UNEXPECTED DIAGONALS

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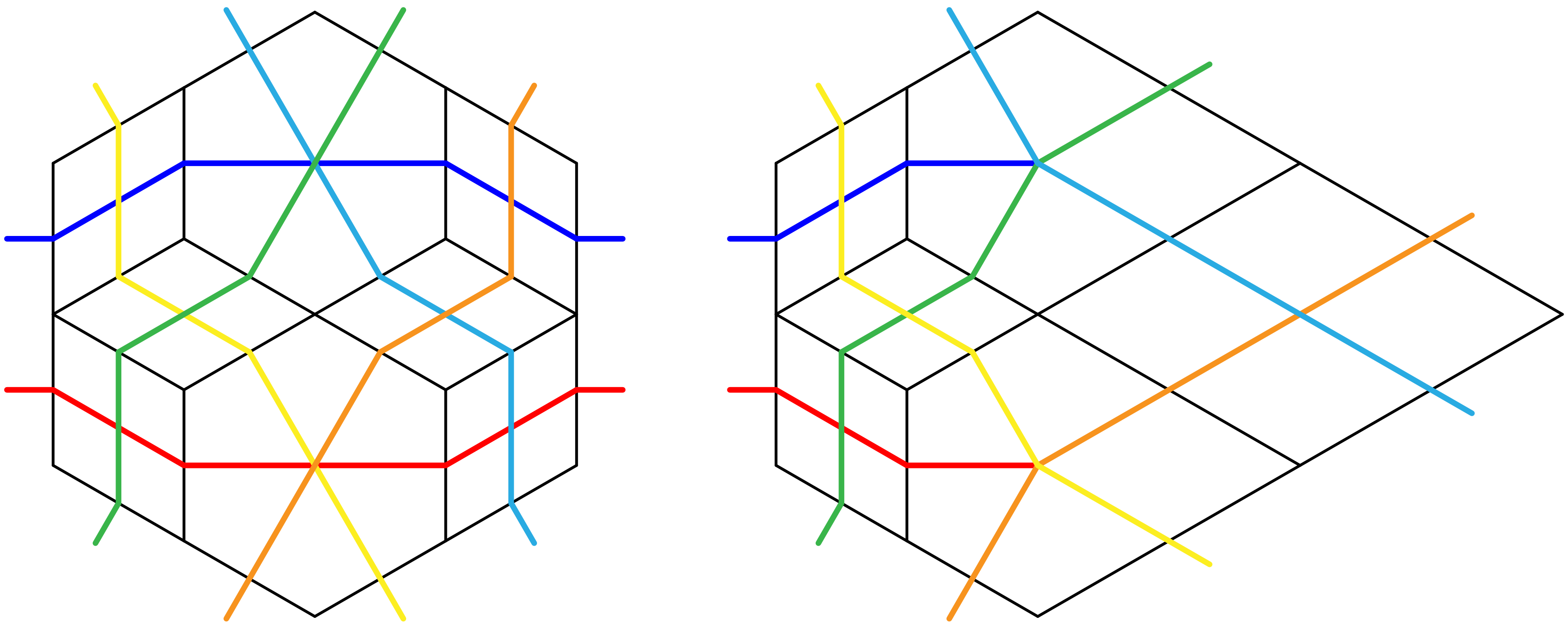
Alin BOSTAN (INRIA)
Frédéric CHYZAK (INRIA)

arXiv:2308.12119

arXiv:2303.10986



slides available at: http://www.lix.polytechnique.fr/~pilaud/documents/presentations/diagonals.pdf

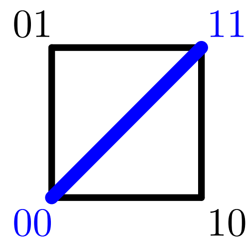
DIAGONALS OF POLYTOPES

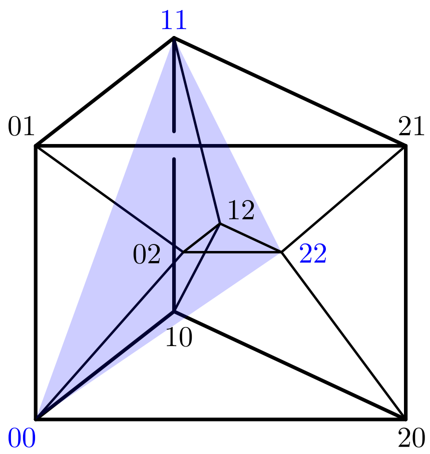
DIAGONALS OF POLYTOPES

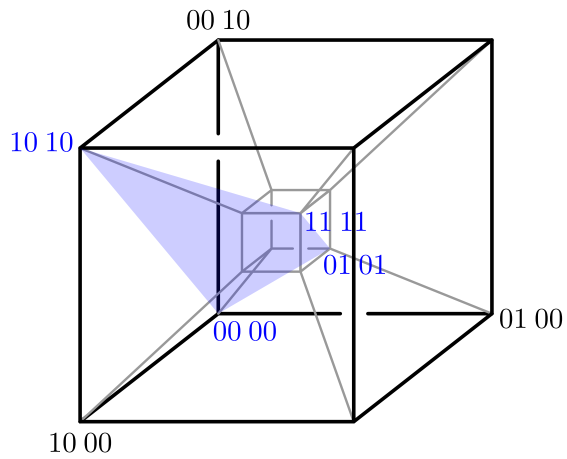
$\mathbb{P}$ polytope in $\mathbb{R}^d$

<u>diagonal</u> of $\mathbb{P} = \delta : \ \mathbb{P} \ \rightarrow \ \mathbb{P} \times \mathbb{P}$

$$p \ \mapsto \ (p, p)$$







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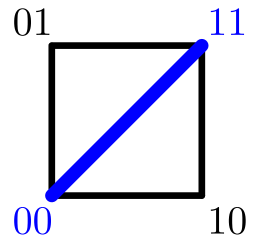
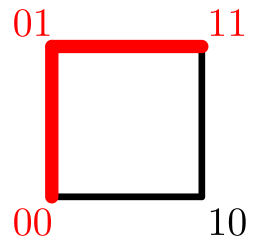


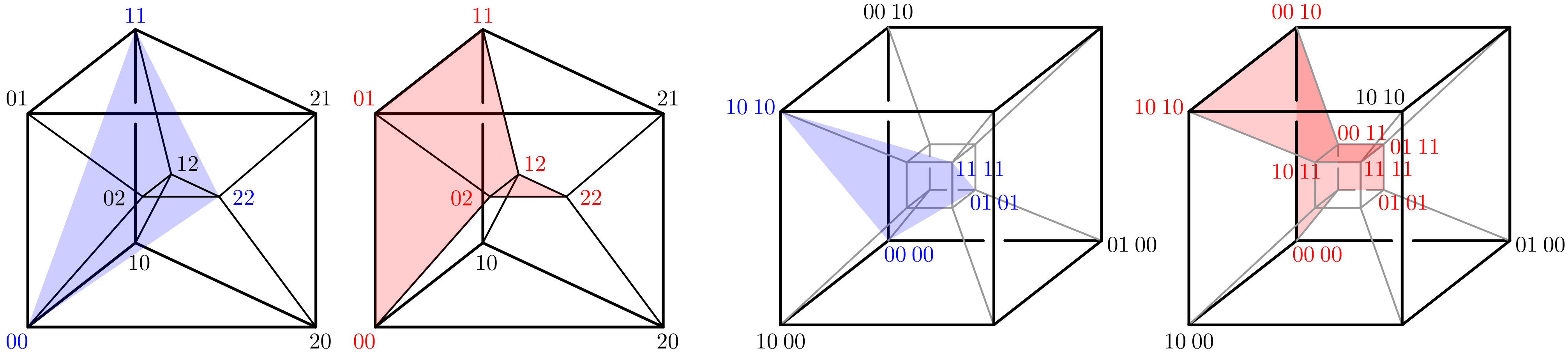
diagonal of $\mathbb{P} = \delta : \mathbb{P} \to \mathbb{P} \times \mathbb{P}$

$$p \mapsto (p, p)$$



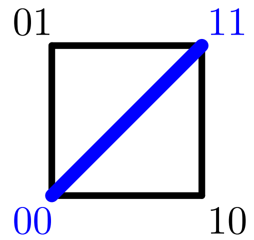
cellular approximation of the diagonal of $\mathbb{P}$ = map $\mathbb{P} \to \mathbb{P} \times \mathbb{P}$ s.t.

- its image is a union of faces of $\mathbb{P} \times \mathbb{P}$
- it agrees with δ on the vertices of $\mathbb{P}$
- it is homotopic to δ



DIAGONALS OF POLYTOPES

$\mathbb{P}$ polytope in $\mathbb{R}^d$



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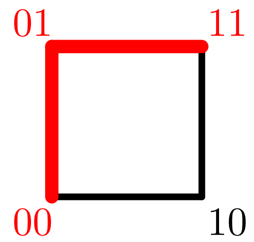

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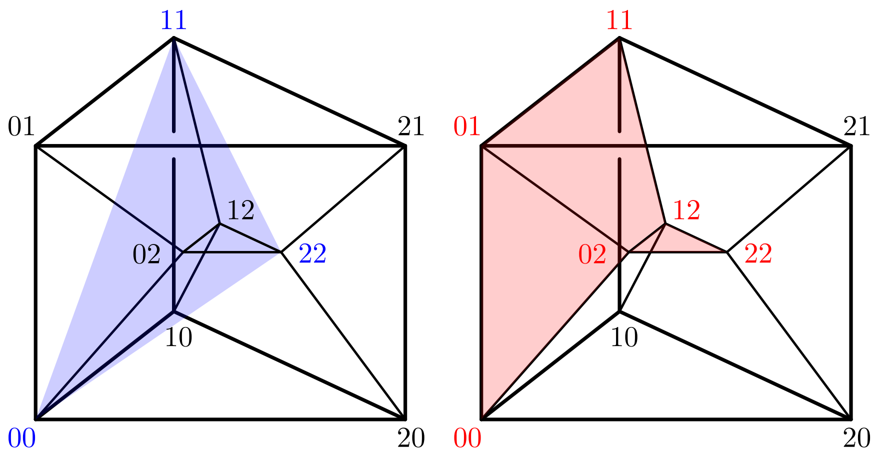
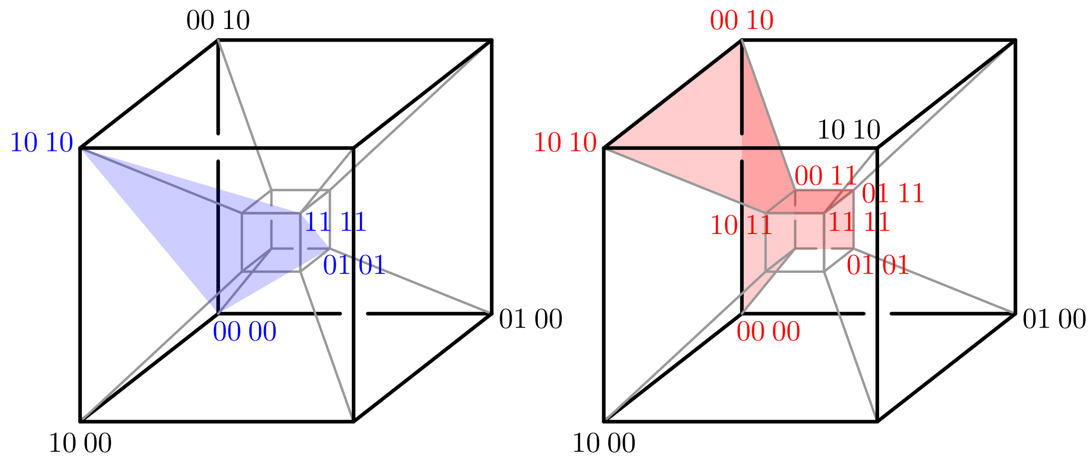


Alexander – Whitney
singular homology



Serre
cubical singular homology

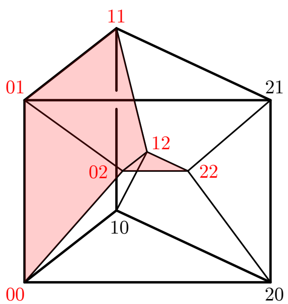
DIAGONALS OF POLYTOPES

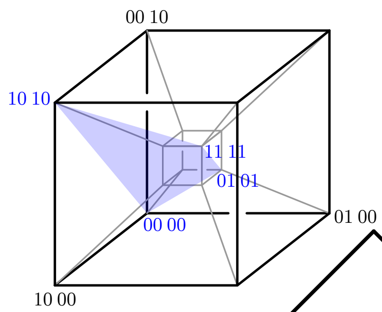
any vertex of the fiber polytope

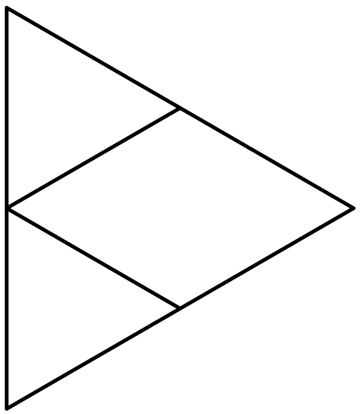
Masuda – Thomas – Tonks – Vallette '21
Laplante-Anfossi '22

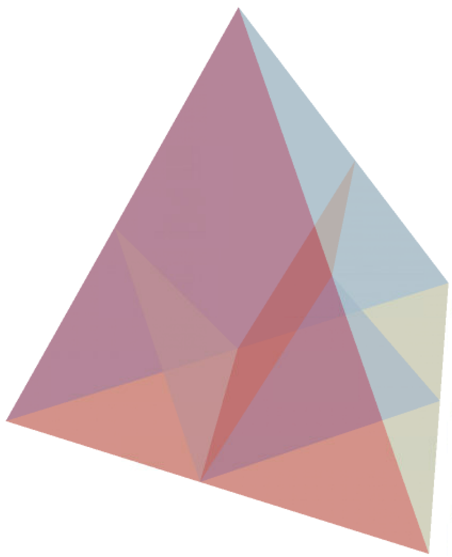
$$\sum \begin{pmatrix} \mathbb{P}\times\mathbb{P} & & (p,q) \\ \downarrow & , & \downarrow \\ \mathbb{P} & & \frac{p+q}{2} \end{pmatrix}$$

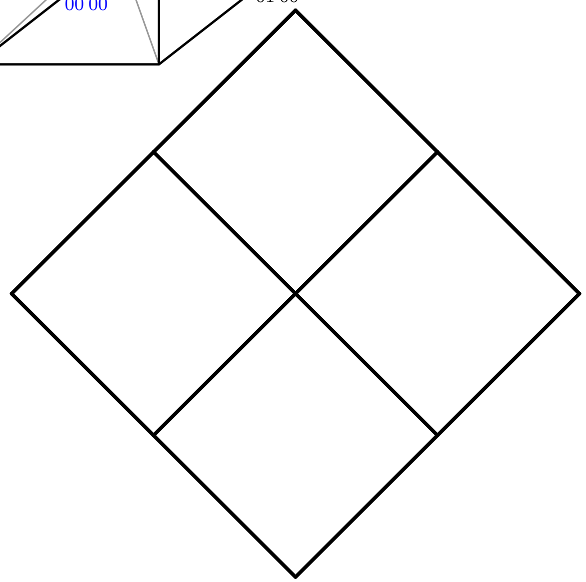
gives a cellular approximation of the diagonal of $\mathbb{P}$
projecting back on $\mathbb{P}$, we see it as a polyhedral subdivision of $\mathbb{P}$

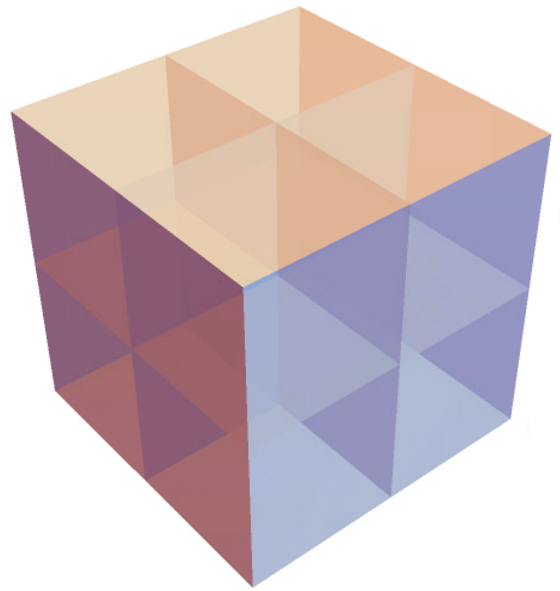












DIAGONALS OF POLYTOPES

Masuda – Thomas – Tonks – Vallette '21
Laplante-Anfossi '22

the vertex of the fiber polytope selected by $(-v, v)$

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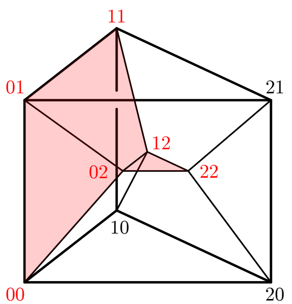
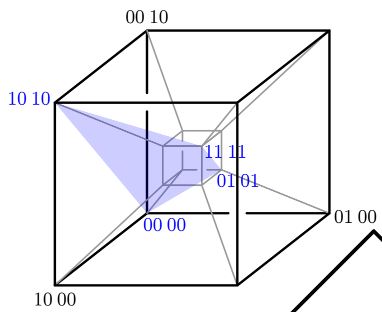
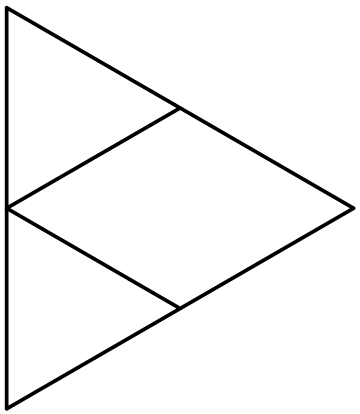
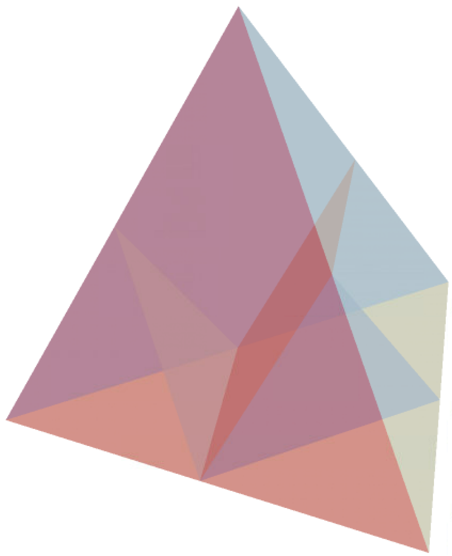
gives a cellular approximation of the diagonal of $\mathbb{P}$

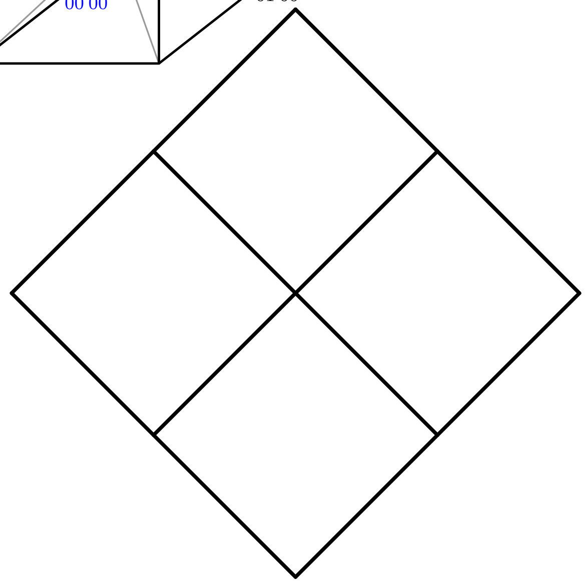
projecting back on $\mathbb{P}$, we see it as a polyhedral subdivision $\Delta_{\mathbb{P},v}$ of $\mathbb{P}$

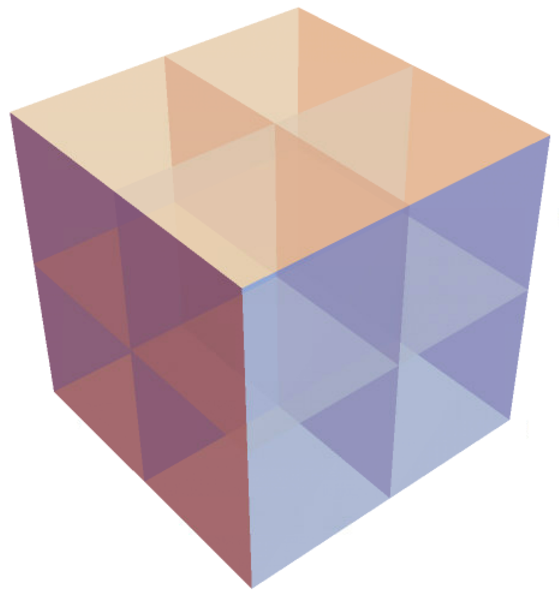












DIAGONALS OF POLYTOPES

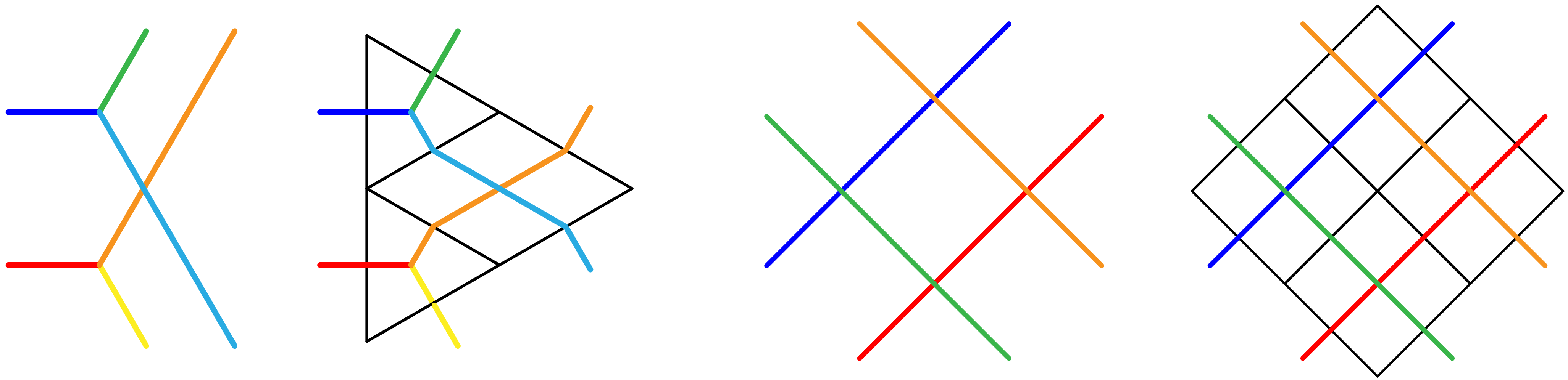
THM.

combinatorics of the diagonal $\Delta_{\mathbb{P},v}$ of $\mathbb{P}$

$$\simeq$$

common refinement of two copies of the normal fan of $\mathbb{P}$ translated by v

Laplante-Anfossi '22

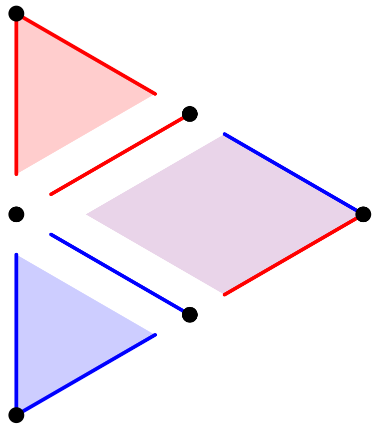


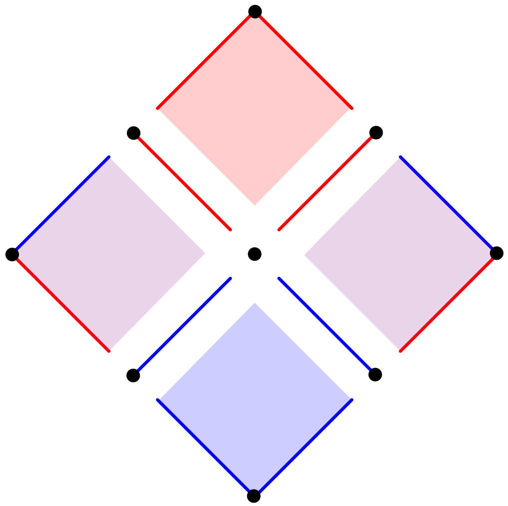
DIAGONALS OF POLYTOPES

THM. Faces of $\triangle_{\mathbb{P},v} \subseteq$ pairs (F, G) such that $\max_v(F) \leq \min_v(G)$

Laplante-Anfossi '22

When these are exactly the faces, it is called "magical formula"
This is the case for simplices, cubes, associahedra, but not permutahedra (see later)





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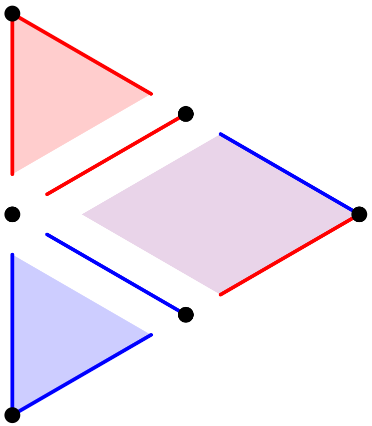
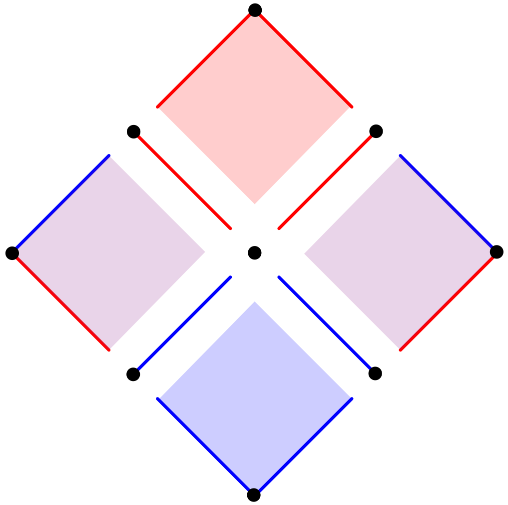




$$f_k(\triangle_{\mathrm{Simplex}(n)}) = (k+1)\binom{n+1}{k+2}$$

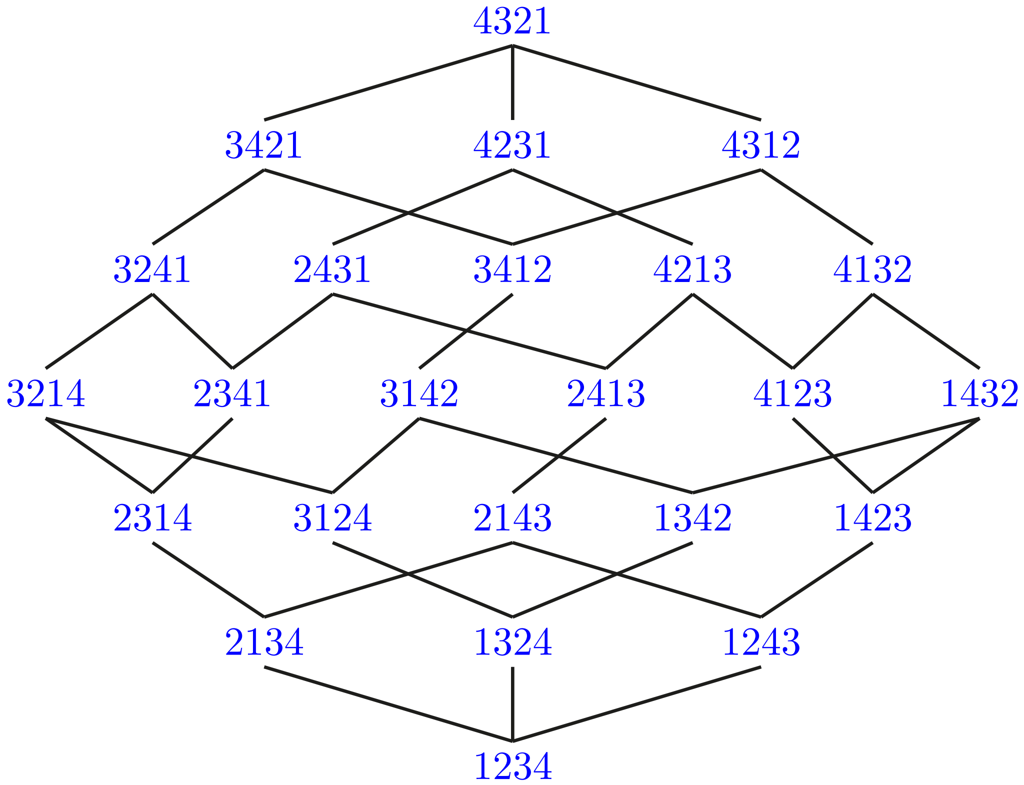
[OEIS, A127717]

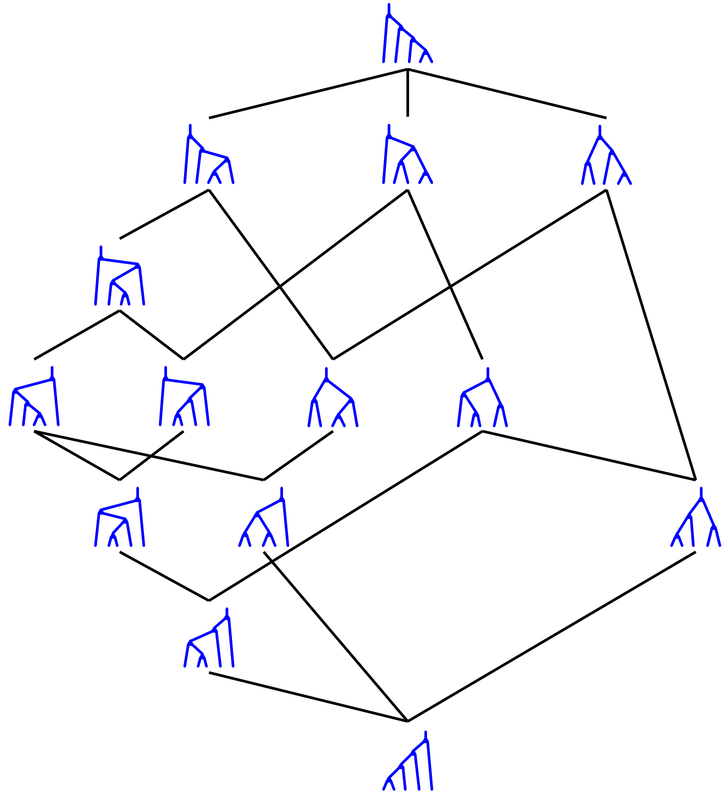
$$f_k(\triangle_{\mathrm{Cube}(n)}) = \binom{n}{k} 2^k 3^{n-k}$$

[OEIS, A038220]

PERMUTAHEDRON & ASSOCIAHEDRON

LATTICES: WEAK ORDER & TAMARI LATTICE





weak order = permutations of $[n]$
ordered by paths of simple transpositions

Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

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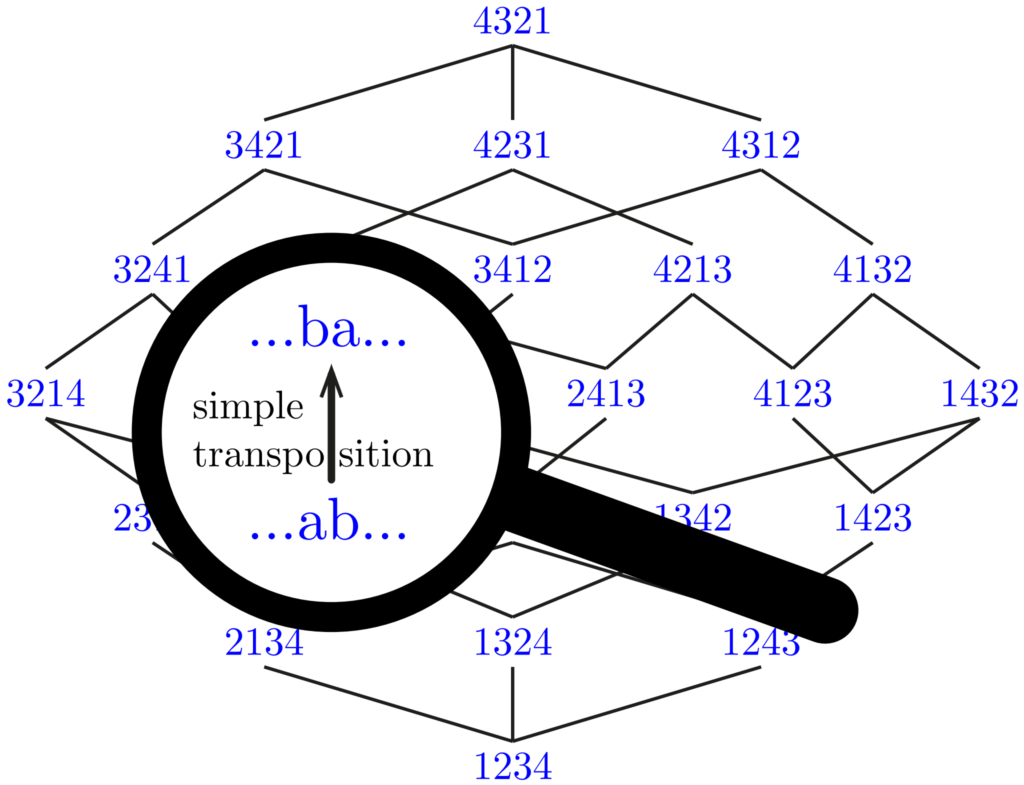

