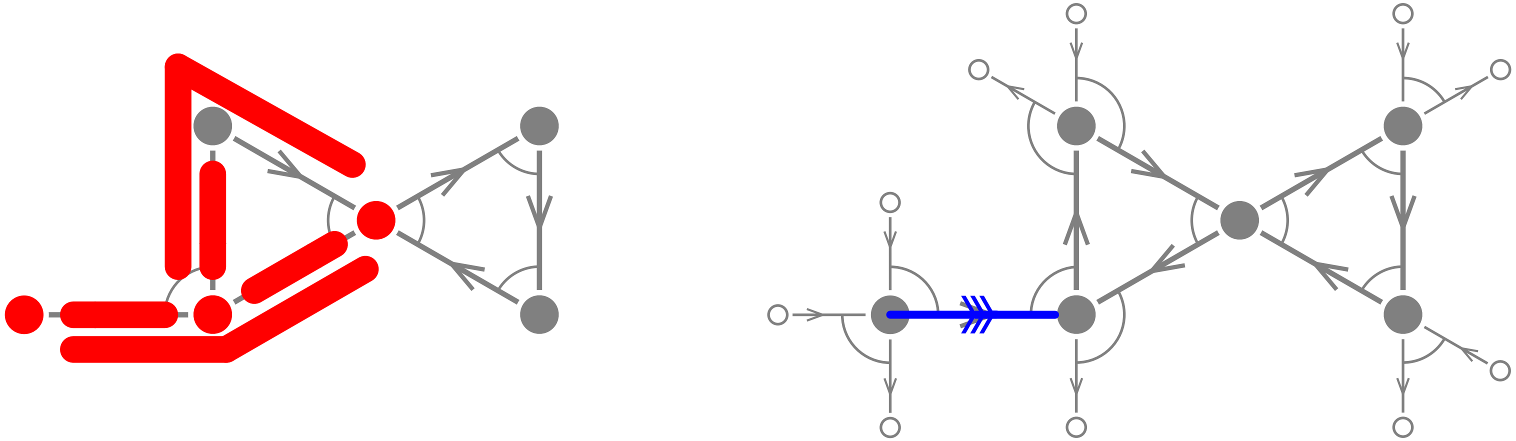

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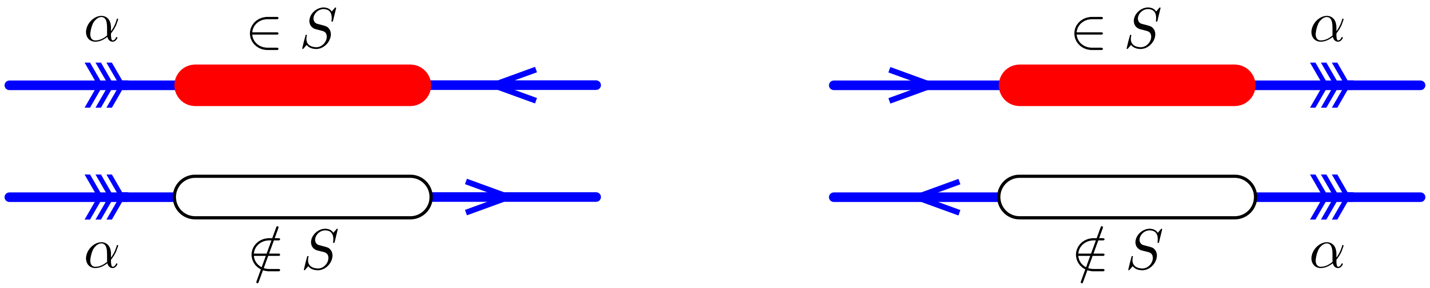

McConville, *Lattice structures of grid Tamari orders* ('17)

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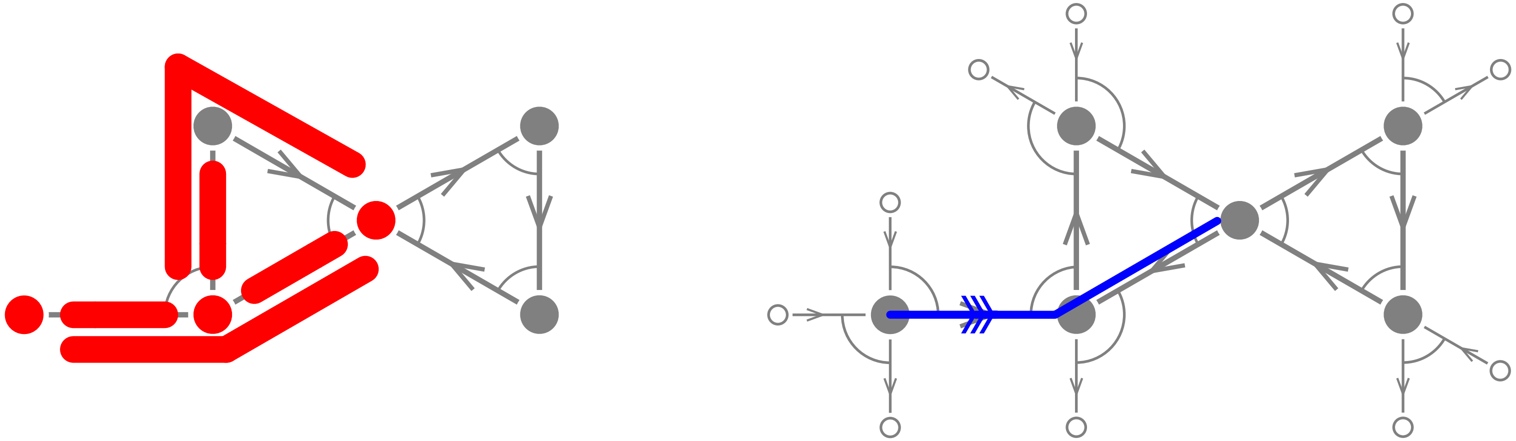

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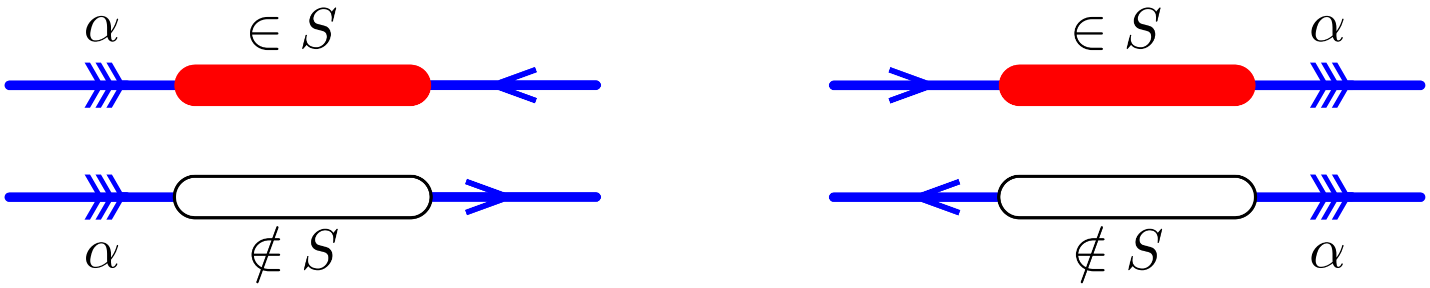

McConville, *Lattice structures of grid Tamari orders* ('17)

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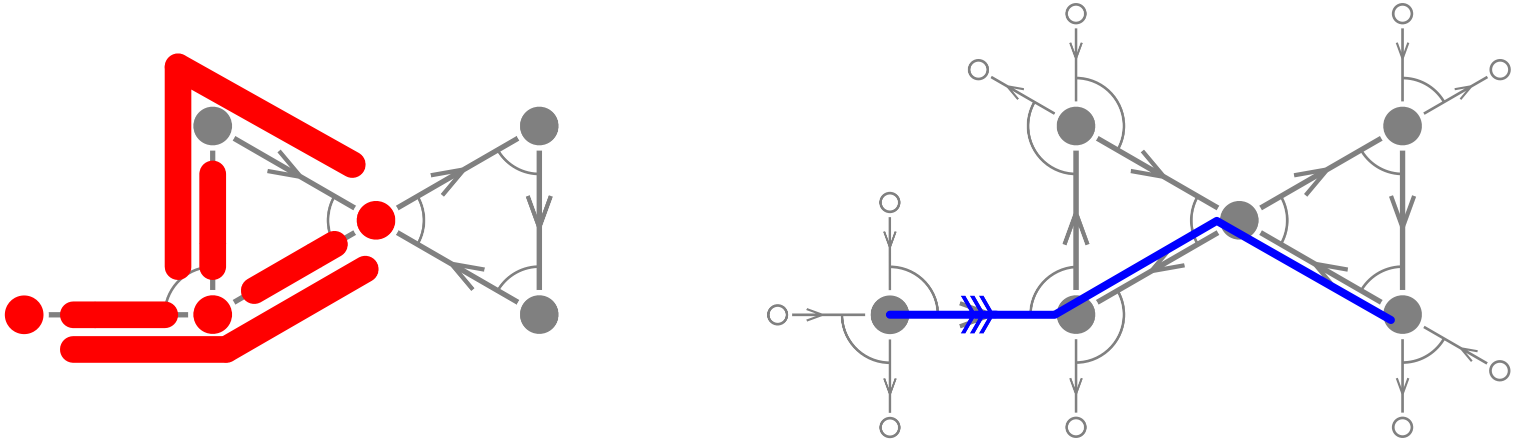

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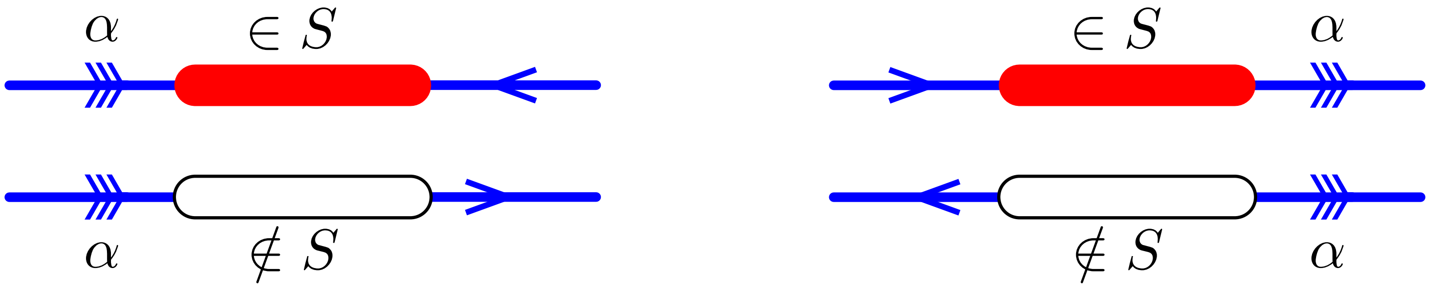

McConville, *Lattice structures of grid Tamari orders* ('17)

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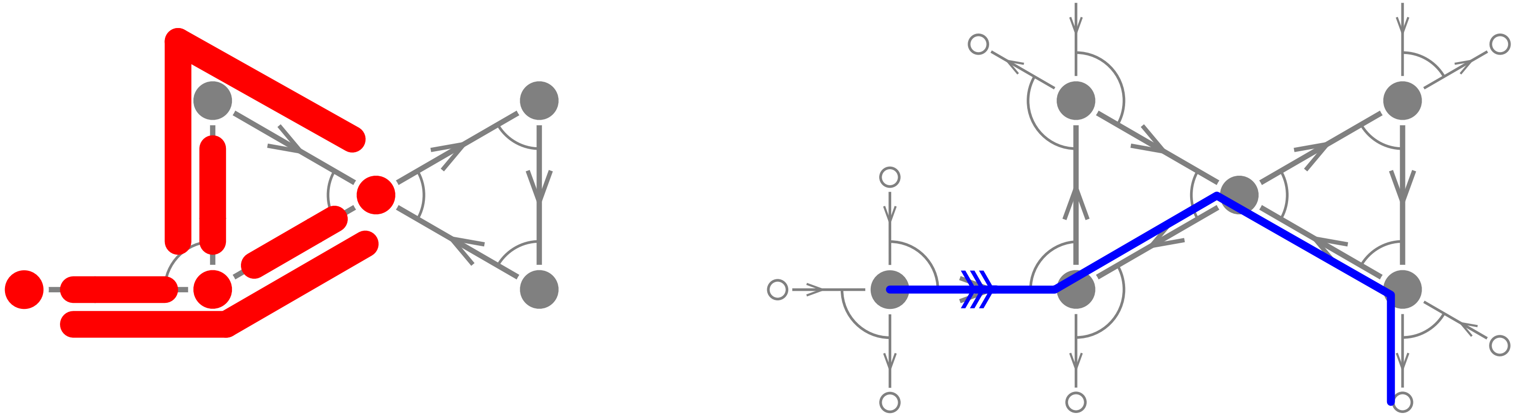

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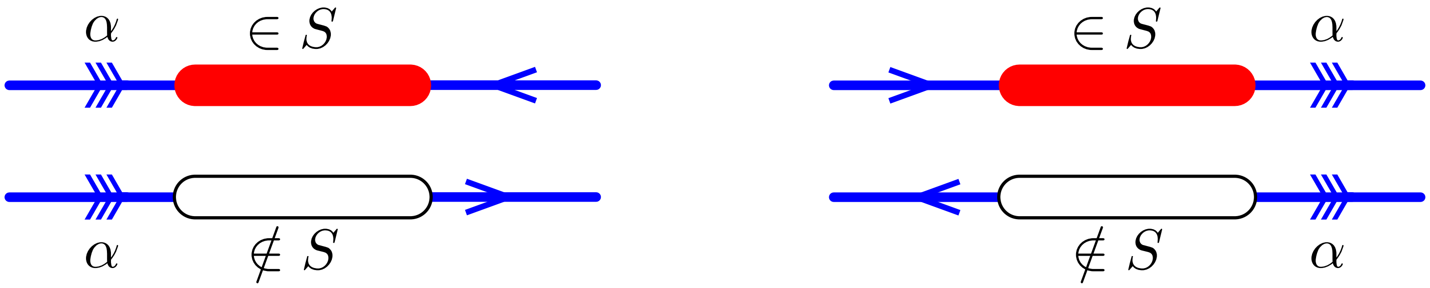

