THE BRICK POLYTOPE

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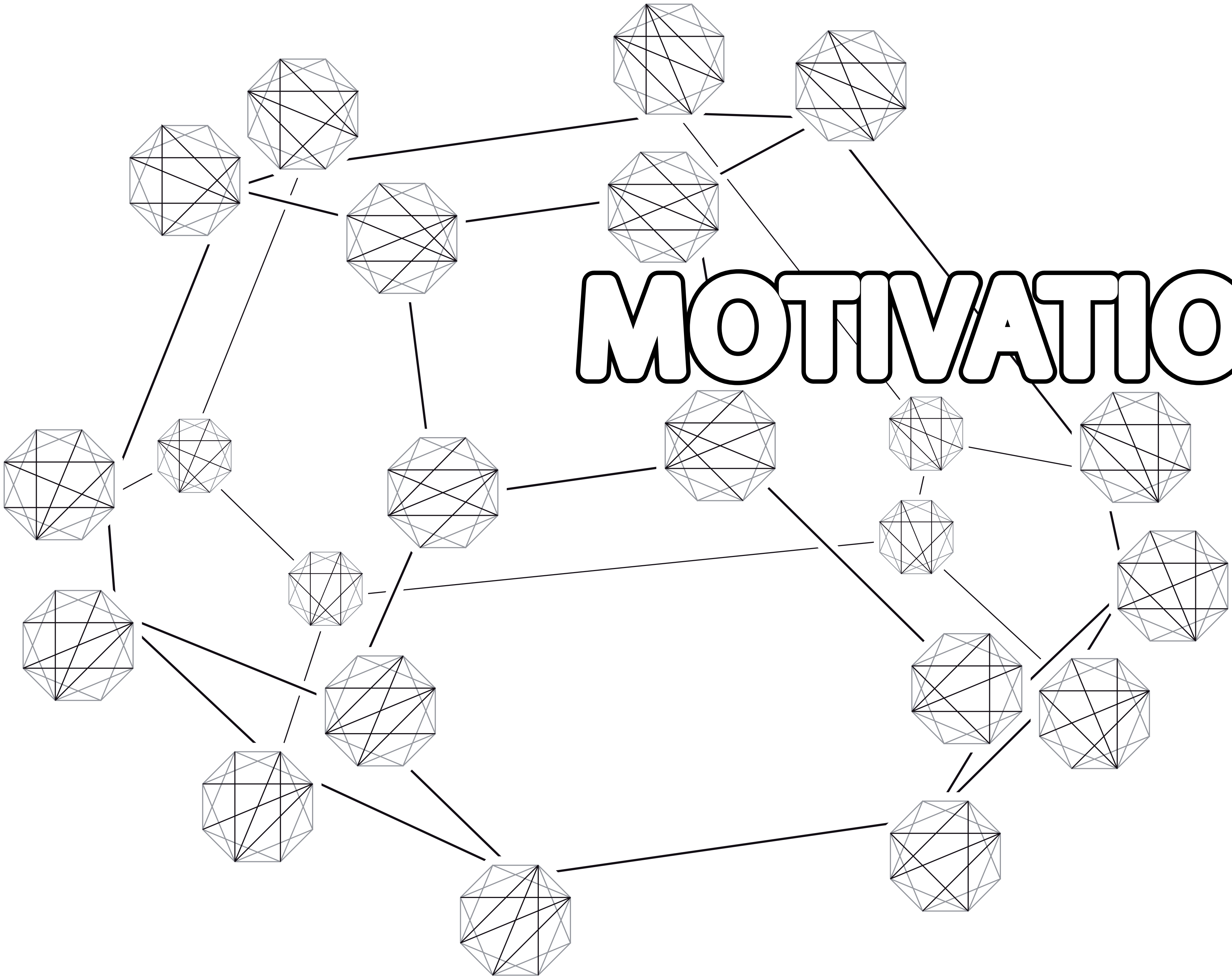
MOTIVATION

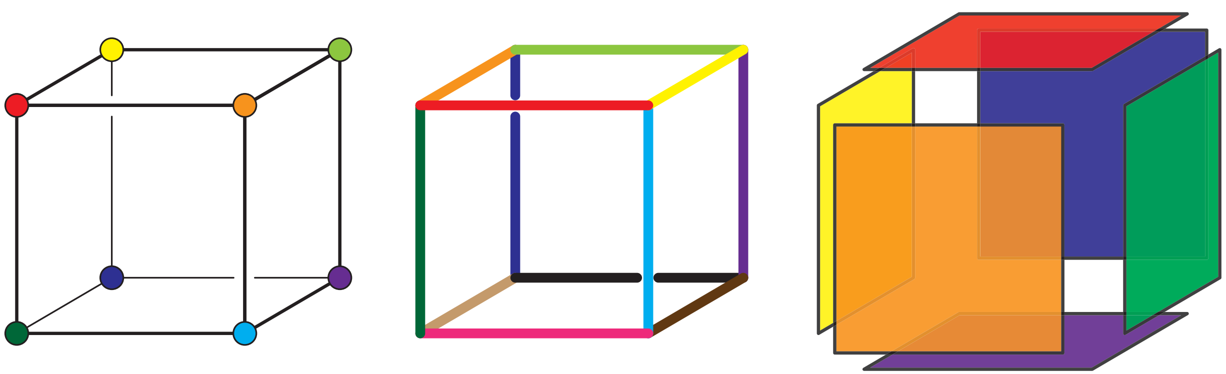


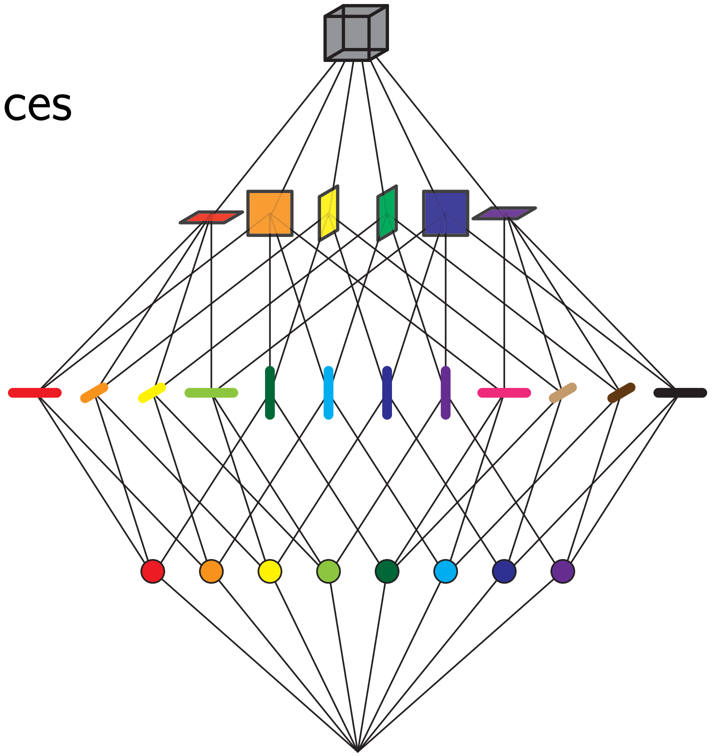
POLYTOPES WITH PRESCRIBED COMBINATORICS

polytope $=$ convex hull of a finite set of $\mathbb{R}^d$
$=$ bounded intersection of finitely many half-spaces