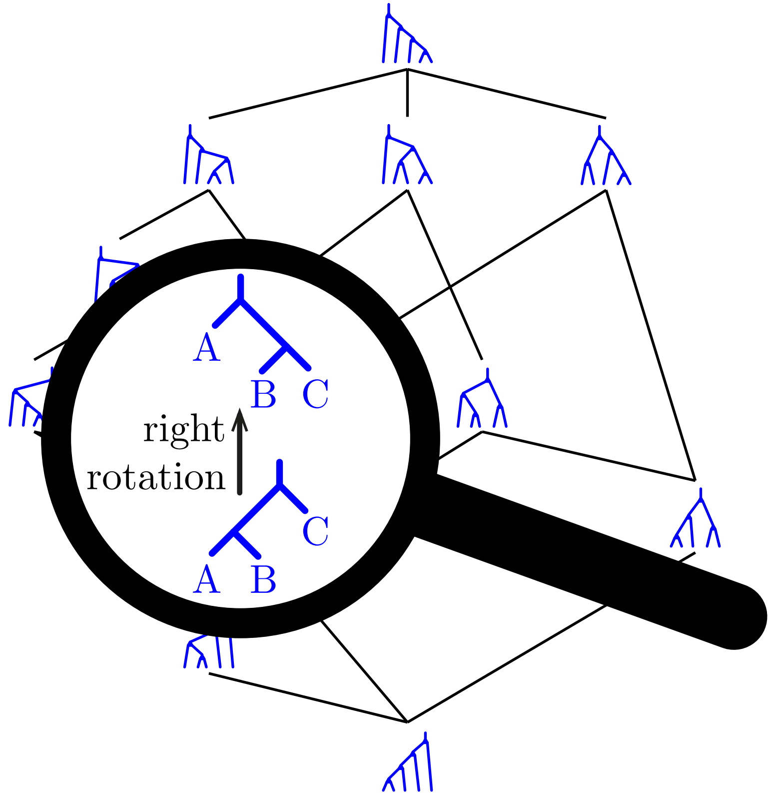

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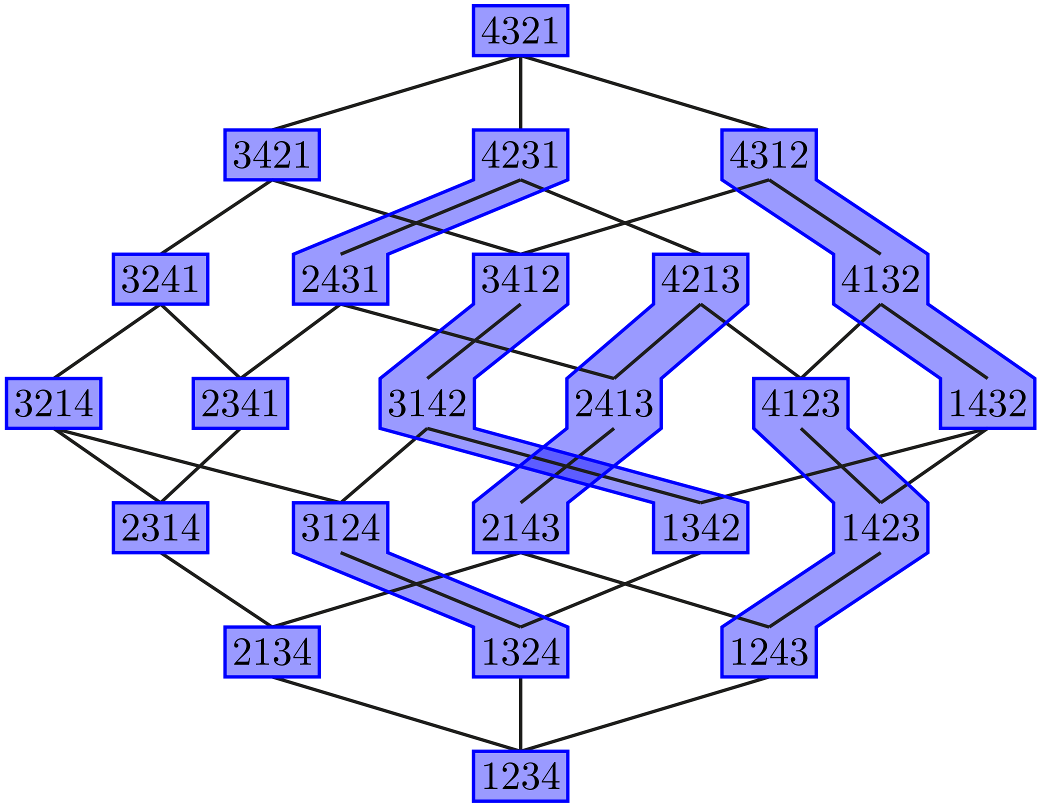

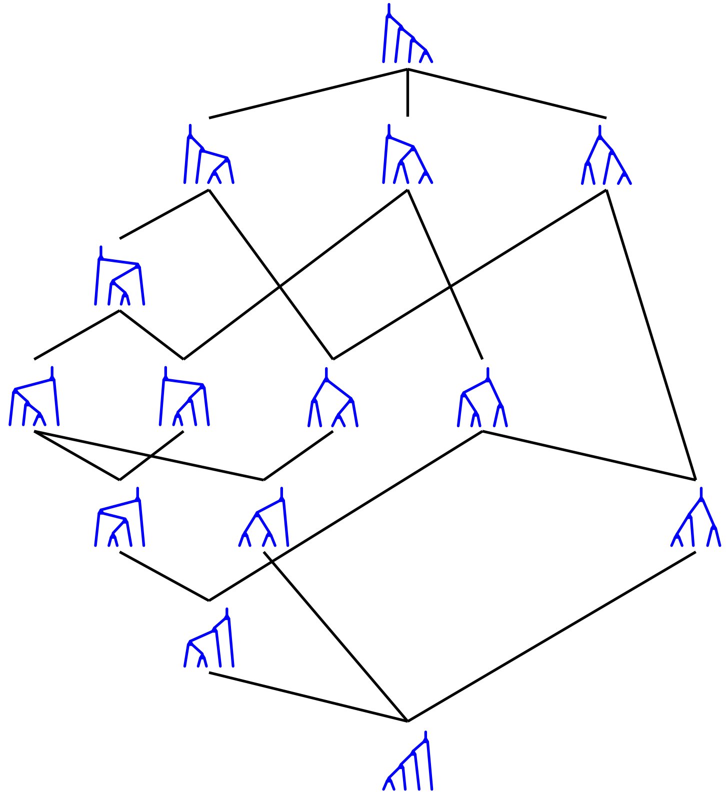

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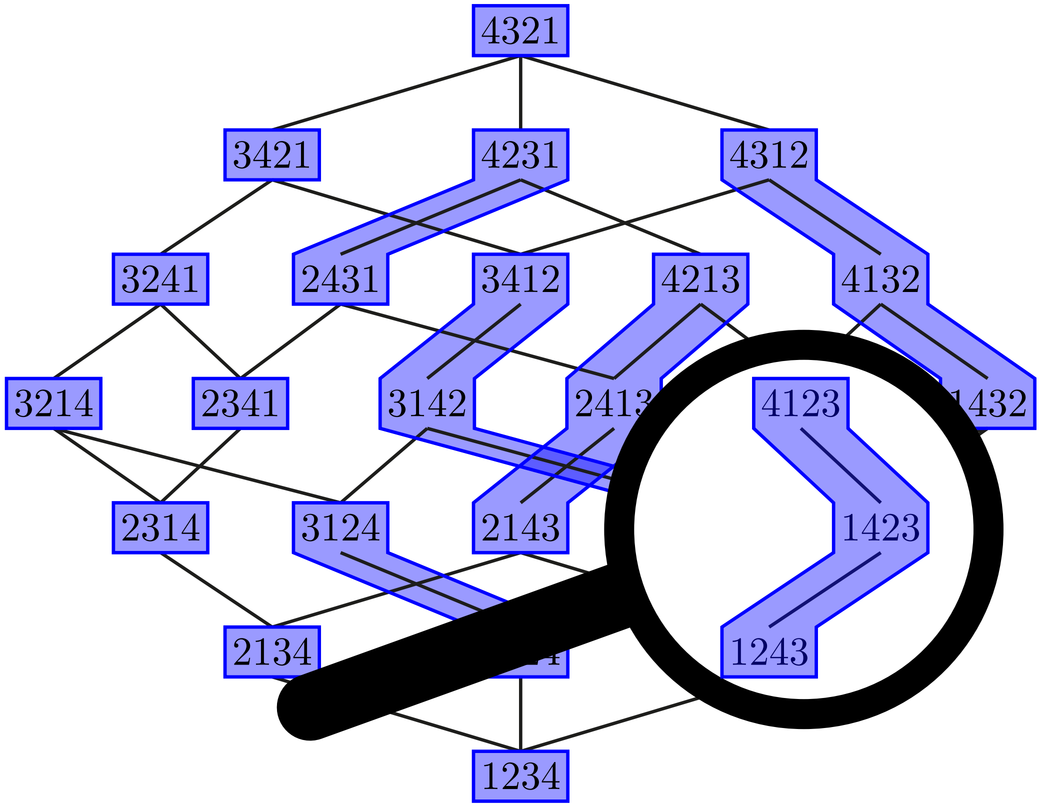
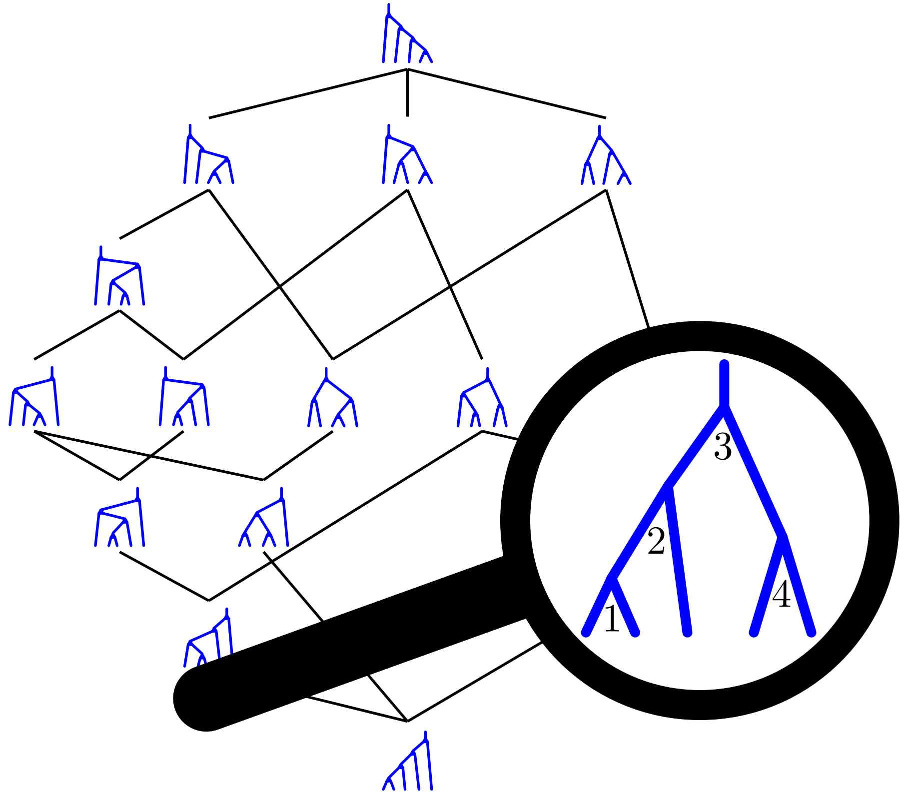
sylvester congruence = equivalence classes are sets of linear extensions of binary trees
= equivalence classes are fibers of BST insertion
= rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$

quotient lattice = lattice on classes with $X \leq Y \iff \exists\, x \in X,\ y \in Y, x \leq y$

LATTICES: WEAK ORDER & TAMARI LATTICE





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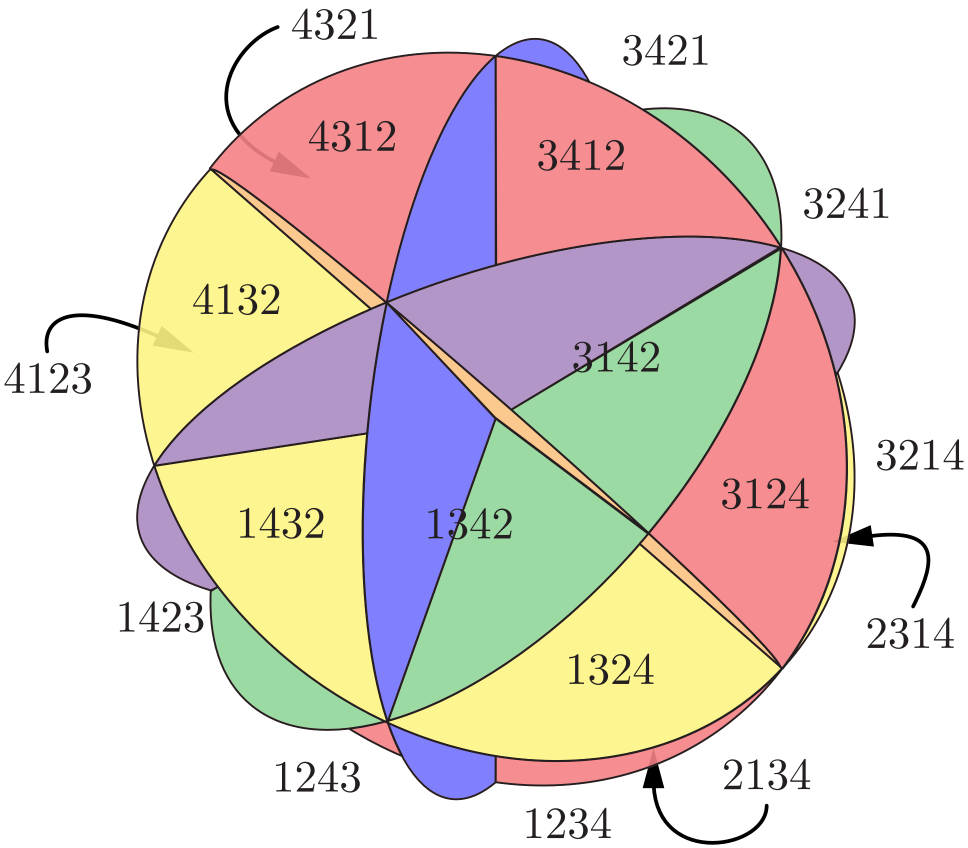
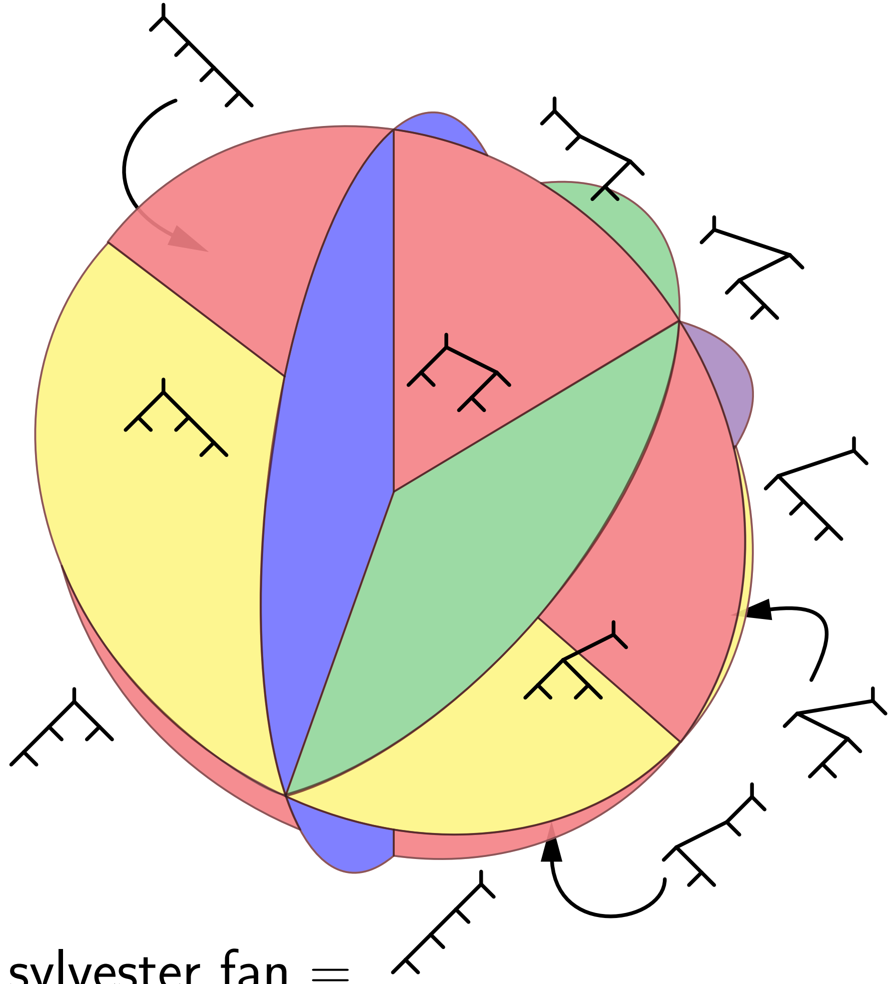
FANS: BRAID FAN & SYLVESTER FAN





braid fan $=$

$\mathbb{C}(\sigma) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \le \cdots \le x_{\sigma(n)}\}$

sylvester fan $=$

$\mathbb{C}(T) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ if } i \to j \text{ in } T\}$

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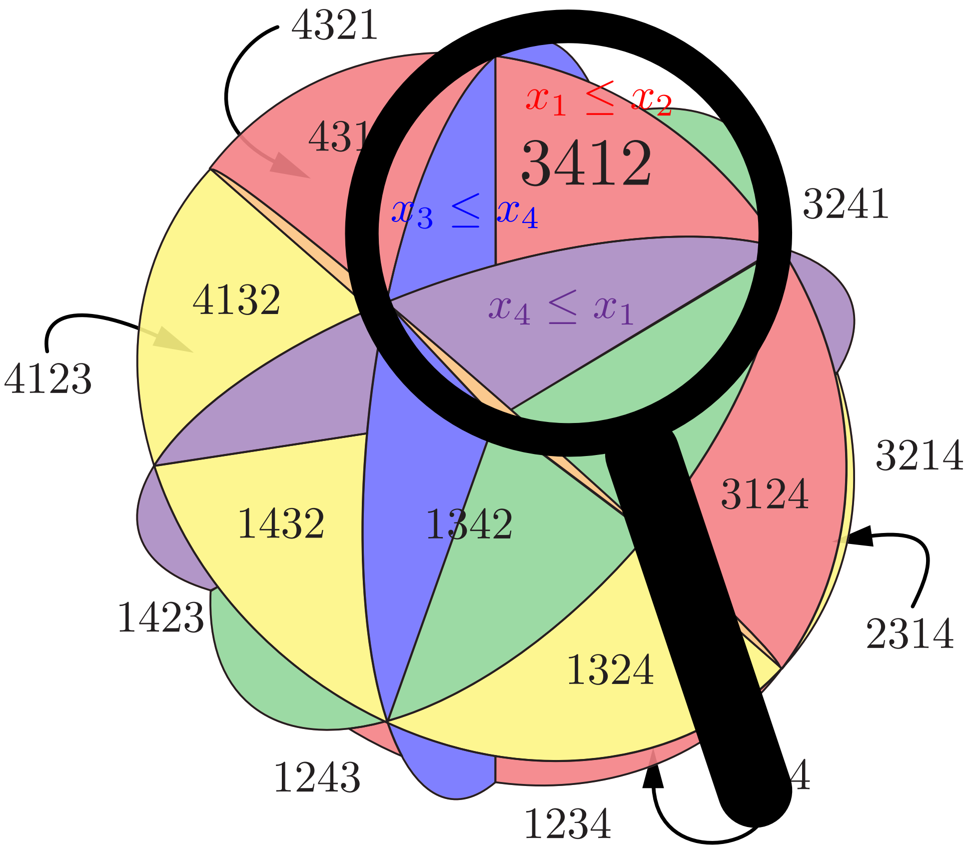

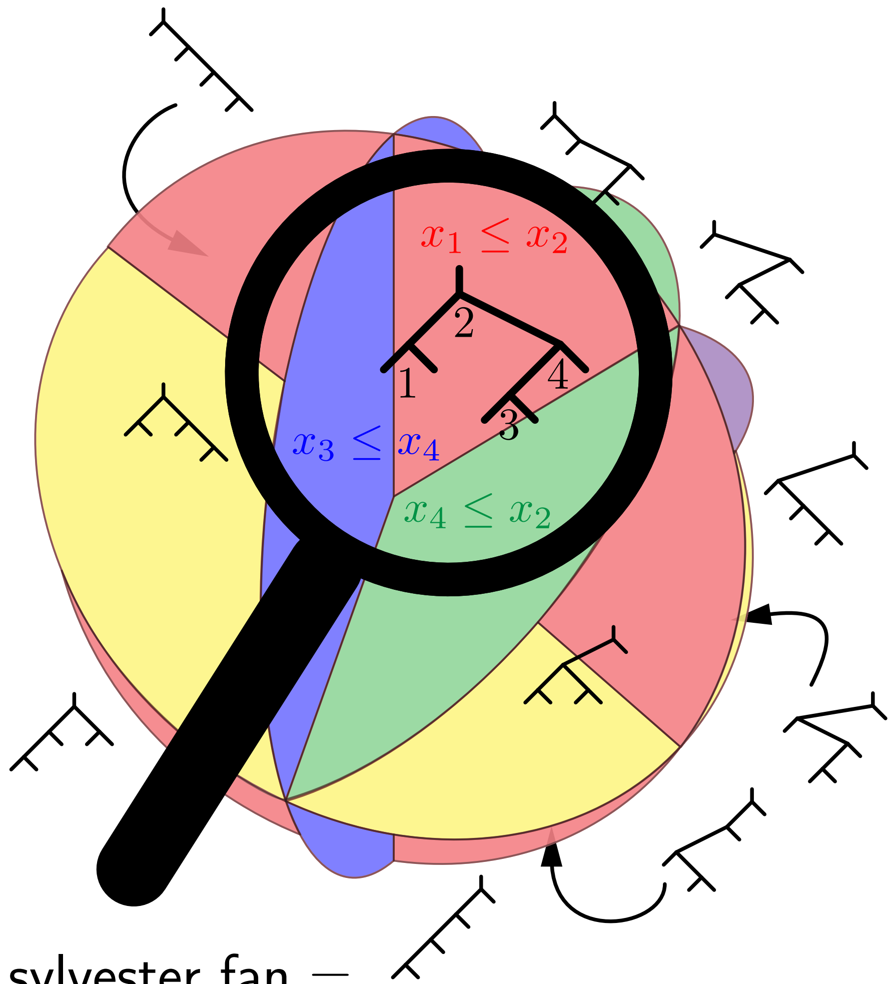

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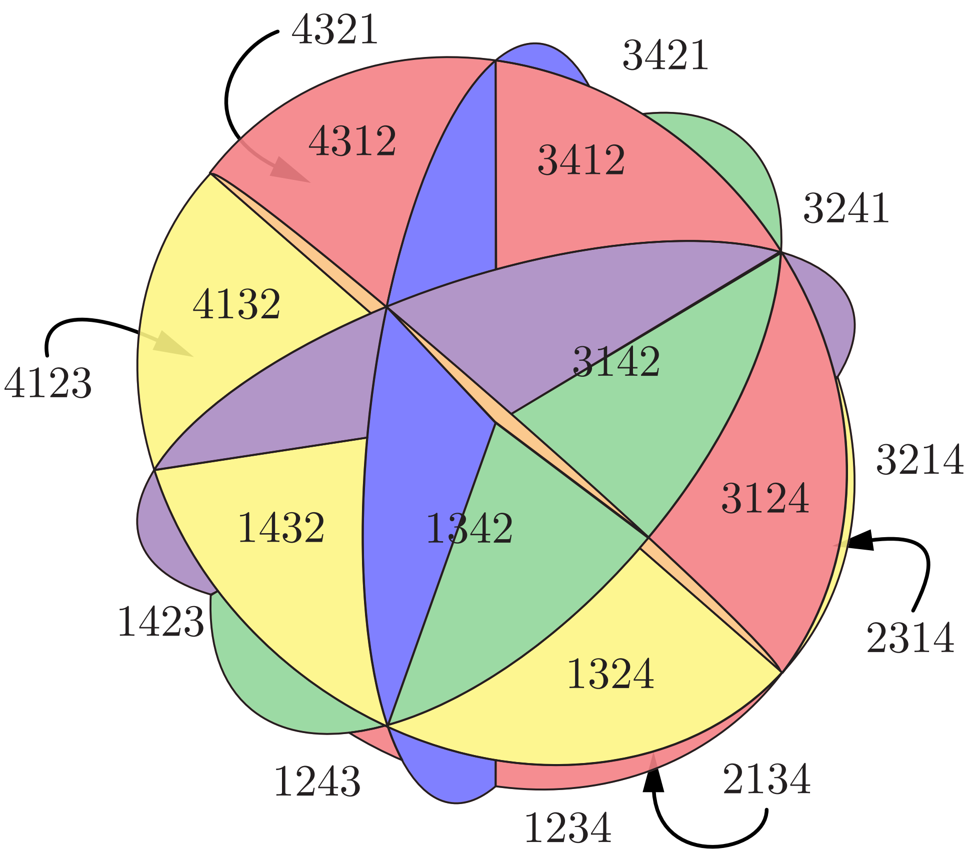

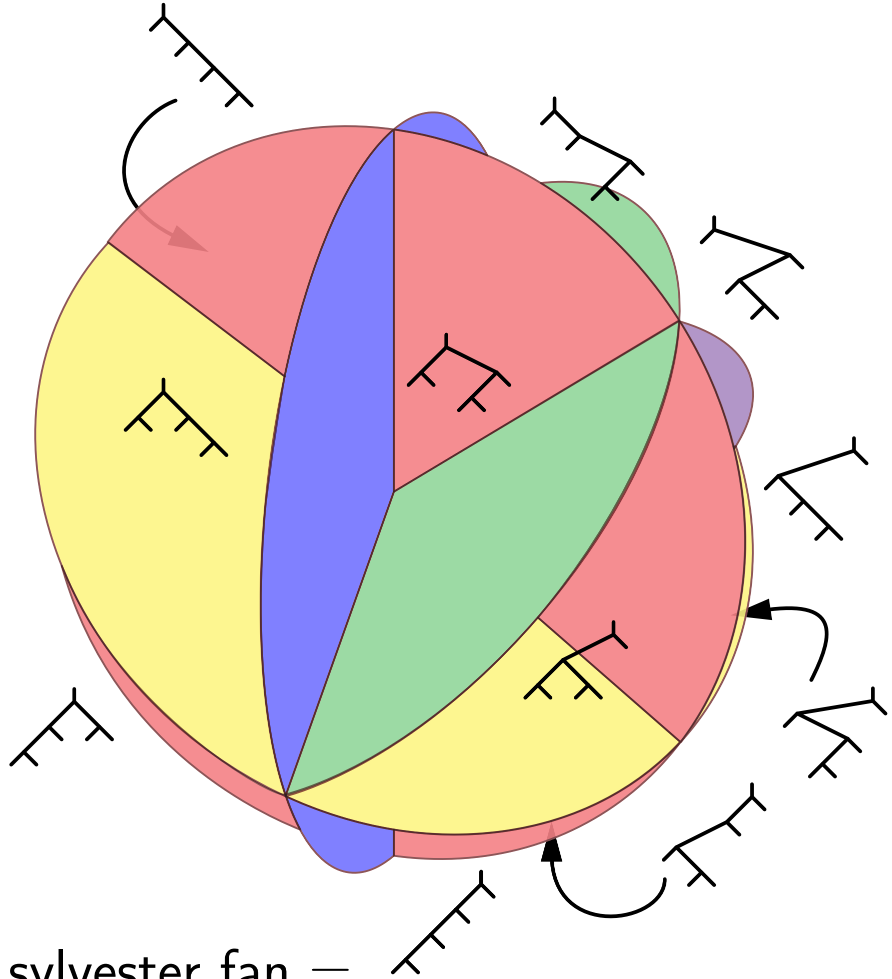

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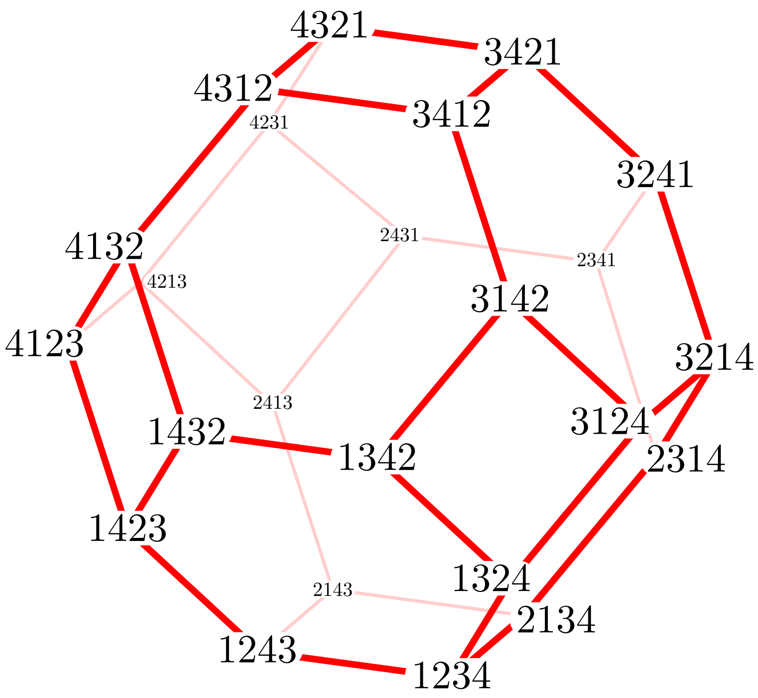
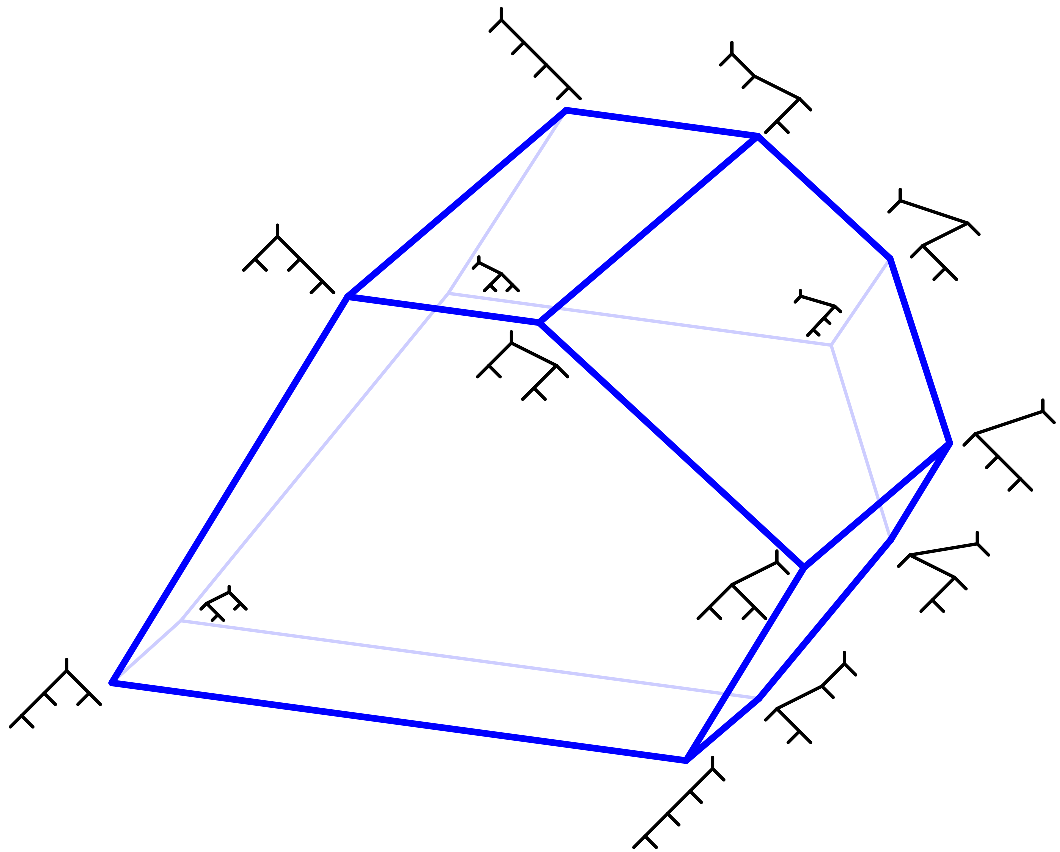
quotient fan $= \mathbb{C}(T)$ is obtained by glueing $\mathbb{C}(\sigma)$ for all linear extensions σ of T

POLYTOPES: PERMUTAHEDRON & ASSOCIAHEDRON





permutahedron $\mathbb{P}\mathsf{erm}(n)$

$$= \operatorname{conv}\left\{ [\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \ne J \subsetneq [n]} \mathbb{H}_J$$

where $\mathbb{H}_J = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j\in J} x_j \ge \binom{|J|+1}{2} \right\}$

associahedron $\mathbb{A}\mathsf{sso}(n)$

$$= \operatorname{conv}\left\{ [\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; T \text{ binary tree} \right\}$$

$$= \mathbb{H} \cap \bigcap_{1\le i<j\le n} \mathbb{H}_{[i,j]}$$

Stasheff ('63)
Shnider – Sternberg ('93)
Loday ('04)

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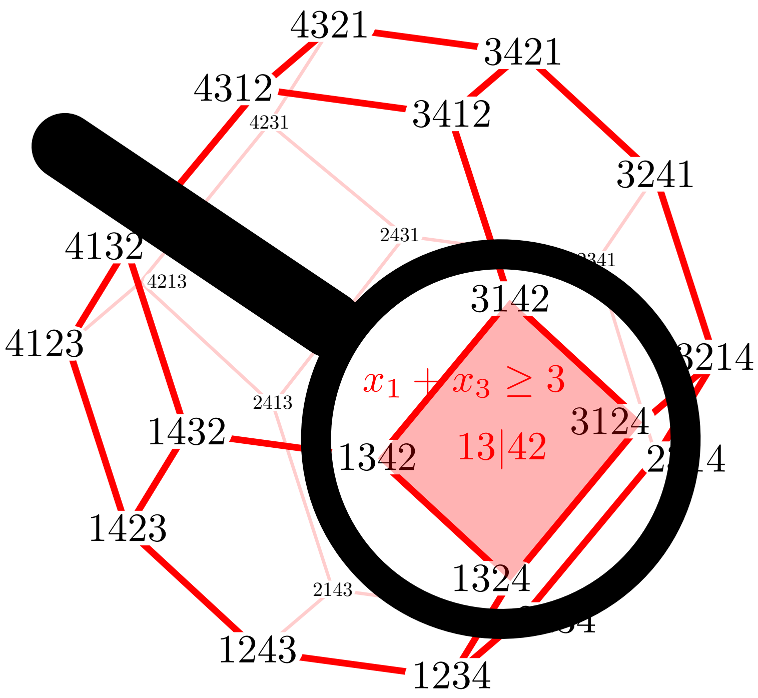