McConville, *Lattice structures of grid Tamari orders* ('17)

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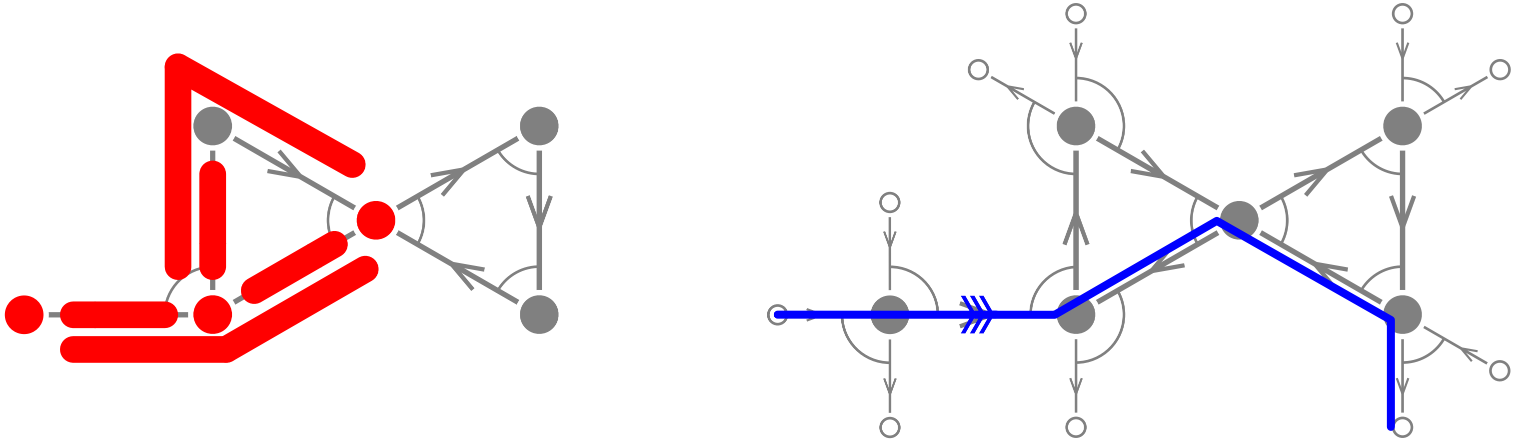

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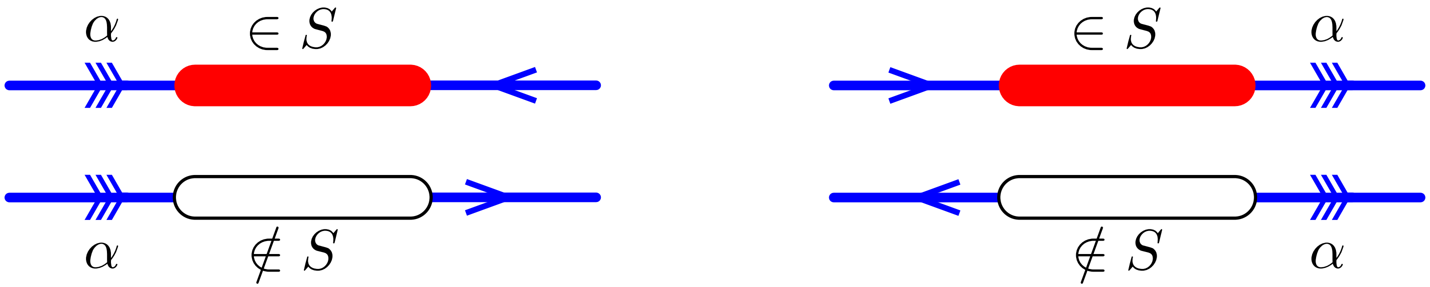

McConville, *Lattice structures of grid Tamari orders* ('17)

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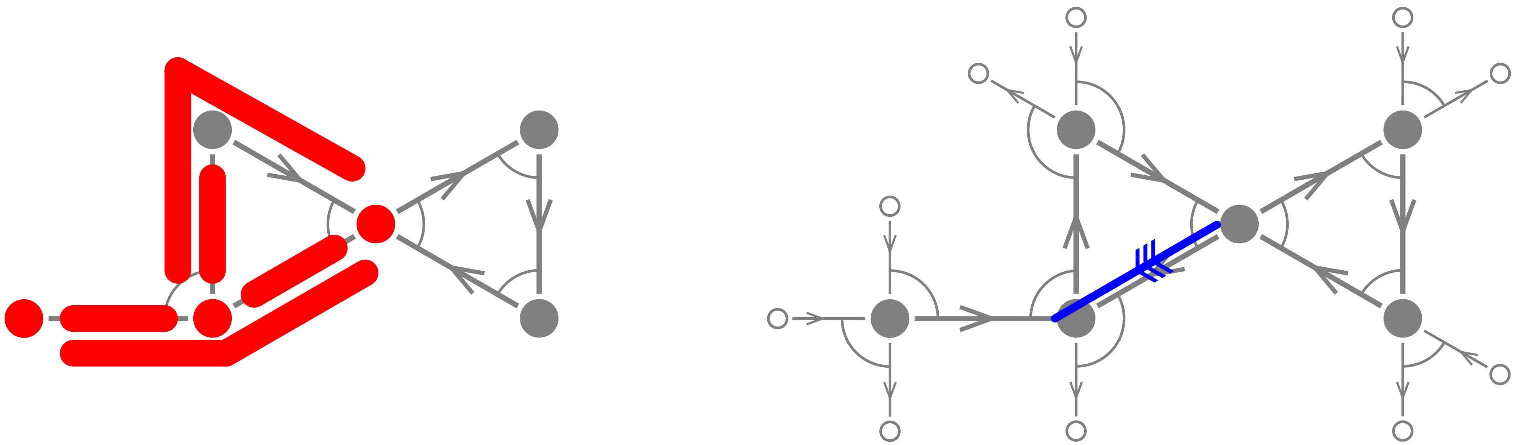
Surjection from biclosed sets of strings to non-kissing facets



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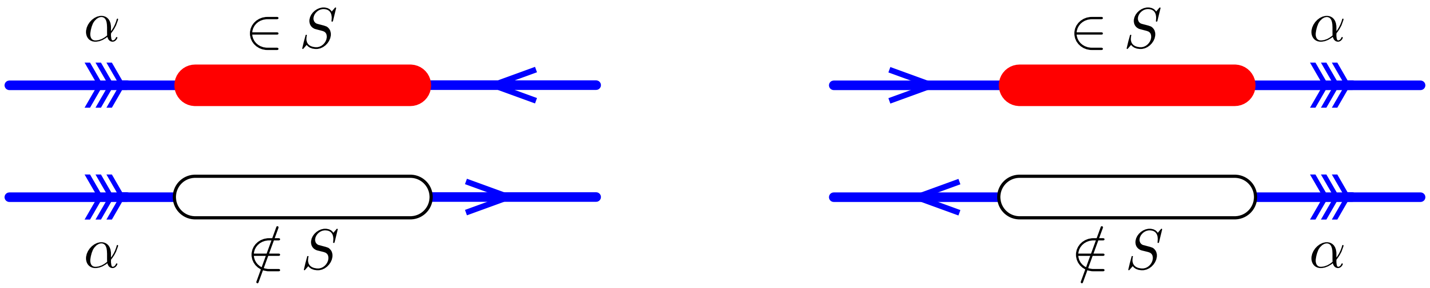

McConville, *Lattice structures of grid Tamari orders* ('17)

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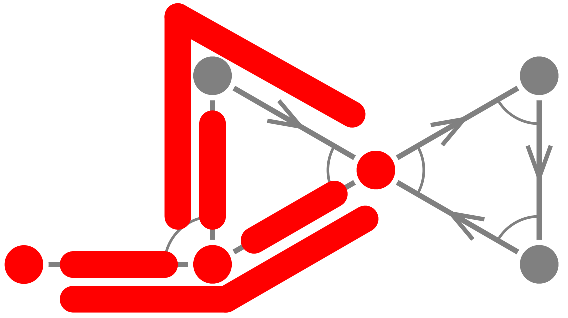
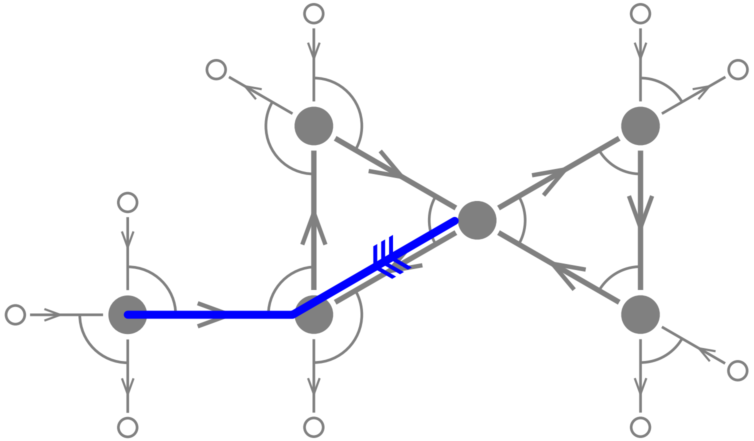
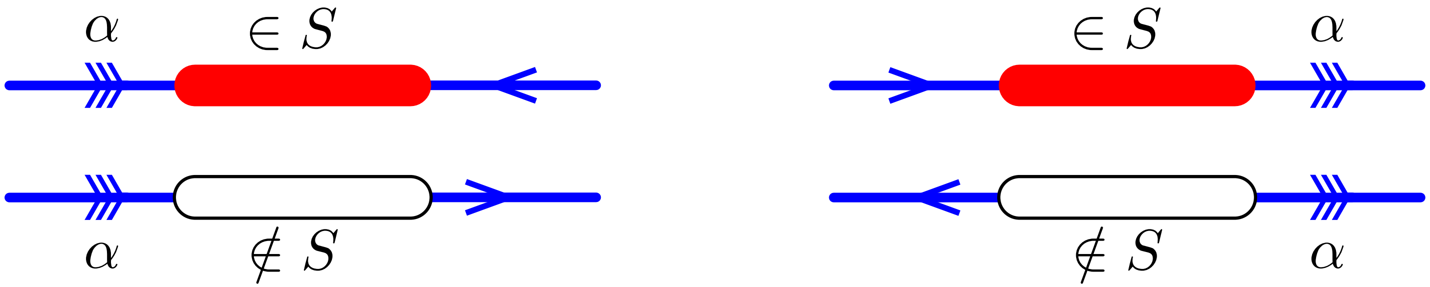
Surjection from biclosed sets of strings to non-kissing facets





S biclosed, $\alpha \in Q_1$

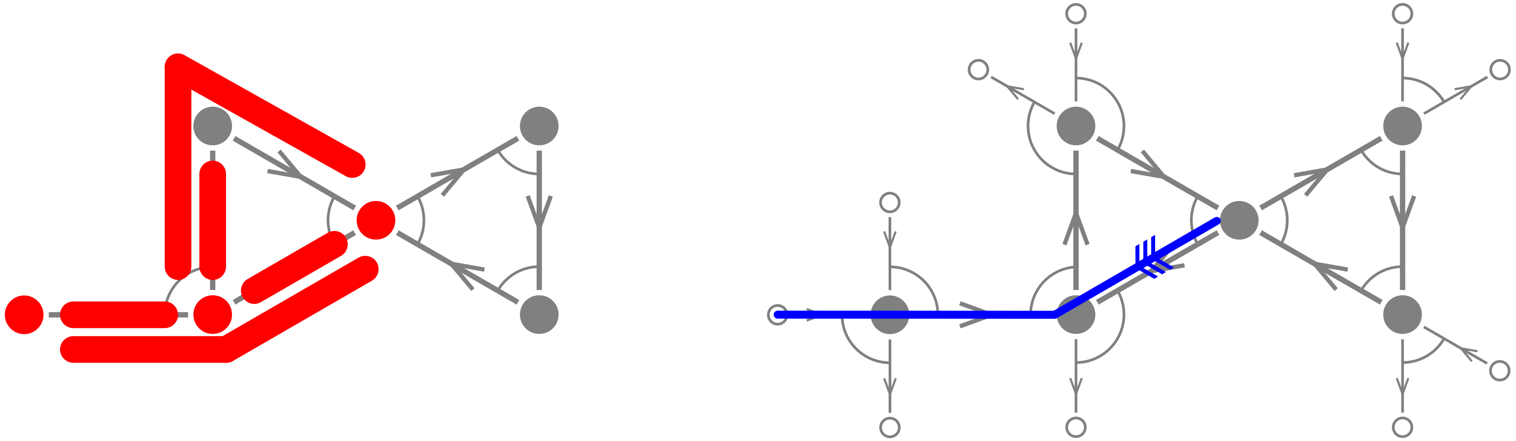
$\omega(\alpha, S) =$ walk constructed with the local rules:



McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets



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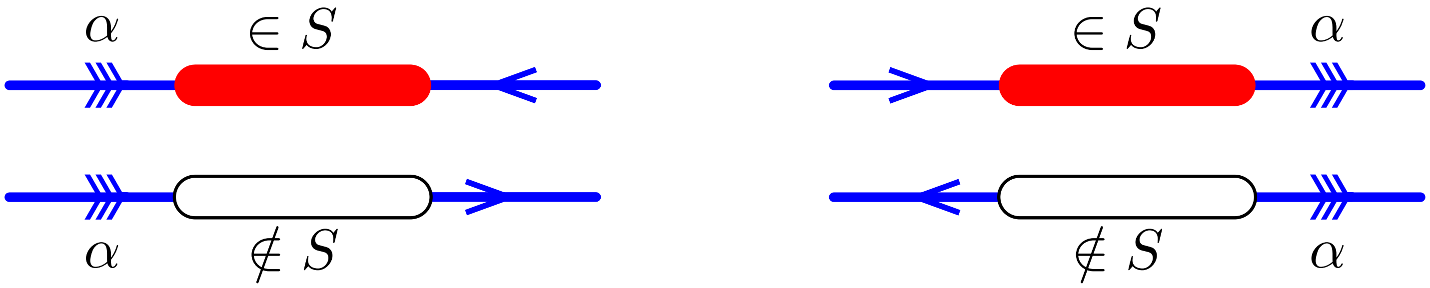