face $=$ intersection with a supporting hyperplane

face lattice $=$ all the faces with their inclusion relations

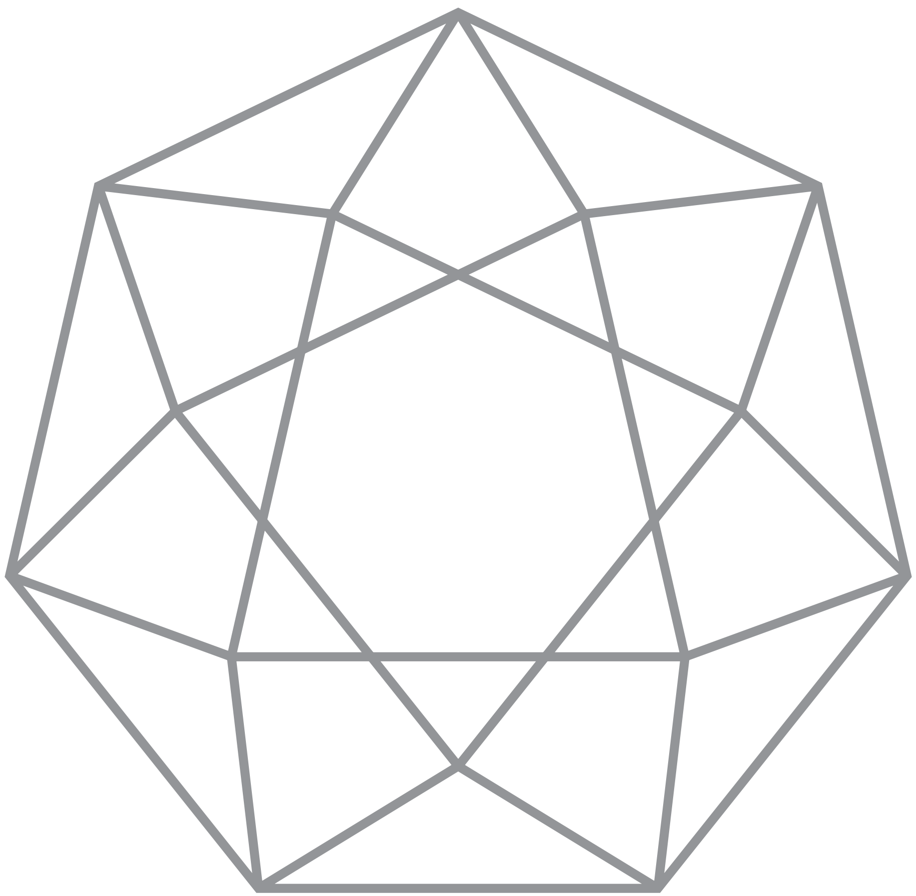


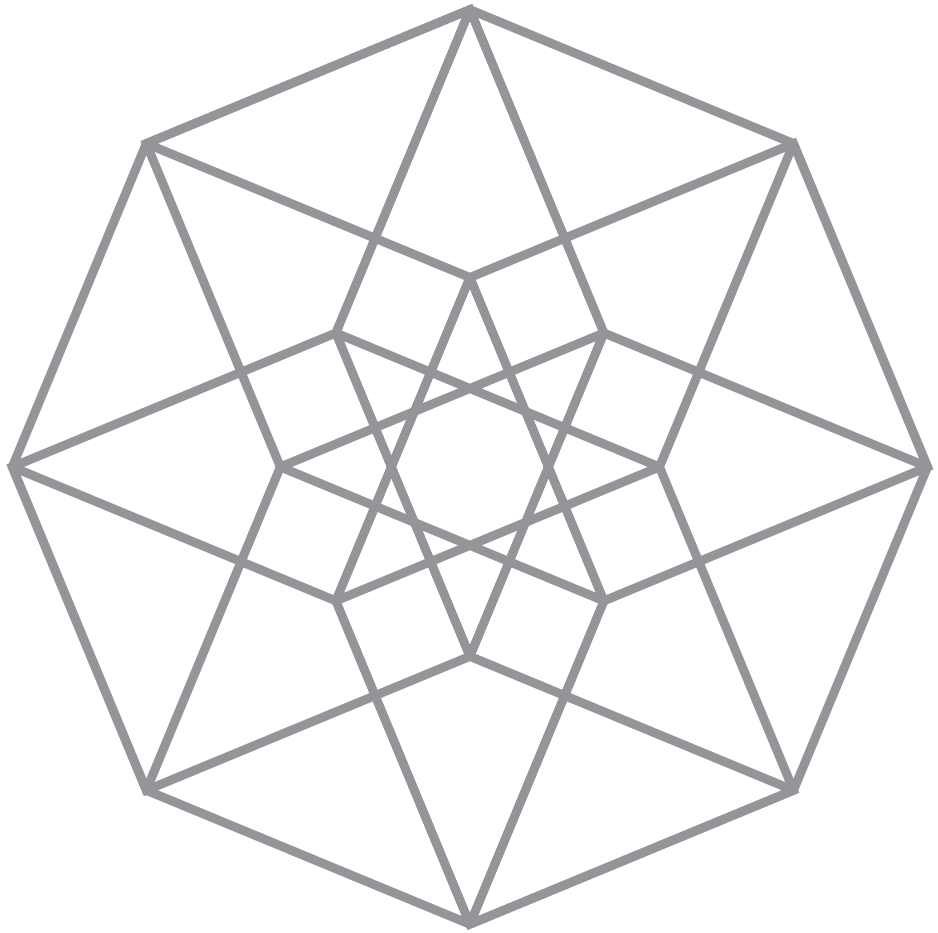


Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?

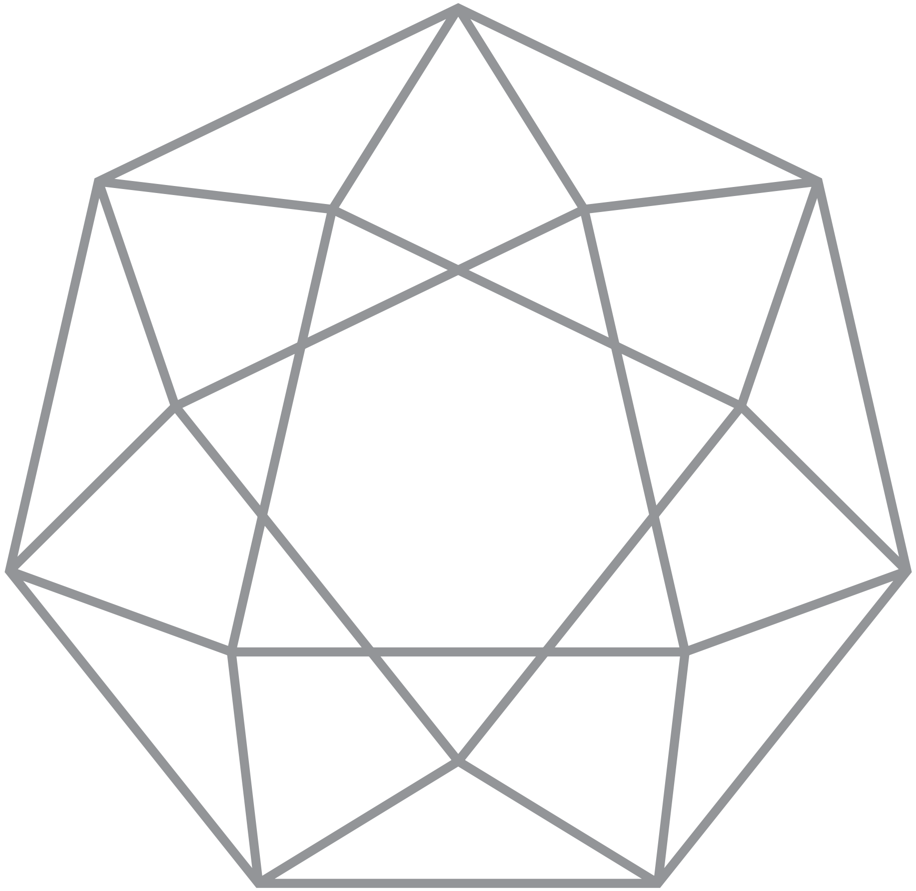
POLYTOPALITY OF GRAPHS

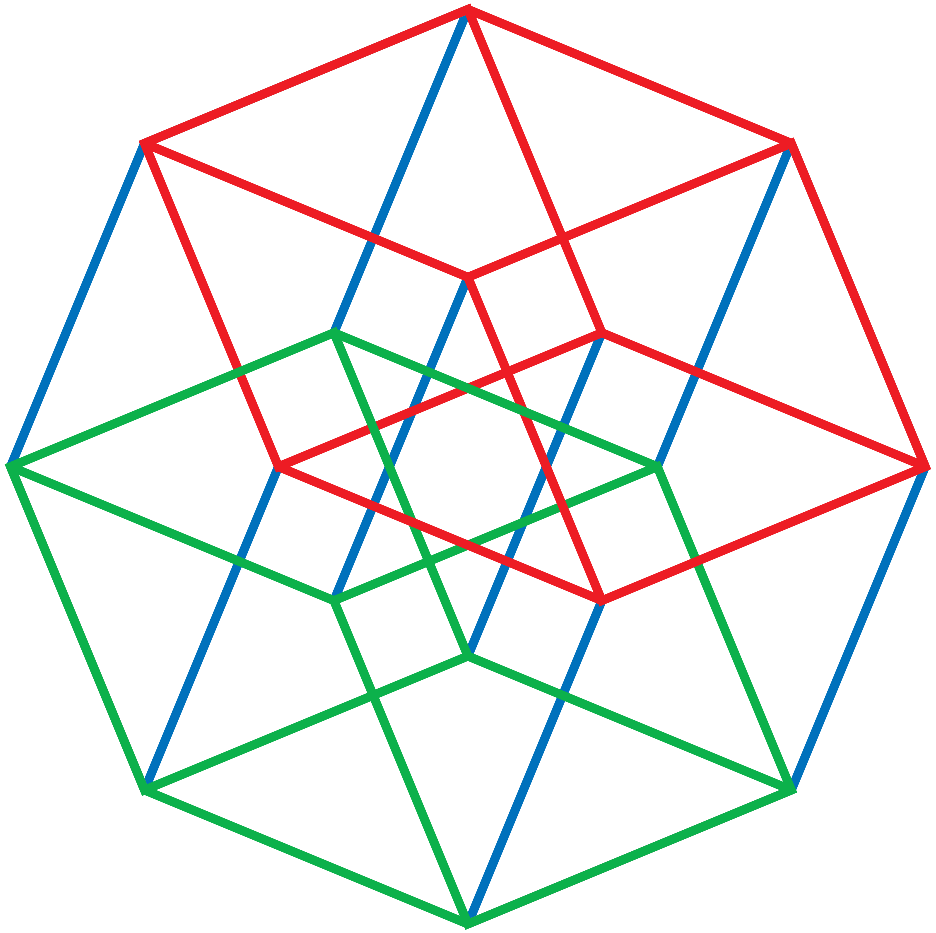




One of these graphs is the 1-skeleton of a polytope. Can you guess which?