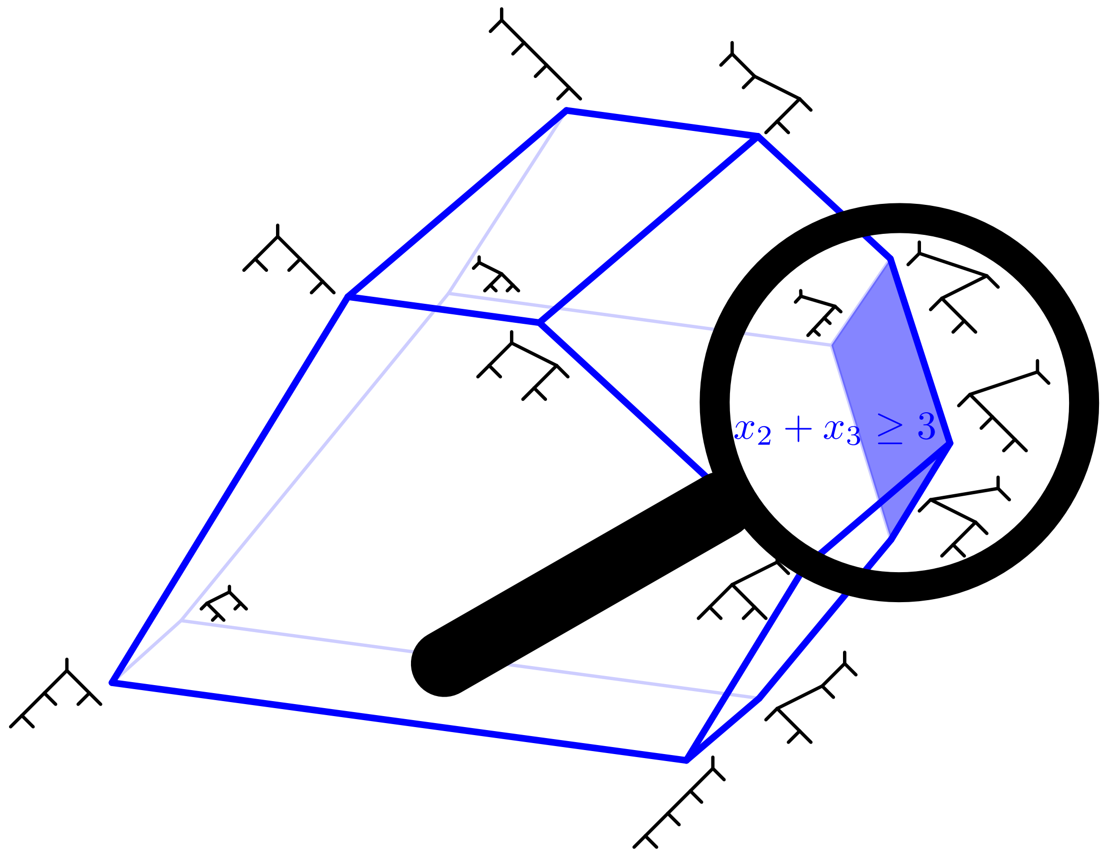

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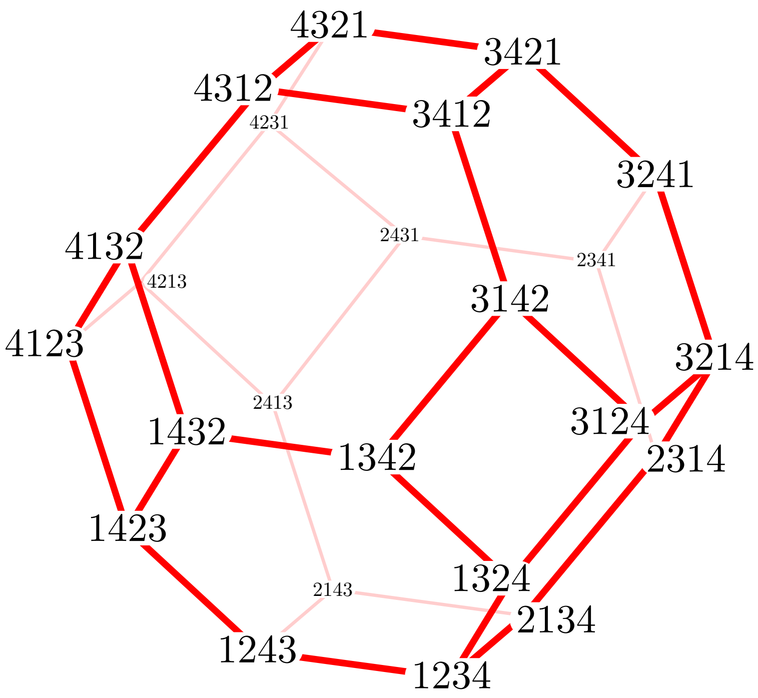

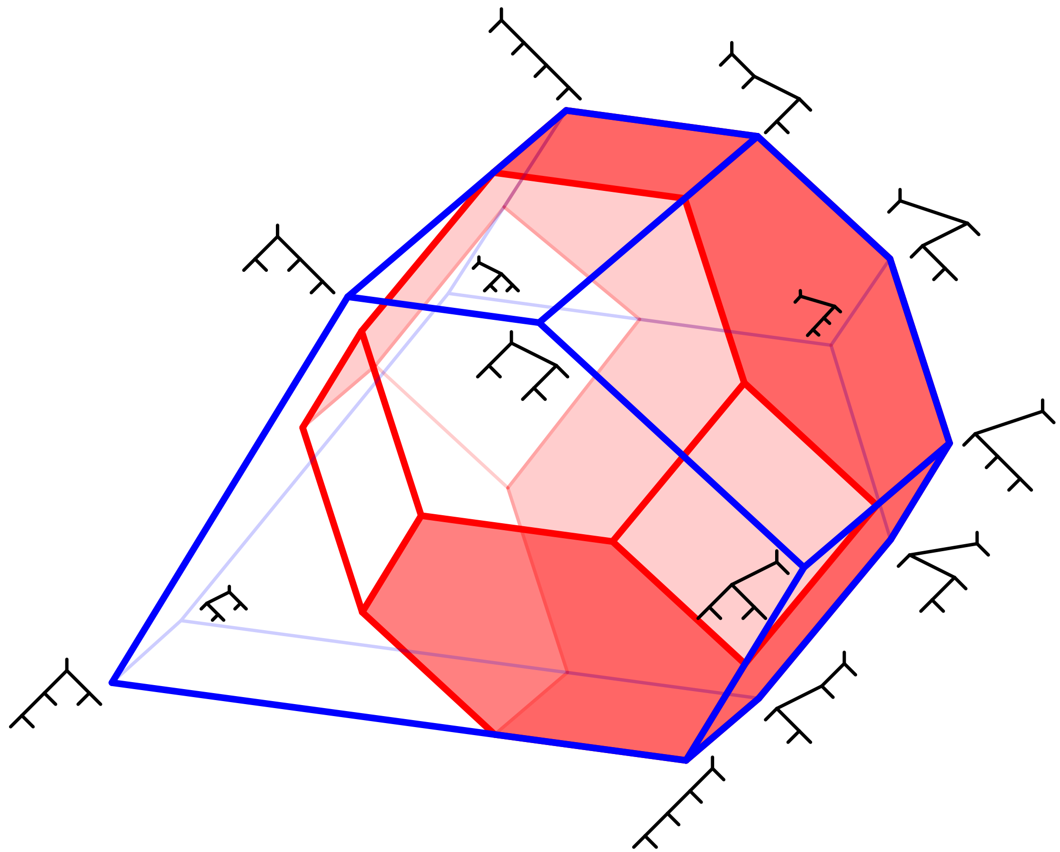

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LATTICES – FANS – POLYTOPES

permutahedron $\mathbb{P}\mathrm{erm}(n)$

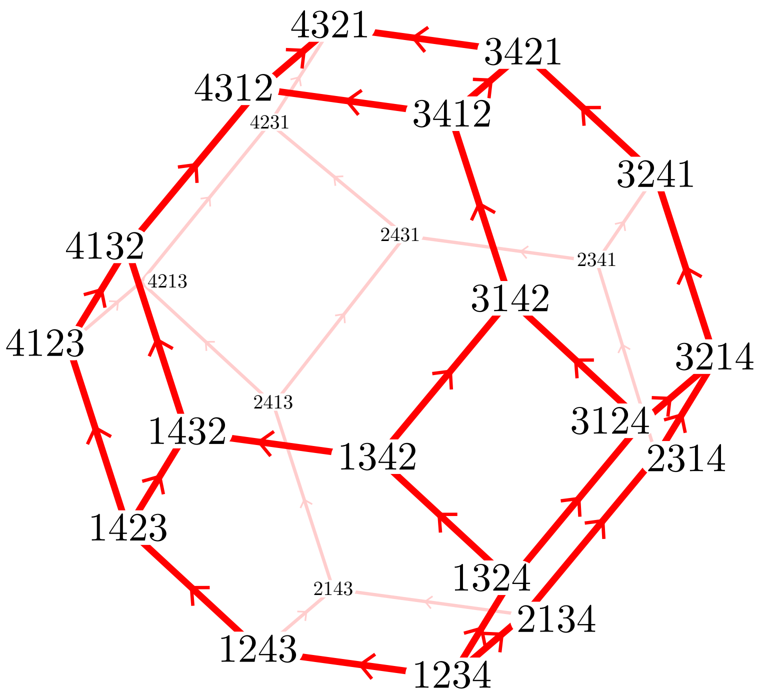
$\Longrightarrow$ braid fan

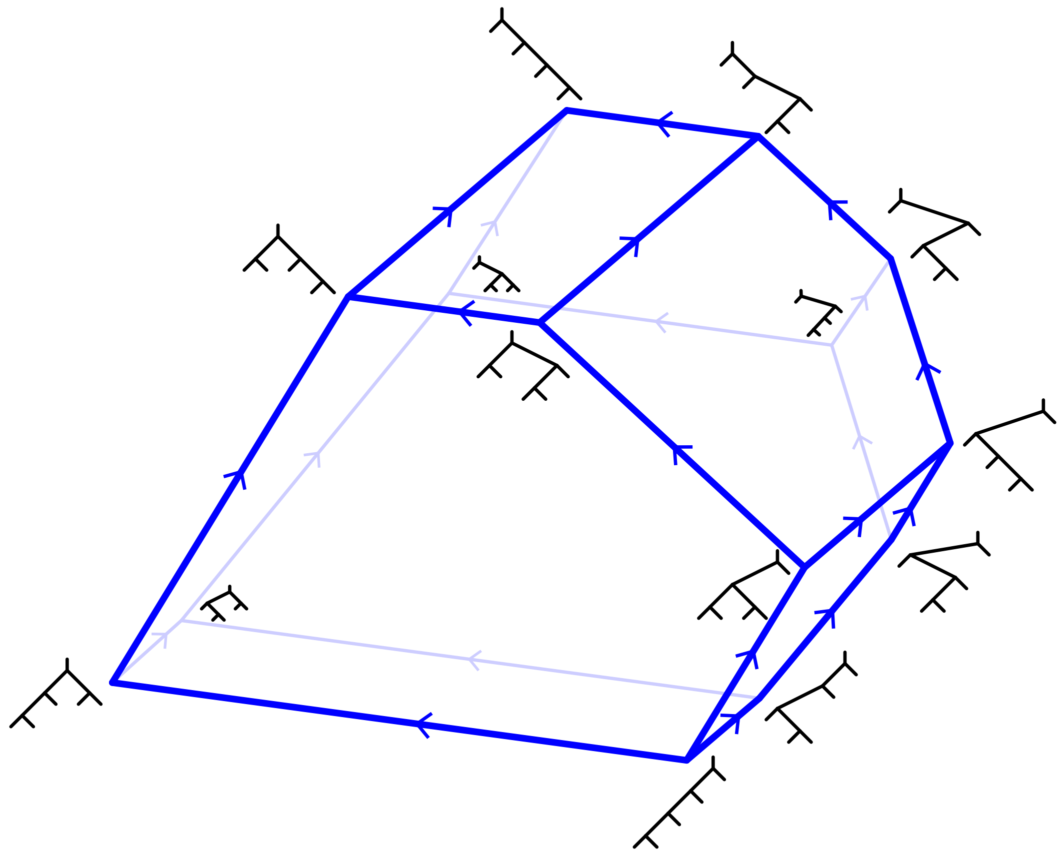
$\Longrightarrow$ weak order on permutations

associahedron $\mathbb{A}\mathrm{sso}(n)$

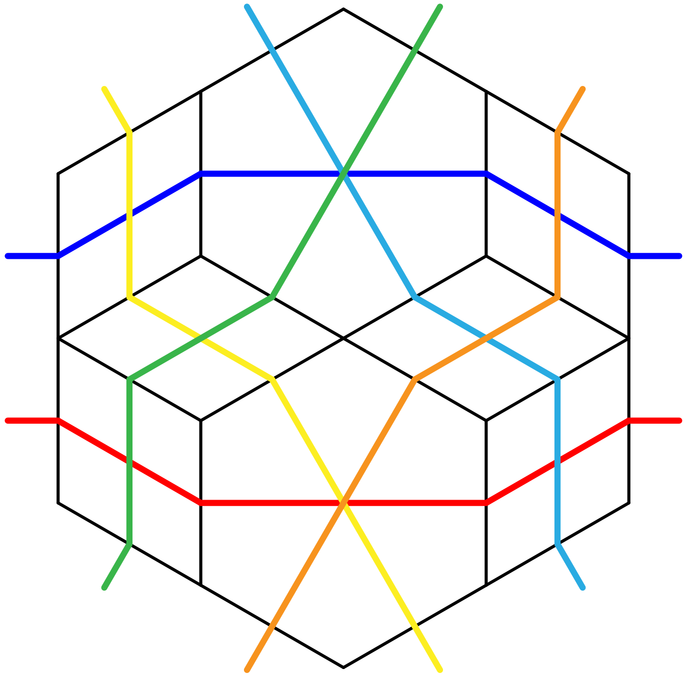
$\Longrightarrow$ Sylvester fan

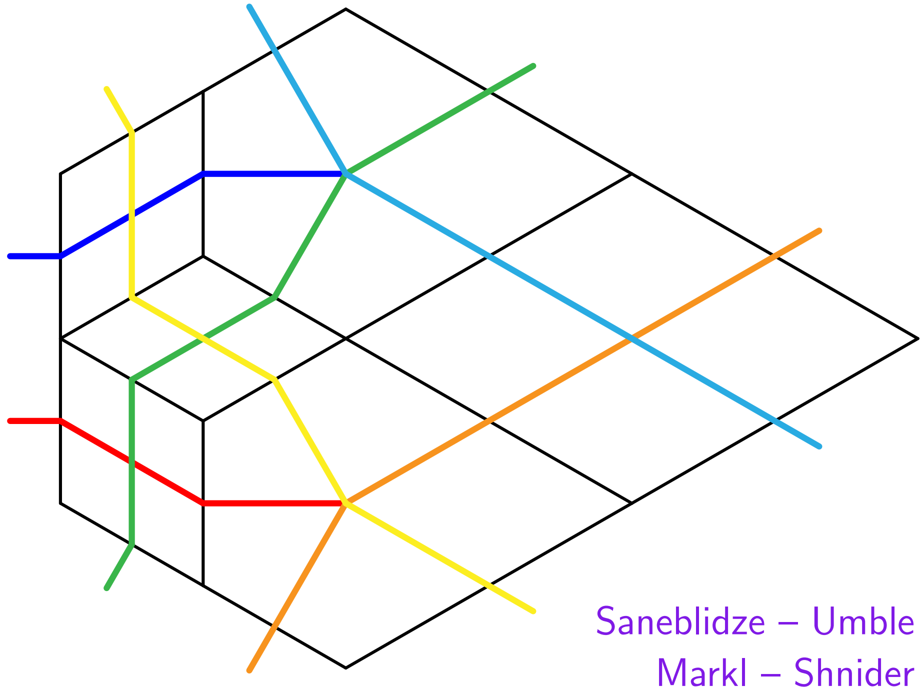
$\Longrightarrow$ Tamari lattice on binary trees



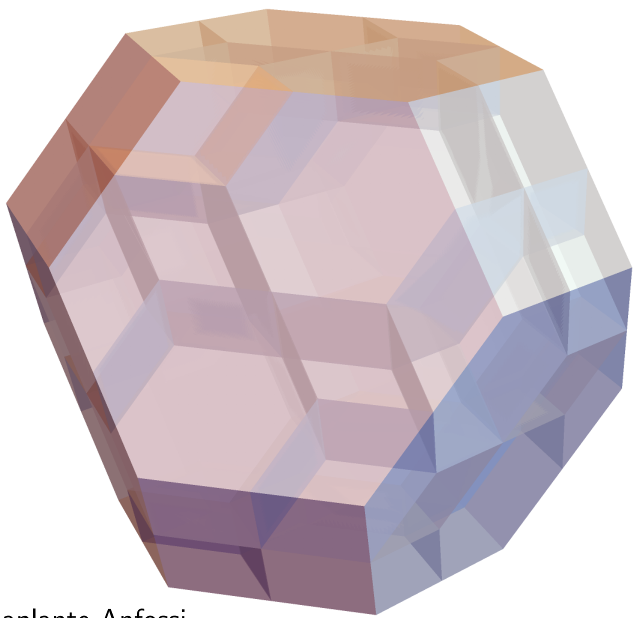


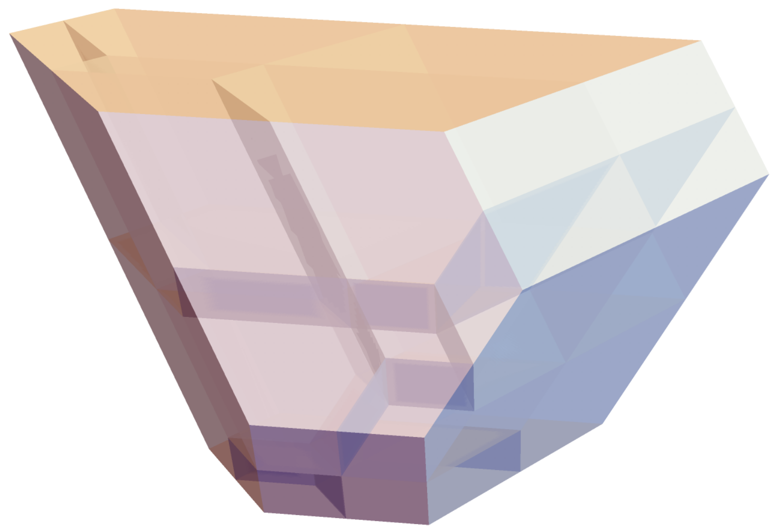
F-VECTOR OF DIAGONALS





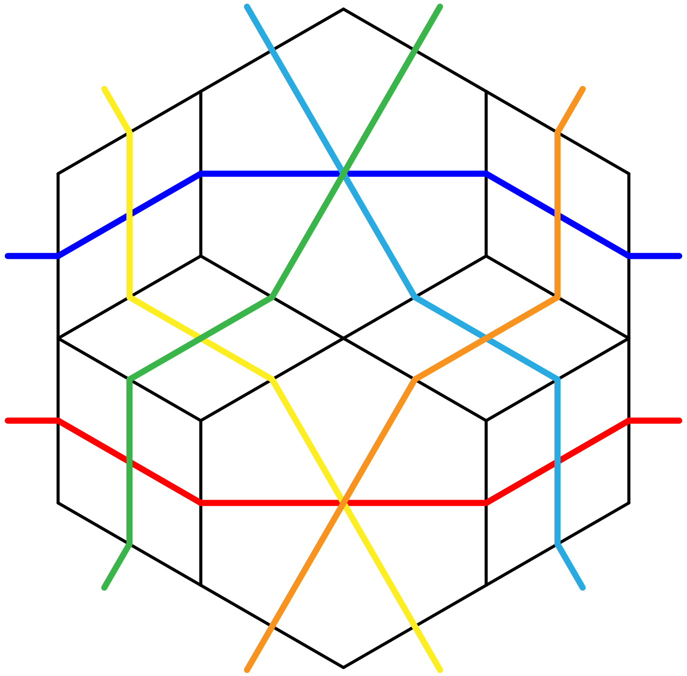
Saneblidze – Umble '04
Markl – Shnider '06
Loday '11

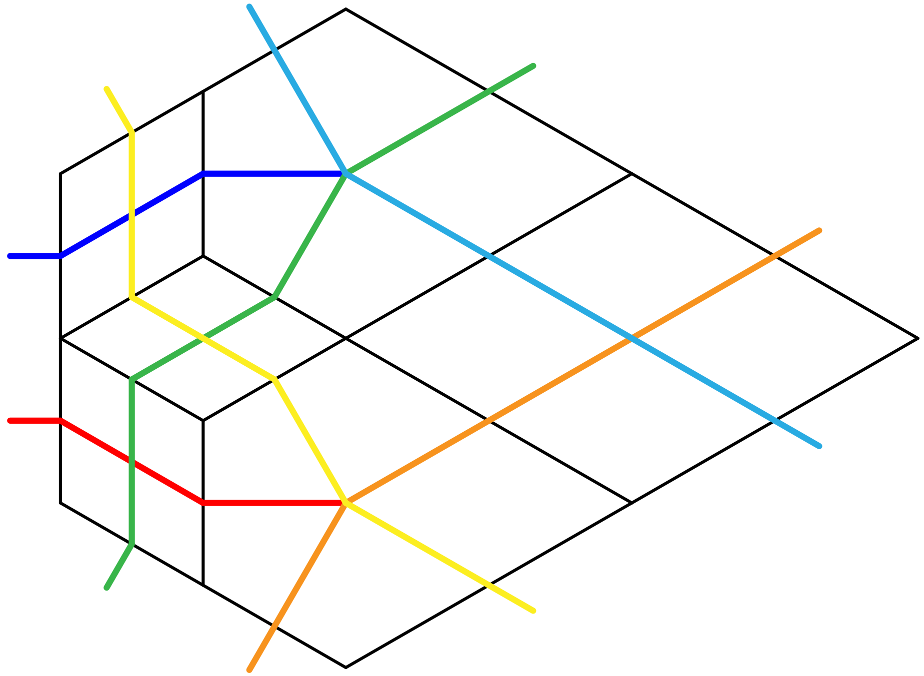




Masuda – Thomas – Tonks – Vallette '21
Laplante-Anfossi '22

F-VECTOR OF DIAGONALS





$$f_k = \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [2]} \prod_{p \in G_i} (\#F_i[p] - 1)!$$

$$f_k = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}$$

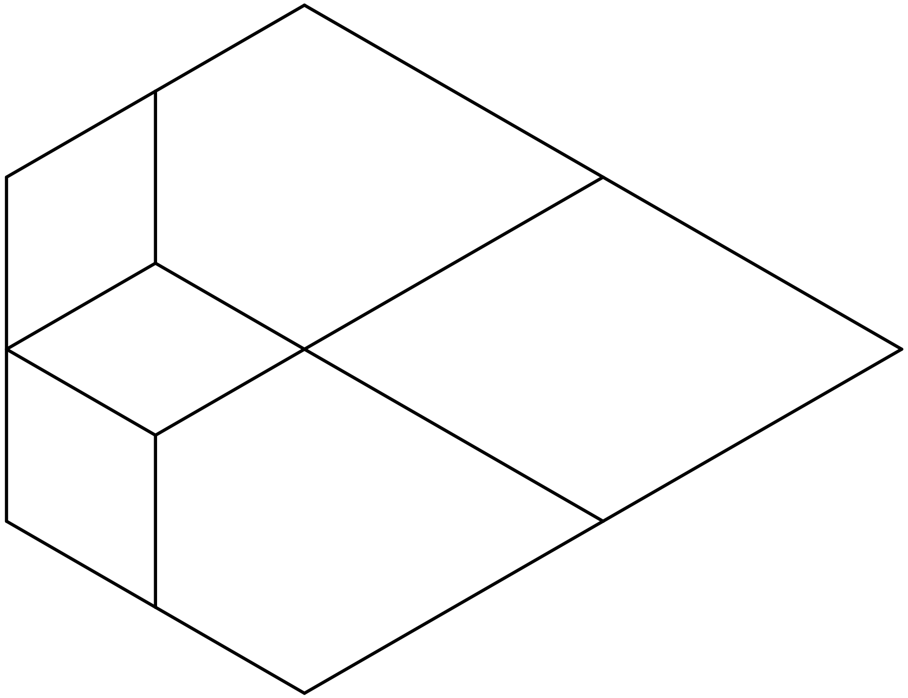
$$f_0 = [x^n] \ \exp\left(\sum_m \frac{x^m}{m(m+1)} \binom{2m}{m} \right)$$

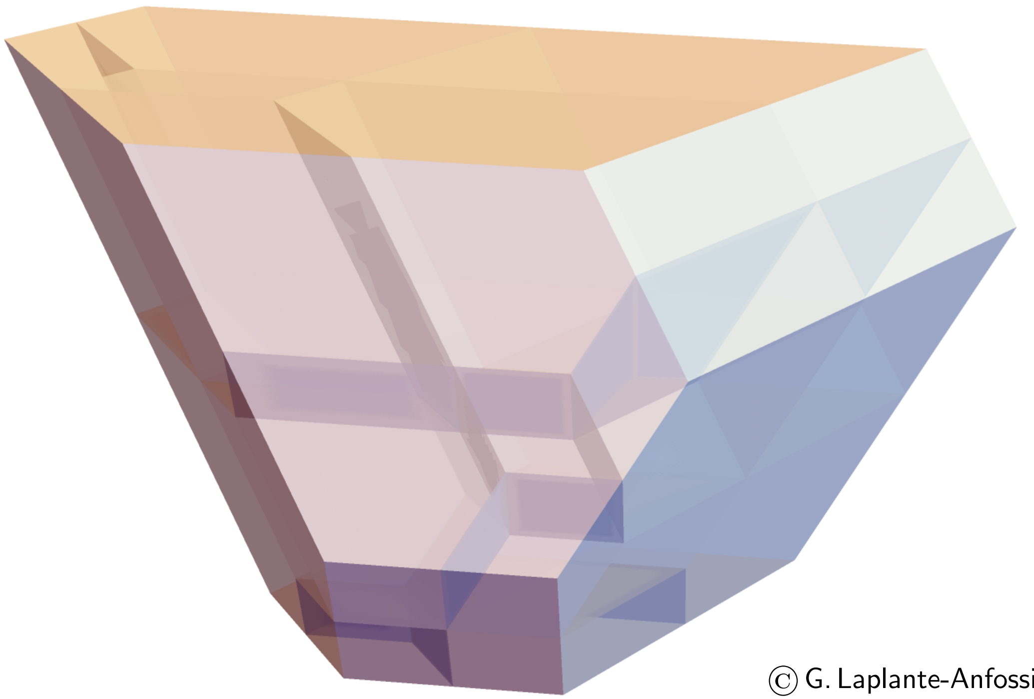
$$f_{n-1} = 2(n+1)^{n-2}$$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23$^+$

Bostan – Chyzak – P. '23$^+$

DIAGONAL OF THE ASSOCIAHEDRON





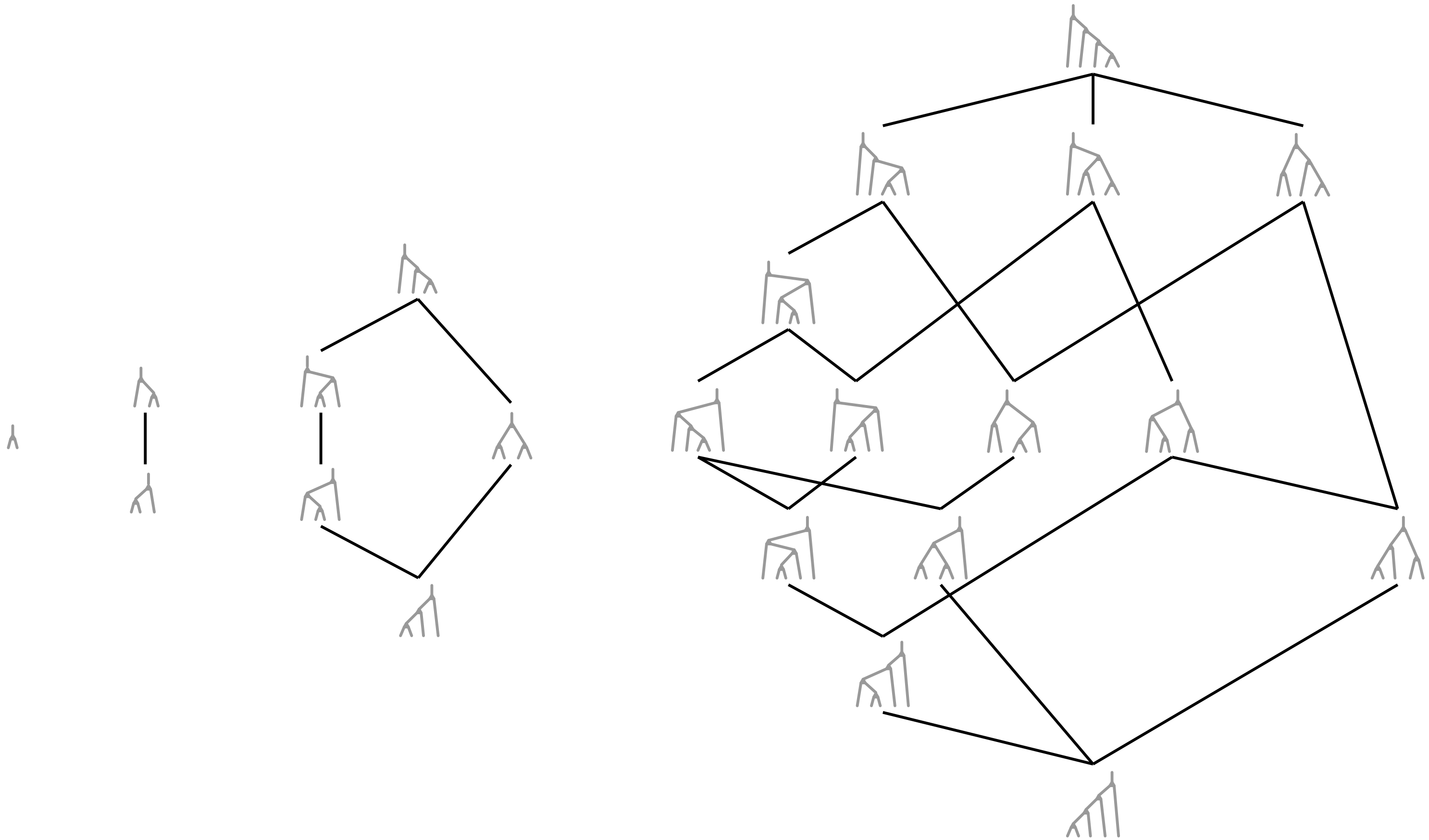
arXiv:2303.10986

with

Alin BOSTAN (INRIA)
Frédéric CHYZAK (INRIA)

NUMBER OF TAMARI INTERVALS

$\mathrm{Tam}(n)$ = Tamari lattice on binary trees with n nodes



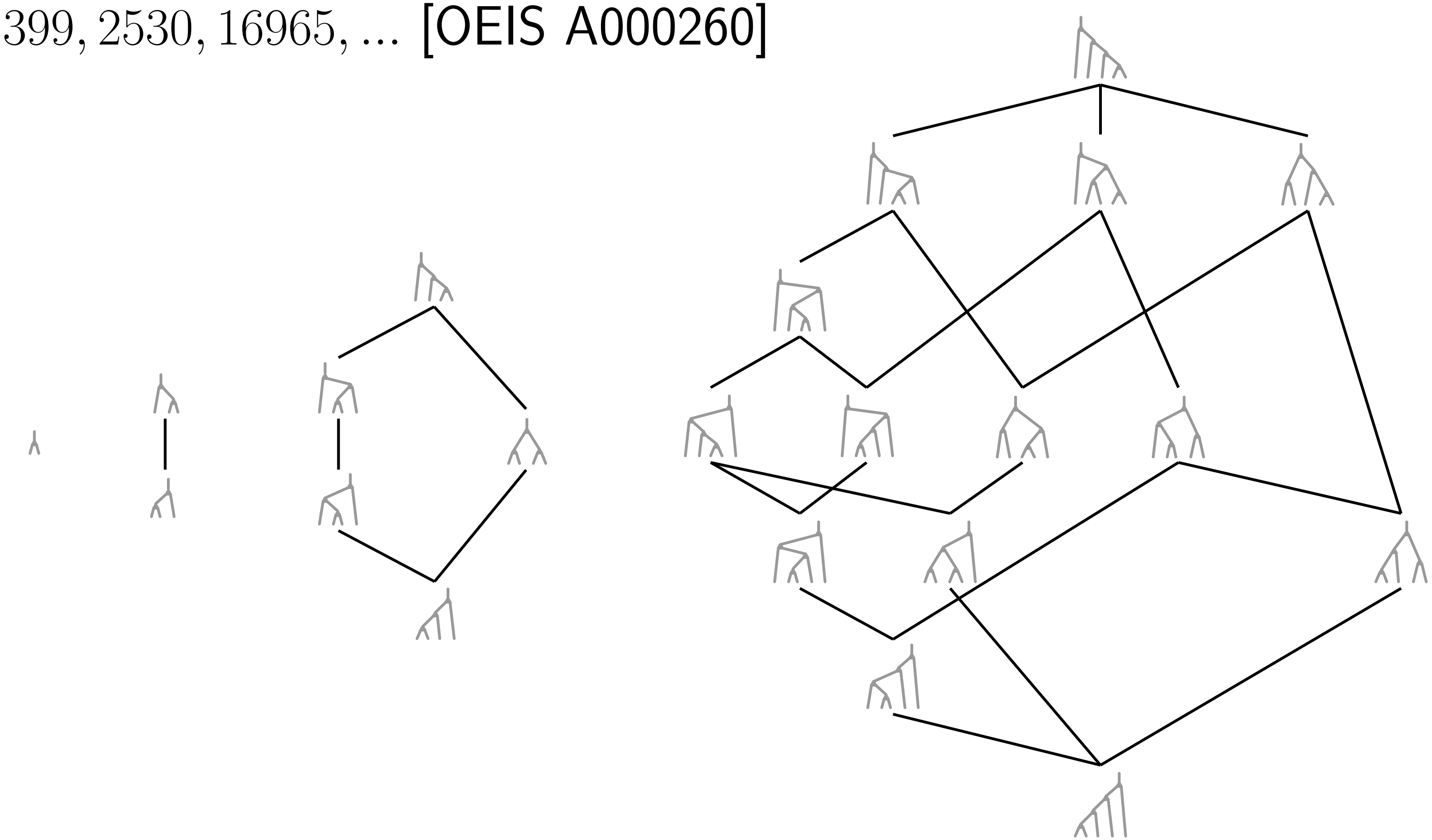
NUMBER OF TAMARI INTERVALS

$\mathrm{Tam}(n) =$ Tamari lattice on binary trees with n nodes

THM. For any $n \geq 1$, Chapoton '07

$$\#\{S \leq T \in \mathrm{Tam}(n)\} = \frac{2}{(3n+1)(3n+2)} \binom{4n+1}{n+1}$$