McConville, *Lattice structures of grid Tamari orders* ('17)

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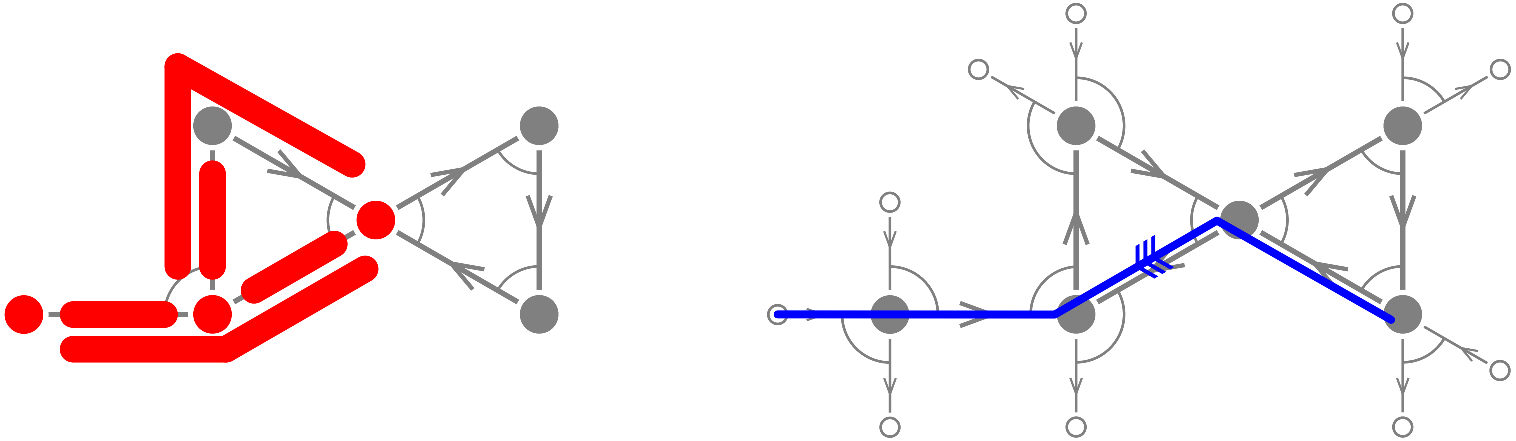

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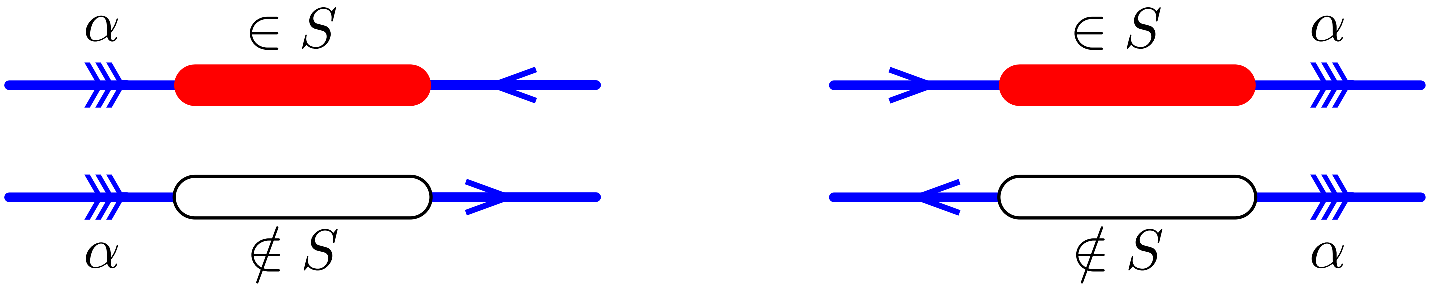

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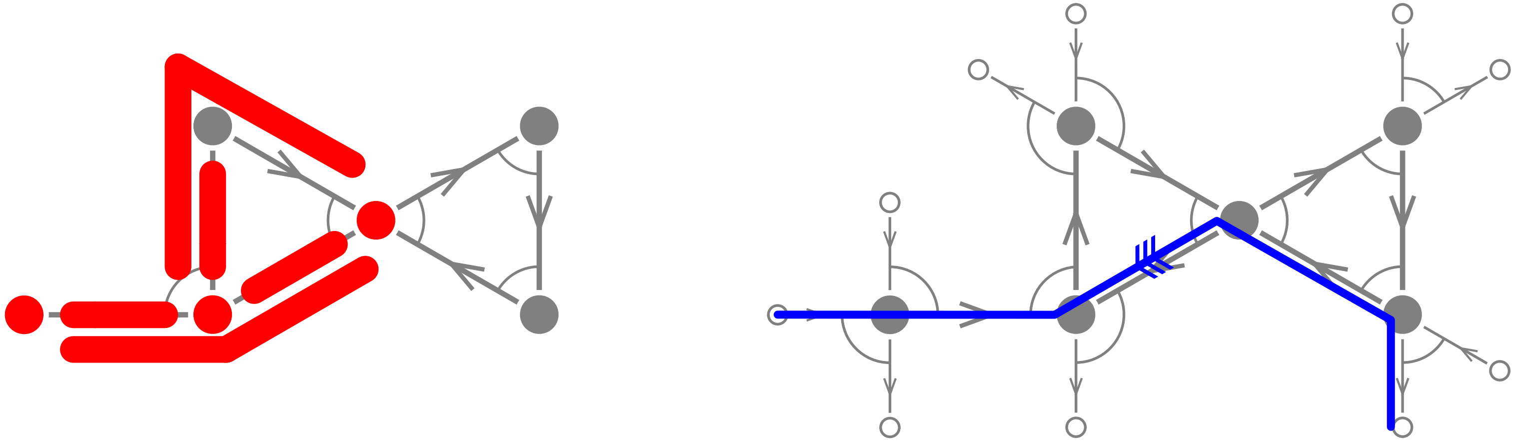

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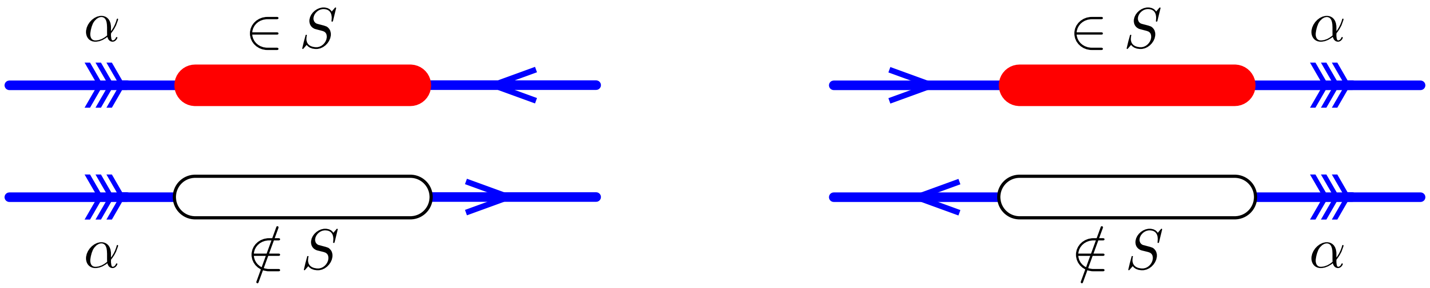

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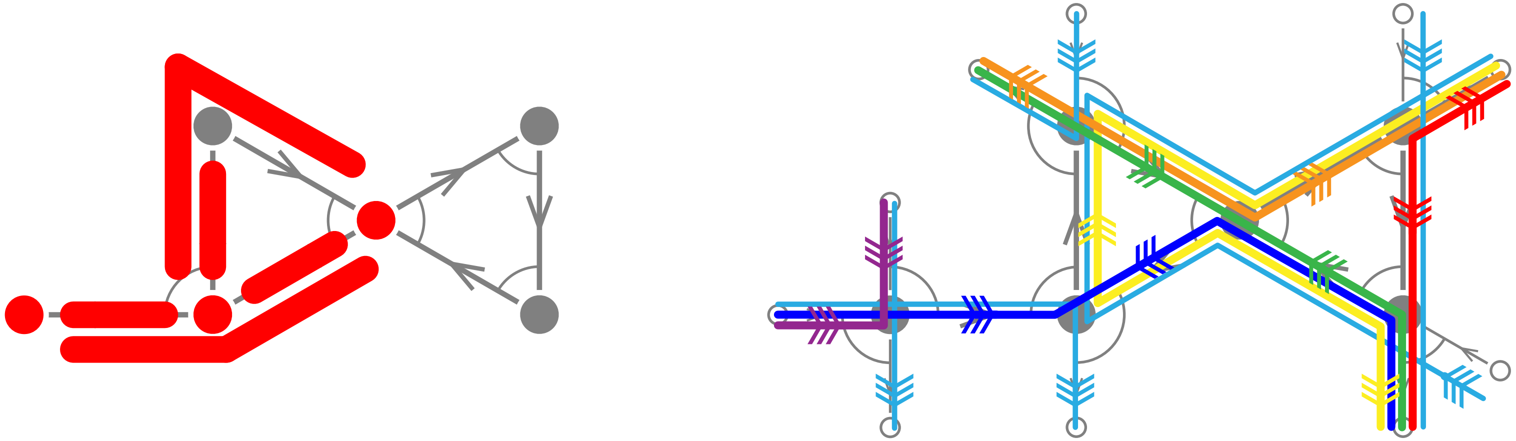
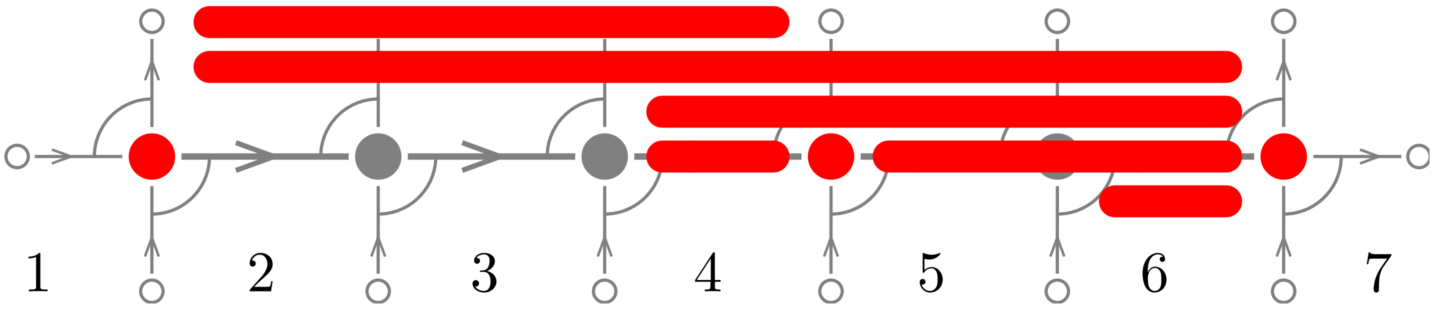


PROP. $\eta(S) \coloneqq \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

McConville, *Lattice structures of grid Tamari orders* ('17)

EXM: BINARY SEARCH TREE INSERTION AGAIN

inversion set of 2751346



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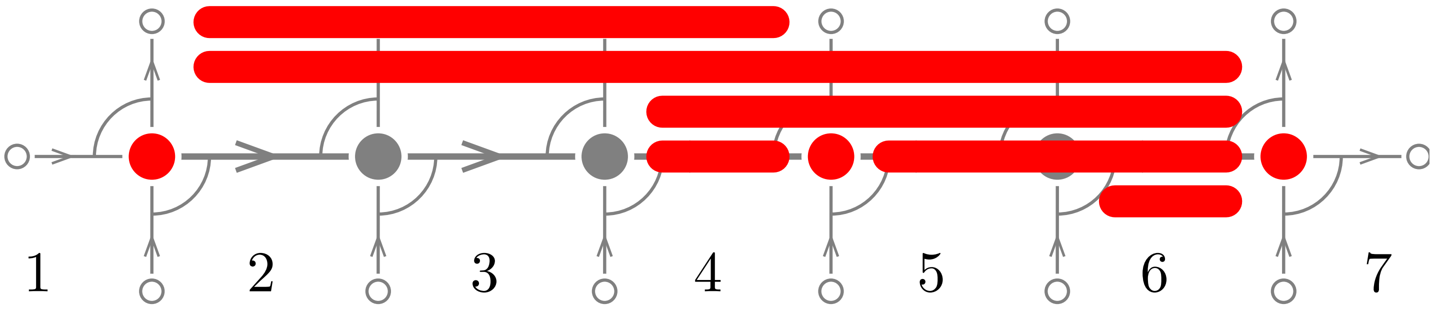