POLYTOPALITY OF GRAPHS





One of these graphs is the 1-skeleton of a polytope. Can you guess which?

POLYTOPES OF DIMENSION ≥ 4

Polytopes of dimension $3 \longleftrightarrow$ planar 3-connected graphs

Various open conjectures in dimension 4:

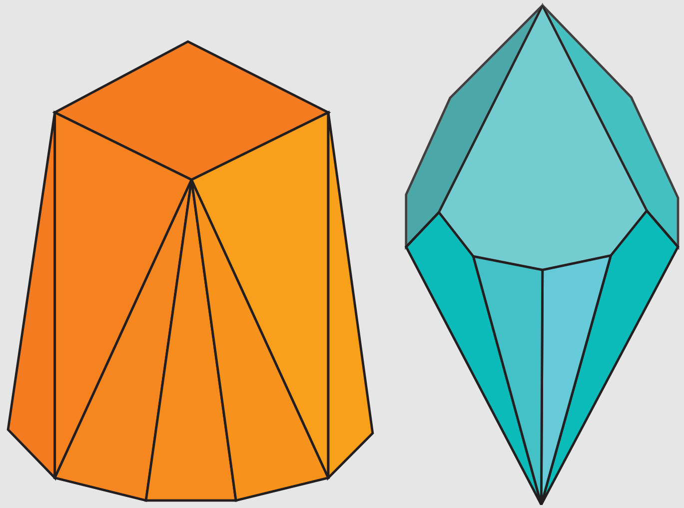
Hirsch conjecture

diameter $\leq$ #facets – dimension (Santos)

complexity of the simplex algorithm

3^d Conjecture (Kalai)

f-vecteur shape (Barany, Ziegler)



Prismatoïdes

"Our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the good questions and in the lack of examples, methods of constructing them, and means of classifying them."

Kalai. Handbook of Discrete and Computational Geometry (2004)

MATCHING POLYTOPE

$G = (V, E)$ finite graph, $\omega : E \mapsto \mathbb{R}$ weight function

Matching polytope of G = conv $\{\mathbb{1}_M \mid M \subset E$ matching of $G\}$

Maximum-weight matching

$$\max \left\{ \sum_{e \in M} \omega(e) \;\middle|\; M \subset E \text{ matching of } G \right\} = \max \left\{ \omega^T x \;\middle|\; x \in \mathsf{MP}(G) \right\}$$

If G is bipartite, the matching polytope is defined by the inequalities

$$x_e \geq 0 \qquad \forall e \in E$$

$$\sum_{e \ni v} x_e \leq 1 \qquad \forall v \in V$$

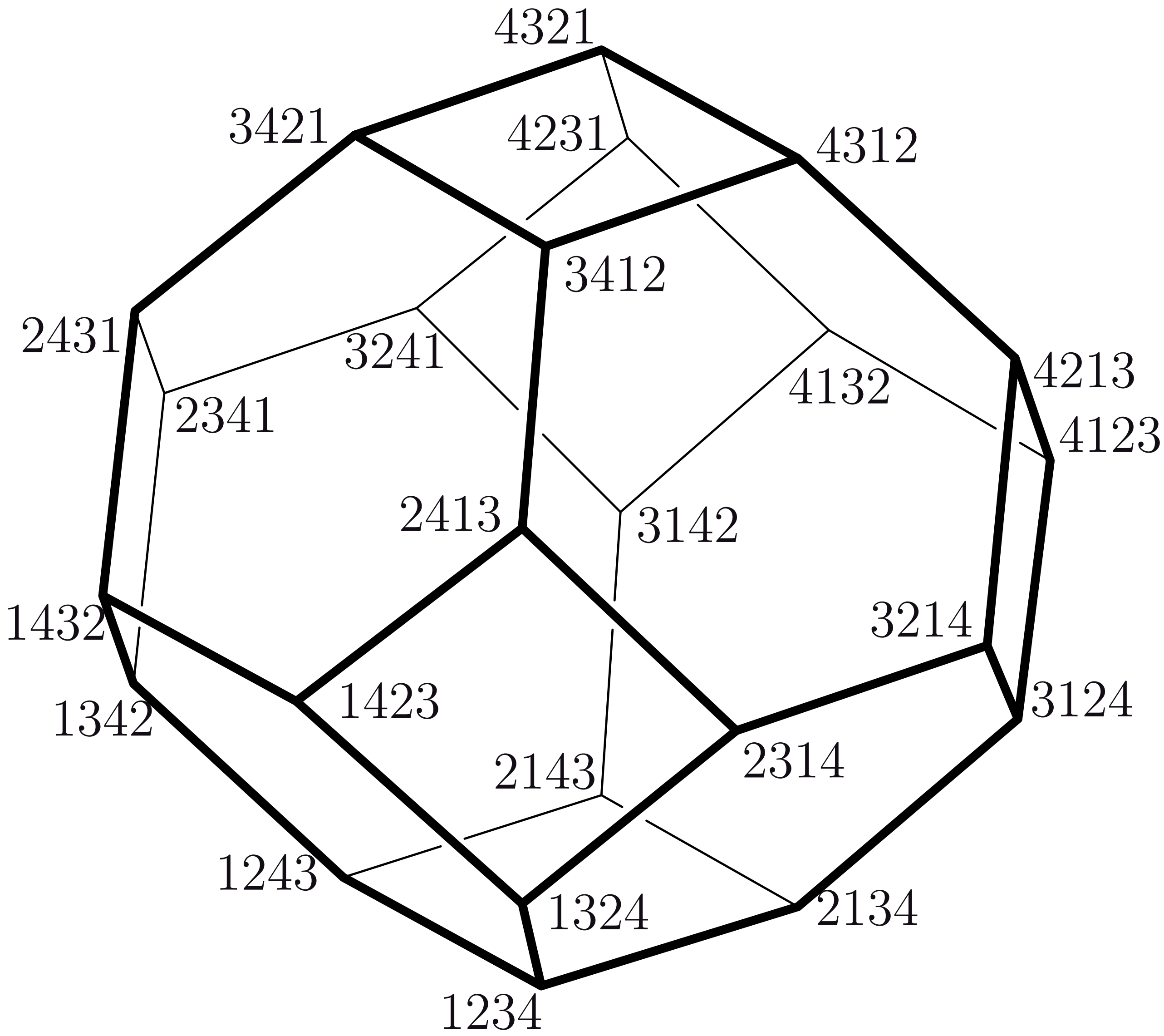
otherwise, we have to add the inequalities

$$\sum_{e \in U} x_e \leq \left\lfloor \frac{1}{2} |U| \right\rfloor \qquad \forall U \subset V,\ |U| \text{odd}$$