$1, 3, 13, 68, 399, 2530, 16965, \ldots$ [OEIS A000260]

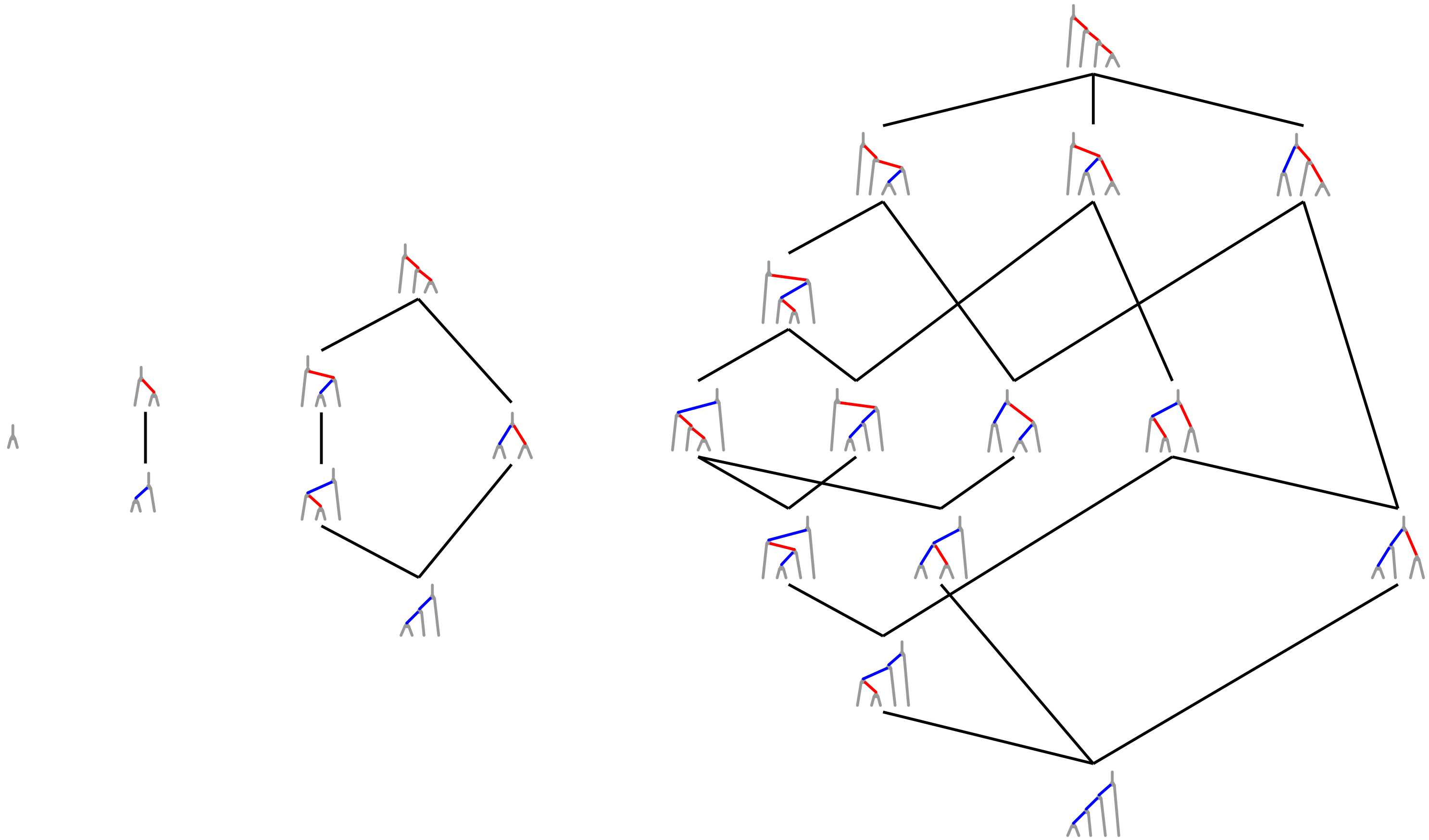


FIRST REFINED FORMULA ON TAMARI INTERVALS

$\mathrm{Tam}(n)$ = Tamari lattice on binary trees with n nodes

$\mathrm{des}(T)$ = number of binary trees covered by T

$\mathrm{asc}(T)$ = number of binary trees covering T



FIRST REFINED FORMULA ON TAMARI INTERVALS

$\mathrm{Tam}(n)$ = Tamari lattice on binary trees with n nodes
$\mathrm{des}(T)$ = number of binary trees covered by T
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THM. For any $n, k \geq 1$, Bostan – Chyzak – P. '23[+]

$$\#\{S \leq T \in \mathrm{Tam}(n) \mid \mathrm{des}(S) + \mathrm{asc}(T) = k\} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}$$

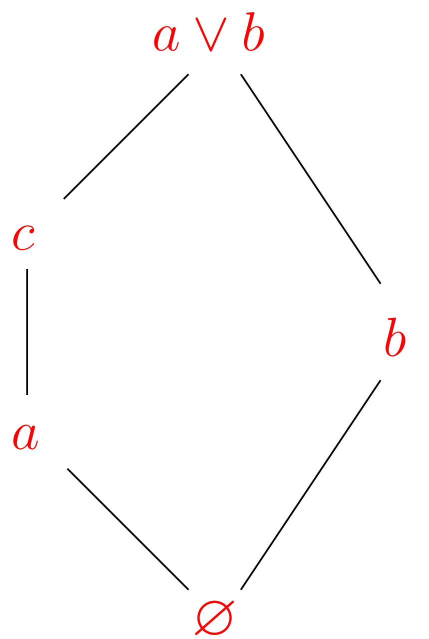
$n \backslash k$	0	1	2	3	4	5	6	7	8	Σ
1	1									1
2	1	2								3
3	1	6	6							13
4	1	12	33	22						68
5	1	20	105	182	91					399
6	1	30	255	816	1020	408				2530
7	1	42	525	2660	5985	5814	1938			16965
8	1	56	966	7084	24794	42504	33649	9614		118668
9	1	72	1638	16380	81900	215280	296010	197340	49335	857956

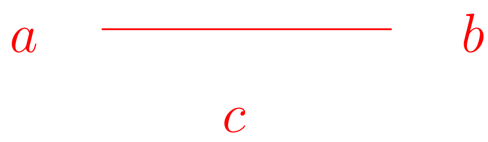
CANONICAL COMPLEX OF THE TAMARI LATTICE

$(L, \leq, \wedge, \vee)$ lattice

join semidistributive $\iff$ $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for all $x, y, z \in L$
$\iff$ any $x \in L$ admits a canonical join representation $x = \bigvee J$

canonical join complex = simplicial complex of canonical join representations
= a simplex J for each element $\bigvee J$ of L





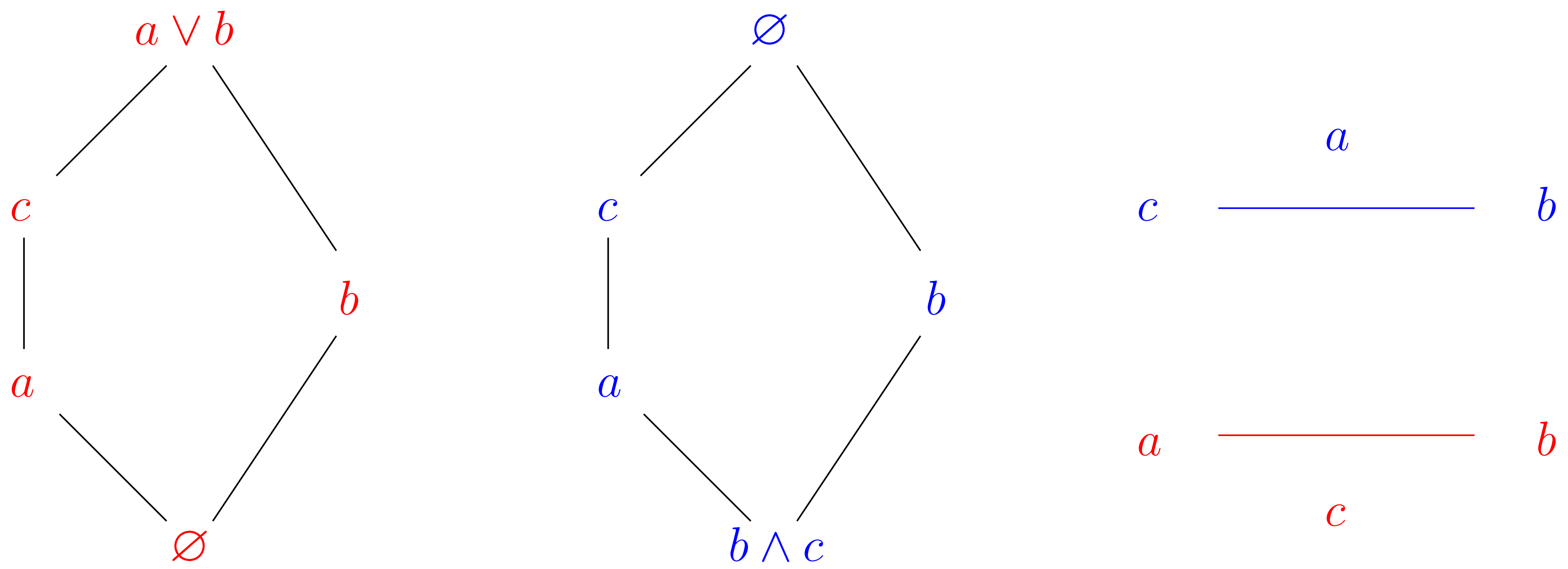
Reading '15
Barnard '19

CANONICAL COMPLEX OF THE TAMARI LATTICE

$(L, \leq, \wedge, \vee)$ lattice

meet semidistributive $\iff$ $x \wedge y = x \wedge z$ implies $x \wedge (y \vee z) = x \wedge y$ for all $x, y, z \in L$
$\iff$ any $x \in L$ admits a canonical meet representation $x = \bigwedge M$

canonical meet complex = simplicial complex of canonical meet representations
= a simplex M for each element $\bigwedge M$ of L



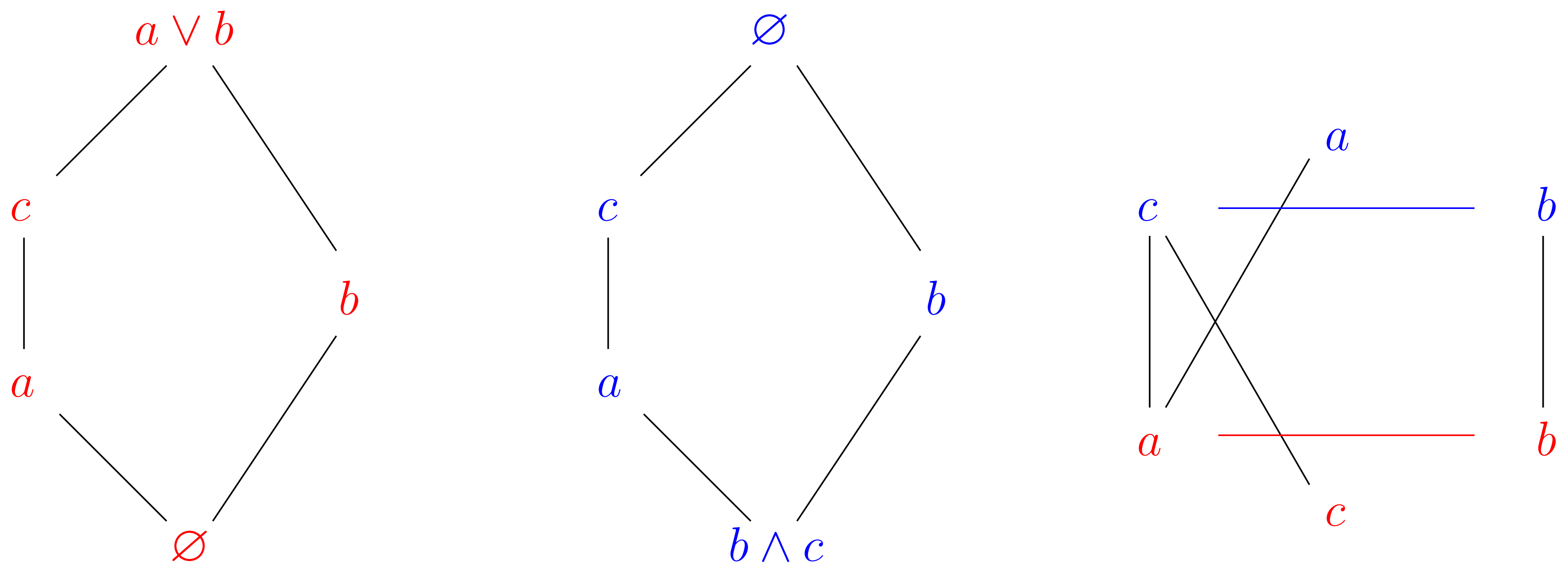
Albertin – P. '22

CANONICAL COMPLEX OF THE TAMARI LATTICE

$(L, \leq, \wedge, \vee)$ lattice

semidistributive $\iff$ join semidistributive and meet semidistributive
$\iff$ any $x \in L$ admits canonical representations $x = \bigvee J = \bigwedge M$

canonical complex = simplicial complex of canonical representations
= a simplex $J \sqcup M$ for each interval $\bigvee J \leq \bigwedge M$ in L



CANONICAL COMPLEX OF THE TAMARI LATTICE

$(L, \le, \wedge, \vee)$ lattice

semidistributive $\iff$ join semidistributive and meet semidistributive

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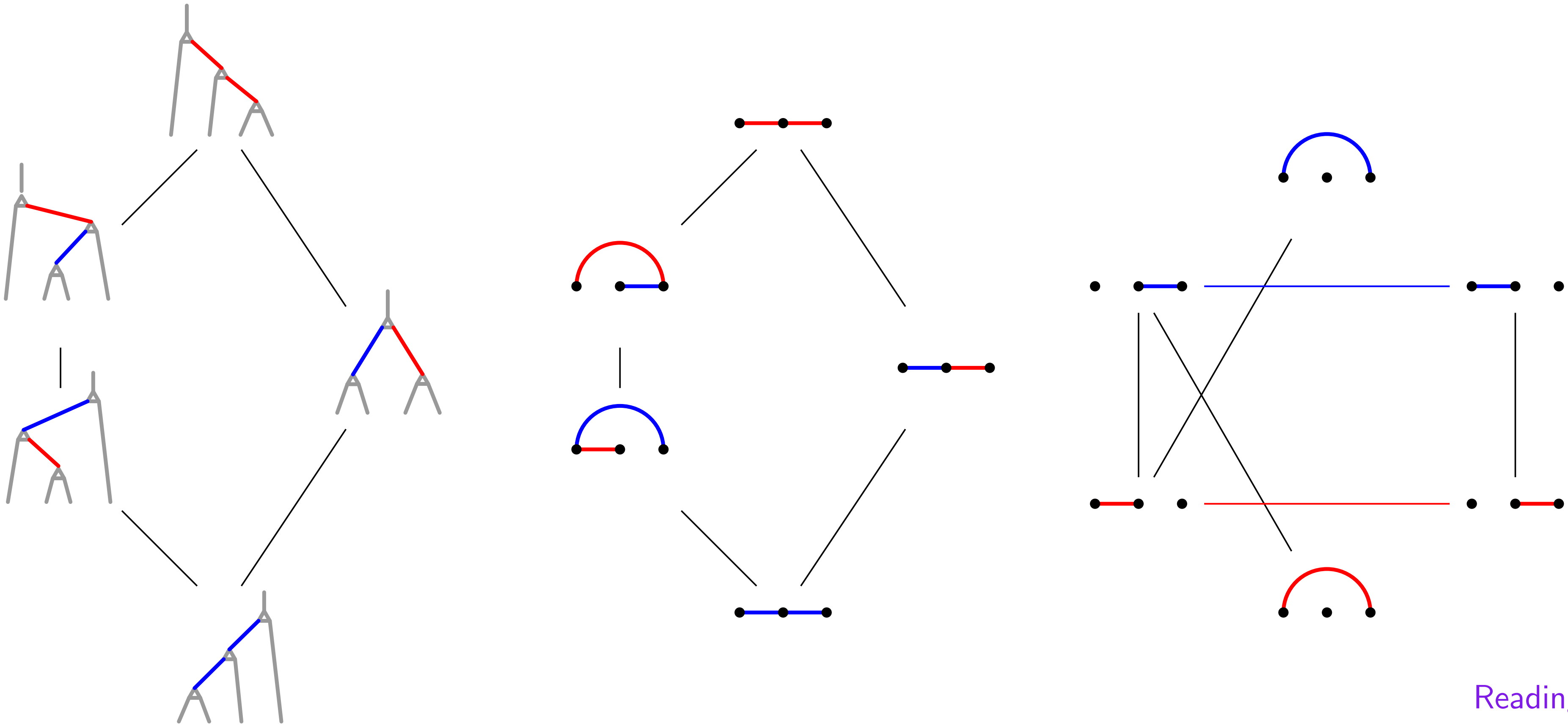


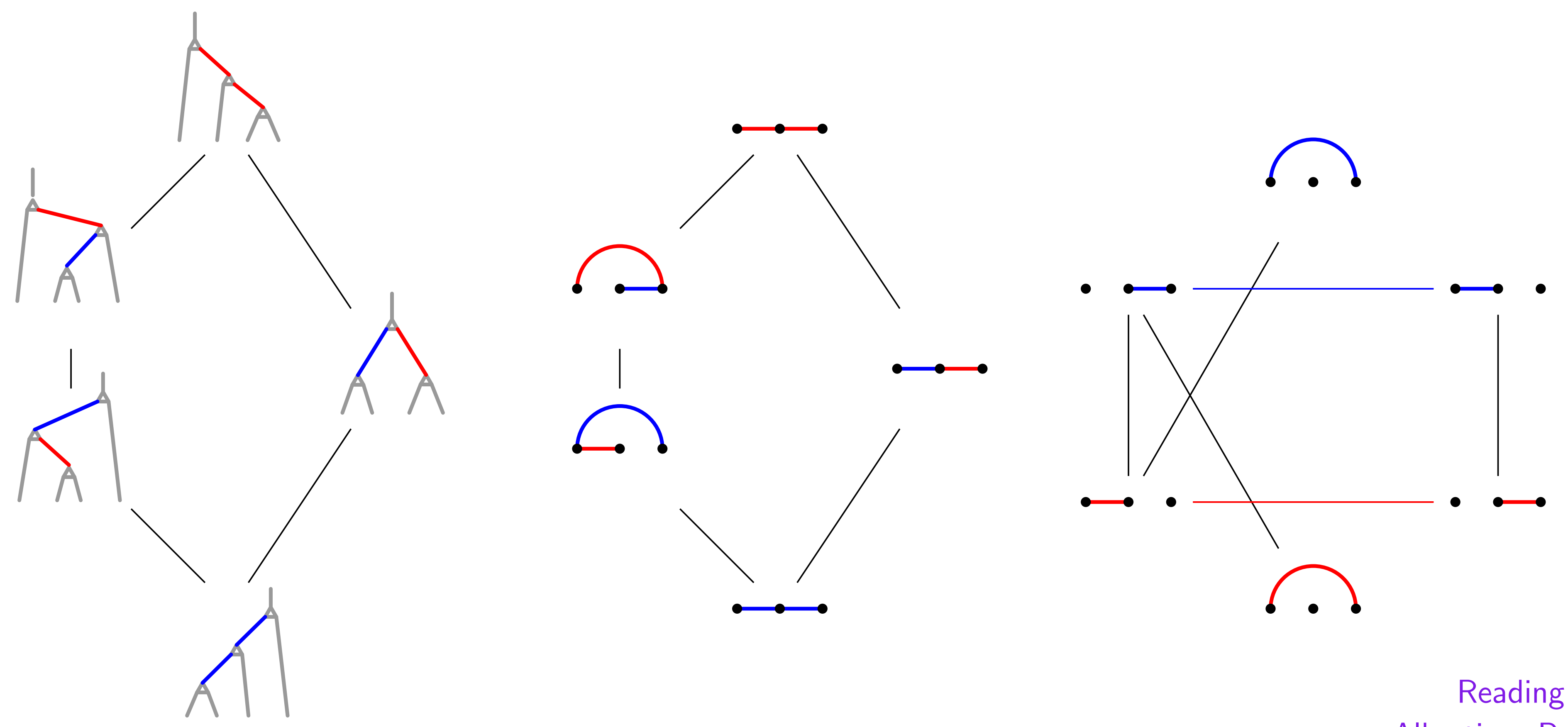
Reading '15

Albertin – P. '22

CANONICAL COMPLEX OF THE TAMARI LATTICE

THM. For any $n, k \geq 1$, — Bostan – Chyzak – P. '23$^+$

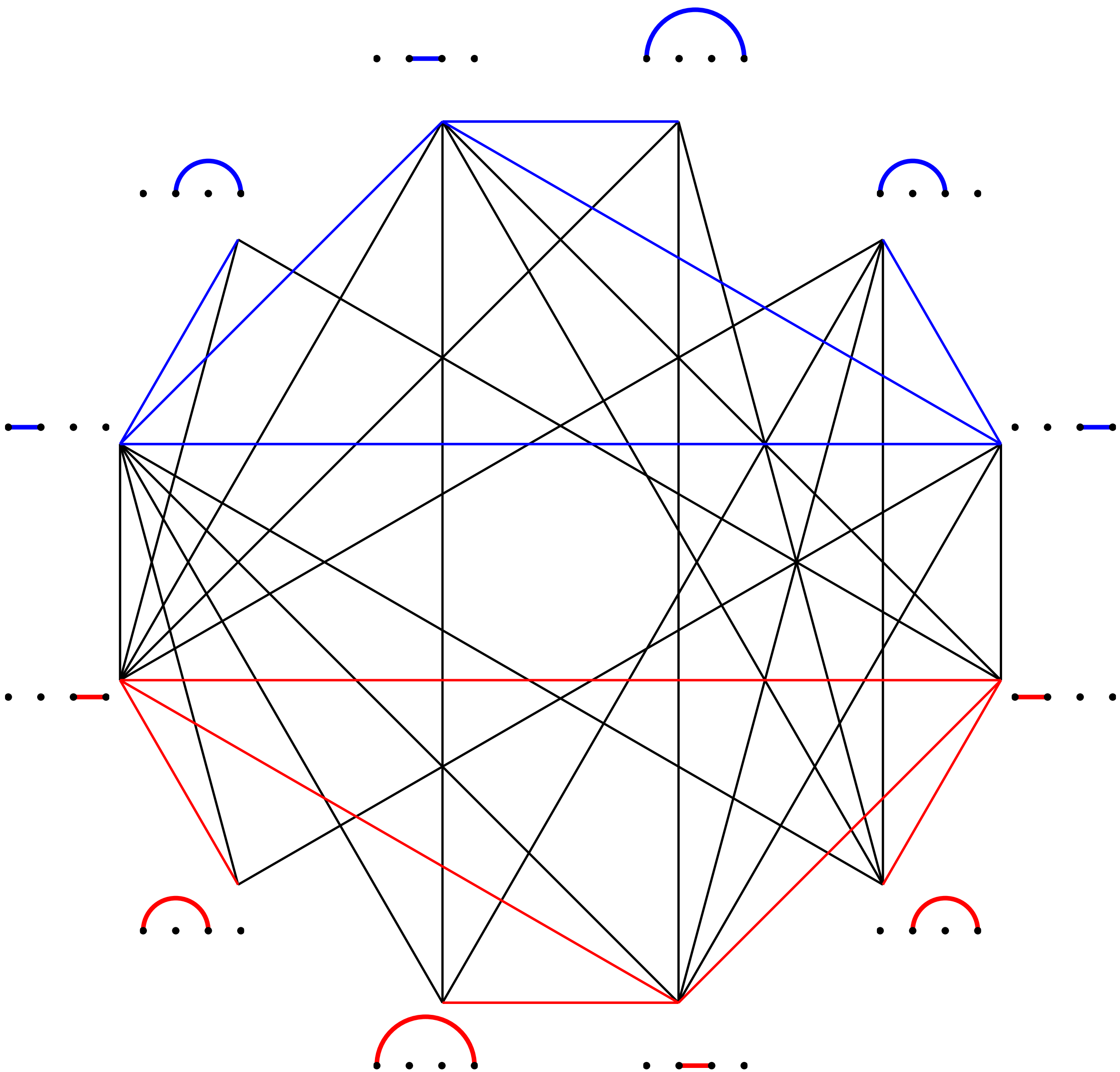
$$f_k(\mathbb{CC}_n) = \#\{S \leq T \in \mathrm{Tam}(n) \mid \mathrm{des}(S) + \mathrm{asc}(T) = k\} = \frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k}$$



Reading '15

Albertin – P. '22

CANONICAL COMPLEX OF THE TAMARI LATTICE



$$1 + 12 + 33 + 22 = 68$$

Reading '15
Albertin – P. '22

SECOND REFINED FORMULA ON TAMARI INTERVALS

$\mathrm{Tam}(n)$ = Tamari lattice on binary trees with n nodes

$\mathrm{des}(T)$ = number of binary trees covered by T

$\mathrm{asc}(T)$ = number of binary trees covering T

THM. For any $n, k \geq 1$,

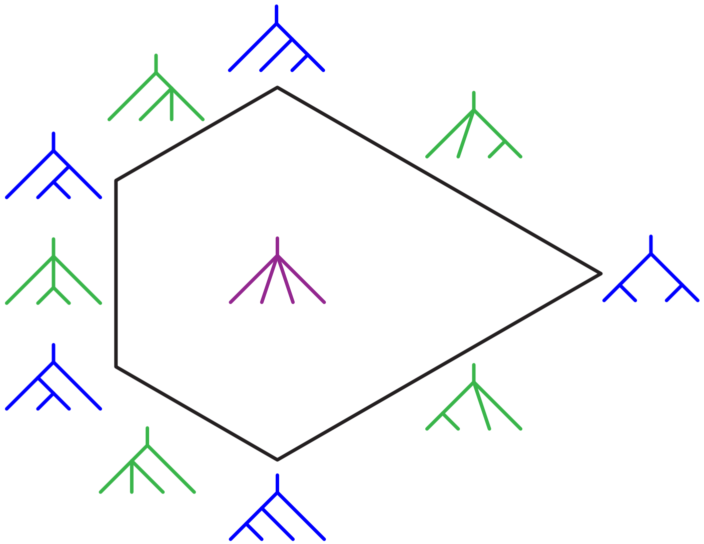
Bostan – Chyzak – P. '23+

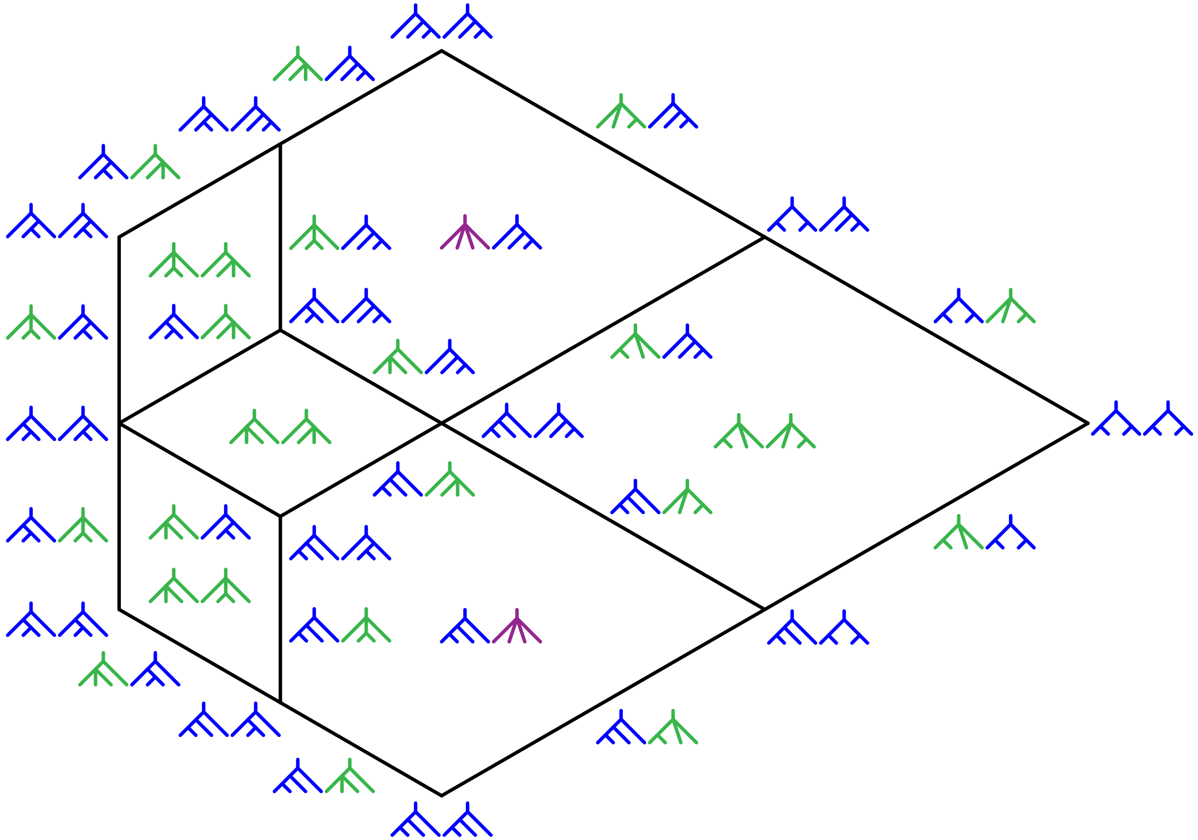
$$\sum_{S \leq T \in \mathrm{Tam}(n)} \binom{\mathrm{des}(S) + \mathrm{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}$$

$n \backslash k$	0	1	2	3	4	5	6	7	8
1	1								
2	3	2							
3	13	18	6						
4	68	144	99	22					
5	399	1140	1197	546	91				
6	2530	9108	12903	8976	3060	408			
7	16965	73710	131625	123500	64125	17442	1938		
8	118668	604128	1302651	1540770	1078539	446292	100947	9614	
9	857956	5008608	12660648	18086640	15958800	8898240	3058770	592020	49335

DIAGONAL OF THE ASSOCIAHEDRON

$\Delta_{\mathbb{A}\mathsf{sso}(n)}$ = diagonal of $(n-1)$-dimensional associahedron





DIAGONAL OF THE ASSOCIAHEDRON

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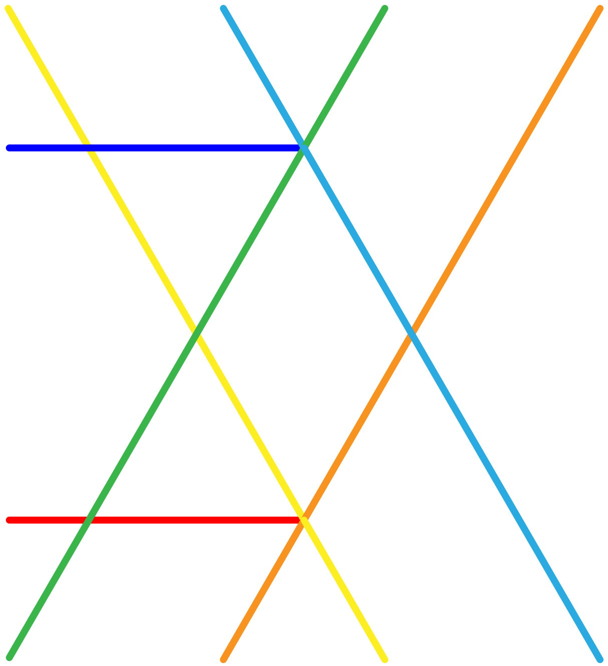