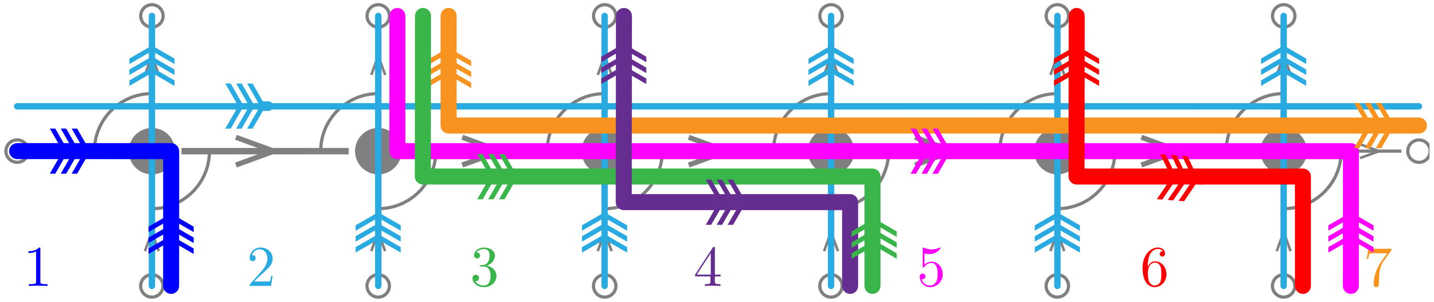

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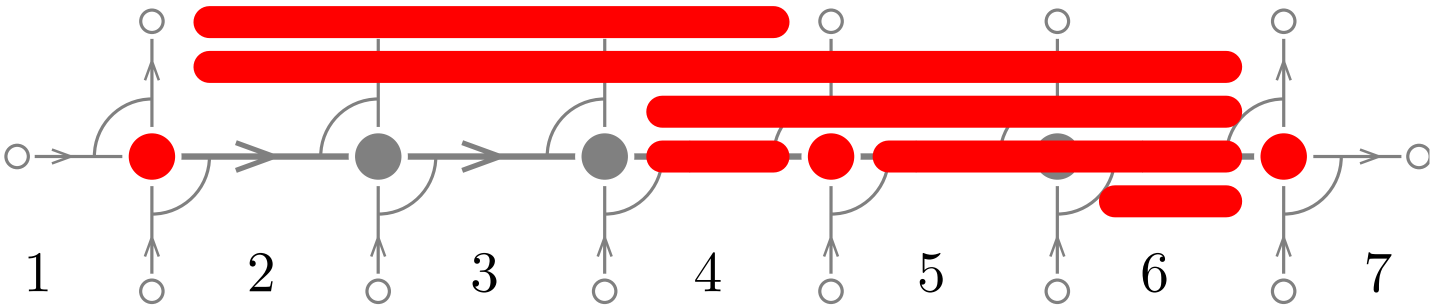
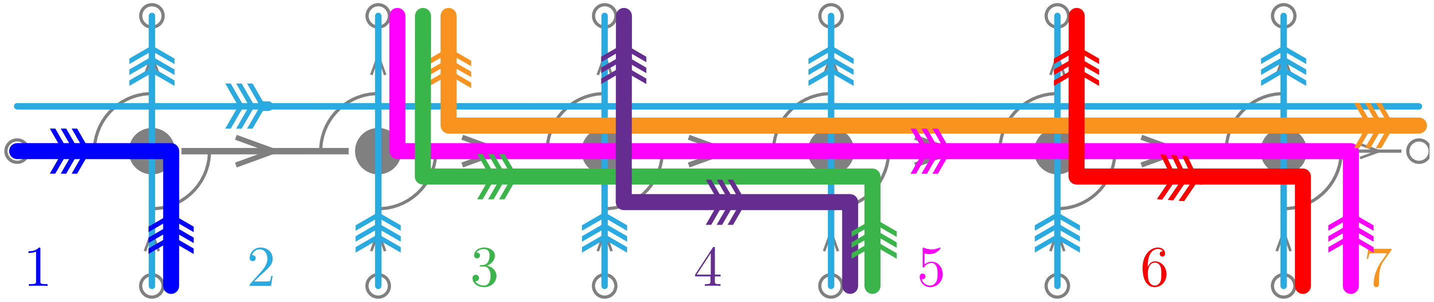
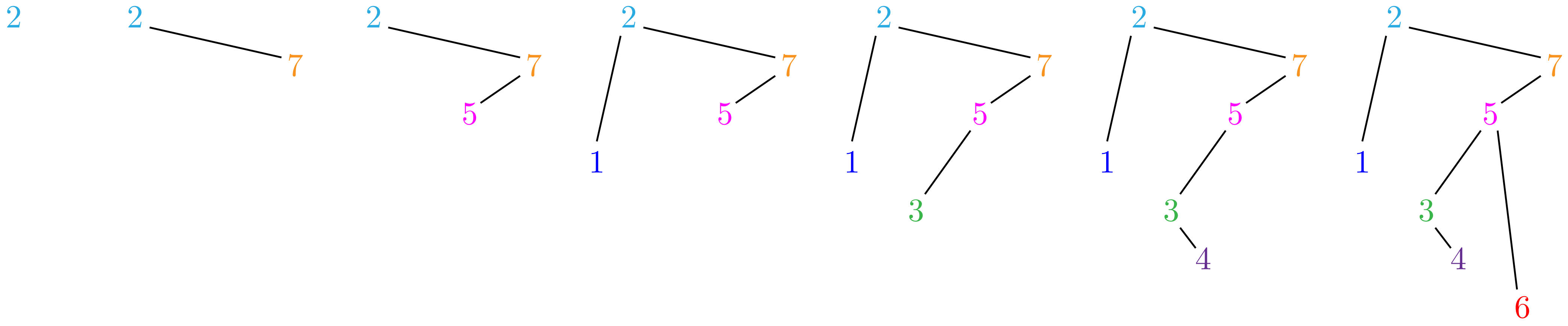
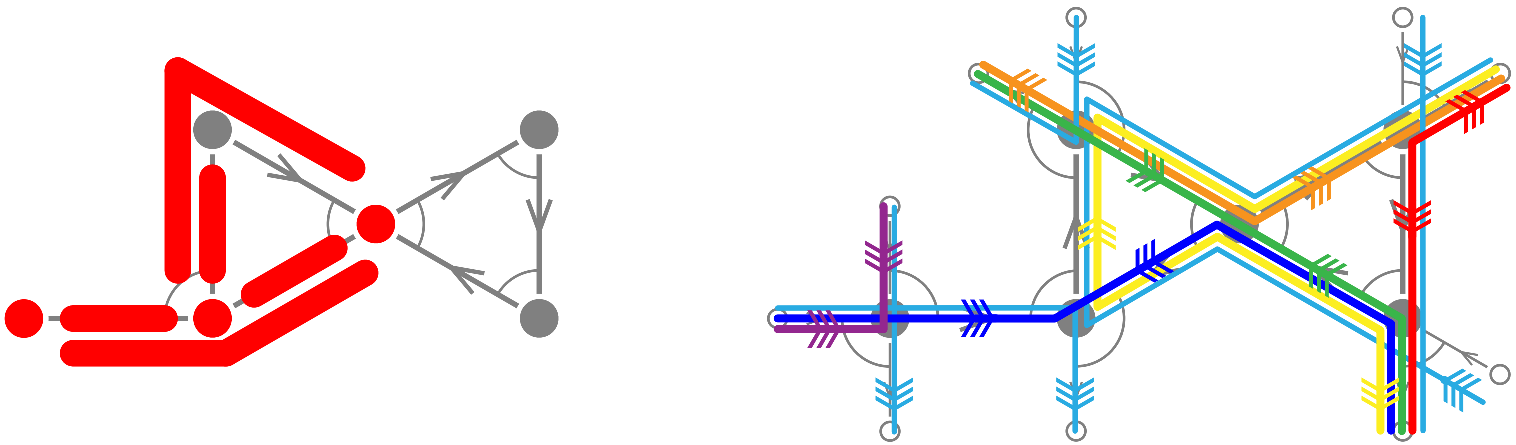






NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets

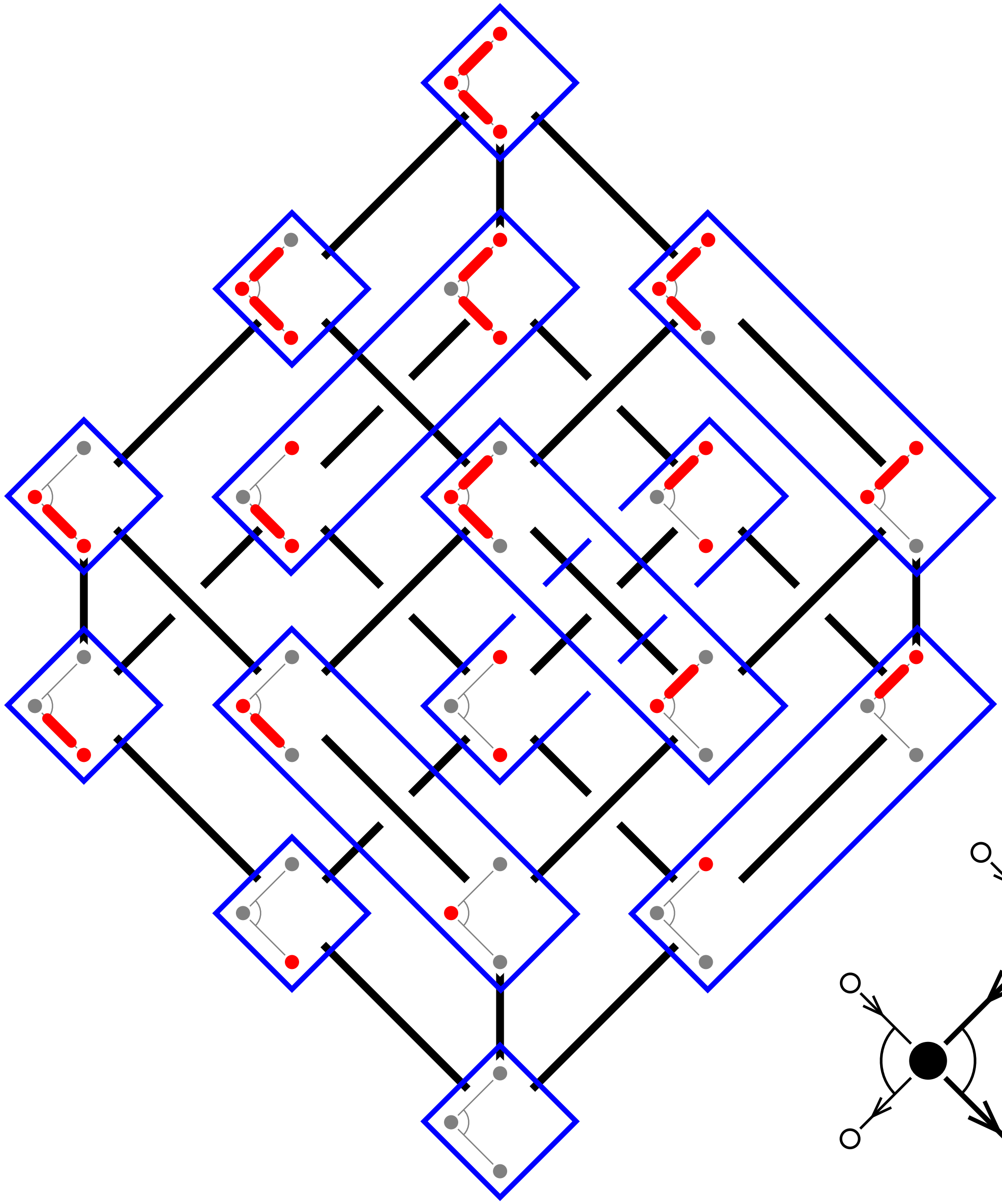


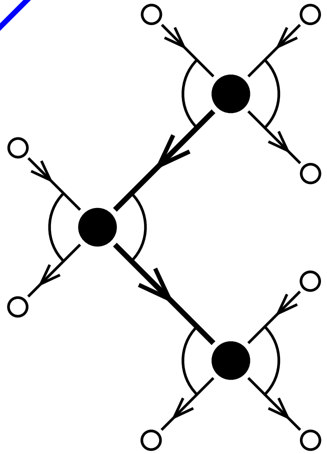
PROP. $\eta(S) \coloneqq \{\omega(\alpha, S) \mid \alpha \in Q_1\}$ is a non-kissing facet.

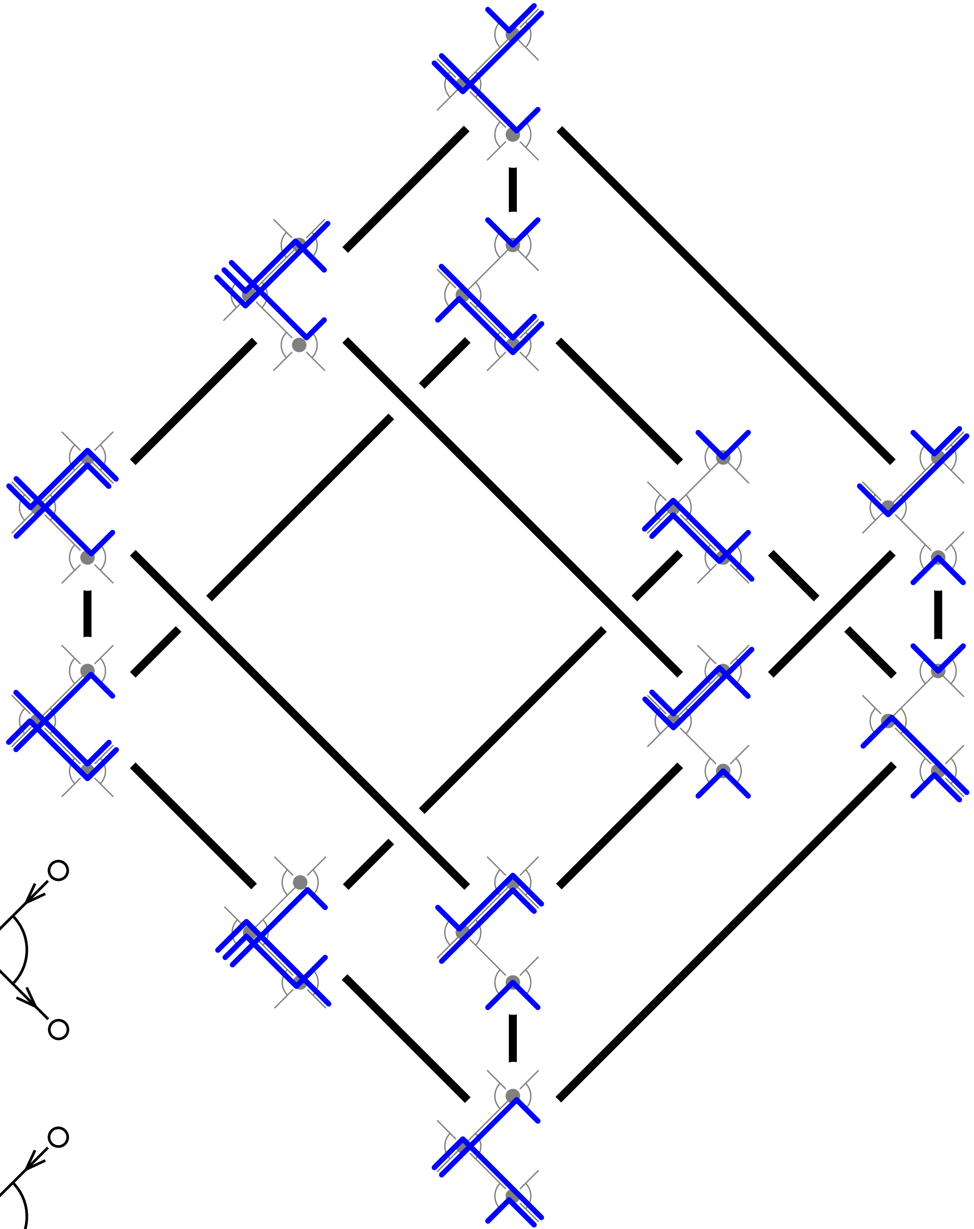
THM. The map η defines a lattice morphism from biclosed sets to non-kissing facets.