PERMUTAHEDRON

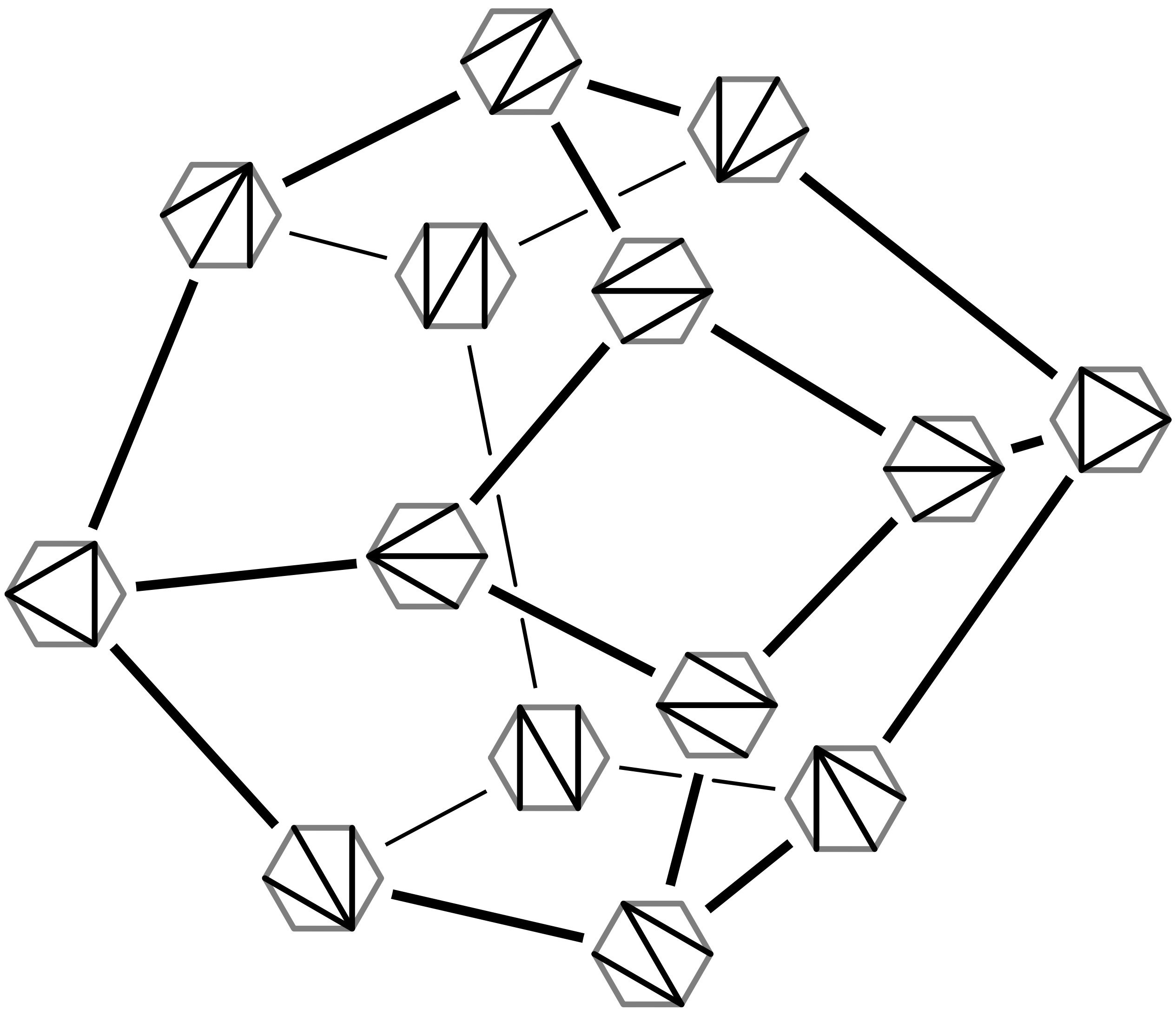
$\Pi_n = \text{conv}\left\{(\sigma(1),\ldots,\sigma(n))^T \;\middle|\; \sigma \in \mathfrak{S}_n\right\} = \sum_{i<j}[e_i, e_j]$

$\partial\Pi_n$ = refinement poset on ordered partitions of $[n]$



ASSOCIAHEDRON

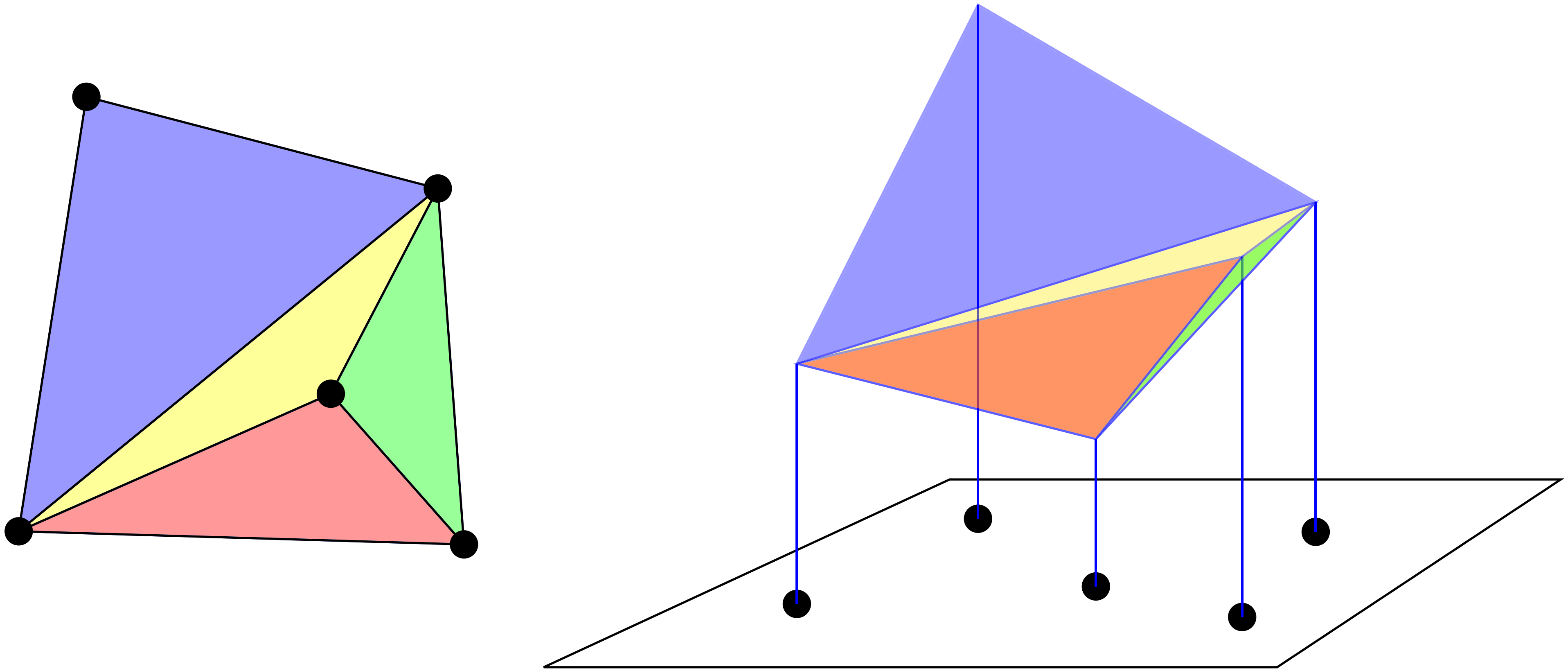
$\partial A_n \quad = \quad$ reverse inclusion poset on non-crossing sets of diagonals of the n-gon



SECONDARY POLYTOPE

P point set

Regular subdivision of P = projection of the lower envelope of a lifting of P



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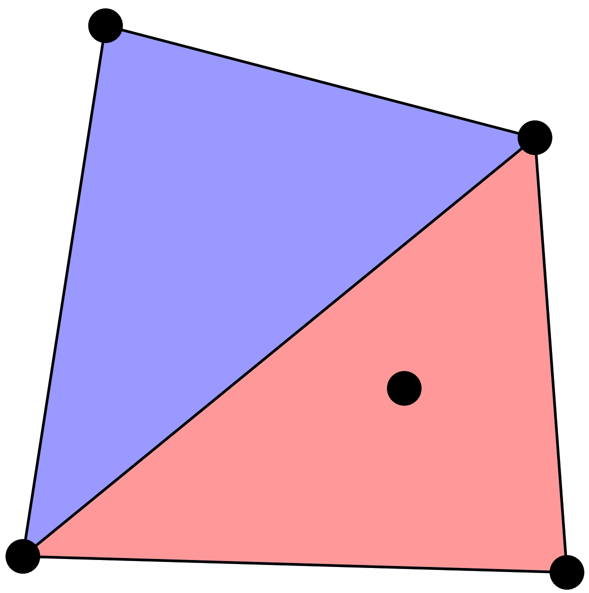