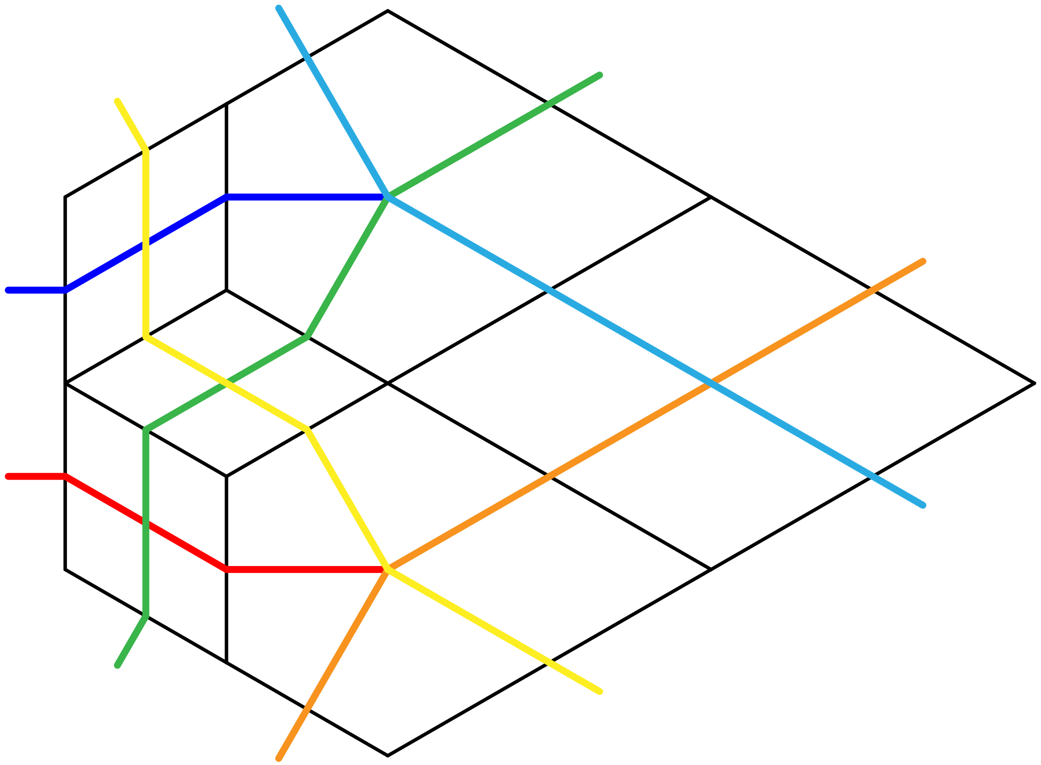

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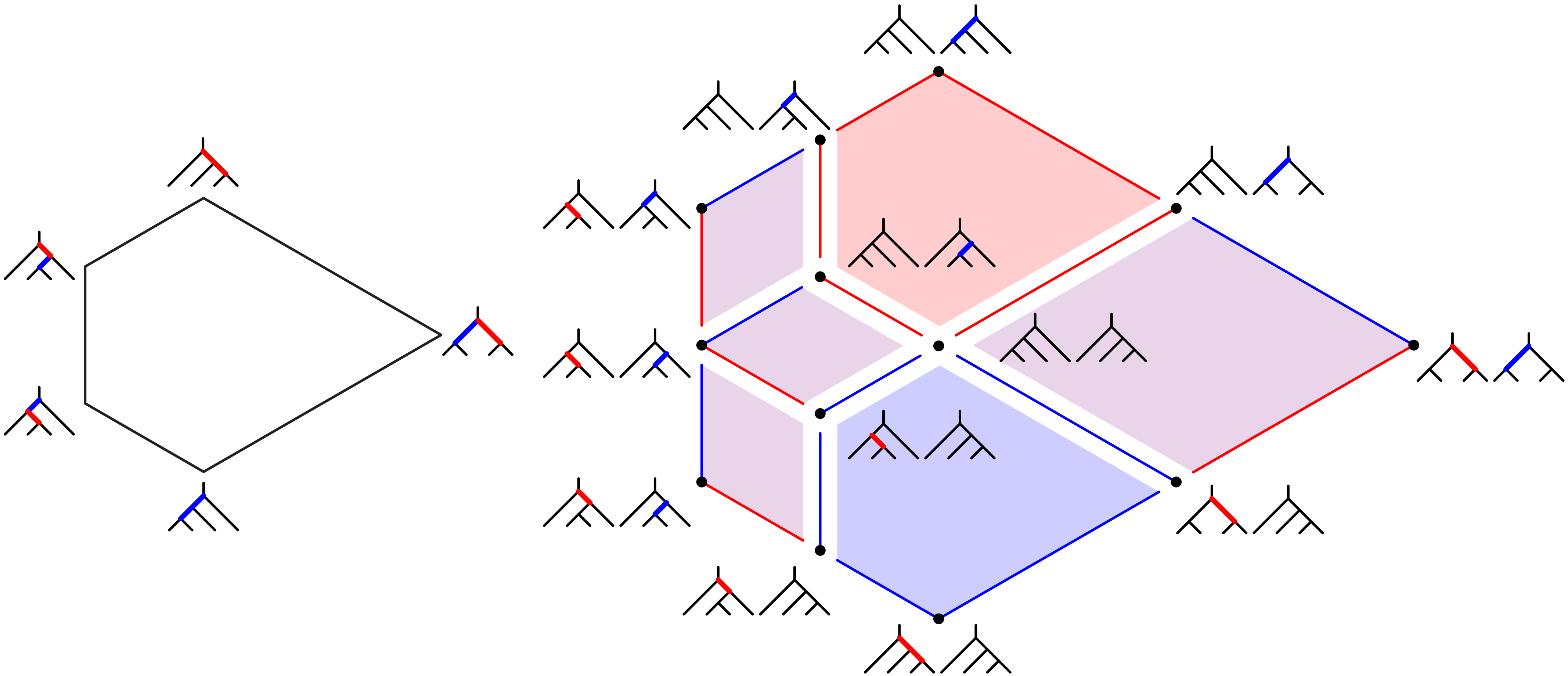
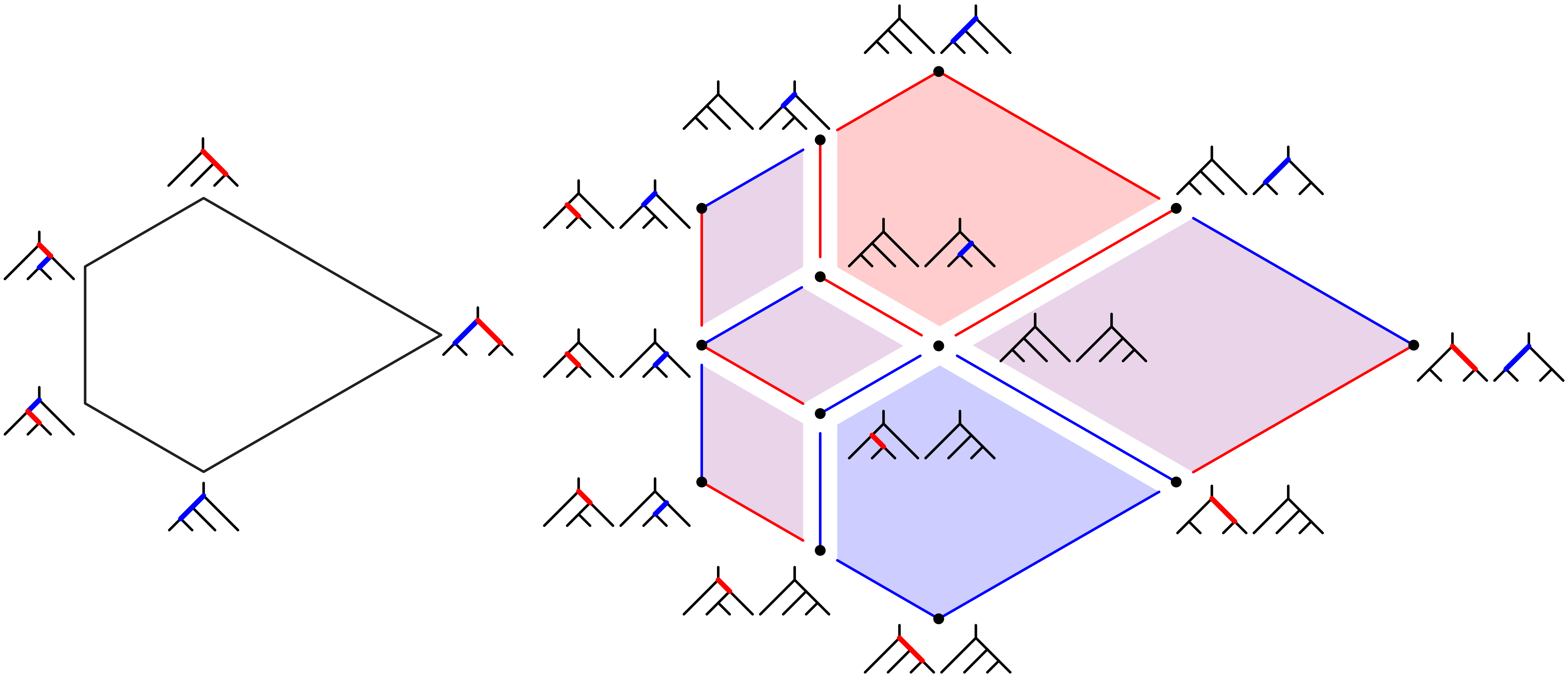


THM. (Magical formula) Masuda – Thomas – Tonks – Vallette '21

k-faces of $\Delta_{\mathbb{A}\mathsf{sso}(n)}$ $\longleftrightarrow$ (F, G) faces of $\mathbb{A}\mathsf{sso}(n)$ with $\dim(F) + \dim(G) = k$ and $\max(F) \leq \min(G)$

DIAGONAL OF THE ASSOCIAHEDRON

$\Delta_{\mathbb{A}\mathrm{sso}(n)}$ = diagonal of $(n-1)$-dimensional associahedron



THM. For any $n, k \geq 1$,   Bostan – Chyzak – P. '23+

$$f_k(\Delta_{\mathbb{A}\mathrm{sso}(n)}) = \sum_{S \leq T \in \mathrm{Tam}(n)} \binom{\mathrm{des}(S) + \mathrm{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}$$

CONNECTION BETWEEN THE TWO FORMULAS

THM. For any $n, k \geq 1$, Bostan – Chyzak – P. '23$^+$

$$f_k(\mathbb{CC}_n) = \#\{S \leq T \in \mathrm{Tam}(n) \mid \mathrm{des}(S) + \mathrm{asc}(T) = k\} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}$$

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$$f_k(\Delta_{\mathbb{A}\mathsf{sso}(n)}) = \sum_{S \leq T \in \mathrm{Tam}(n)} \binom{\mathrm{des}(S) + \mathrm{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}$$

CONNECTION BETWEEN THE TWO FORMULAS

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$$f_k(\mathbb{CC}_n) = \#\left\{S \leq T \in \mathrm{Tam}(n) \mid \mathrm{des}(S) + \mathrm{asc}(T) = k\right\} = \frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k}$$

THM. For any $n, k \geq 1$, Bostan – Chyzak – P. '23$^+$

$$f_k(\Delta_{\mathbb{A}\mathsf{sso}(n)}) = \sum_{S \leq T \in \mathrm{Tam}(n)} \binom{\mathrm{des}(S) + \mathrm{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)}\binom{n-1}{k}\binom{4n+1-k}{n+1}$$

Second formula follows from the first since ...

THM. For any $n, k, r \in \mathbb{N}$, Bostan – Chyzak – P. '23$^+$

$$\sum_{\ell=k}^{n-1}\binom{n+1}{\ell+2}\binom{r}{\ell}\binom{\ell}{k} = \frac{n(n+1)}{(r+1)(r+2)}\binom{n-1}{k}\binom{r+n+1-k}{n+1}.$$

... by application of Chu – Vandermonde equality

QUADRATIC EQUATION

$n(T)$ = number of nodes of T

$\ell(T)$ = number of bounded edges on the left branch of T

$$\mathbb{A}(u, v, t, z) \coloneqq \sum_{S \le T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\mathrm{des}(S) + \mathrm{asc}(T)}$$

QUADRATIC EQUATION

$n(T)$ = number of nodes of T
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$$\mathbb{A}(u,v,t,z) \coloneqq \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)}$$

We want to compute

$$A \coloneqq A(t,z) \coloneqq \sum_{S \leq T} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)} = \mathbb{A}(1,1,t,z)$$

we will use u and v as catalytic variables ...

QUADRATIC EQUATION

$n(T)$ = number of nodes of T

$\ell(T)$ = number of bounded edges on the left branch of T

$$\mathbb{A}(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)}$$

We want to compute

$$A := A(t, z) := \sum_{S \leq T} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)} = \mathbb{A}(1, 1, t, z)$$

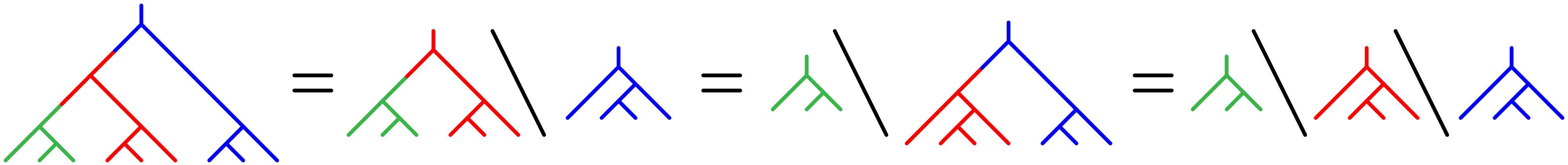
we will use u and v as catalytic variables ...

PROP. The generating functions $A_u := \mathbb{A}(u, 1, t, z)$ and $A_1 := \mathbb{A}(1, 1, t, z)$ satisfy the quadratic functional equation

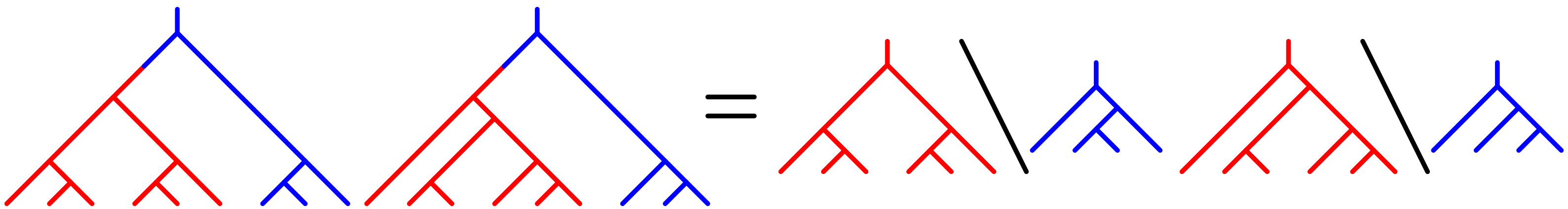
$$(u-1)A_u = t\big(u - 1 + u(u + z - 1)A_u - zA_1\big)\big(1 + uzA_u\big)$$

GRAFTING DECOMPOSITIONS

$S \backslash T =$ binary tree obtained by grafting S on the leftmost leaf of T
$S = S_0 \backslash S_1 \backslash \dots \backslash S_k$ grafting decomposition



> LEM. If $S = S_0 \backslash S_1 \backslash \dots \backslash S_k$ and $T = T_0 \backslash T_1 \backslash \dots \backslash T_k$ are s.t. $n(S_i) = n(T_i)$ for all $i \in [k]$, then $S \le T \iff S_i \le T_i$ for all $i \in [k]$ Chapoton '07



> LEM. If $S \le T$, then we can write $S = S_0 \backslash S_1 \backslash \dots \backslash S_\ell$ and $T = T_0 \backslash T_1 \backslash \dots \backslash T_\ell$ where $\ell = \ell(T)$ and $n(S_i) = n(T_i)$ for all $i \in [\ell]$ Chapoton '07

$\ell(T) =$ number of bounded edges on the left branch of T

QUADRATIC EQUATION

$n(T) =$ number of nodes of T
$\ell(T) =$ number of bounded edges on the left branch of T

$$\mathbb{A}(u, v, t, z) := \sum_{S \le T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\mathrm{des}(S) + \mathrm{asc}(T)}$$

Consider

$$A_u(t, z) := \mathbb{A}(u, 1, t, z) \quad \text{and} \quad A_u^\circ(t, z) := \mathbb{A}(u, 0, t, z)$$

$= $ all Tamari intervals $\qquad\qquad = $ indecomposable intervals

QUADRATIC EQUATION

$A_u = A_u(t, z) =$ all Tamari intervals

$A_u^\circ = A_u^\circ(t, z) =$ indecomposable intervals

$$\sum_{S \leq T} u^{\ell(S)} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)}$$

Chapoton '07

QUADRATIC EQUATION

$A_u = A_u(t,z) =$ all Tamari intervals

$A_u^\circ = A_u^\circ(t,z) =$ indecomposable intervals

$$\sum_{S \leq T} u^{\ell(S)} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)}$$

1. all intervals $=$ indecomposable intervals $\setminus$ (ε $+$ all intervals)

$$A_u = A_u^\circ \left(1 + uzA_u \right)$$

QUADRATIC EQUATION

$A_u = A_u(t,z) =$ all Tamari intervals
$A_u^\circ = A_u^\circ(t,z) =$ indecomposable intervals

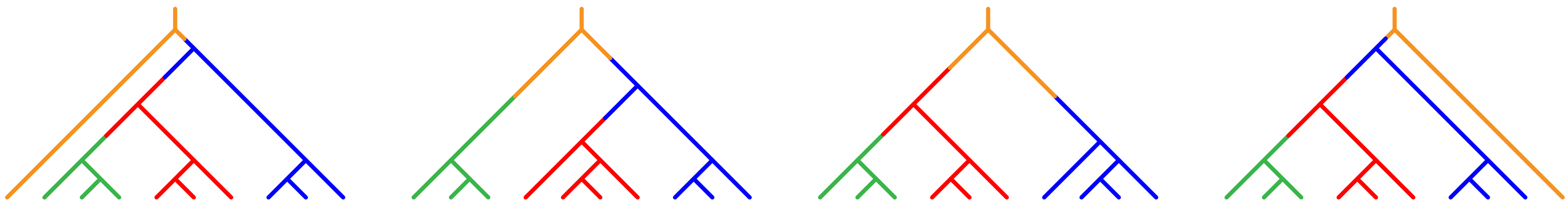
$$\sum_{S \le T} u^{\ell(S)} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)}$$

1. all intervals $=$ indecomposable intervals $\backslash$ (ε $+$ all intervals)

$$A_u = A_u^\circ \, (1 + uzA_u)$$

2. from any Tamari interval (S,T) where $S = S_0/S_1/\dots/S_{\ell(S)}$, we can construct $\ell(S)+2$ indecomposable Tamari intervals (S'_k, T') for $0 \le k \le \ell(S)+1$, where

$$S'_k = \big(S_0/\dots/S_{k-1}\big)/Y\backslash\big(S_k/\dots/S_{\ell(S)}\big) \qquad \text{and} \qquad T' = Y\backslash T$$



$S'_0 = Y/(S_0/S_1/S_2)$ $\quad$ $S'_1 = S_0/Y\backslash(S_1/S_2)$ $\quad$ $S'_2 = (S_0/S_1)/Y\backslash S_2$ $\quad$ $S'_3 = (S_0/S_1/S_2)/Y$

... and all indecomposable intervals are obtained this way

$$A_u^\circ = t\left(1 + z\frac{uA_u - A_1}{u-1} + uA_u\right)$$

Chapoton '07

QUADRATIC EQUATION

$A_u = A_u(t,z) =$ all Tamari intervals $\sum_{S \le T} u^{\ell(S)} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)}$

$A_u^\circ = A_u^\circ(t,z) =$ indecomposable intervals

1. $$A_u = A_u^\circ(1 + uzA_u)$$

2. $$A_u^\circ = t\left(1 + z\frac{uA_u - A_1}{u-1} + uA_u\right)$$

PROP. The generating functions $A_u \coloneqq \mathbb{A}(u,1,t,z)$ and $A_1 \coloneqq \mathbb{A}(1,1,t,z)$ satisfy the quadratic functional equation

$$(u-1)A_u = t\big(u - 1 + u(u+z-1)A_u - zA_1\big)\big(1 + uzA_u\big)$$

QUADRATIC METHOD

PROP. The generating functions $A_u := \mathbb{A}(u,1,t,z)$ and $A_1 := \mathbb{A}(1,1,t,z)$ satisfy the quadratic functional equation

$$(u-1)A_u = t\big(u - 1 + u(u+z-1)A_u - zA_1\big)\big(1 + uzA_u\big)$$

Quadratic equation with a catalytic variable... quadratic method

The discriminant of this quadratic polynomial must have multiple roots, hence, its own discriminant vanishes

CORO. The generating function $A = A(t,z)$ is a root of the polynomial

$$\begin{aligned}
& t^3z^6X^4 \\
& + t^2z^4(tz^2 + 6tz - 3t + 3)X^3 \\
& + tz^2(6t^2z^3 + 9t^2z^2 - 12t^2z + 2tz^2 + 3t^2 - 6tz + 21t + 3)X^2 \\
& + (12t^3z^4 - 4t^3z^3 - 9t^3z^2 - 10t^2z^3 + 6t^3z + 26t^2z^2 \\
& \qquad - t^3 + 6t^2z + tz^2 + 3t^2 - 12tz - 3t + 1)X \\
& + t(8t^2z^3 - 12t^2z^2 + 6t^2z - tz^2 - t^2 + 10tz + 2t - 1)
\end{aligned}$$

REPARAMETRIZATION

CORO. The generating function $A = A(t,z)$ is a root of the polynomial

$$\begin{aligned}
&t^3z^6X^4\\
&+ t^2z^4(tz^2+6tz-3t+3)X^3\\
&+ tz^2(6t^2z^3+9t^2z^2-12t^2z+2tz^2+3t^2-6tz+21t+3)X^2\\
&+ (12t^3z^4-4t^3z^3-9t^3z^2-10t^2z^3+6t^3z+26t^2z^2\\
&\qquad -t^3+6t^2z+tz^2+3t^2-12tz-3t+1)X\\
&+ t(8t^2z^3-12t^2z^2+6t^2z-tz^2-t^2+10tz+2t-1)
\end{aligned}$$

Reparametrize by

$$t = \frac{s}{(s+1)(sz+1)^3} \qquad X = s - zs^2 - zs^3$$

CORO. The generating function $A = A(t,z)$ can be written

$$A = S - zS^2 - zS^3 \qquad \text{where} \qquad t = \frac{S}{(S+1)(Sz+1)^3}$$

LAGRANGE INVERSION

CORO. The generating function $A = A(t, z)$ can be written

$$A = S - zS^2 - zS^3 \qquad \text{where} \qquad t = \frac{S}{(S+1)(Sz+1)^3}$$

THM. (Lagrange inversion) If $S = t\psi(S)$, then $[t^n]\, S^r = \frac{r}{n}\, [s^{n-r}]\, \phi(s)^n$ for any $r \geq 1$

Here $\phi(s) := (s+1)(sz+1)^3$

Hence $[s^a]\, \phi(s)^n = [s^a](s+1)^n(sz+1)^{3n} = \sum_{i+j=a} \binom{n}{i}\binom{3n}{j} z^j$

Hence $[t^n z^k] S^r = \frac{r}{n}[s^{n-r} z^k]\phi(s)^n = \frac{r}{n}\binom{n}{n-r-k}\binom{3n}{k} = \frac{r}{n}\binom{n}{k+r}\binom{3n}{k}$

Finally,

$$[t^n z^k]A = [t^n z^k]S - [t^n z^{k-1}]S^2 - [t^n z^{k-1}]S^3 = \frac{2}{n(n+1)}\binom{3n}{k}\binom{n+1}{k+2}$$

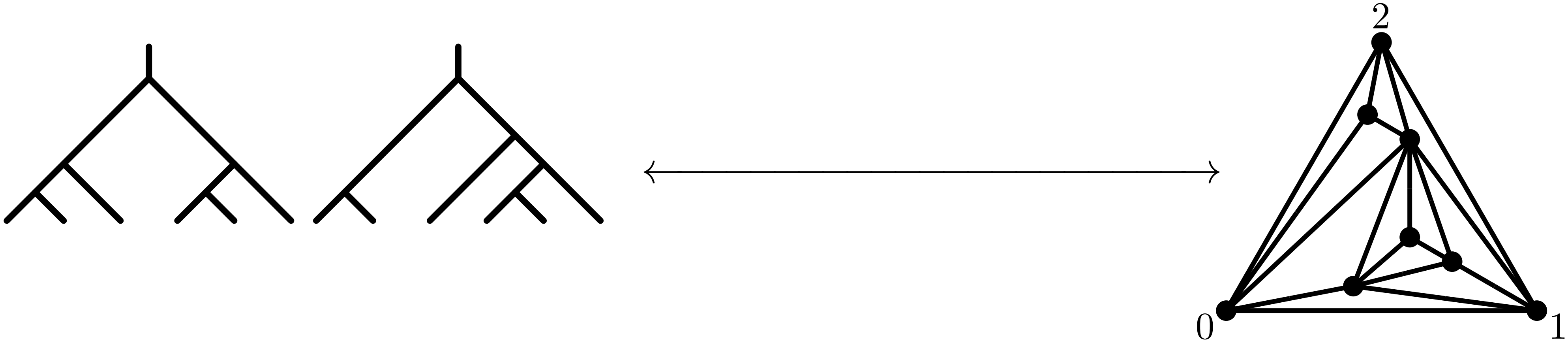
BIJECTIONS TO PLANAR TRIANGULATIONS

$\mathrm{Tam}(n) =$ Tamari lattice on binary trees with n nodes

THM. For any $n \geq 1$, Chapoton '07

$$\#\{S \leq T \in \mathrm{Tam}(n)\} = \frac{2}{(3n+1)(3n+2)}\binom{4n+1}{n+1}$$

Also counts rooted 3-connected planar triangulations with $2n+2$ faces Tutte



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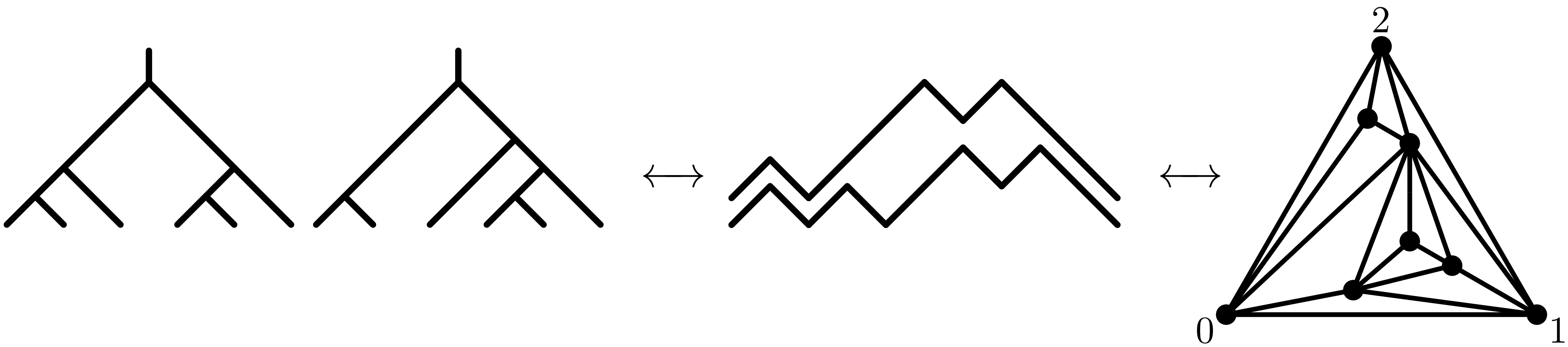

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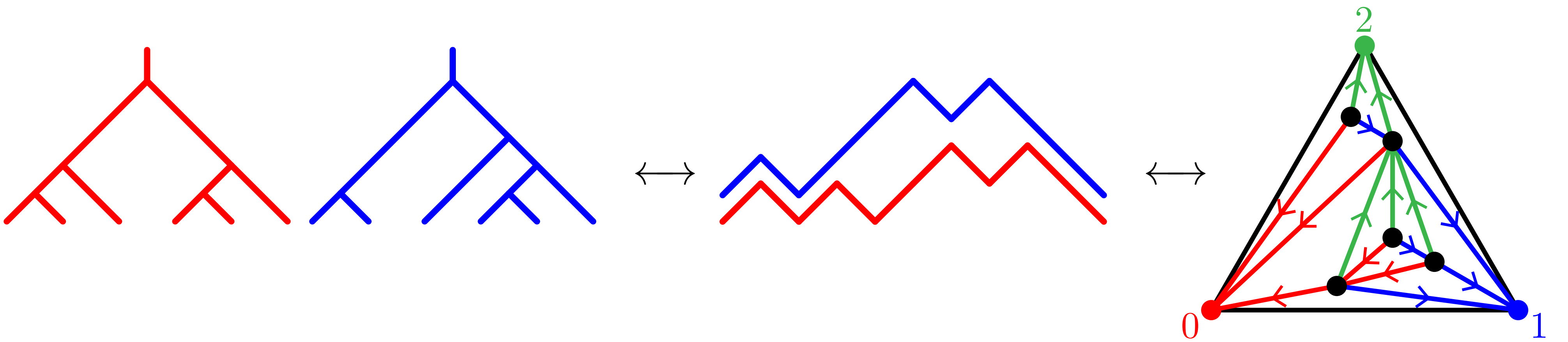


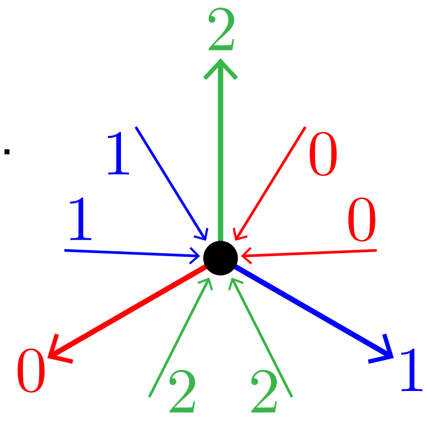
Bernardi – Bonichon, '09

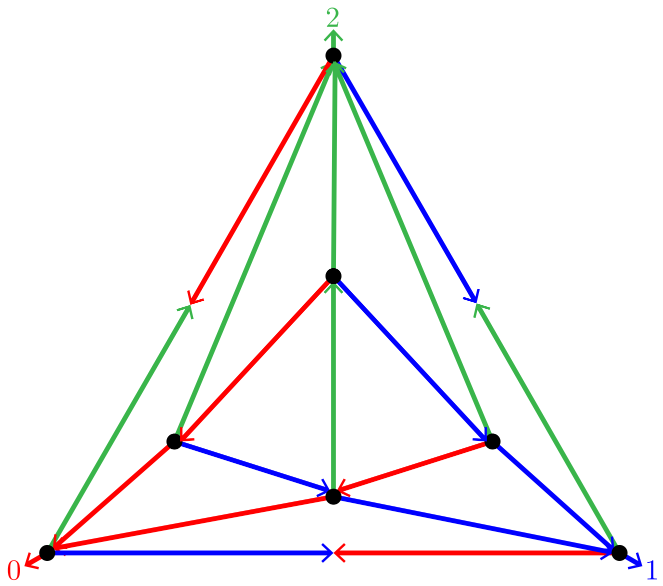
SCHNYDER WOODS

M planar triangulation with external vertices v_0, v_1, v_3
n internal nodes, $3n$ internal edges, $2n+1$ internal triangles

Schnyder wood $=$ color (with $0, 1, 2$) and orient the internal edges s.t.

- the edges colored i form a spanning tree oriented towards v_i
- each vertex satisfies the vertex rule:





Schnyder '89

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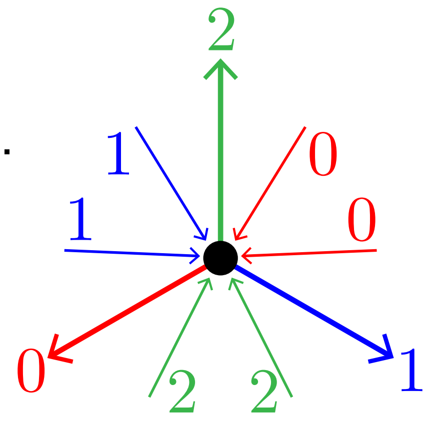
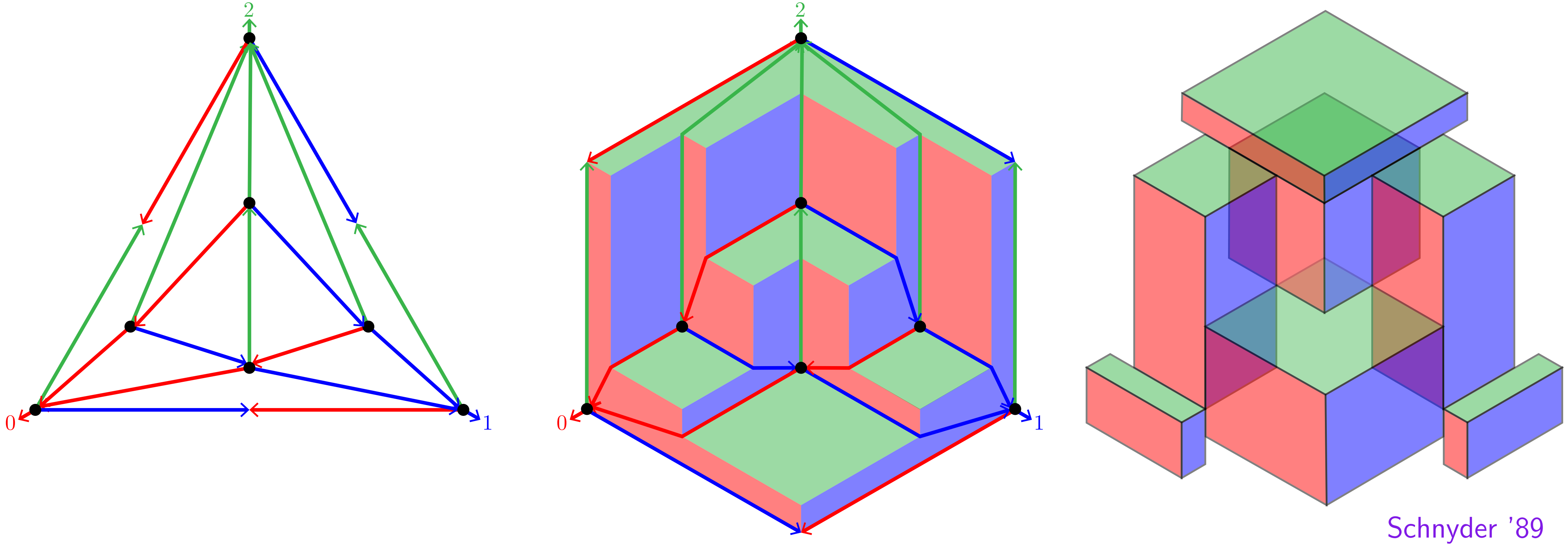


Used for graph drawing and representations:



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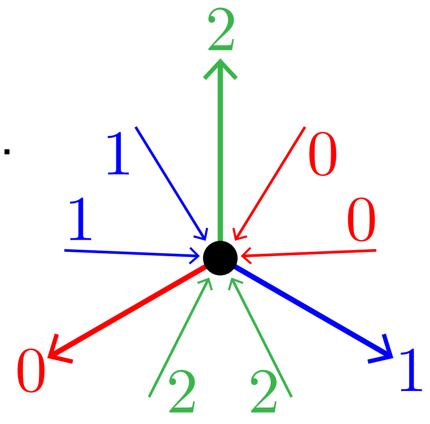
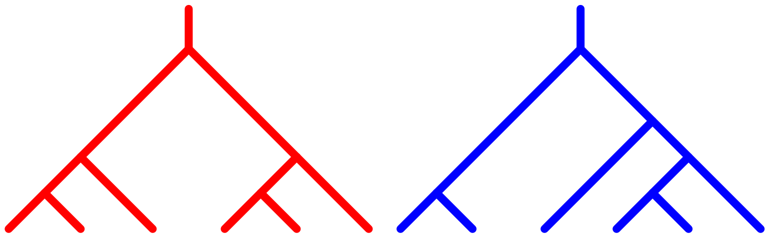


THM. The Schnyder woods of a planar triangulation form a lattice structure under reorientations of clockwise essential cycles

CORO. Any planar triangulation admits a unique Schnyder wood with no clockwise cycle

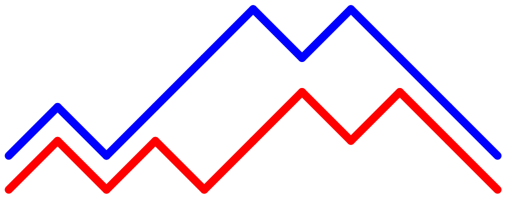
Ossona de Mendez '94
Propp '97
Felsner '04