McConville, *Lattice structures of grid Tamari orders* ('17)

NON-KISSING LATTICE







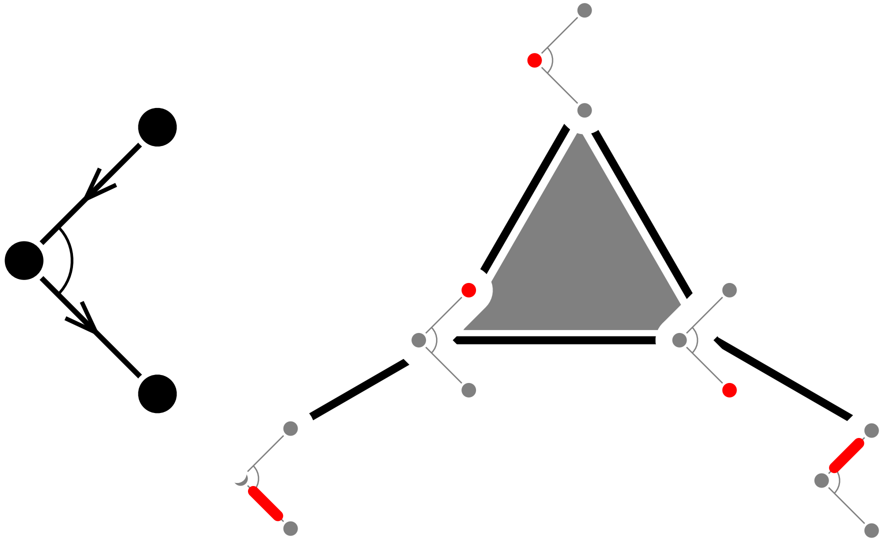
NON-KISSING LATTICE

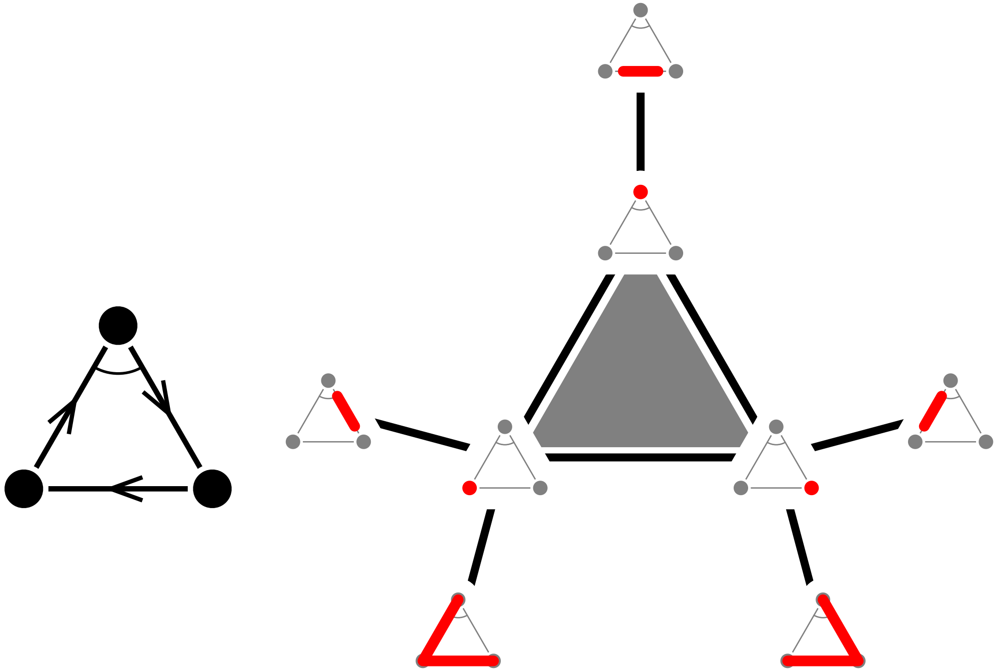
THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.

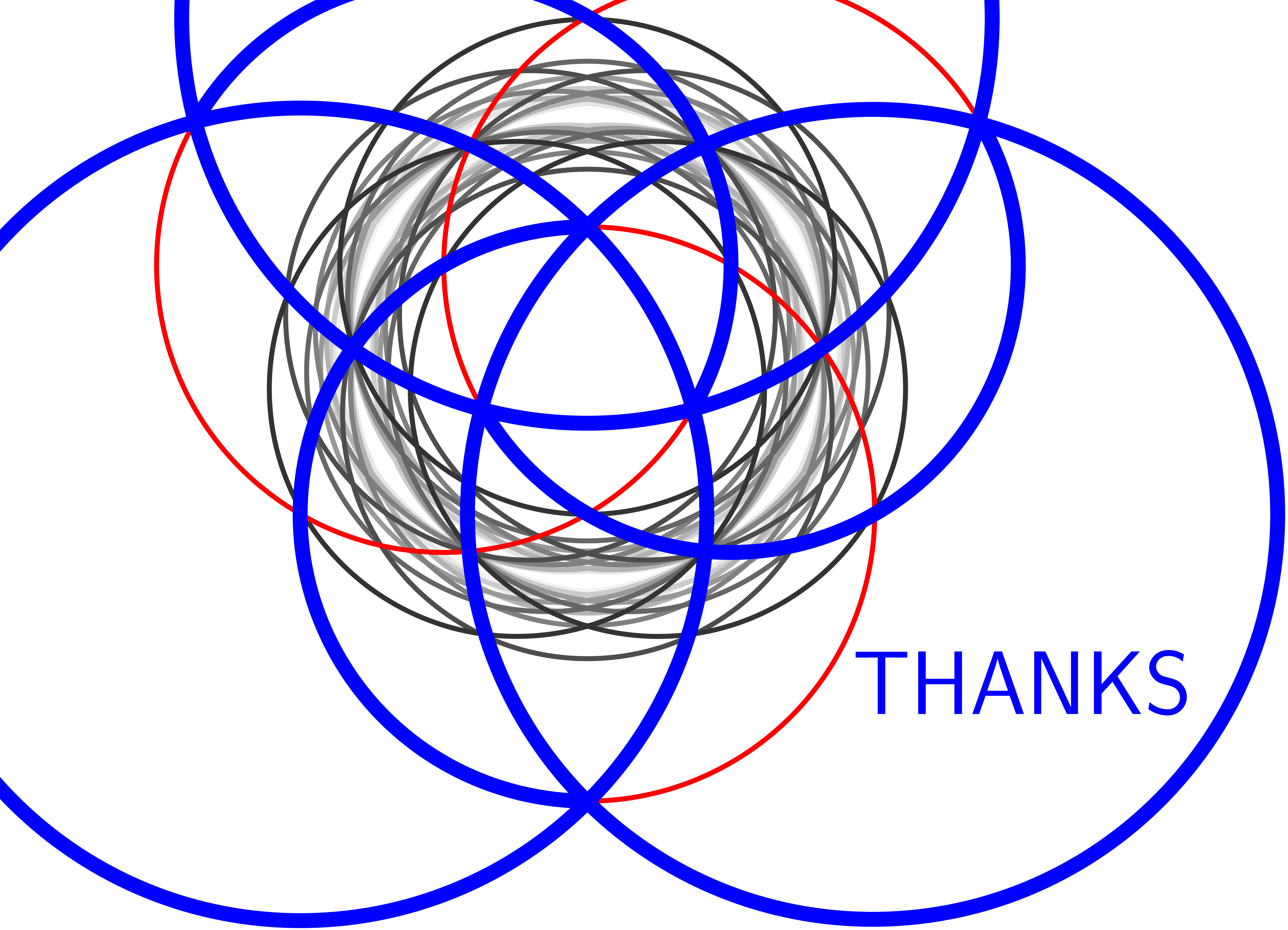
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle algebras* ('17$^+$)

Much more nice combinatorics:

- join-irreducible elements of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ are in bijection with distinguishable strings
- canonical join complex of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ is a generalization of non-crossing partitions







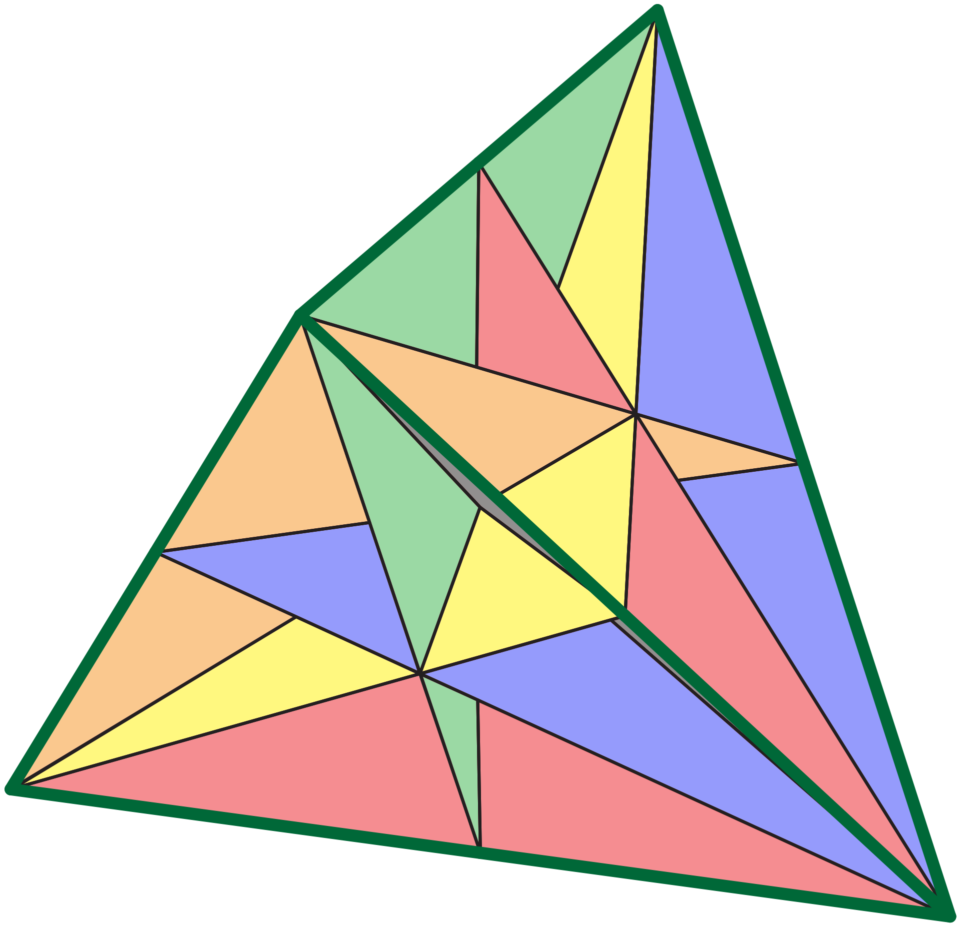
THANKS

FINITE COXETER GROUPS

Humphreys, *Reflection groups and Coxeter groups* ('90)
Bjorner-Brenti, *Combinatorics of Coxeter groups* ('05)