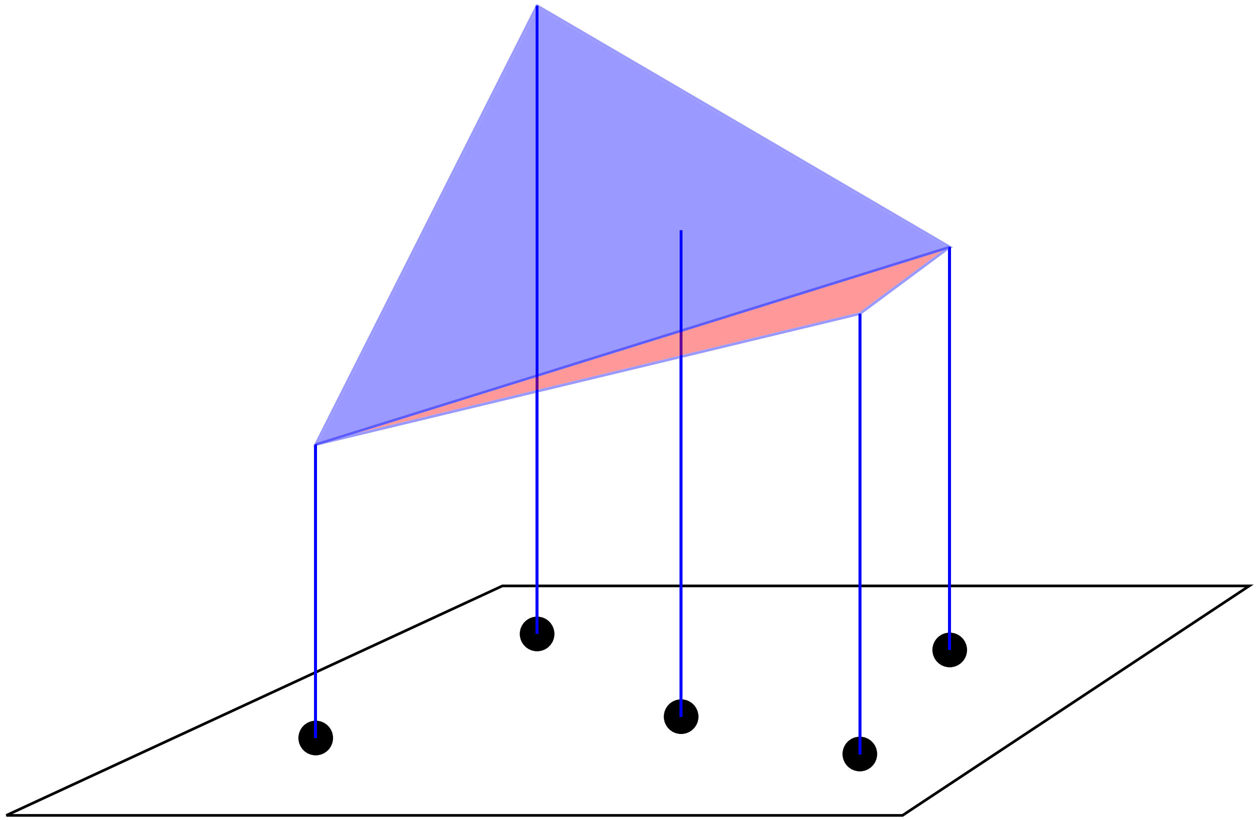

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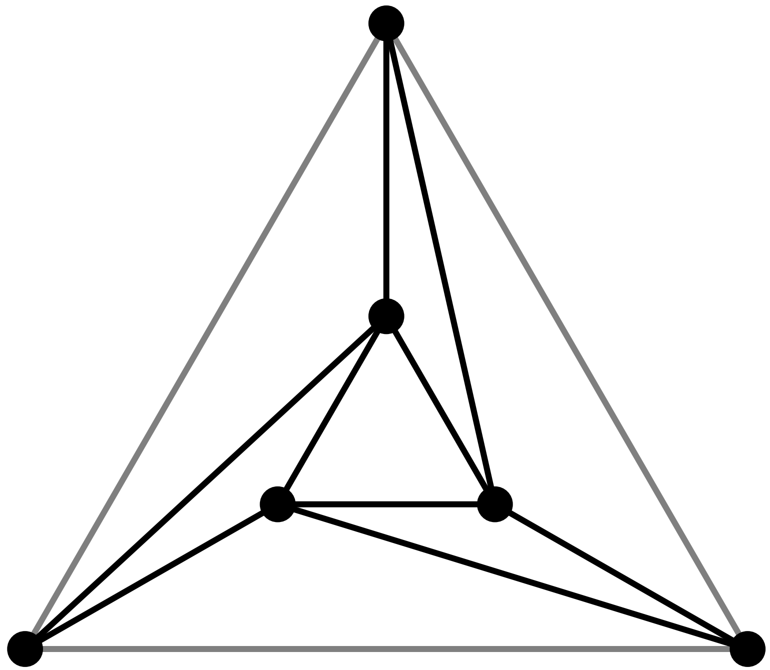
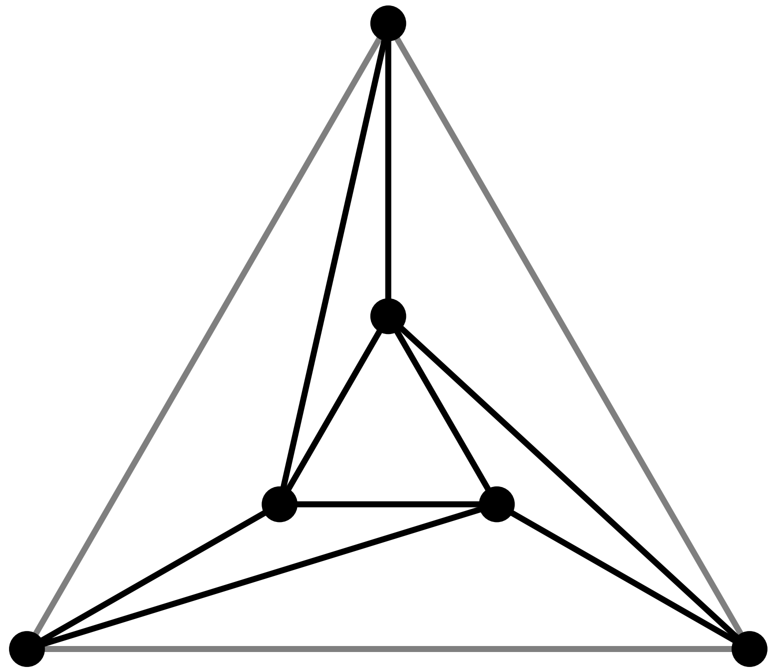
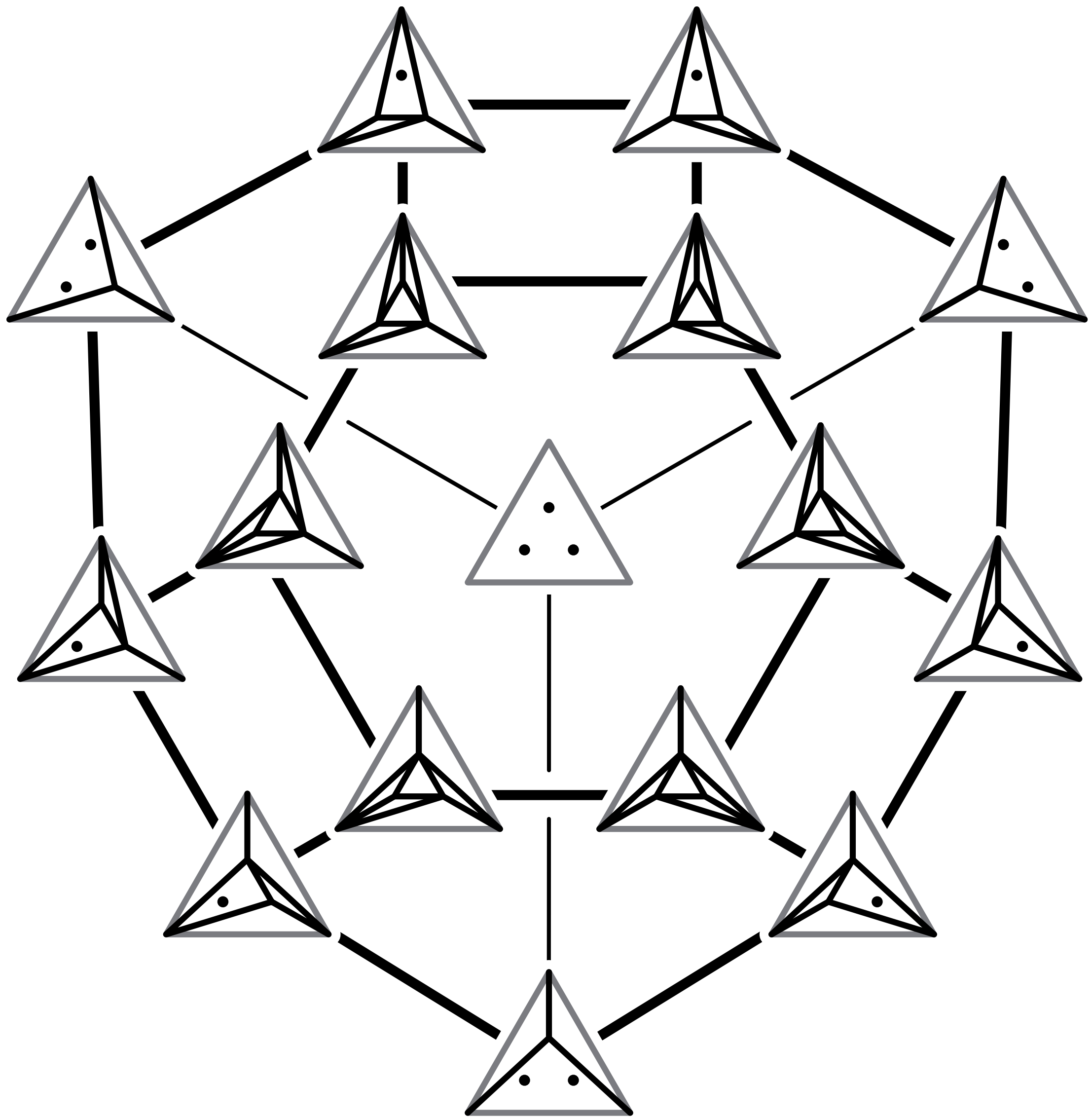