BERNARDI – BONICHON BIJECTION



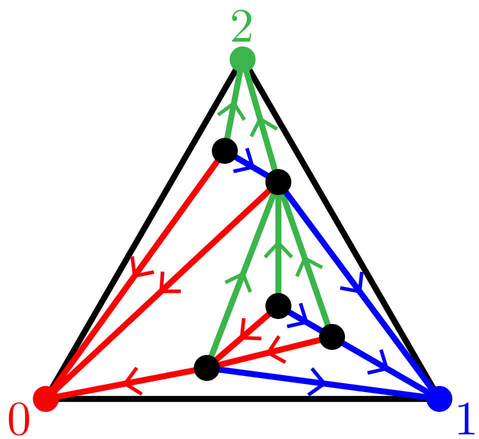
binary trees $S \leq T$
with n nodes

contour of T
transform $\searrow$ to ╱
and $\nwarrow$ to ╲



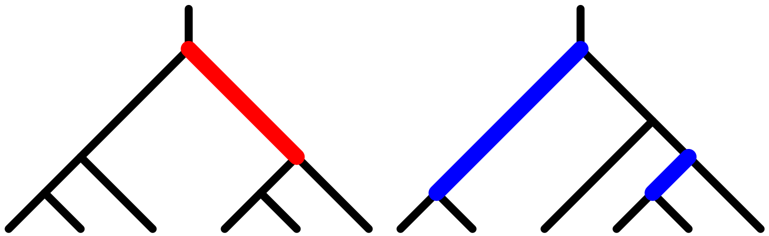
Dyck paths $\mu \leq \nu$
with semilength n

contour of T_0
transform $\leftarrow\bullet$ to ╱ and ╱
$\rightarrow\bullet$ to ╲
and $\rightarrow\bullet$ to ╲

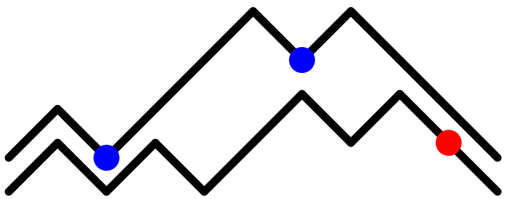


planar triangulations
with n internal vertices

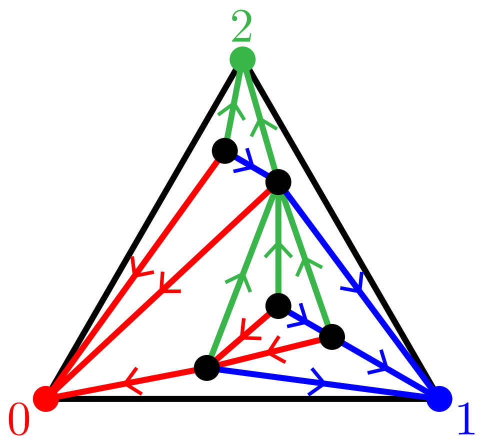
BERNARDI – BONICHON BIJECTION



binary trees $S \leq T$ with n nodes — descents of S — ascents of T



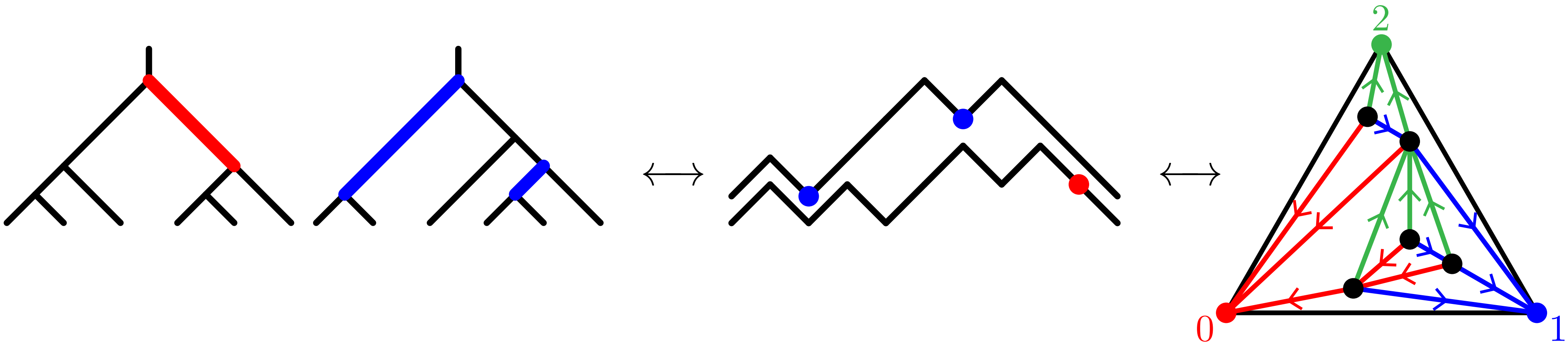
Dyck paths $\mu \leq \nu$ with semilength n — double falls of μ — valleys of ν



planar triangulations with n internal vertices — intermediate red vertices — intermediate blue vertices

Bernardi – Bonichon, '09

COUNTING INTERNAL DEGREES



THM. The generating function $F := F(u,v,w) := \sum_{S \le T} u^{\diagup\diagup} v^{\diagdown\diagdown} w^{\diagdown\diagup}$ is given by

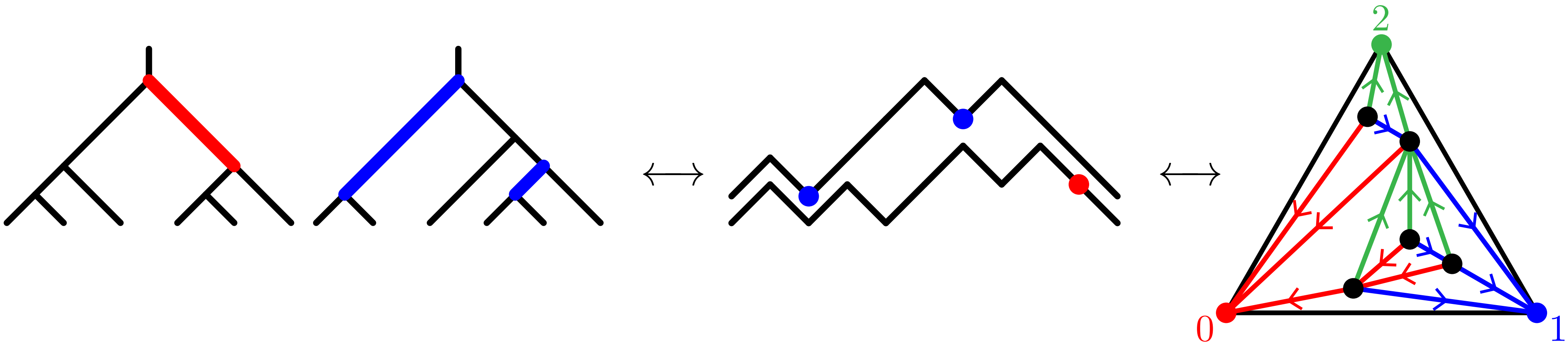
$$uvF = uU + vV + wUV - \frac{UV}{(1+U)(1+V)}$$

where the series $U := U(u,v,w)$ and $V := V(u,v,w)$ satisfy the system

$$U = (v + wU)(1+U)(1+V)^2$$
$$V = (u + wV)(1+V)(1+U)^2$$

Fusy – Humbert '19

COUNTING INTERNAL DEGREES



CORO. The function $A := A(t,z) := \sum_{S \le T} t^{n(S)} z^{\mathrm{des}(S)+\mathrm{asc}(T)} = tF(tz,tz,t)$ is given by

$$tz^2 A = 2tzS + tS^2 - \frac{S^2}{(1+S)^2}$$

where the series $S := S(t,z)$ satisfies

$$S = t(z+S)(1+S)^3$$

... and Lagrange inversion again

(thanks to Éric Fusy)

CANOPY

T binary tree with n nodes, labeled in inorder and oriented towards its root.

canopy of T = vector $\mathrm{can}(T) \in \{-,+\}^{n-1}$ with $\mathrm{can}(T)_i = -$

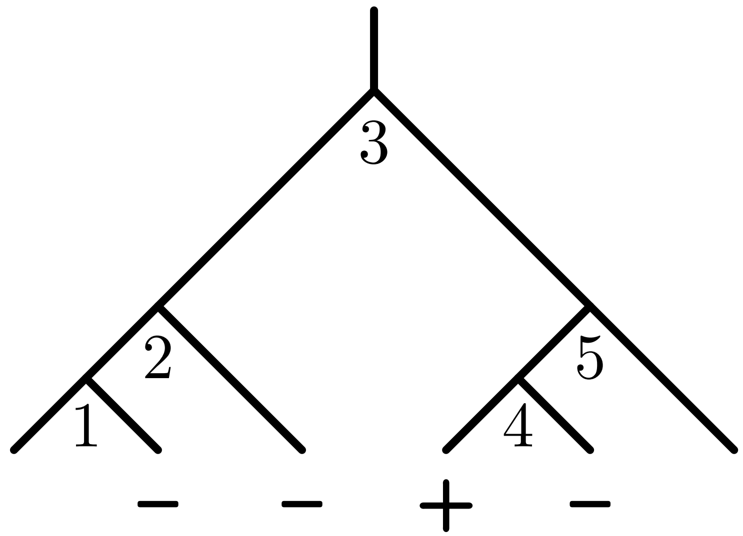
$\iff$ $(j+1)$st leaf of T is a right leaf

$\iff$ there is an oriented path joining its jth node to its $(j+1)$st node

$\iff$ the jth node of T has an empty right subtree

$\iff$ the $(j+1)$st node of T has a non-empty left subtree

$\iff$ the cone corresponding to T is located in the halfspace $x_j \leq x_{j+1}$



CANOPY AGREEMENTS

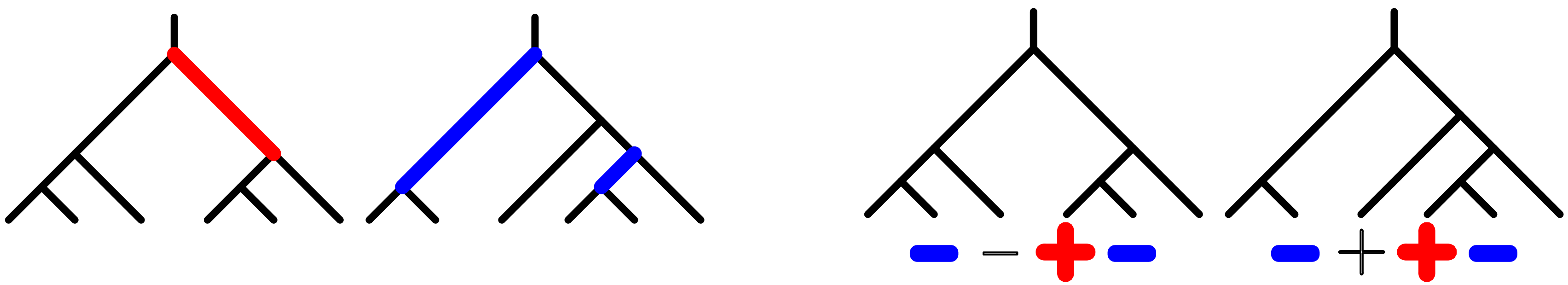
T binary tree with n nodes, labeled in inorder and oriented towards its root.

canopy of T = vector $\mathrm{can}(T) \in \{-,+\}^{n-1}$ with $\mathrm{can}(T)_i = -$
$\iff$ the jth node of T has an empty right subtree
$\iff$ the $(j+1)$st node of T has a non-empty left subtree

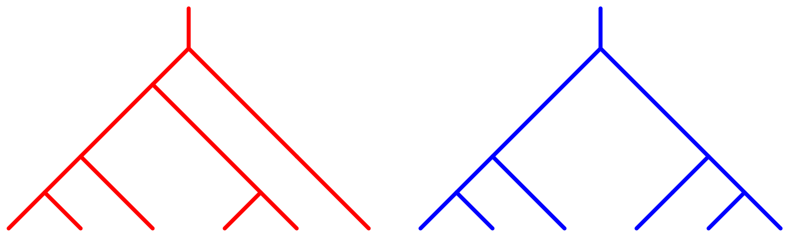
LEM. $\mathrm{asc}(T) = \#\{i \mid \mathrm{can}(T)_i = -\}$ and $\mathrm{des}(T) = \#\{i \mid \mathrm{can}(T)_i = +\}$

LEM. If $S \leq T$, then
- $\mathrm{can}(S) \leq \mathrm{can}(T)$ componentwise
- $\mathrm{des}(S) = \#\{i \mid \mathrm{can}(S)_i = \mathrm{can}(T)_i = +\}$ and $\mathrm{asc}(S) = \#\{i \mid \mathrm{can}(S)_i = \mathrm{can}(T)_i = -\}$

CORO. $\mathrm{des}(S) + \mathrm{asc}(T) = \#$canopy agreements between S and T

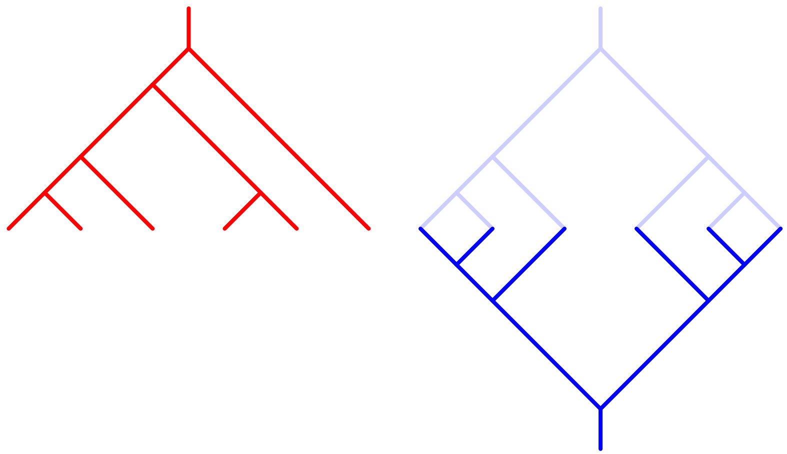


FANG – FUSY – NADEAU BIJECTION



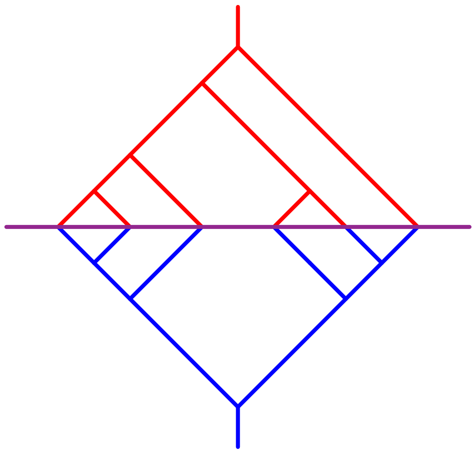
Fang – Fusy – Nadeau '23+

FANG – FUSY – NADEAU BIJECTION



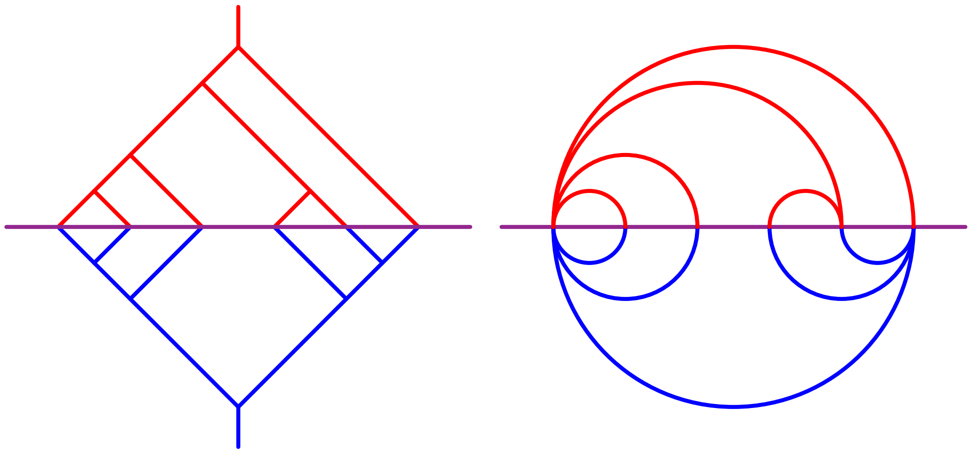
Fang – Fusy – Nadeau '23^{+}

FANG – FUSY – NADEAU BIJECTION



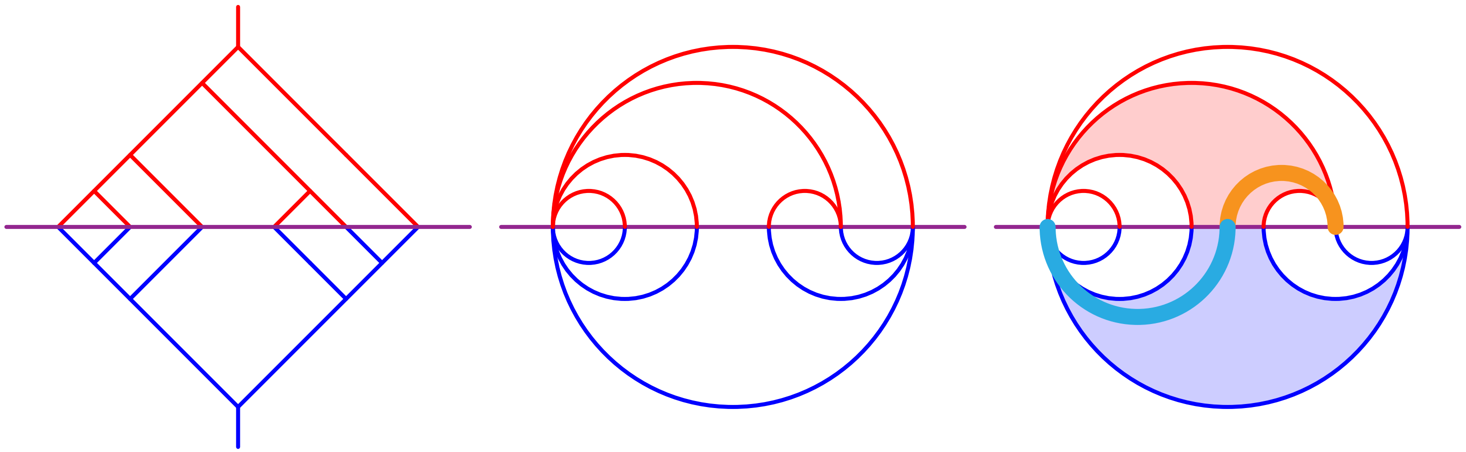
Fang – Fusy – Nadeau '23$^+$

FANG – FUSY – NADEAU BIJECTION



Fang – Fusy – Nadeau '23+

FANG – FUSY – NADEAU BIJECTION



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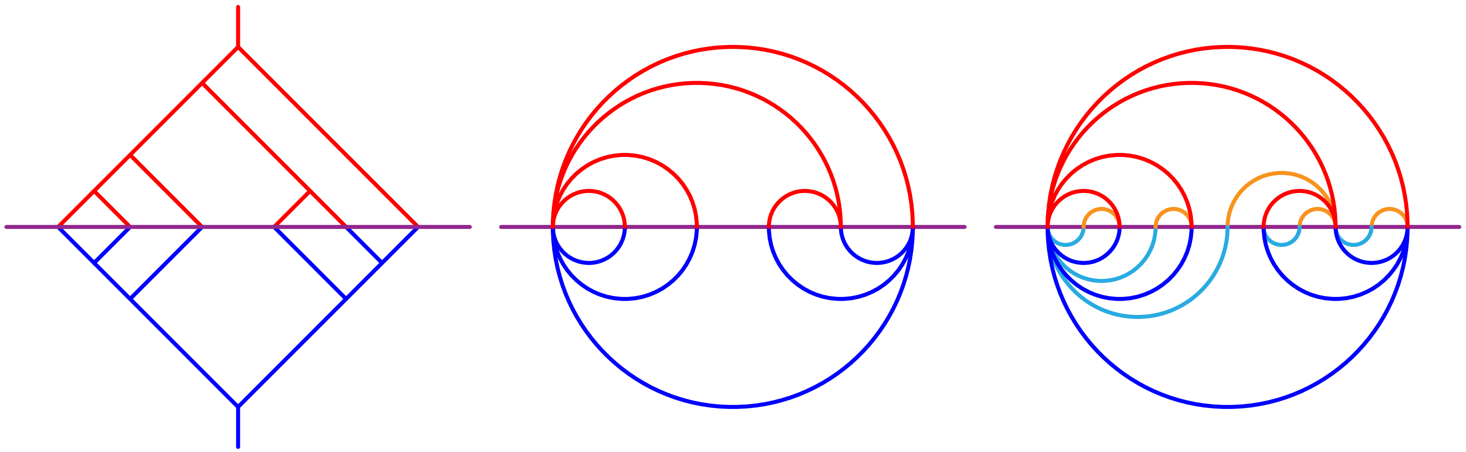

Fang – Fusy – Nadeau '23+

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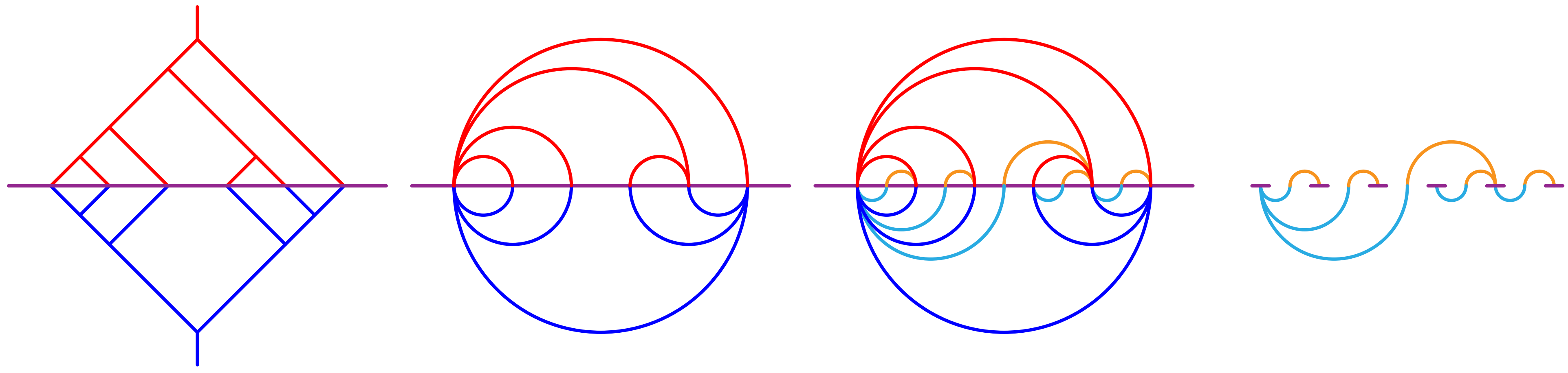
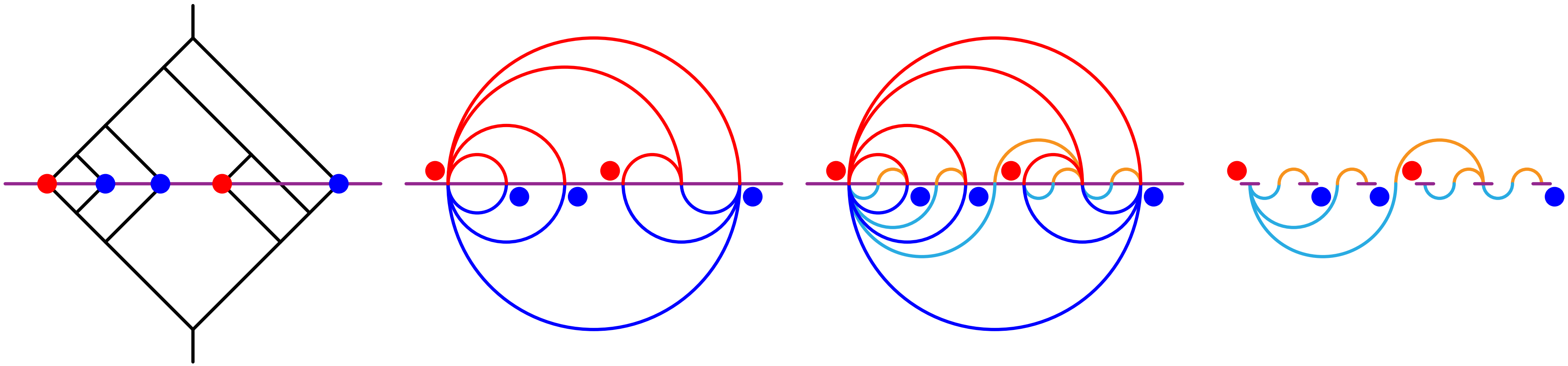


Fang – Fusy – Nadeau '23^{+}

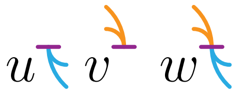
FANG – FUSY – NADEAU BIJECTION

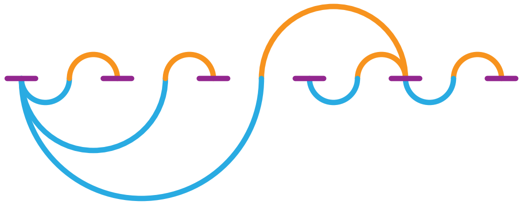


Fang – Fusy – Nadeau '23^{+}

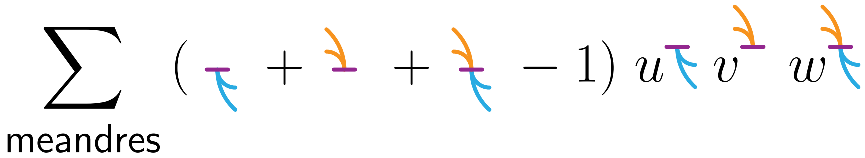
FANG – FUSY – NADEAU BIJECTION

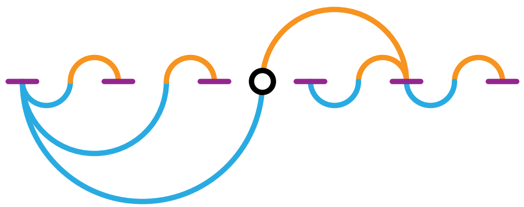
$$\sum_{\text{meandres}}$$





FANG – FUSY – NADEAU BIJECTION





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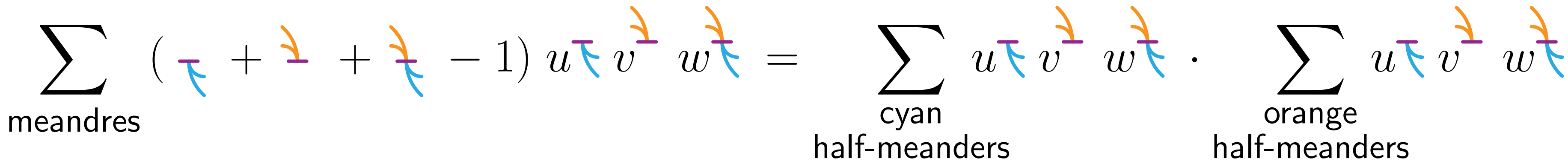
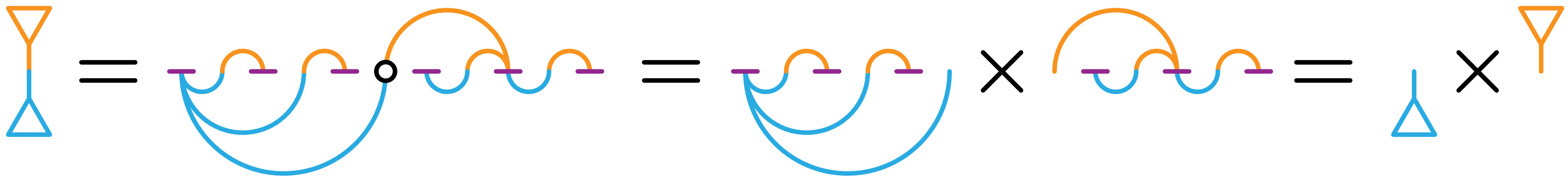
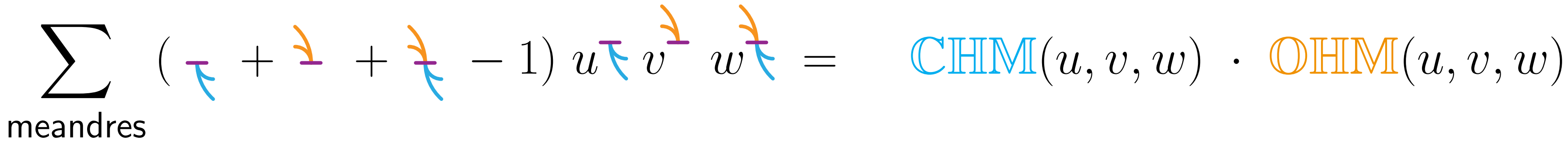
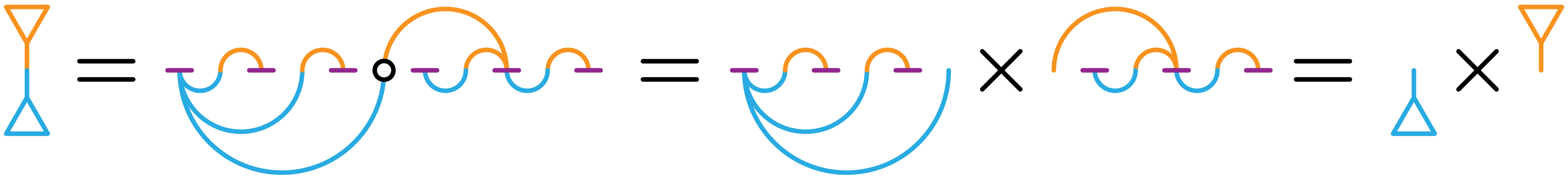




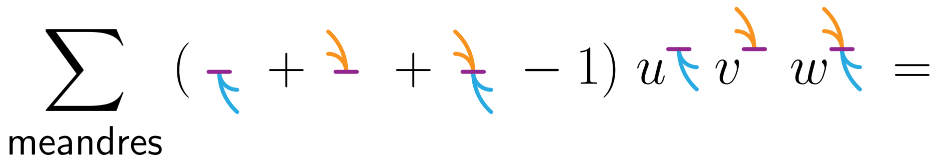
Fang – Fusy – Nadeau '23+

FANG – FUSY – NADEAU BIJECTION

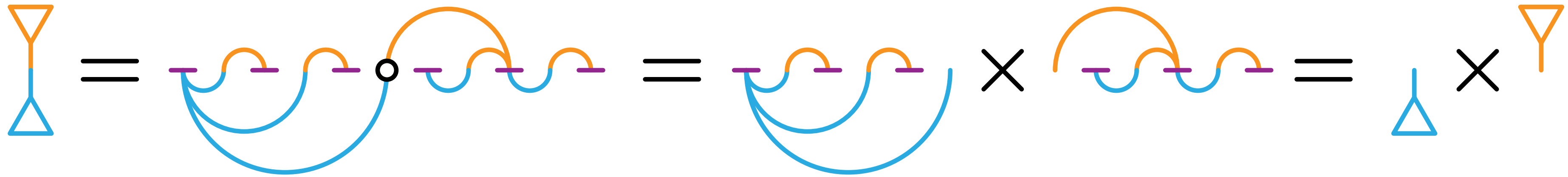




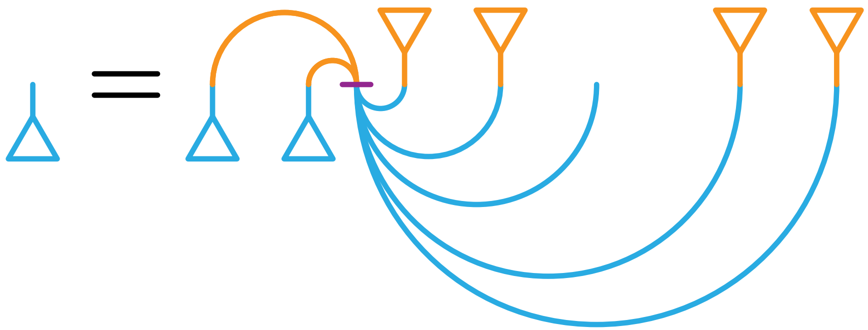
FANG – FUSY – NADEAU BIJECTION



$$= \mathbb{CHM}(u,v,w) \cdot \mathbb{OHM}(u,v,w)$$

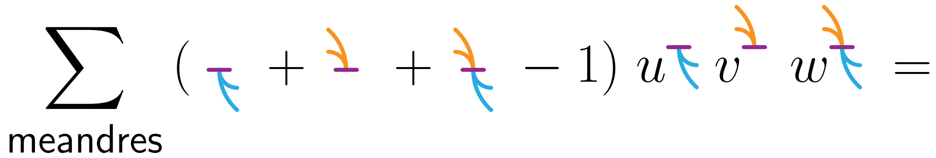


$$\mathbb{CHM} = \frac{1}{(1-\mathbb{CHM})^2}\left(u + \frac{w\,\mathbb{OHM}}{1-\mathbb{OHM}}\right)$$

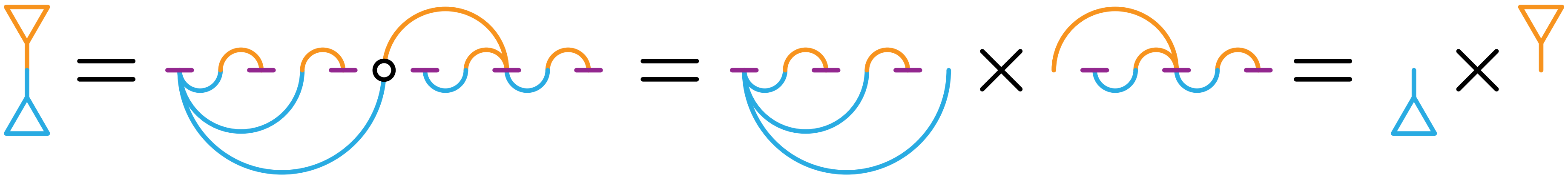


Fang – Fusy – Nadeau '23^{+}

FANG – FUSY – NADEAU BIJECTION



$$= \quad \mathbb{CHM}(u, v, w) \cdot \mathbb{OHM}(u, v, w)$$

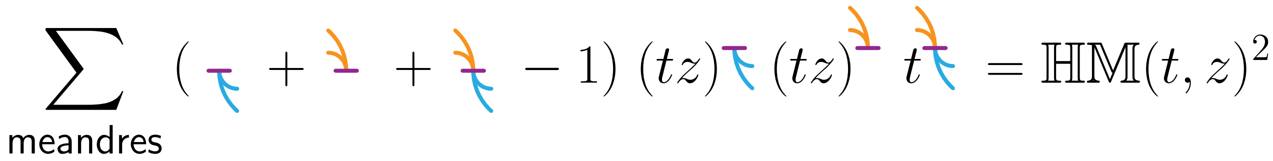


$$\mathbb{CHM} = \frac{1}{(1-\mathbb{CHM})^2}\left(u+\frac{w\,\mathbb{OHM}}{1-\mathbb{OHM}}\right)$$

and

$$\mathbb{OHM} = \frac{1}{(1-\mathbb{OHM})^2}\left(v+\frac{w\,\mathbb{CHM}}{1-\mathbb{CHM}}\right)$$