FINITE COXETER GROUPS

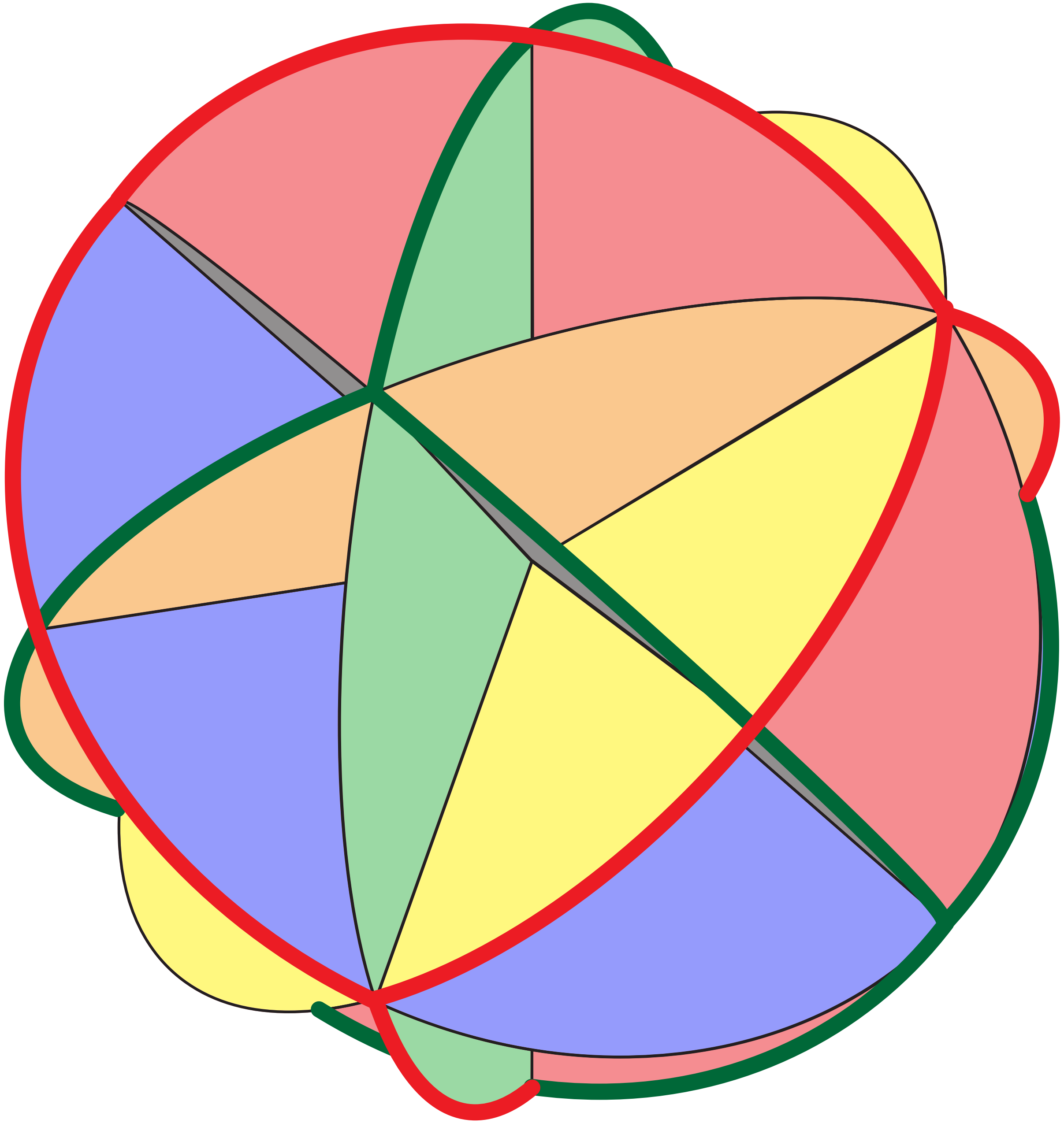
W = finite Coxeter group



FINITE COXETER GROUPS

W = finite Coxeter group

Coxeter fan



FINITE COXETER GROUPS

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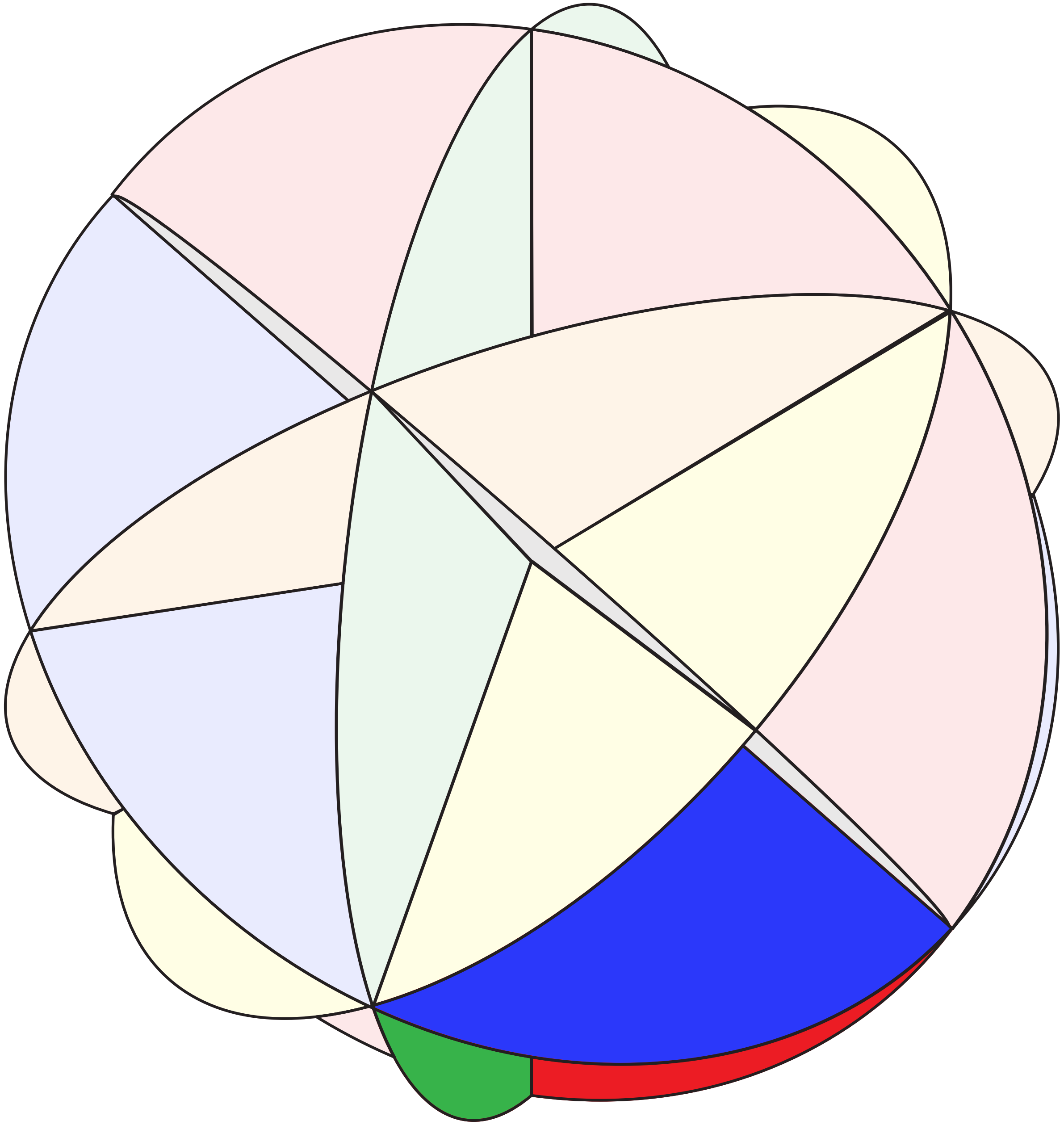
fundamental chamber



FINITE COXETER GROUPS

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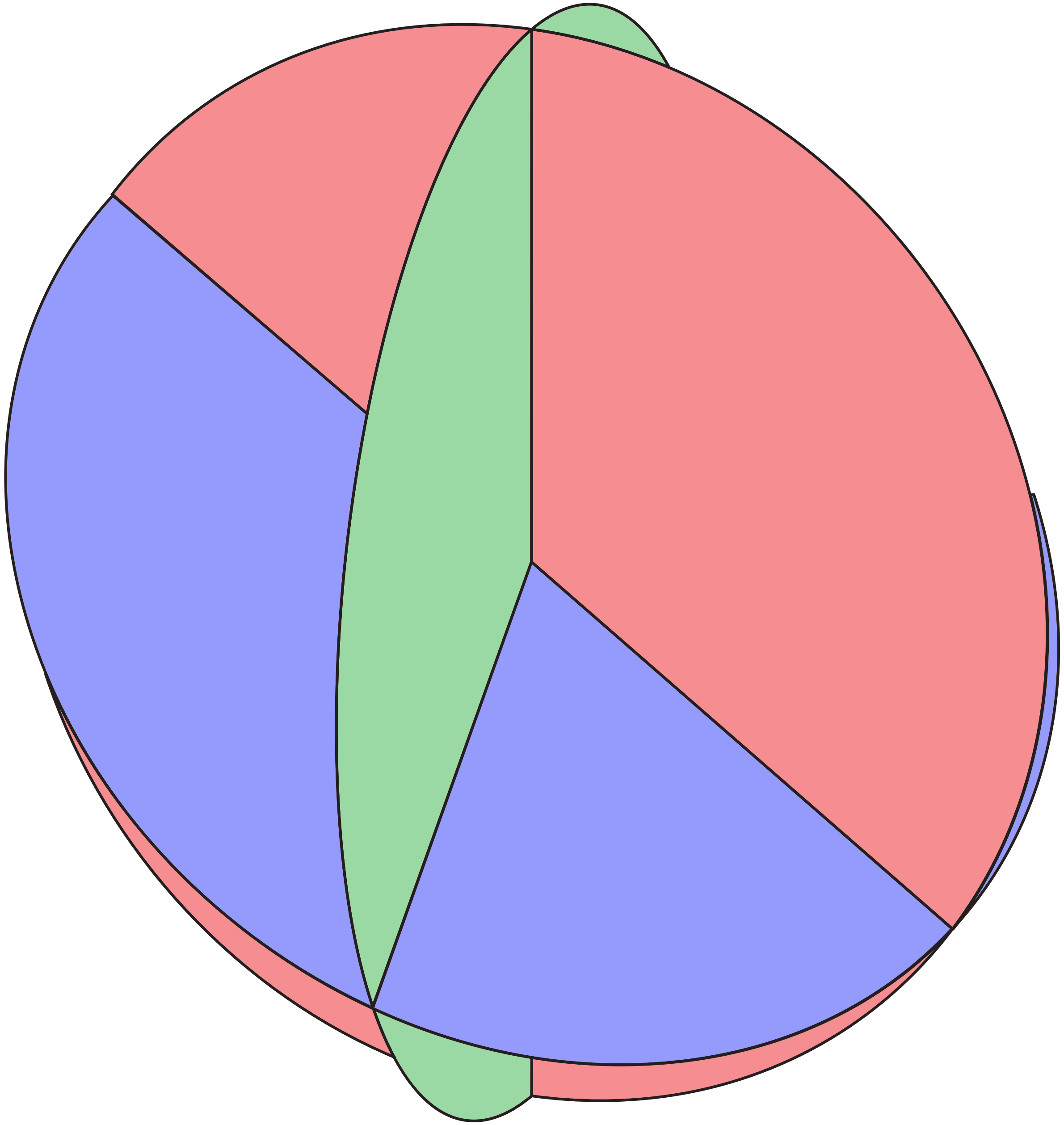

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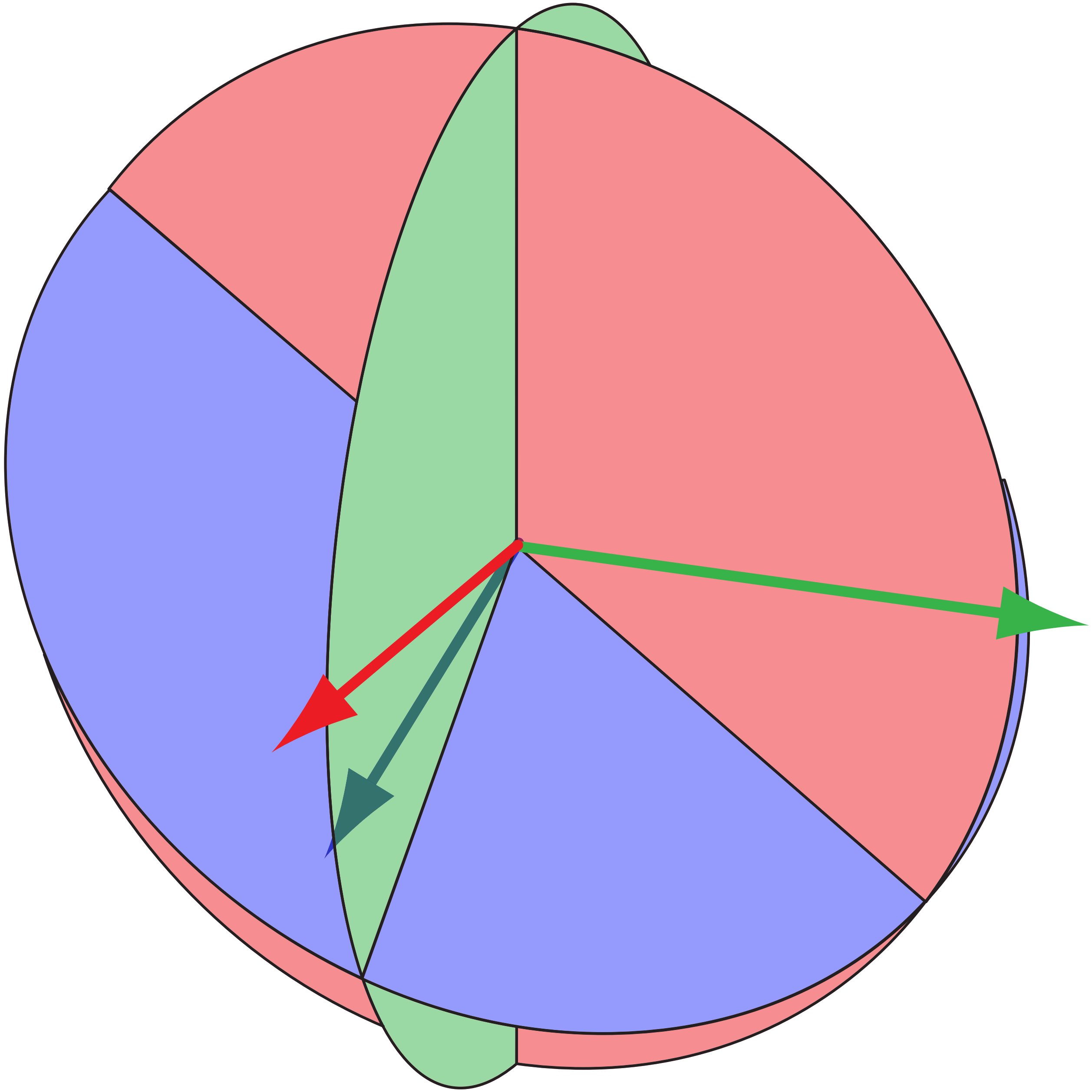

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$\Delta = \{\alpha_s \mid s \in S\}$ = simple roots