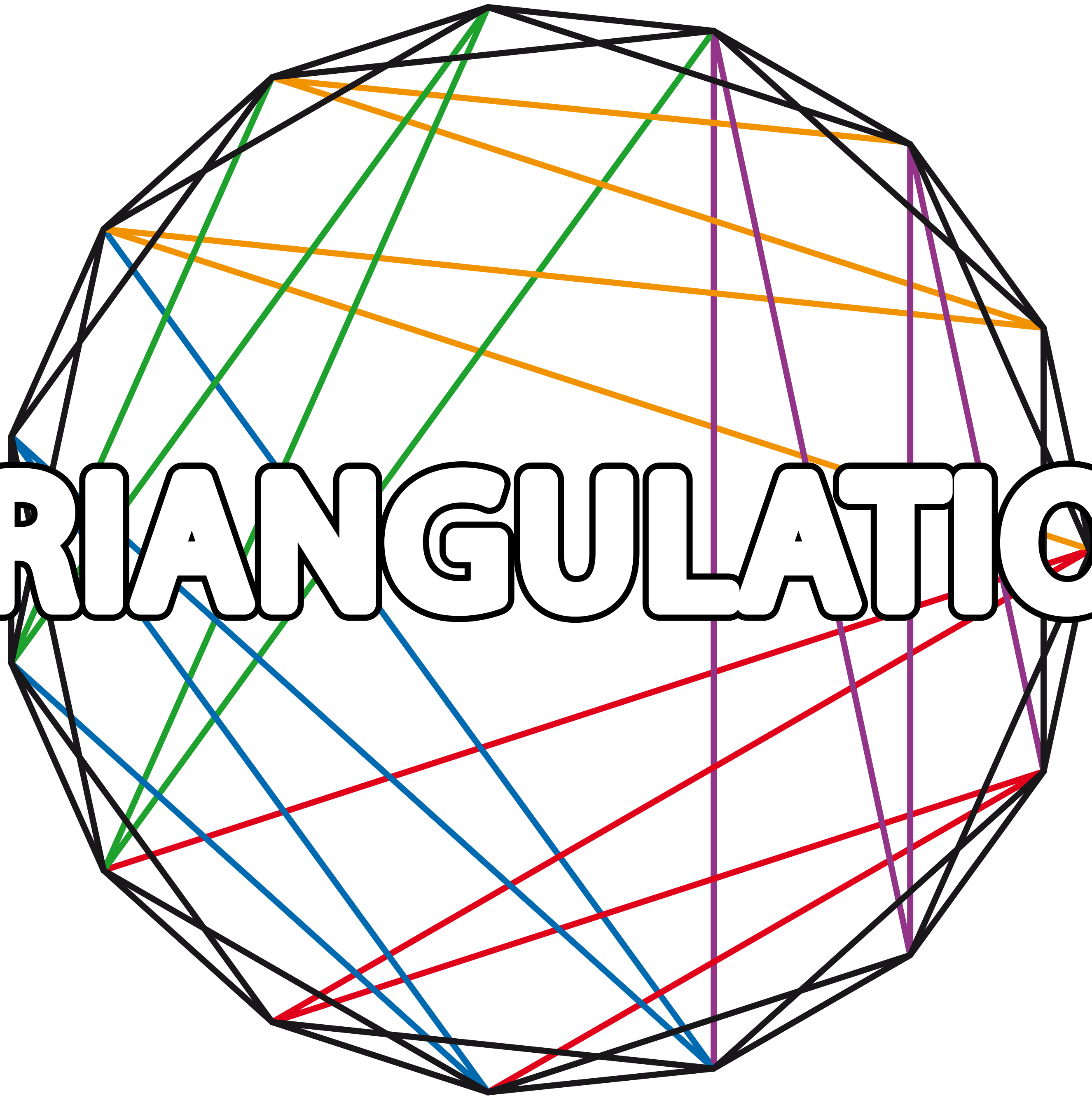
SECONDARY POLYTOPE

$\Sigma(P) = \text{conv}\left\{\sum_{p\in P} \text{vol}(T,p)e_p \;\middle|\; T \text{ triang. } P\right\}$

$\partial\Sigma(P) =$ refinement poset on regular polyhedral subdivisions of P

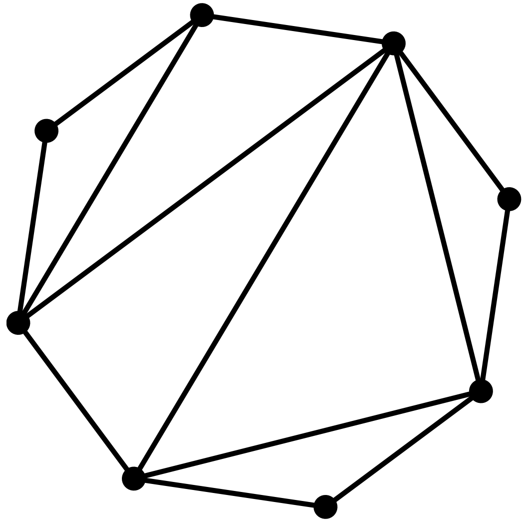


...TRIANGULATIONS

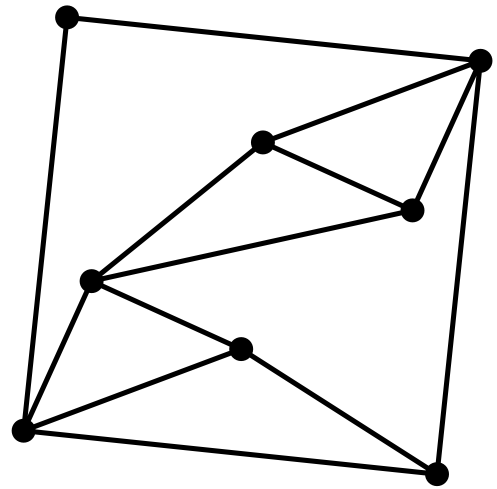


THREE GEOMETRIC STRUCTURES

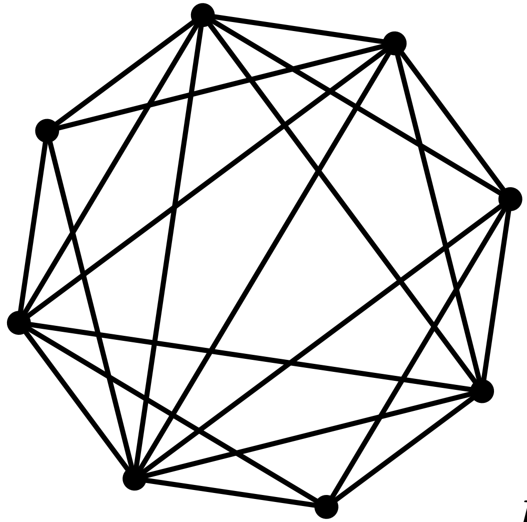
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

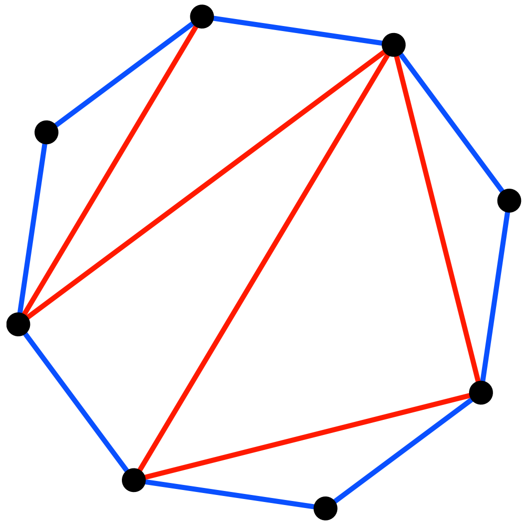
triangulation = maximal crossing-free set of edges,

pseudotriangulation = maximal crossing-free pointed set of edges,

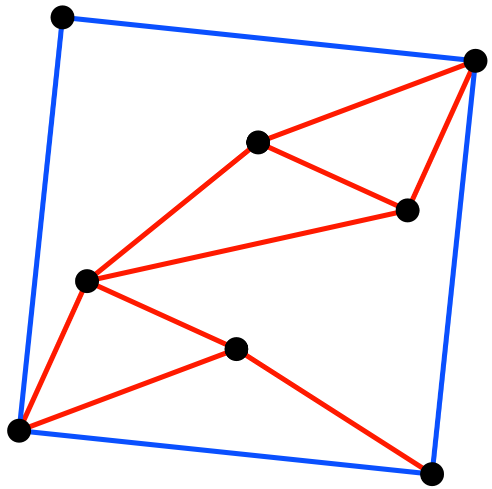
k-triangulation = maximal $(k+1)$-crossing-free set of edges,