Fang – Fusy – Nadeau '23^{+}



where $$\mathbb{HM} = \frac{t}{(1-\mathbb{HM})^2}\left(z + \frac{\mathbb{HM}}{1-\mathbb{HM}}\right)$$

Fang – Fusy – Nadeau '23^{+}

FANG – FUSY – NADEAU BIJECTION

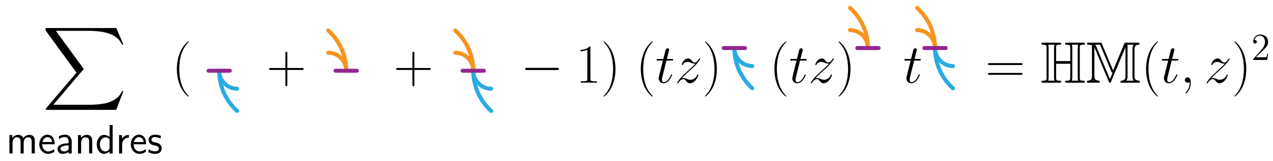
$\sum_{\text{meandres}} ($  $+$  $+$  $- 1)\,(tz)^{}$ $(tz)^{}$ $t^{}$ $= \mathbb{HM}(t, z)^2$

where $$\mathbb{HM} = \frac{t}{(1 - \mathbb{HM})^2}\left(z + \frac{\mathbb{HM}}{1 - \mathbb{HM}}\right)$$

Lagrange inversion again:

$$[t^n z^k]\, \mathbb{HM}^2 = \frac{2}{n}\, [s^{n-2} z^k]\, \frac{1}{(1-s)^{2n}}\left(z + \frac{s}{1-s}\right)^n = \frac{2}{n}\binom{n}{k}\, [s^{n-2}]\, \frac{s^{n-k}}{(1-s)^{3n-k}}$$

$$= \frac{2}{n}\binom{n}{k}\, [s^{k-2}]\, \frac{1}{(1-s)^{3n-k}} = \frac{2}{n}\binom{n}{k}\binom{3n-3}{k-2}$$



where $$\mathbb{HM} = \frac{t}{(1-\mathbb{HM})^2}\left(z + \frac{\mathbb{HM}}{1-\mathbb{HM}}\right)$$

Lagrange inversion again:

$$[t^n z^k]\ \mathbb{HM}^2 = \frac{2}{n}\,[s^{n-2}z^k]\,\frac{1}{(1-s)^{2n}}\left(z+\frac{s}{1-s}\right)^n = \frac{2}{n}\binom{n}{k}\ [s^{n-2}]\,\frac{s^{n-k}}{(1-s)^{3n-k}}$$

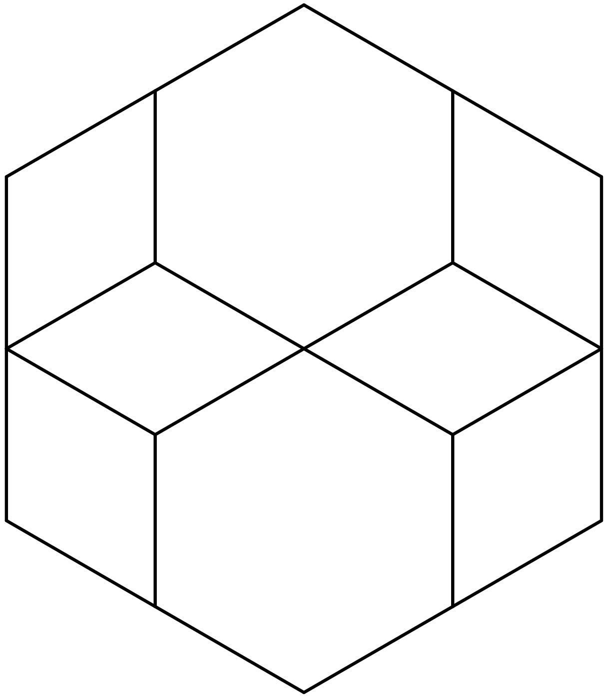
$$= \frac{2}{n}\binom{n}{k}\ [s^{k-2}]\,\frac{1}{(1-s)^{3n-k}} = \frac{2}{n}\binom{n}{k}\binom{3n-3}{k-2}$$

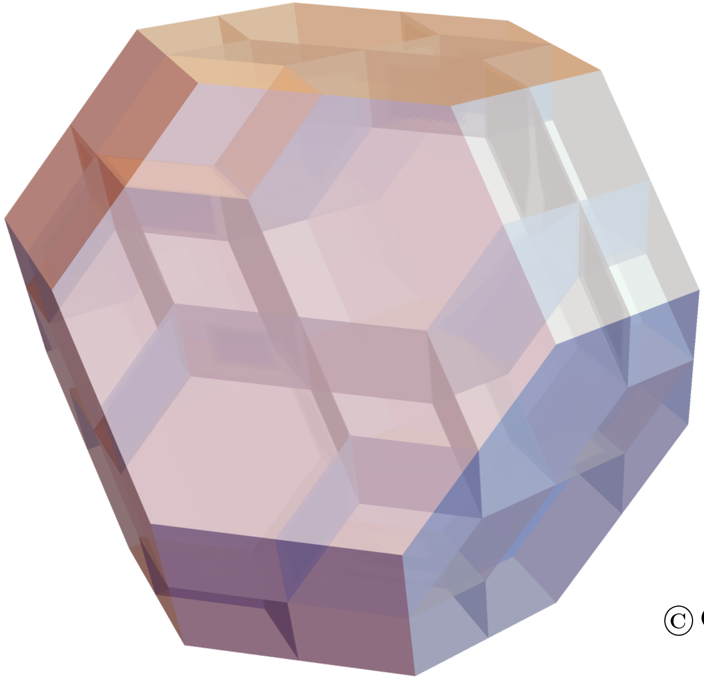
Hence

$$[t^n z^k]\ A(t,z) = \frac{1}{n+1}\,[t^{n+1}z^{k+2}]\ \mathbb{HM}^2 = \frac{2}{n(n+1)}\binom{n+1}{k+2}\binom{3n}{k}$$

Fang – Fusy – Nadeau '23^{+}

DIAGONAL OF THE PERMUTAHEDRON

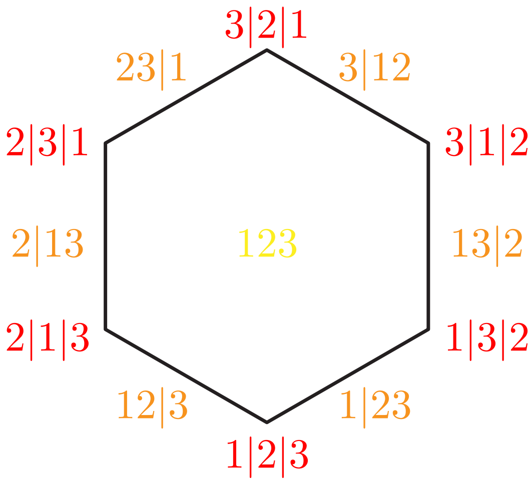


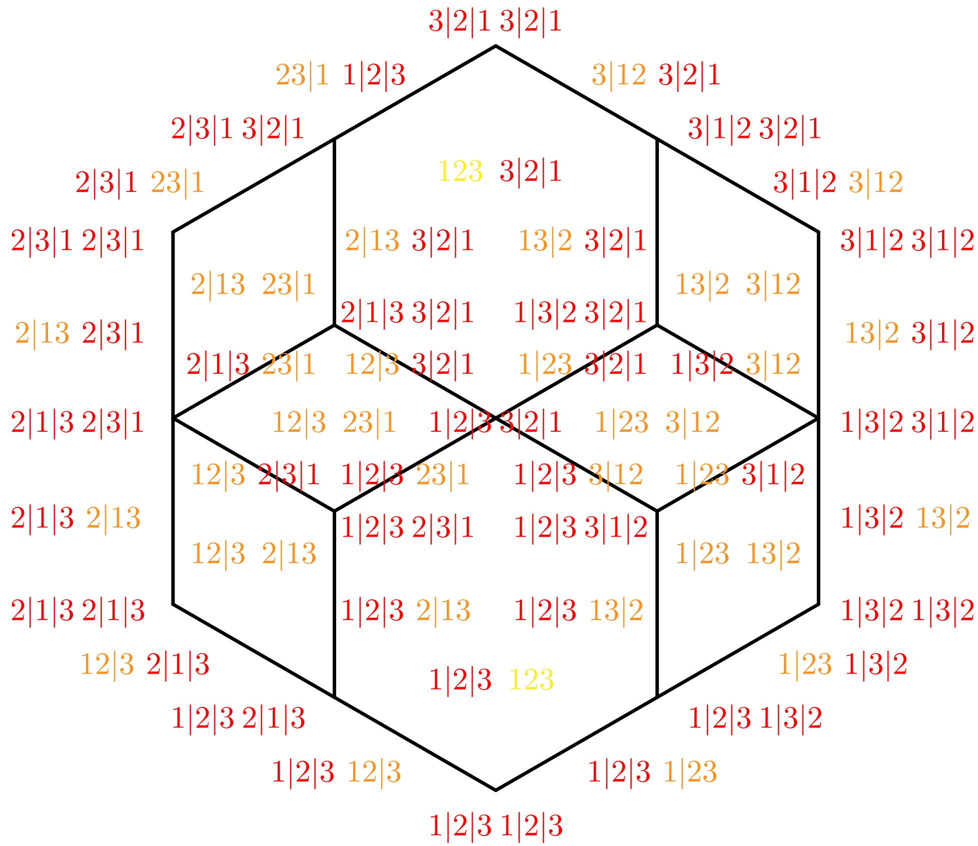


with
Bérénice DELCROIX-OGER (Univ. Montpellier)
Matthieu JOSUAT-VERGÈS (CNRS & Univ. Paris Cité)
Guillaume LAPLANTE-ANFOSSI (Univ. Melbourne)
Kurt STOECKL (Univ. Melbourne)

DIAGONAL OF THE PERMUTAHEDRON

$\Delta_{\mathbb{P}\mathrm{erm}(n)}$ = diagonal of $(n-1)$-dimensional permutahedron





THM. k-faces of $\Delta_{\mathbb{P}\mathrm{erm}(n)}$ $\longleftrightarrow$ (μ, ν) ordered partitions of $[n]$ such that

$$\forall (I, J) \in D(n),\ \exists k \in [n],\ \#\mu_{[k]} \cap I > \#\mu_{[k]} \cap J$$

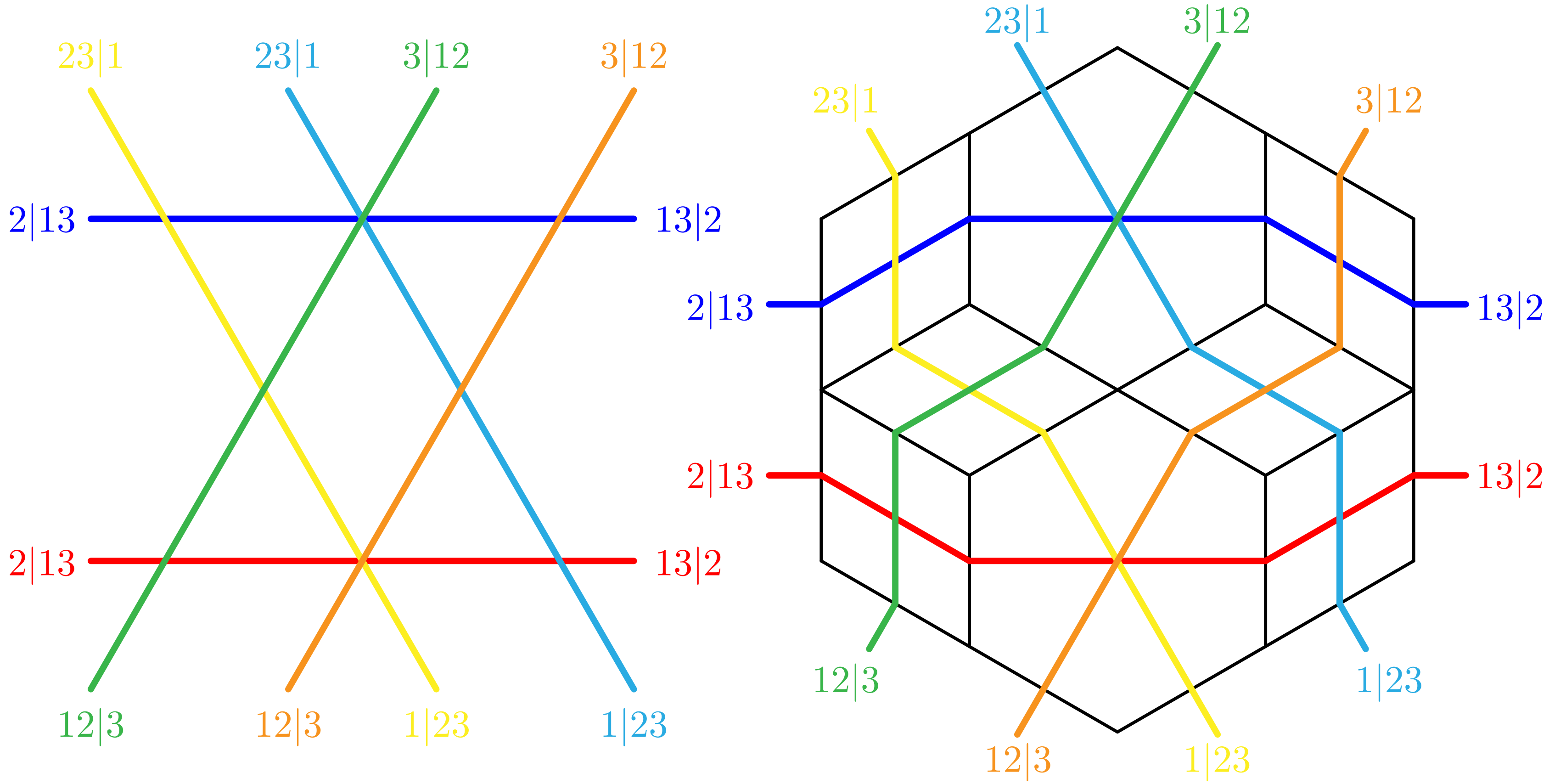
$$\text{or } \exists \ell \in [n],\ \#\mu_{[\ell]} \cap I < \#\mu_{[\ell]} \cap J$$

Laplante-Anfossi '22

where $D(n) := \{(I, J) \mid I, J \subseteq [n],\ \#I = \#J,\ I \cap J = \varnothing, \min(I \cup J) \in I\}$

DIAGONAL OF THE PERMUTAHEDRON

$\Delta_{\mathbb{P}\text{erm}(n)}$ = diagonal of $(n-1)$-dimensional permutahedron



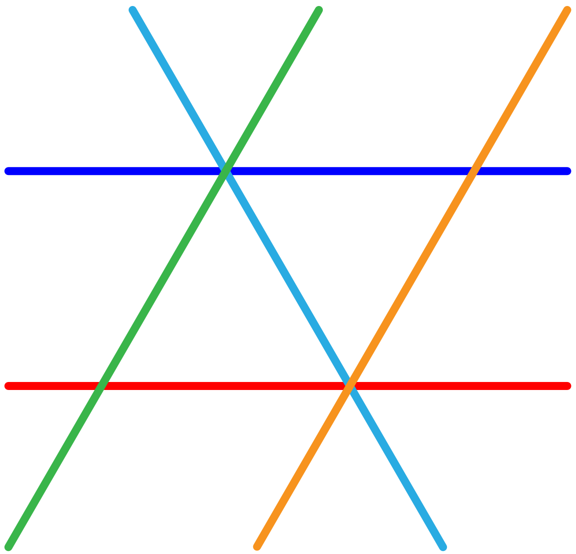
PROP. $\mathcal{B}_n^2$ = two generically translated copies of the braid arrangement

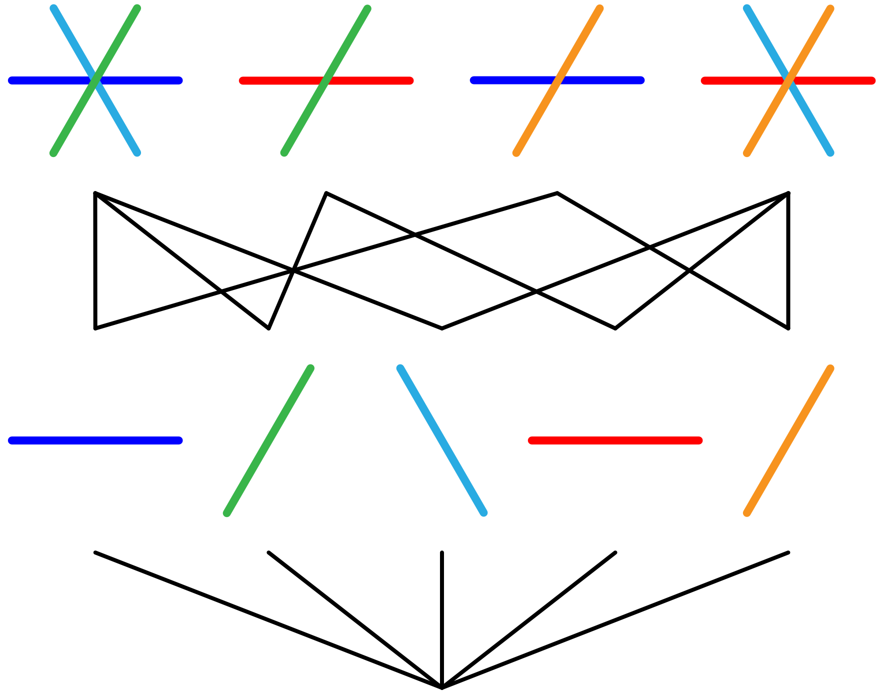
$$f_k\left(\Delta_{\mathbb{P}\text{erm}(n)}\right) = f_{n-k-1}\left(\mathcal{B}_n^2\right)$$

Laplante-Anfossi '22

FLAT POSET & ZASLAVSKY'S THEOREM

flat poset $\boldsymbol{Fl}(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A} =$ reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$





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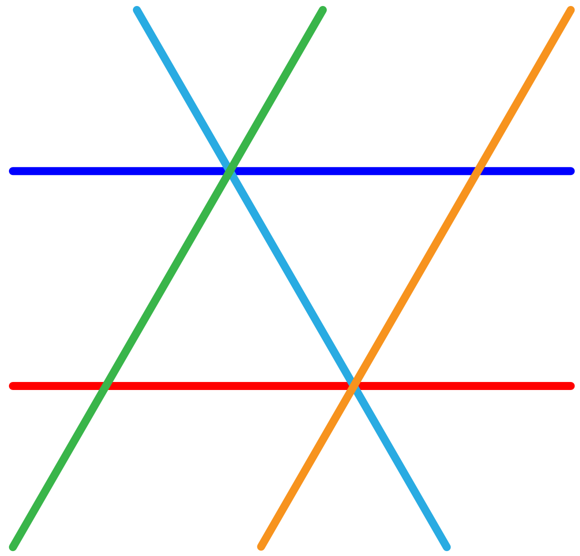

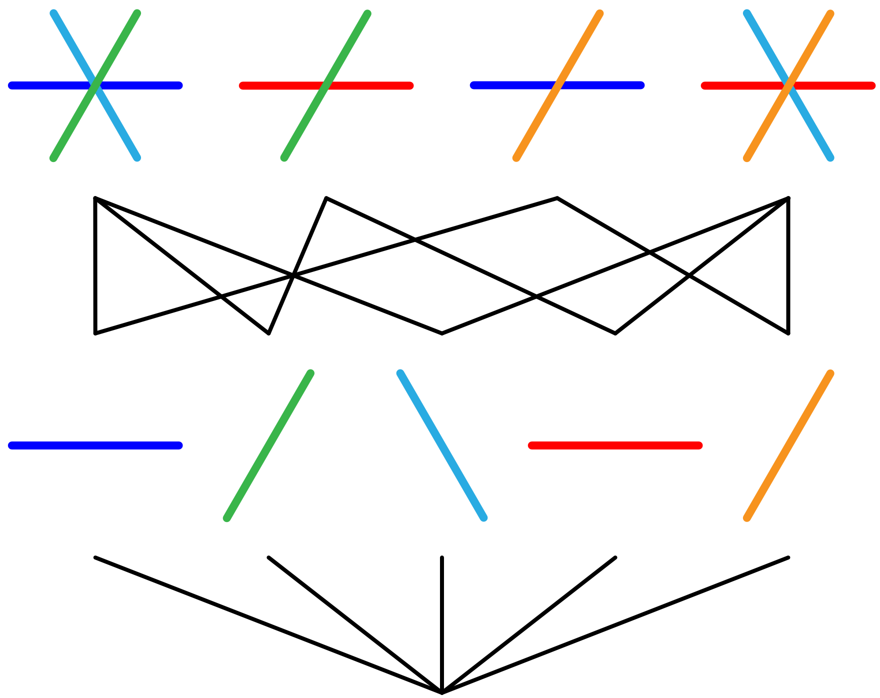

EXM. flat poset of braid arrangement $\mathcal{B}_n$ $\longleftrightarrow$ refinement poset on partitions of $[n]$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x_i = x_j \text{ for all } i, j \text{ in} \\ \text{the same part of } \pi \end{array} \right\}$$

$$\uparrow$$

$$\pi$$

FLAT POSET & ZASLAVSKY'S THEOREM

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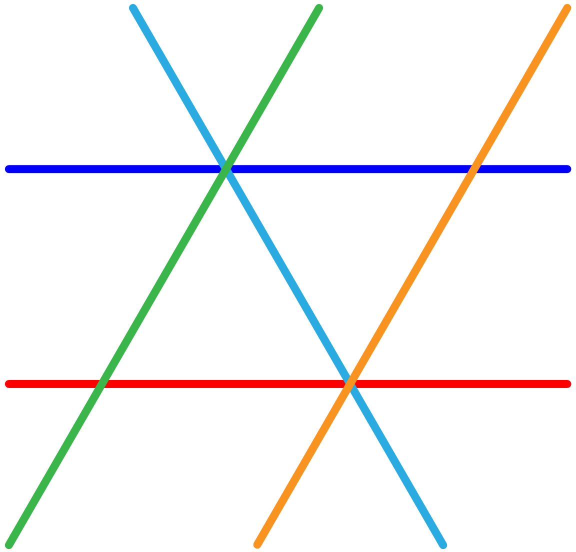

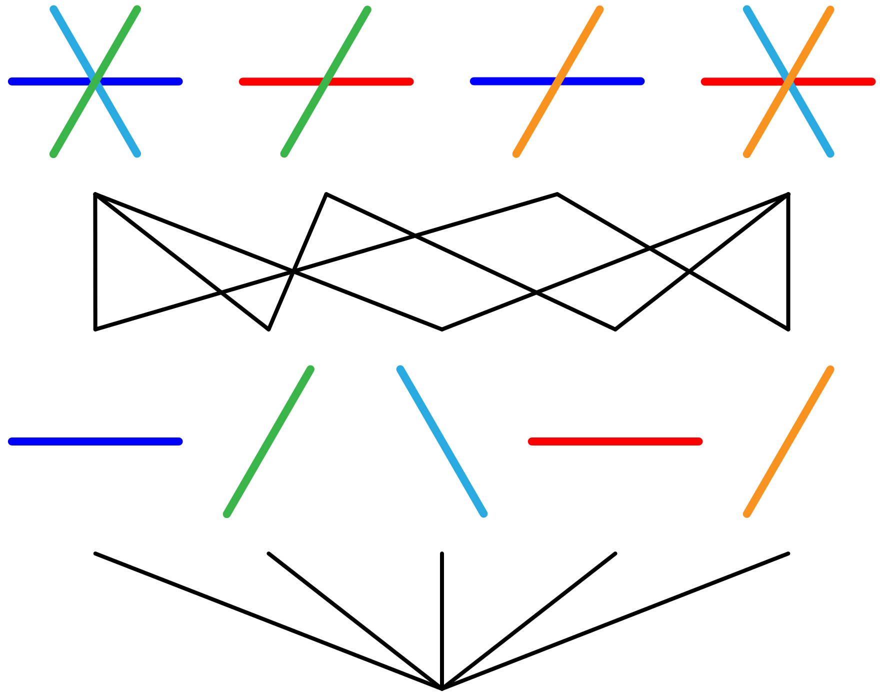

Möbius function μ of a poset: $\mu(x, x) = 1$ and $\sum_{x \le y \le z} \mu(x, y) = 0$ for all $x < z$

FLAT POSET & ZASLAVSKY'S THEOREM

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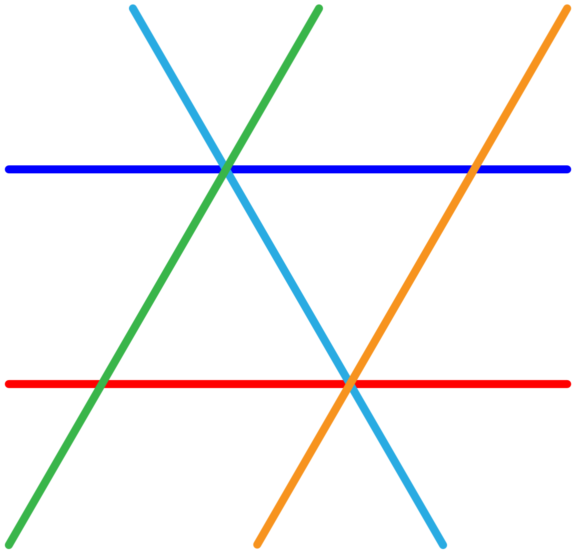

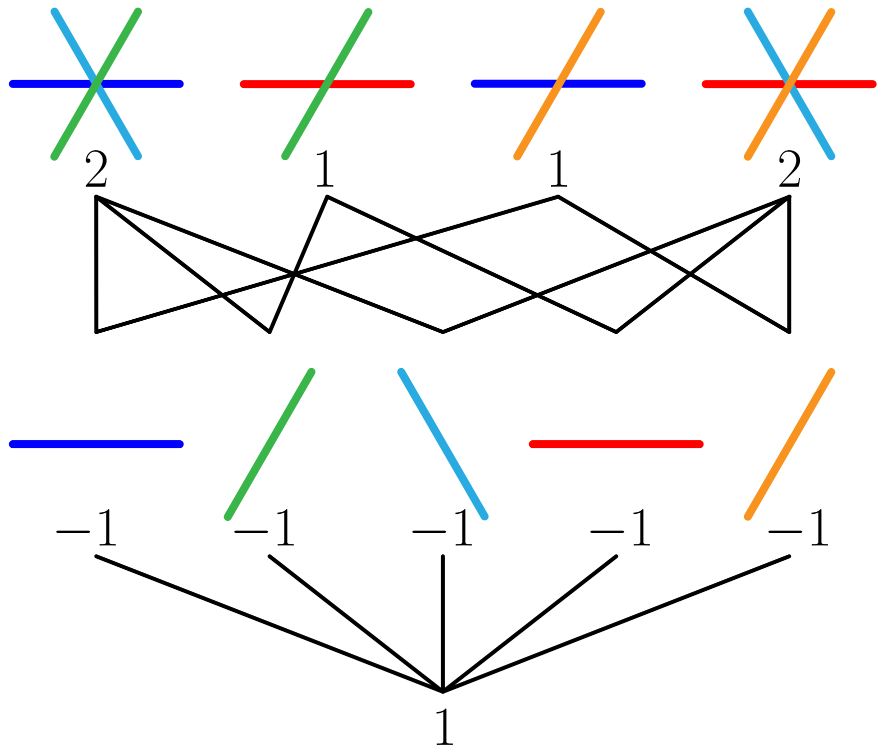

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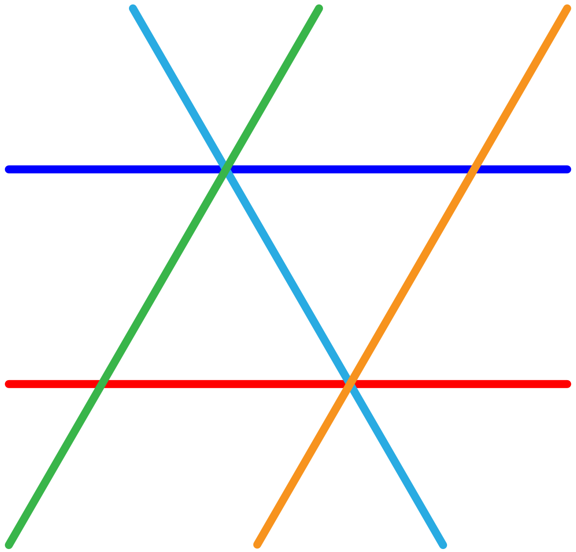
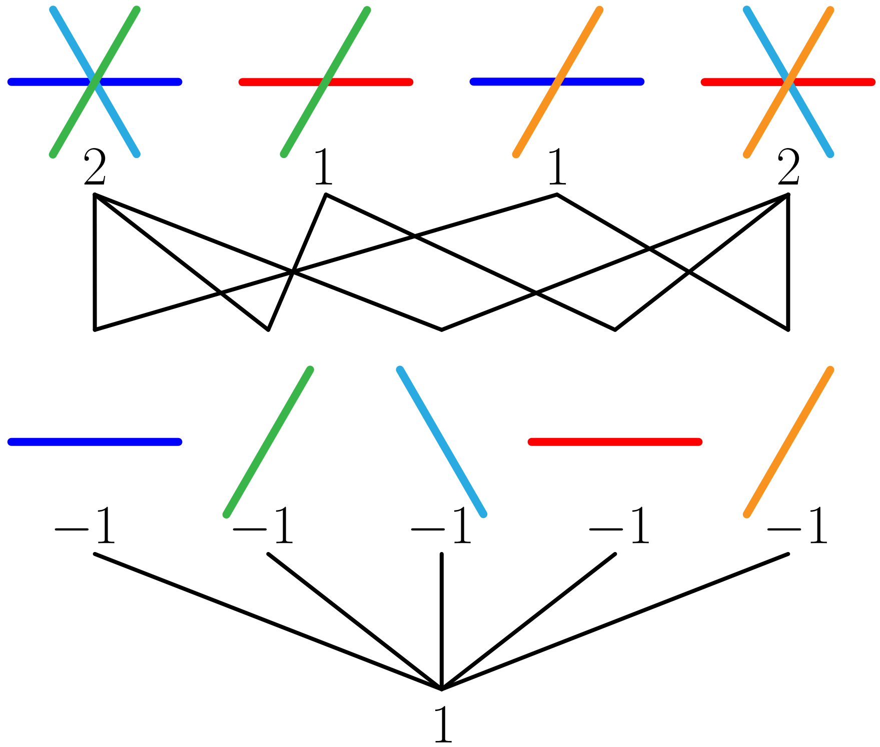




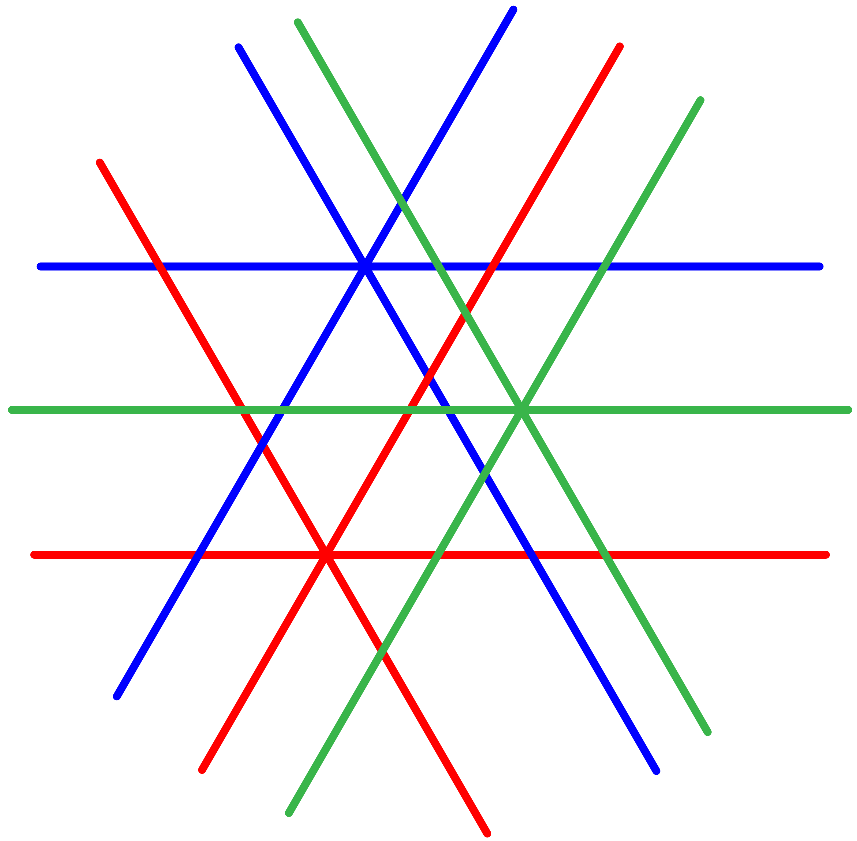
Möbius function μ of a poset: $\mu(x, x) = 1$ and $\sum_{x \le y \le z} \mu(x, y) = 0$ for all $x < z$

Möbius polynomial $\boldsymbol{\mu}_{\mathcal{A}}(x, y) = \sum_{F \le G} \mu(F, G)\, x^{\dim(F)}\, y^{\dim(G)}$

THM. $\boldsymbol{f}_{\mathcal{A}}(x) = \boldsymbol{\mu}_{\mathcal{A}}(-x, -1)$ and $\boldsymbol{b}_{\mathcal{A}}(x) = \boldsymbol{\mu}_{\mathcal{A}}(-x, 1)$ Zaslavsky '75