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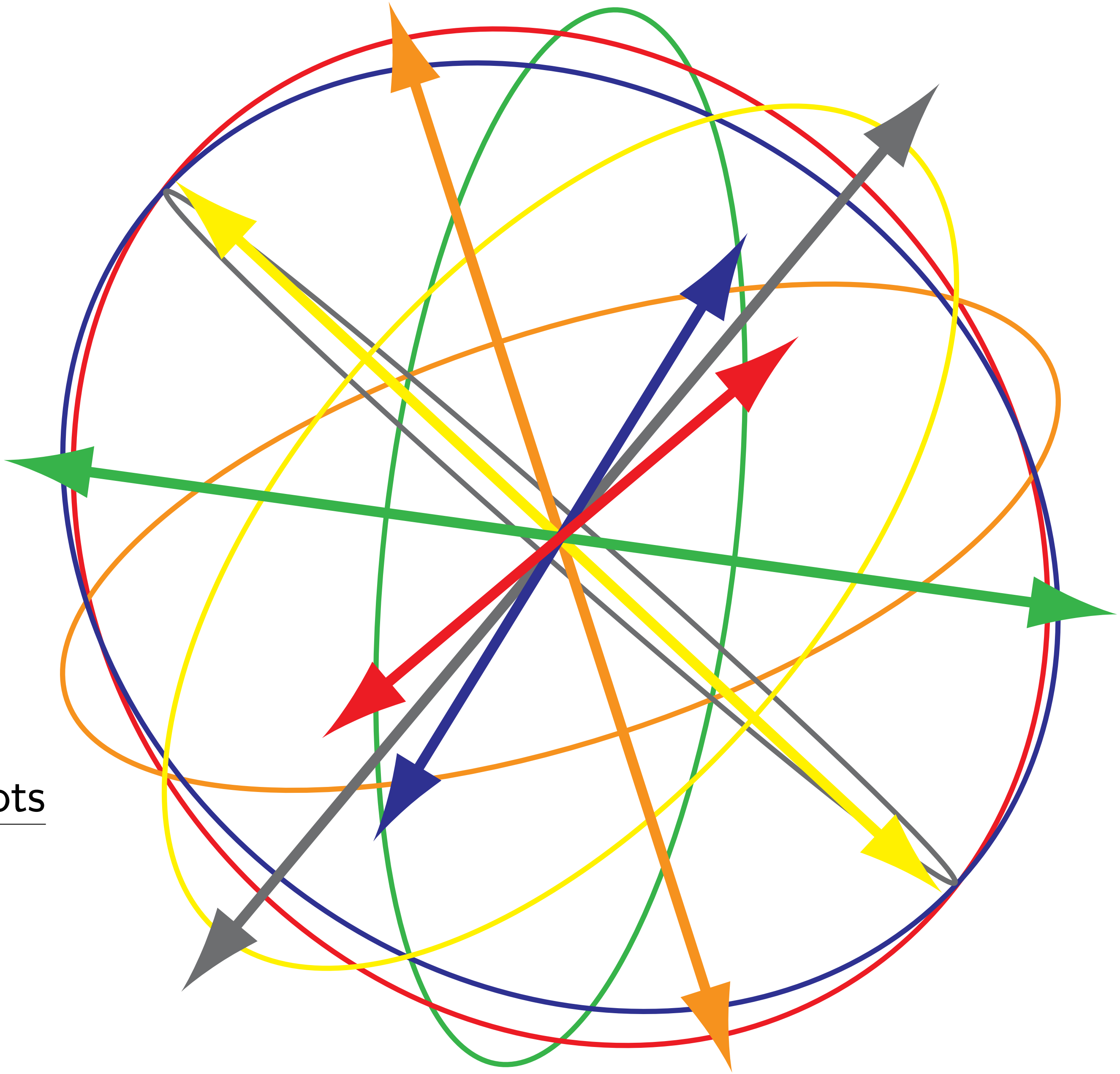

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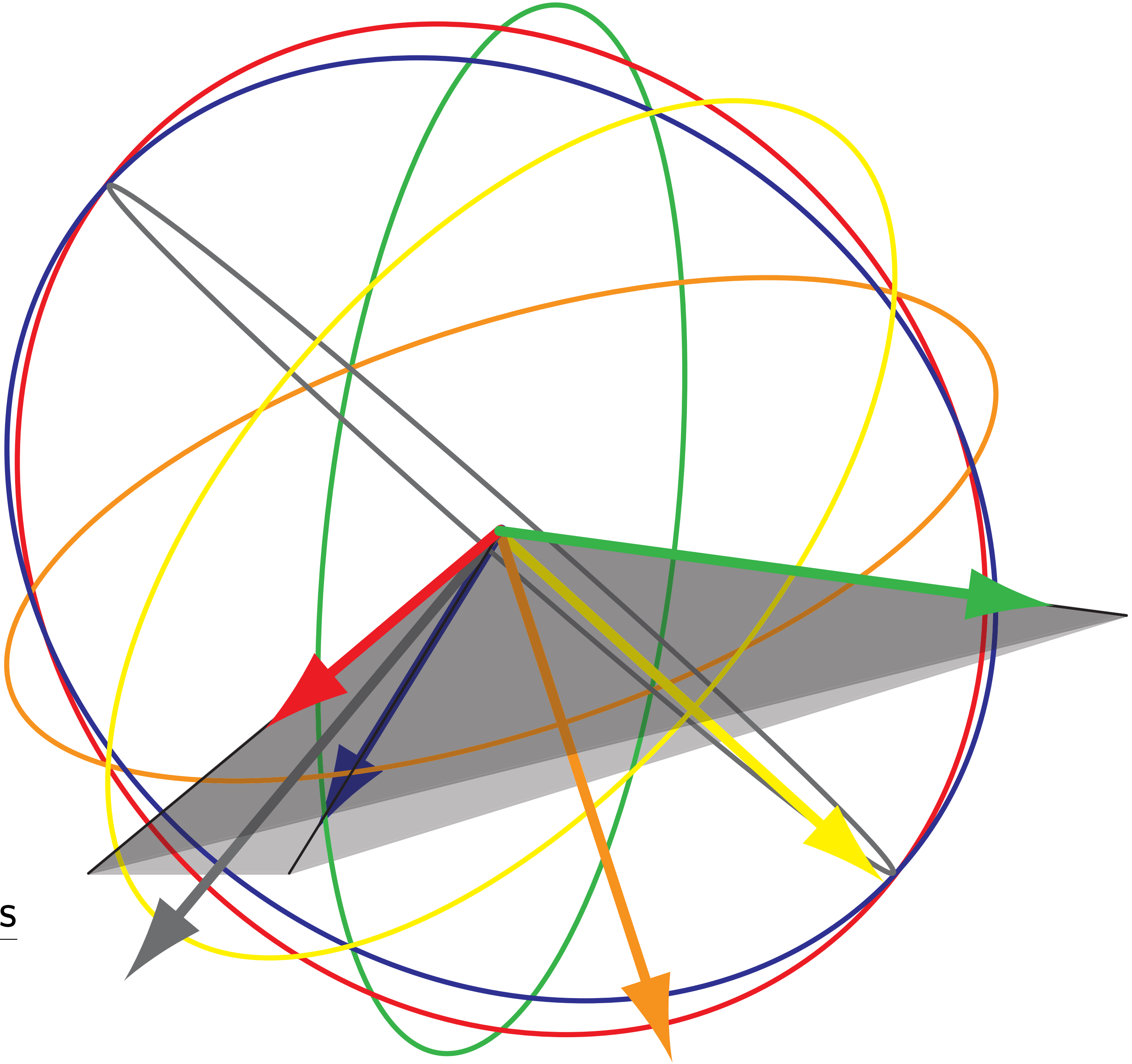

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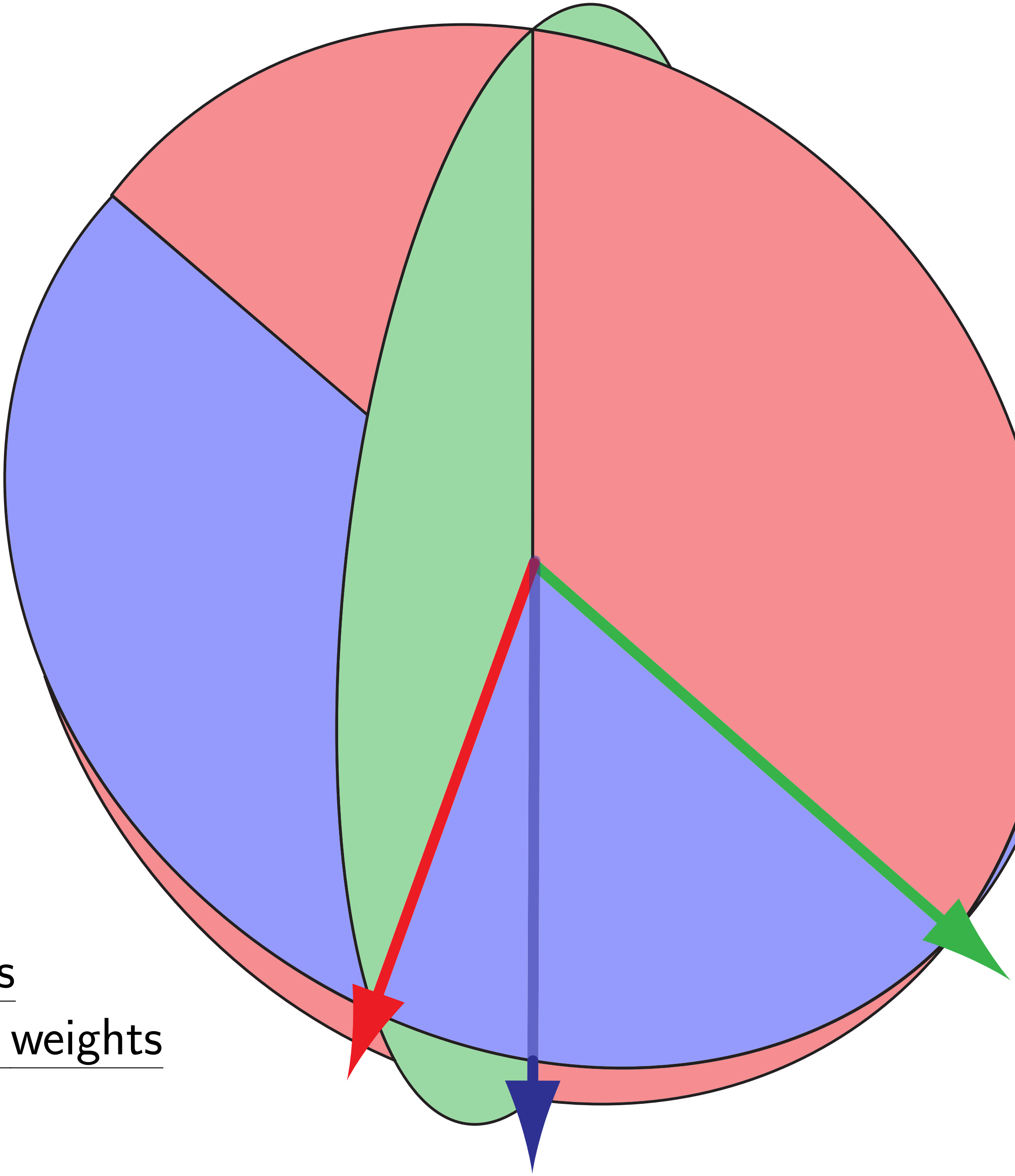

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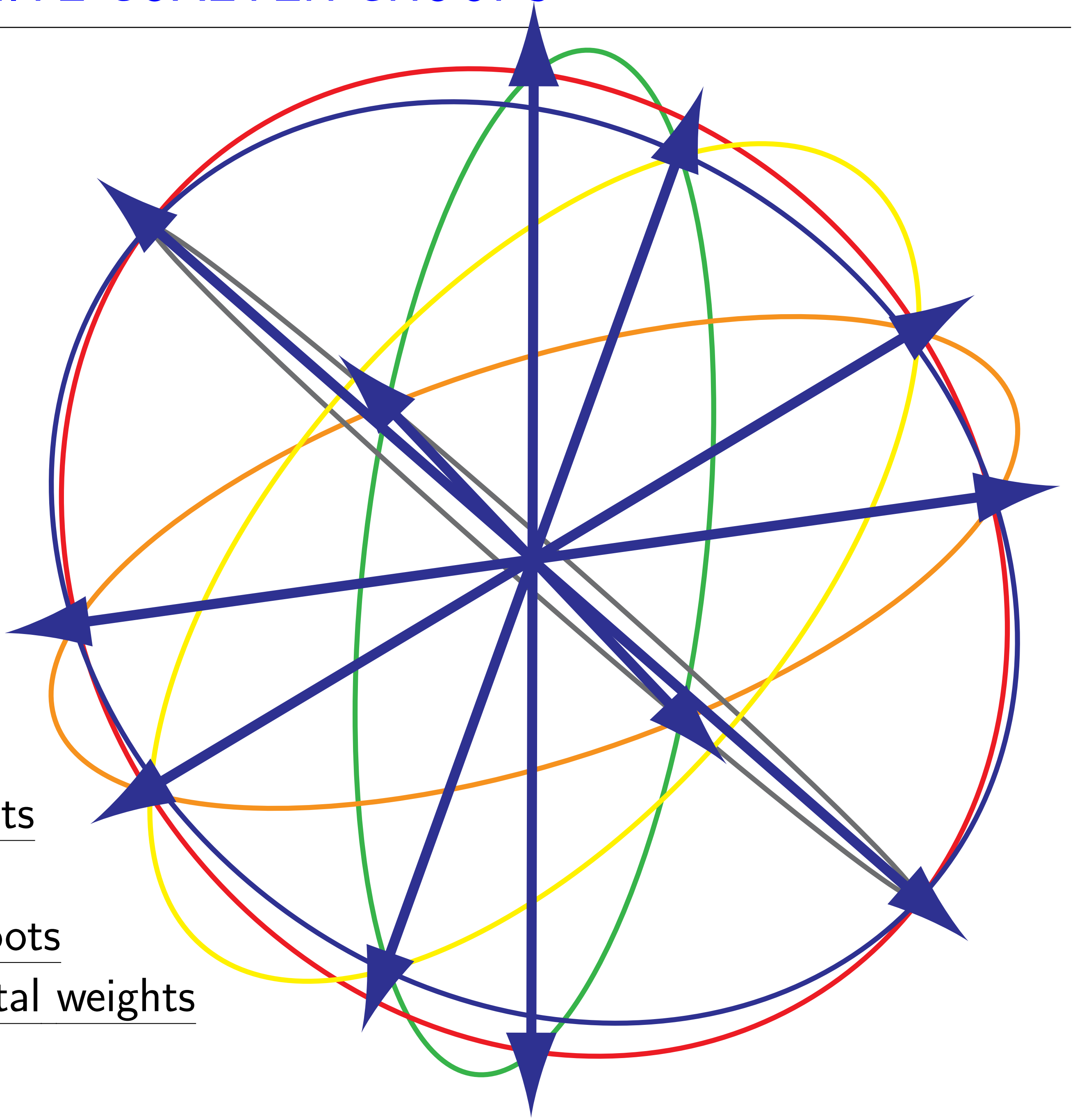