ℓ-BRAID ARRANGEMENT & PARTITION FORESTS

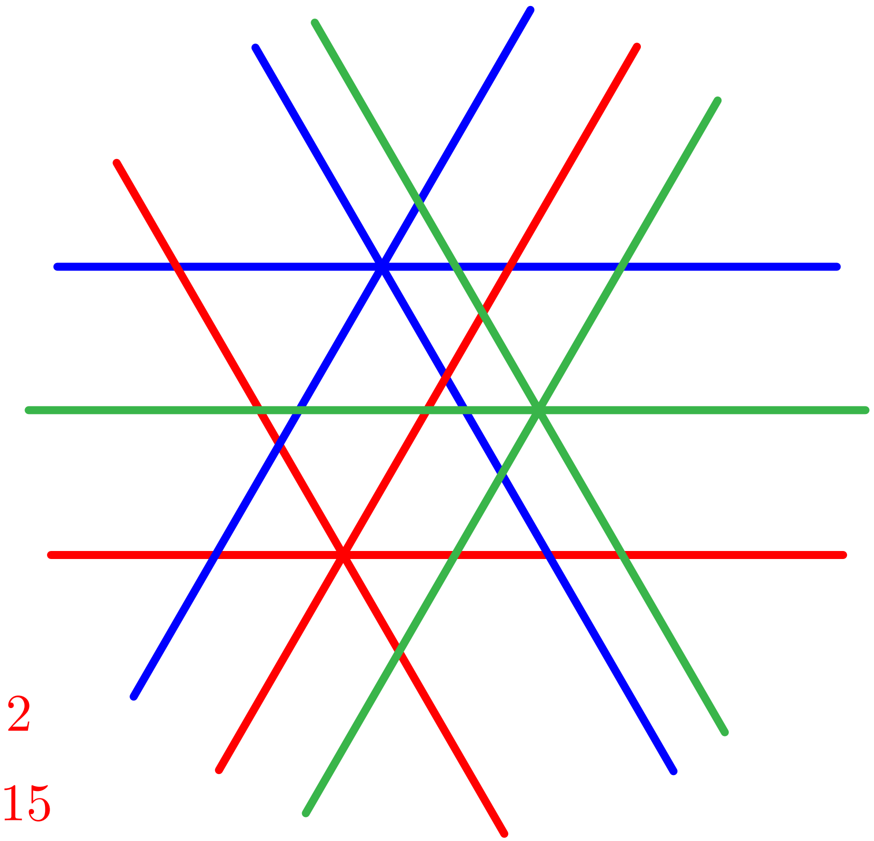
$\mathcal{B}_n^\ell$ = union of ℓ generically translated copies of the braid arrangement

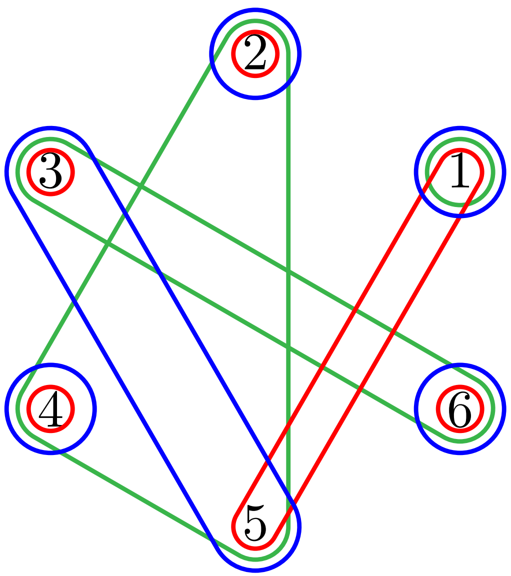


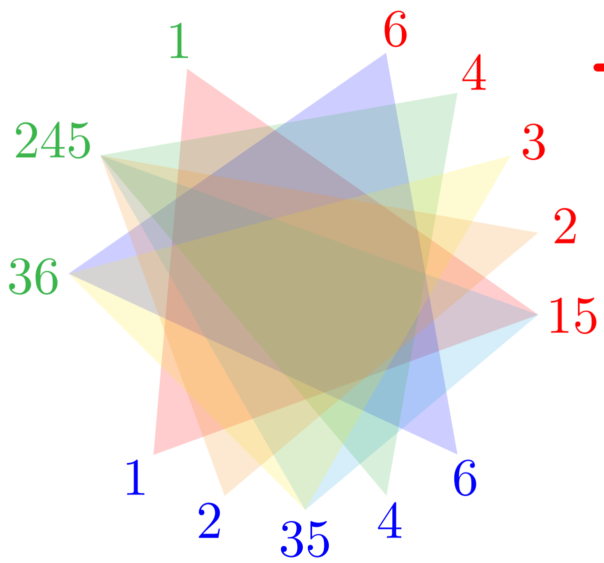
ℓ-BRAID ARRANGEMENT & PARTITION FORESTS

$\mathcal{B}_n^\ell$ = union of ℓ generically translated copies of the braid arrangement

(ℓ, n) partition forest =
ℓ-tuple of partitions of $[n]$ whose intersection hypergraph is a forest







PROP. Intersection poset of $\mathcal{B}_n^\ell$ $\longleftrightarrow$ refinement poset on (ℓ, n) partition forests

ℓ-BRAID ARRANGEMENT & PARTITION FORESTS

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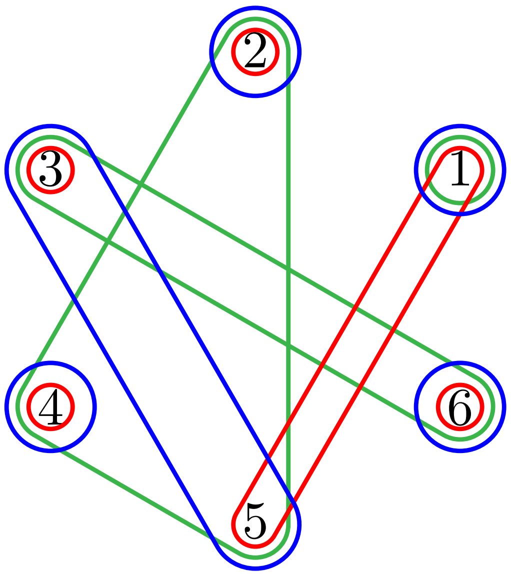

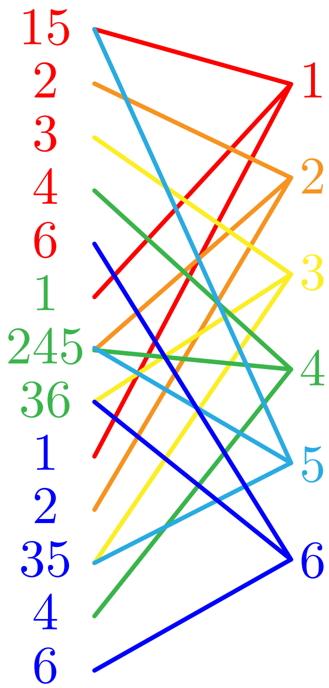

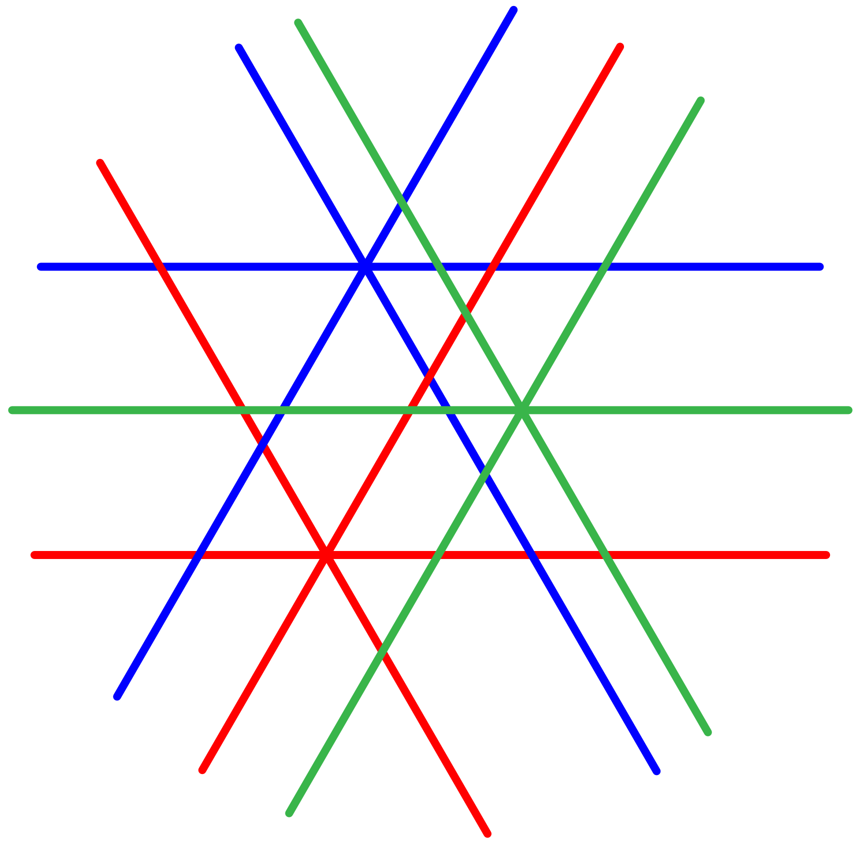

PROP. Intersection poset of $\mathcal{B}_n^\ell$ $\longleftrightarrow$ refinement poset on (ℓ, n) partition forests

MÖBIUS POLYNOMIAL

$\mathbb{P}_p$ = refinement poset on partitions of $[p]$
$\mathbb{PF}_n^\ell$ = refinement poset on (ℓ, n) partition forests

FACT 1. The Möbius function of $\mathbb{P}_p$ is $\mu(\hat{0}, \hat{1}) = (-1)^{p-1}(p-1)!$

FACT 2. In $\mathbb{P}_p$, $$[F, G] \simeq \prod_{p \in G} \mathbb{P}_{\#F[p]}$$ where $F[p]$ = restriction of F to p

FACT 2. $$[\boldsymbol{F}, \boldsymbol{G}] \simeq \prod_{i \in [\ell]} [F_i, G_i]$$ for $\boldsymbol{F} = (F_1, \dots, F_\ell)$ and $\boldsymbol{G} = (G_1, \dots, G_\ell)$ in $\mathbb{PF}_n^\ell$

FACT 4. Möbius is multiplicative $$\mu_{P \times Q}\big((p, q), (p', q')\big) = \mu_P(p, p') \cdot \mu_Q(q, q')$$

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FACT 4. Möbius is multiplicative $\mu_{P \times Q}\big((p, q), (p', q')\big) = \mu_P(p, p') \cdot \mu_Q(q, q')$

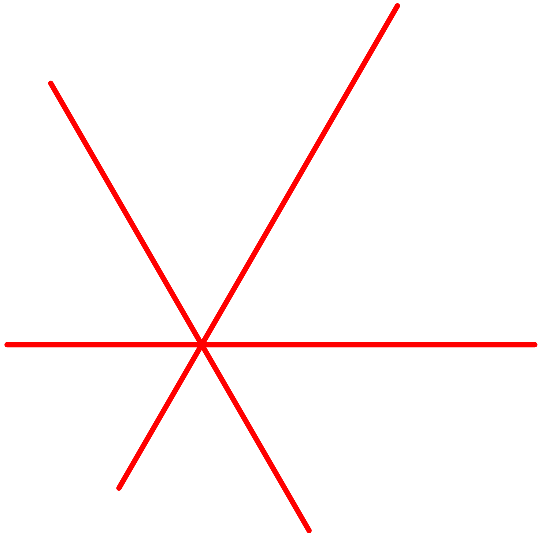
THM. $$\boldsymbol{\mu}_{\mathcal{B}_n^\ell} = x^{n-1-\ell n} y^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\#F_i} y^{\#G_i} \prod_{p \in G_i} (-1)^{\#F_i[p]-1} (\#F_i[p] - 1)!$$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23$^+$

FACE POLYNOMIAL

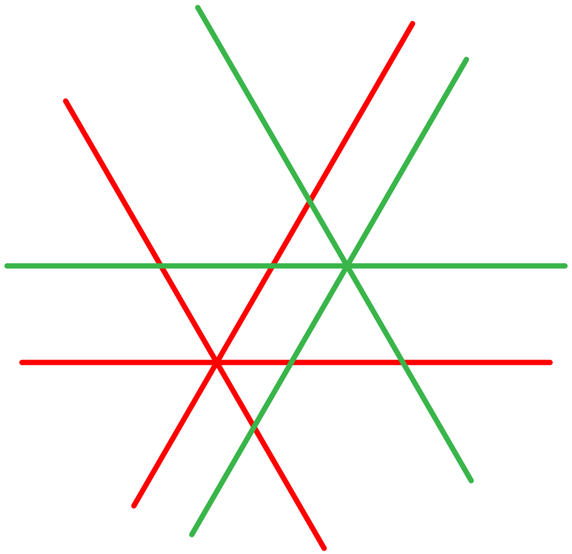
THM. $$\boldsymbol{f}_{\mathcal{B}_n^\ell}(x) = x^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\# F_i} \prod_{p \in G_i} (\# F_i[p] - 1)!$$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23+



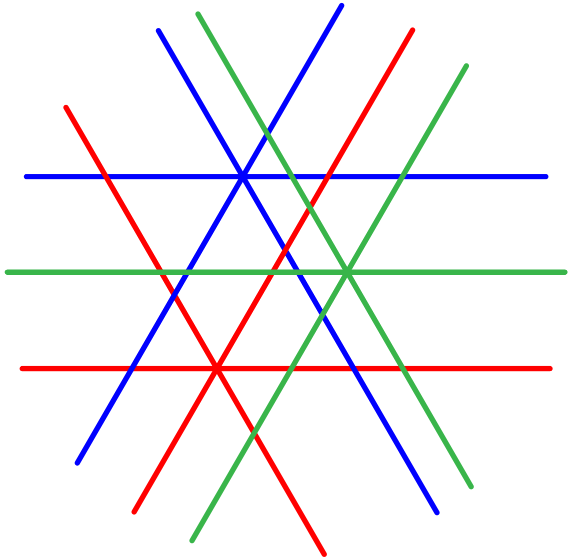
$n \backslash k$	0	1	2	3	Σ
1	1				1
2	2	1			3
3	6	6	1		13
4	24	36	14	1	75

$\ell = 1$



$n \backslash k$	0	1	2	3	Σ
1	1				1
2	3	2			5
3	17	24	8		49
4	149	324	226	50	749

$\ell = 2$



$n \backslash k$	0	1	2	3	Σ
1	1				1
2	4	3			7
3	34	54	21		109
4	472	1152	924	243	2791

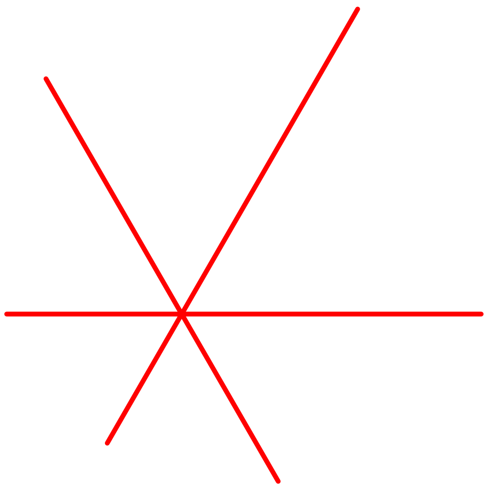
$\ell = 3$

BOUNDED FACE POLYNOMIAL

THM.

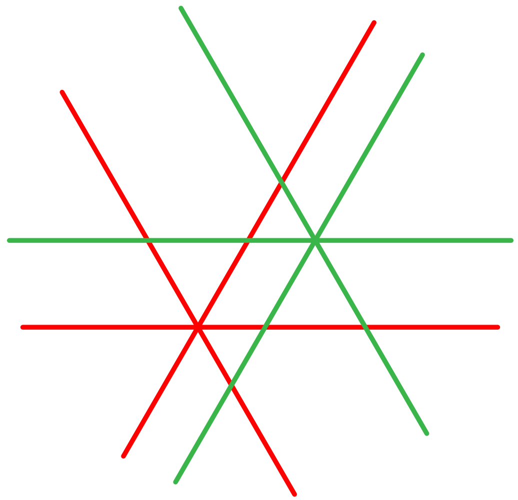
$$\boldsymbol{b}_{\mathcal{B}_n^\ell}(x) = (-1)^\ell x^{n-1-\ell n} \sum_{\boldsymbol{F} \le \boldsymbol{G}} \prod_{i \in [\ell]} x^{\# F_i} \prod_{p \in G_i} -(\# F_i[p] - 1)!$$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl ’23$^+$



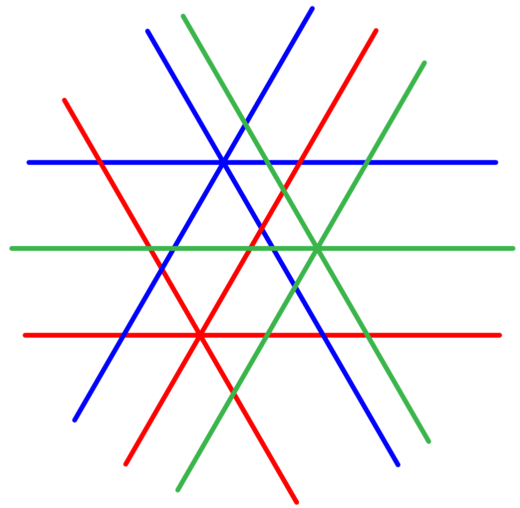
$n \backslash k$	0	1	2	3	Σ
1	1				1
2	0	1			1
3	0	0	1		1
4	0	0	0	1	1

$\ell = 1$



$n \backslash k$	0	1	2	3	Σ
1	1				1
2	1	2			3
3	5	12	8		25
4	43	132	138	50	363

$\ell = 2$



$n \backslash k$	0	1	2	3	Σ
1	1				1
2	2	3			5
3	16	36	21		73
4	224	684	702	243	1853

$\ell = 3$

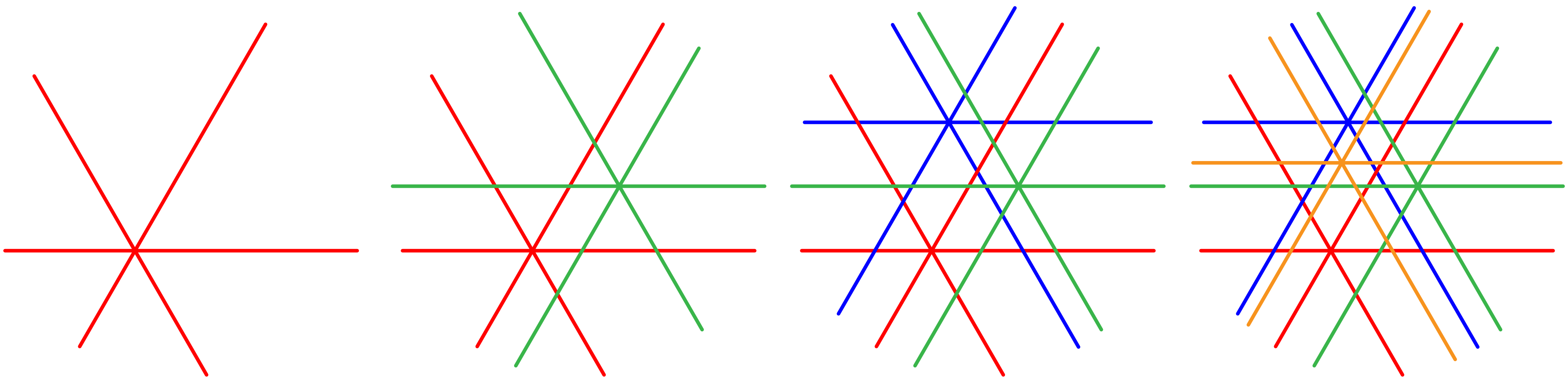
VERTICES

THM. $f_0(\mathcal{B}_n^\ell) = \#\{(\ell, n) \text{ partition trees}\} = \ell\big(n(\ell-1)+1\big)^{n-2}$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23$^+$

$n \backslash \ell$	1	2	3	4	5	6	
1	1	1	1	1	1	1	$\leftarrow 1$
2	1	2	3	4	5	6	$\leftarrow \ell$
3	1	8	21	40	65	96	$\leftarrow \ell(3\ell-2)$ [OEIS, A000567]
4	1	50	243	676	1445	2646	

$1 \rightharpoonup$ (column $\ell=1$) $\leftharpoonup 2(n+1)^{n-2}$ [OEIS, A007334] (column $\ell=2$)

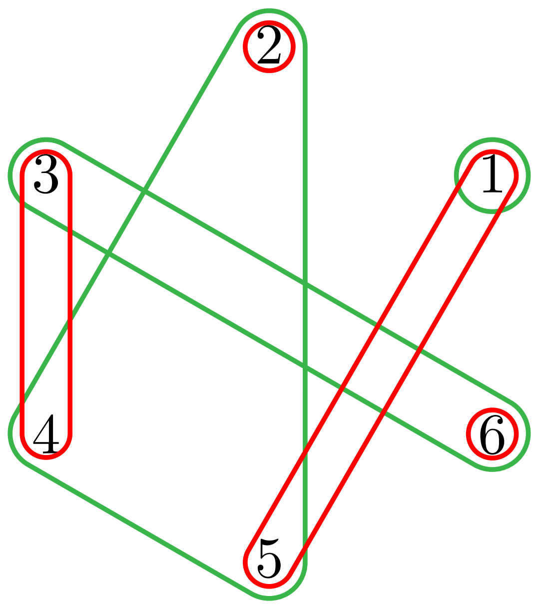


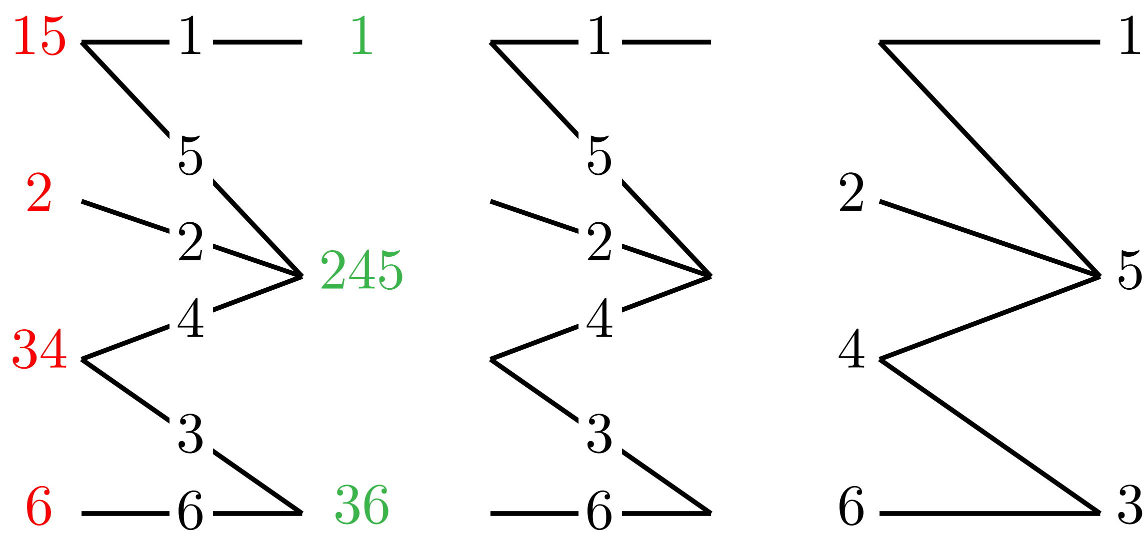
VERTICES

THM. $f_0(\mathcal{B}_n^2) = \#\{(2,n)$ partition trees$\} = \#$spanning trees of K_{n+1} with 01

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23$^+$

$1,\ 2,\ 8,\ 50,\ 432,\ 4802,\ 65536,\ 1062882,\ 20000000,\ 428717762,\ \dots$ [OEIS, A007334]



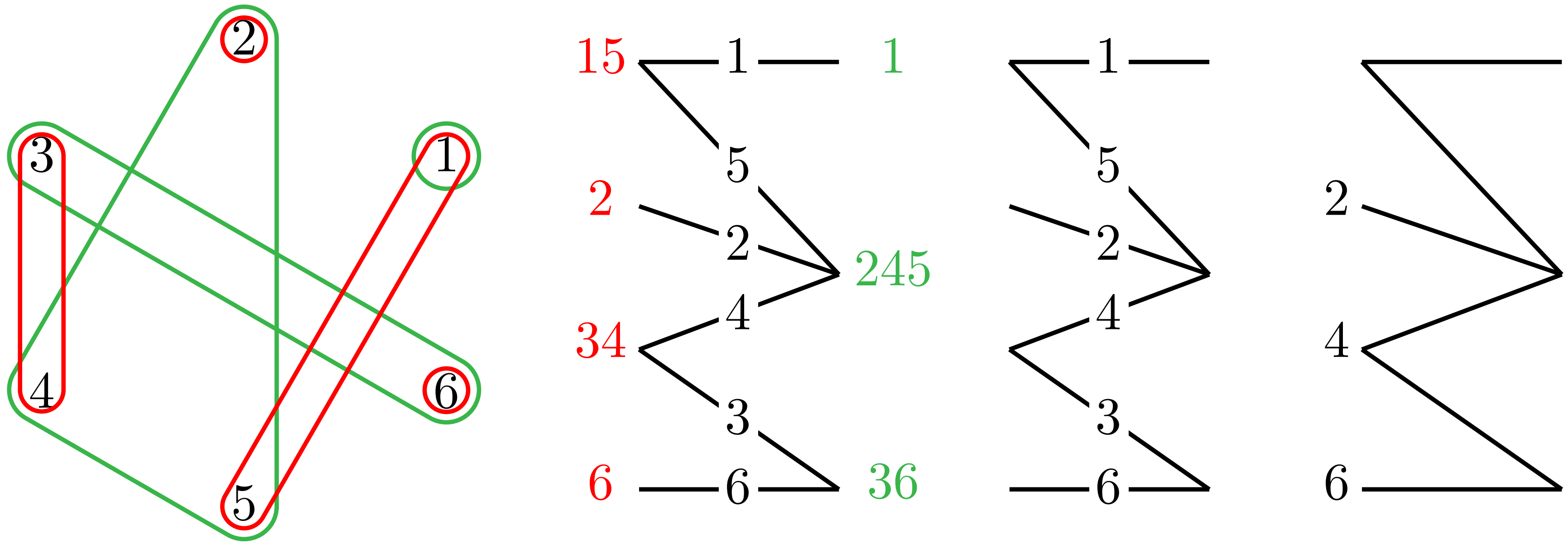


VERTICES

THM. $f_0(\mathcal{B}_n^2) = \#\{(2,n)$ partition trees$\} = \#$spanning trees of $K_{n+1} \cdot \frac{2}{n+1} = 2(n+1)^{n-2}$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23+

$1, 2, 8, 50, 432, 4802, 65536, 1062882, 20000000, 428717762, \dots$ [OEIS, A007334]



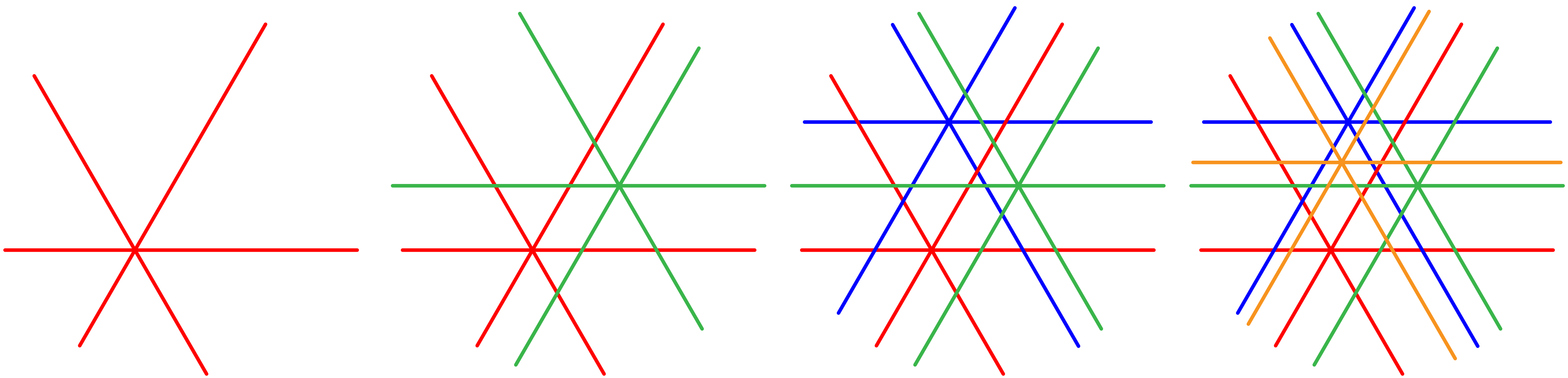
REGIONS

THM. $f_{n-1}(\mathcal{B}_n^\ell) = n!\,[z^n] \exp\left(\sum_{m\geq 1} \frac{F_{\ell,m}\, z^m}{m}\right)$ where $F_{\ell,m} = \frac{1}{(\ell-1)m+1}\binom{\ell m}{m}$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23$^+$

$n \backslash \ell$	1	2	3	4	5	6	
1	1	1	1	1	1	1	$\leftarrow 1$
2	2	3	4	5	6	7	$\leftarrow \ell + 1$
3	6	17	34	57	86	121	$\leftarrow 3\ell^2 + 2\ell + 1$ [OEIS, A056109]
4	24	149	472	1089	2096	3589	

$n!$ ↗ ↖ [OEIS, A213507]



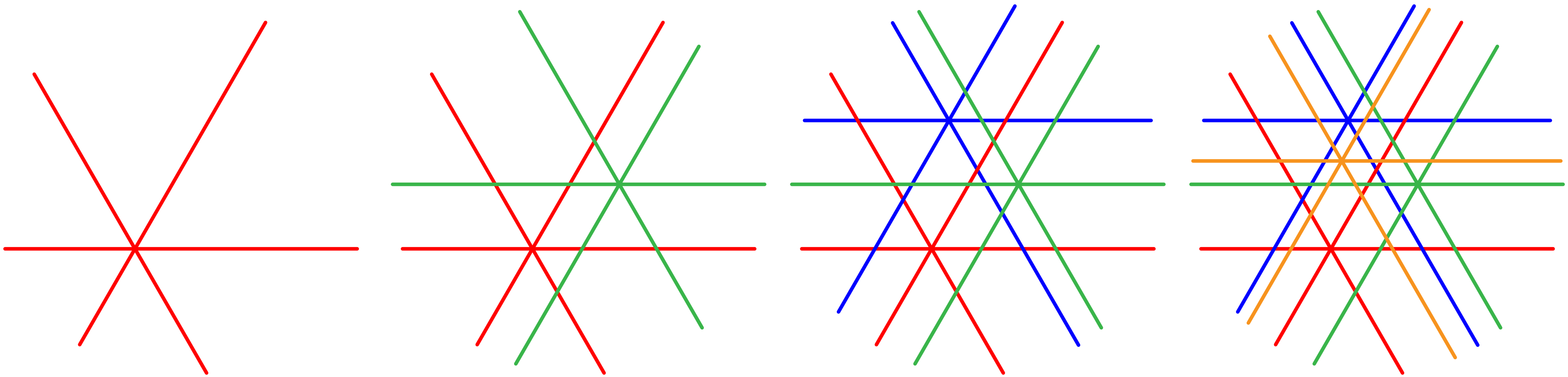
BOUNDED REGIONS

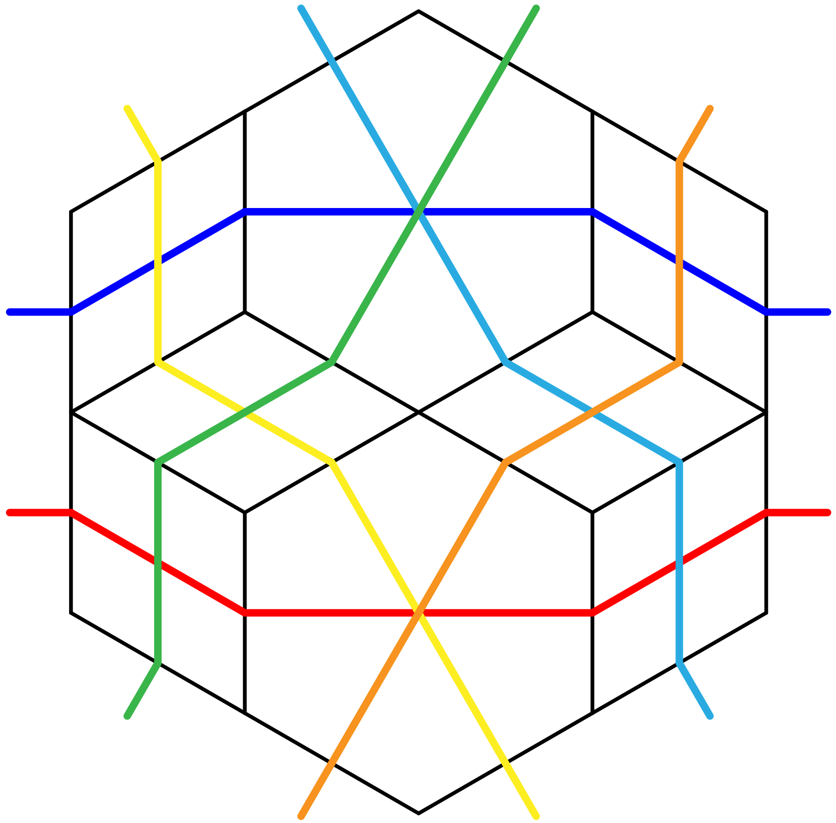
THM. $b_{n-1}(\mathcal{B}_n^\ell) = (n-1)!\,[z^{n-1}]\exp\left((\ell-1)\sum_{m\geq 1} F_{\ell,m}\, z^m\right)$

Delcroix-Oger – Josuat-Vergès – Laplante-Anfossi – P. – Stoeckl '23+

$n \backslash \ell$	1	2	3	4	5	6	
1	1	1	1	1	1	1	$\leftarrow 1$
2	0	1	2	3	4	5	$\leftarrow \ell - 1$
3	0	5	16	33	56	85	$\leftarrow 3\ell^2 - 4\ell + 1$ [OEIS, A045944]
4	0	43	224	639	1384	2555	$= 3(\ell-1)^2 + 2(\ell-1)$

0 ⬏ ⬑ [OEIS, A251568]





THANKS