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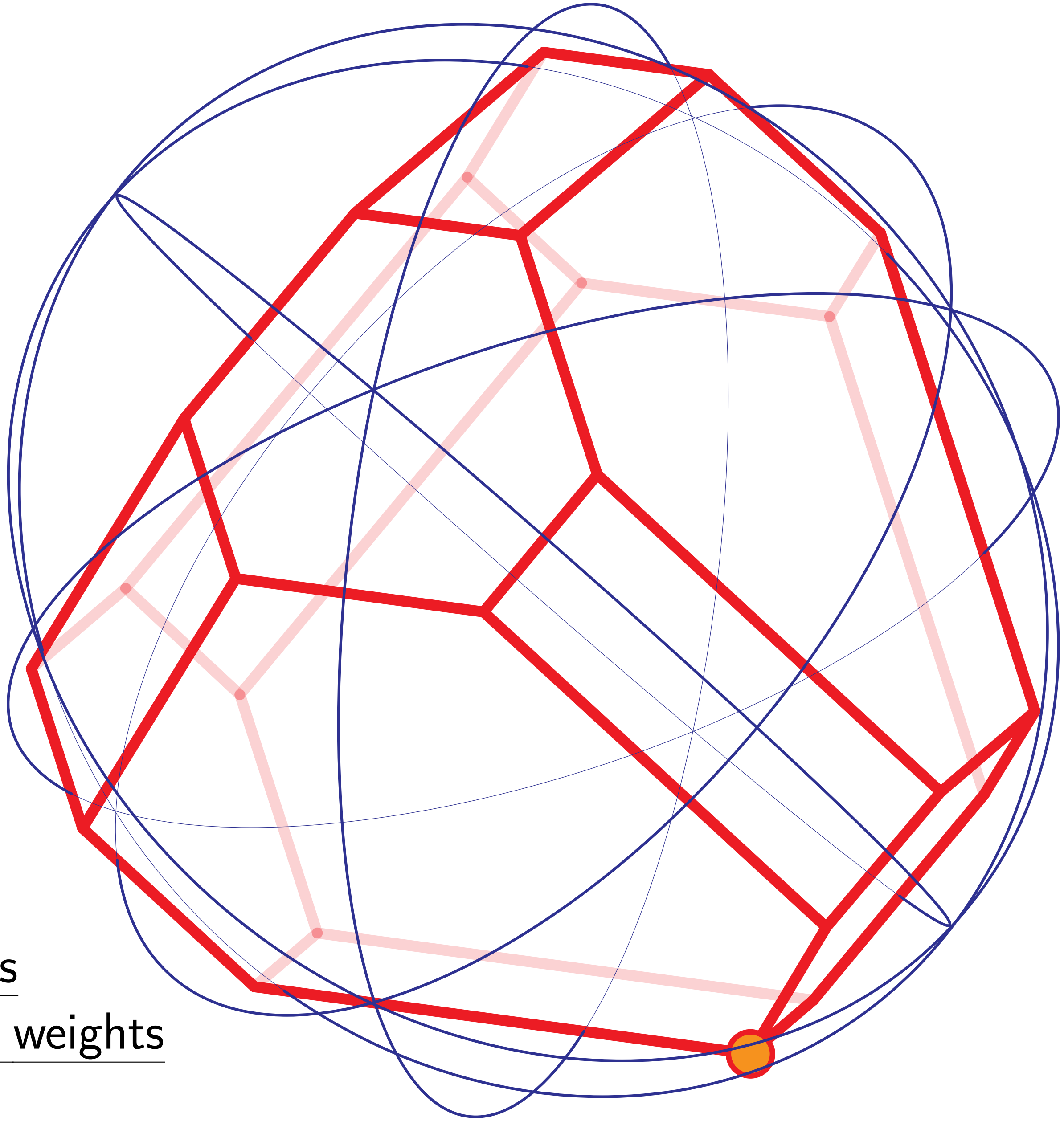

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permutahedron

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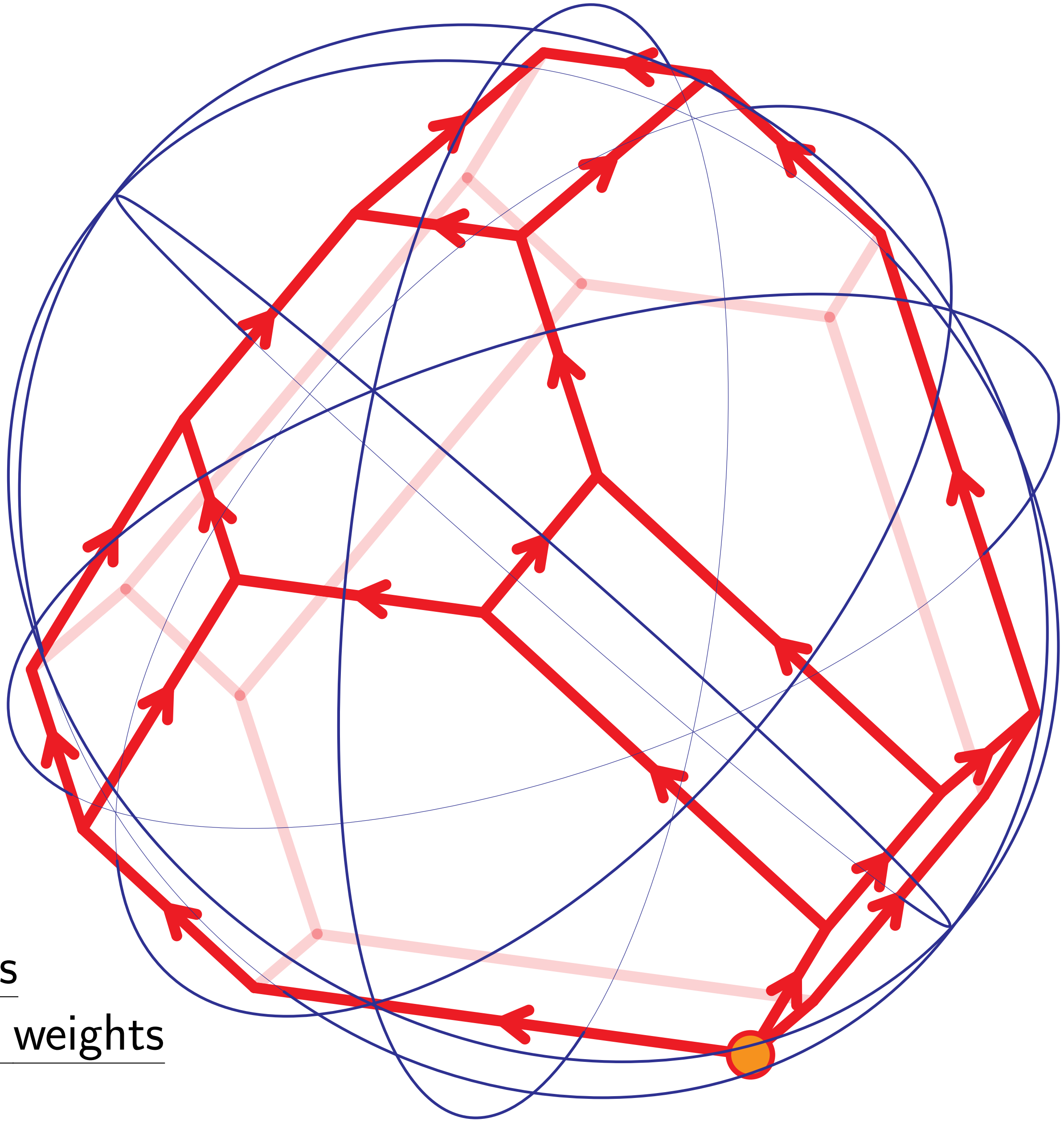

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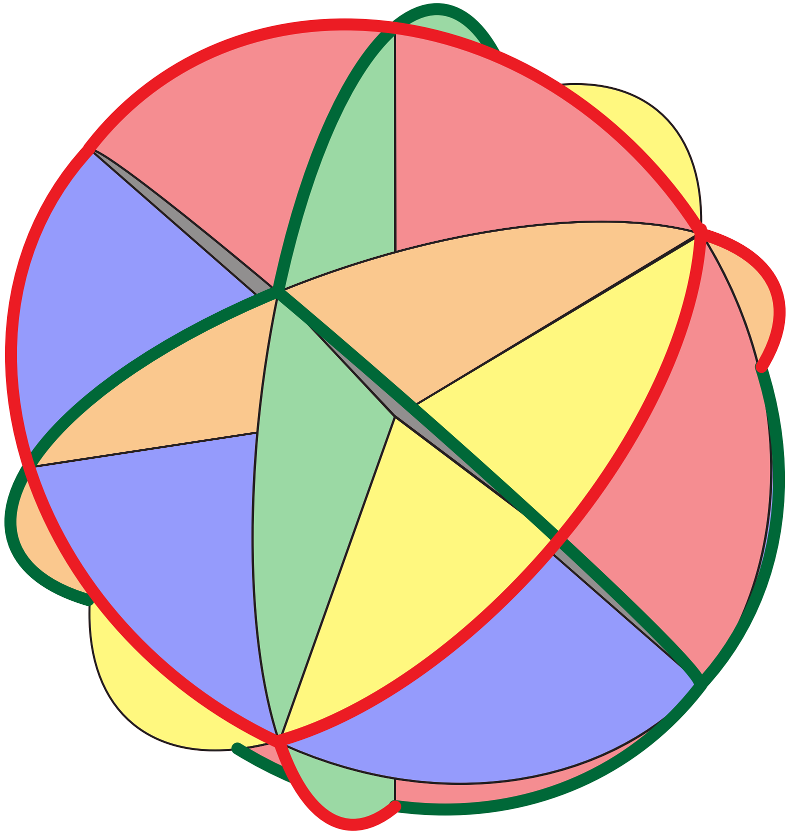
permutahedron

weak order = $u \leq w \iff \exists v \in W,\ uv = w$ and $\ell(u) + \ell(v) = \ell(w)$

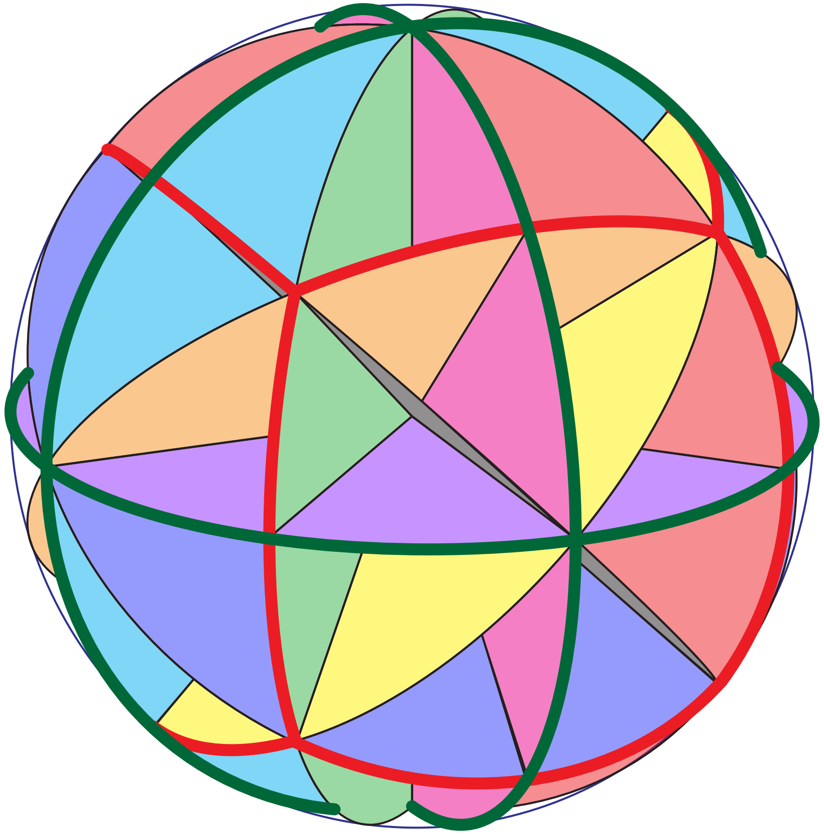
EXAMPLES: TYPE A AND B

TYPE A_n $=$ symmetric group $\mathfrak{S}_{n+1}$



$$S = \{(i, i+1) \mid i \in [n]\}$$
$$\Delta = \{e_{i+1} - e_i \mid i \in [n]\}$$
$$\text{roots} = \{e_i - e_j \mid i, j \in [n+1]\}$$
$$\nabla = \left\{\sum_{j>i} e_j \;\middle|\; i \in [n]\right\}$$

TYPE B_n $=$ semidirect product $\mathfrak{S}_n \rtimes (\mathbb{Z}_2)^n$



$$S = \{(i, i+1) \mid i \in [n-1]\} \cup \{\chi\}$$
$$\Delta = \{e_{i+1} - e_i \mid i \in [n-1]\} \cup \{e_1\}$$
$$\text{roots} = \{\pm e_i \pm e_j \mid i, j \in [n]\} \cup \{\pm e_i \mid i \in [n]\}$$
$$\nabla = \left\{\sum_{j\ge i} e_j \;\middle|\; i \in [n]\right\}$$

back to cluster algebras