THREE GEOMETRIC STRUCTURES

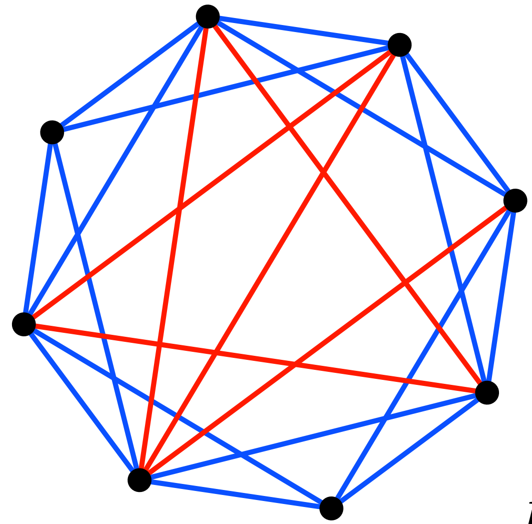
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

triangulation $=$ maximal crossing-free set of edges,

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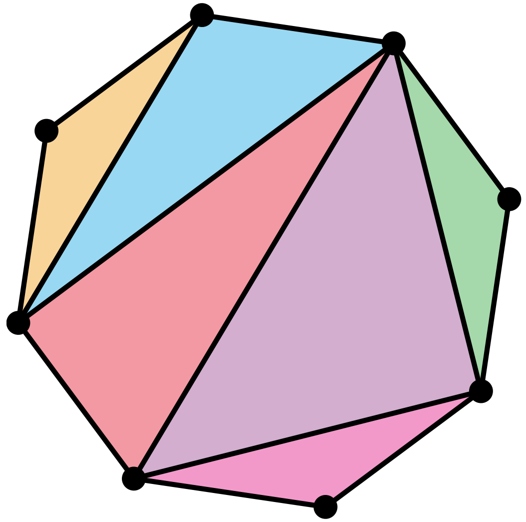
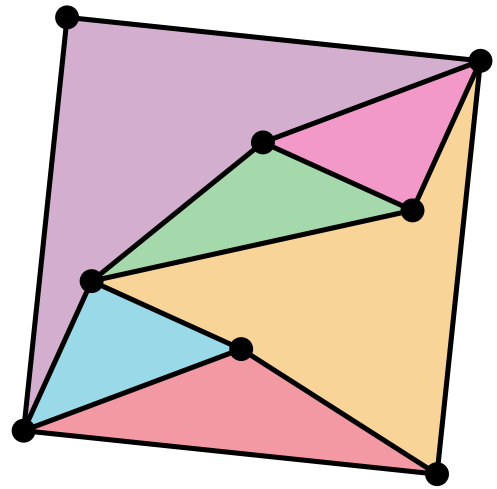
THREE GEOMETRIC STRUCTURES

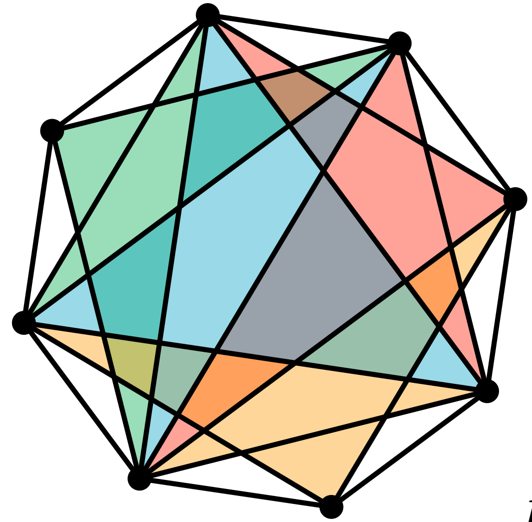
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

triangulation $=$ maximal crossing-free set of edges,
$=$ decomposition into triangles.

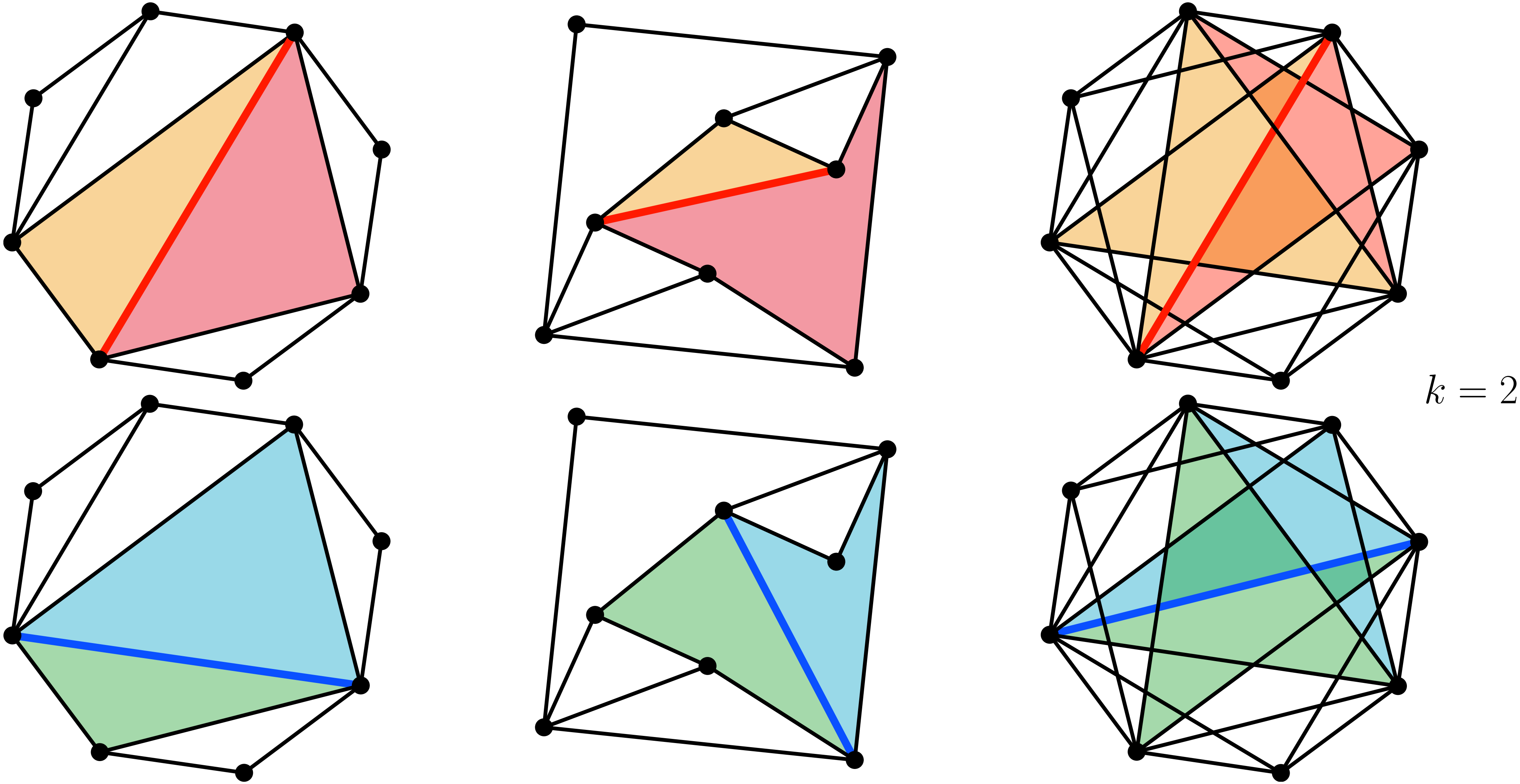
pseudotriangulation $=$ maximal crossing-free pointed set of edges,
$=$ decomposition into pseudotriangles.

k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges,
$=$ decomposition into k-stars.

VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

THREE GEOMETRIC STRUCTURES

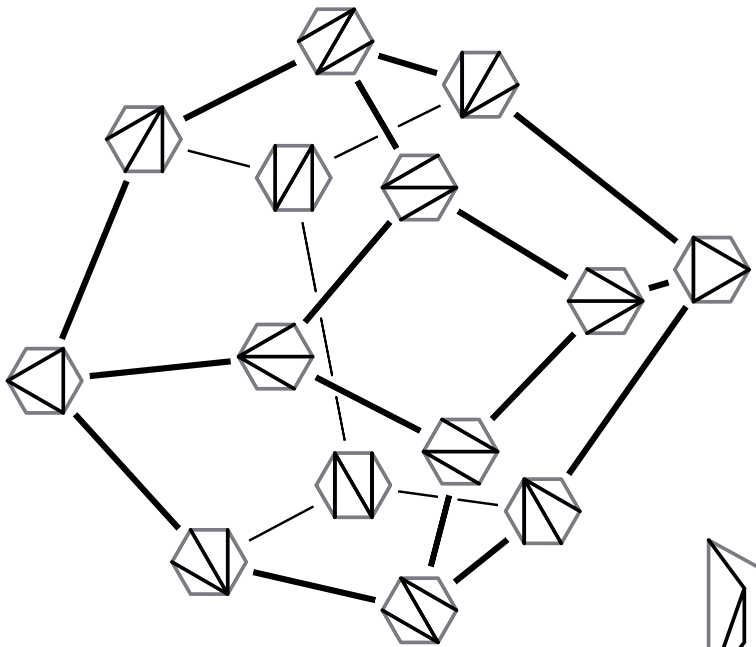
Triangulations | Pseudotriangulations | Multitriangulations

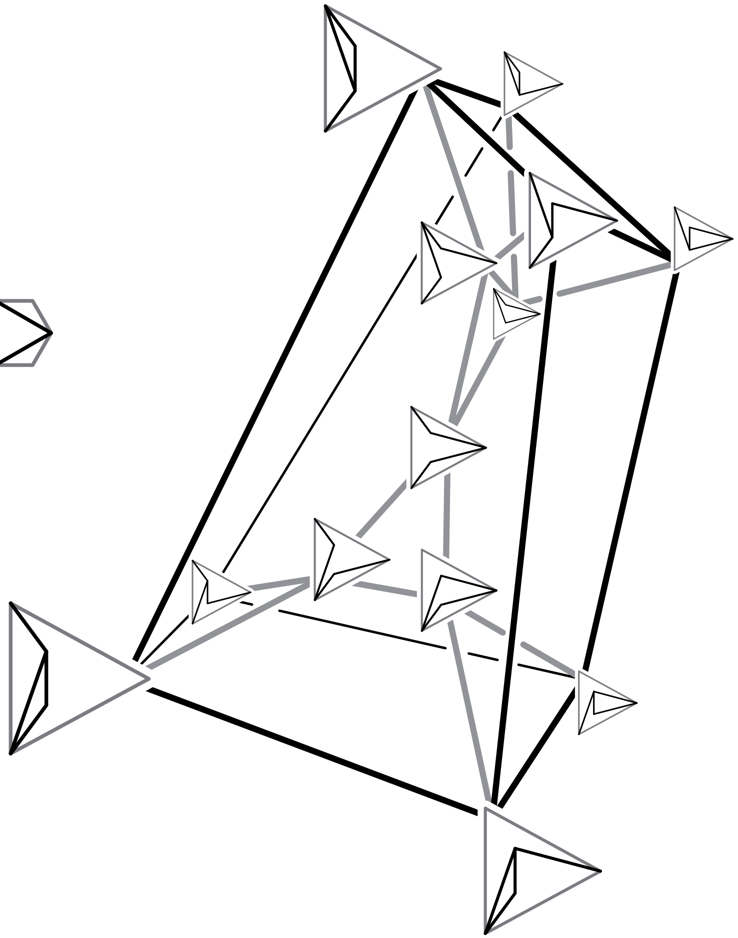


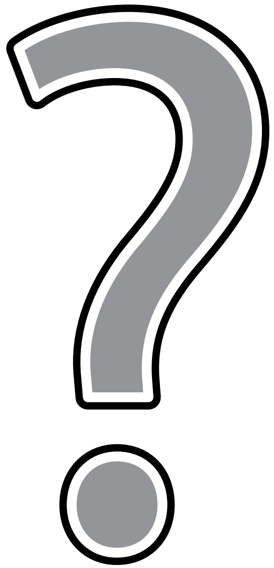
flip = exchange an internal edge with the common bisector of the two adjacent cells.

THREE GEOMETRIC STRUCTURES

Triangulations	Pseudotriangulations	Multitriangulations



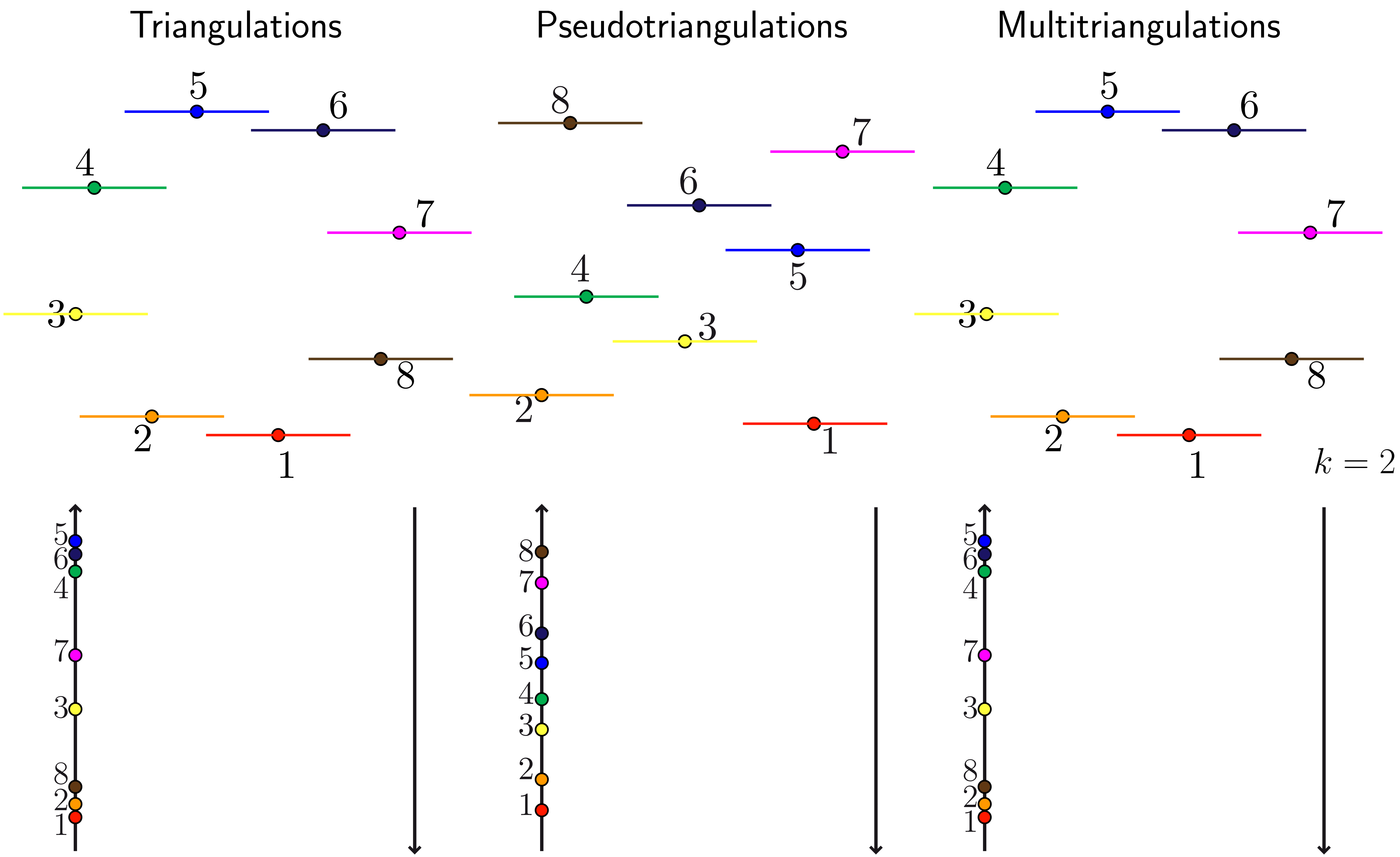




associahedron	$\longleftrightarrow$	crossing-free sets of internal edges.
pseudotriangulations polytope	$\longleftrightarrow$	pointed crossing-free sets of internal edges.
multiassociahedron	$\longleftrightarrow$	$(k+1)$-crossing-free sets of k-internal edges.

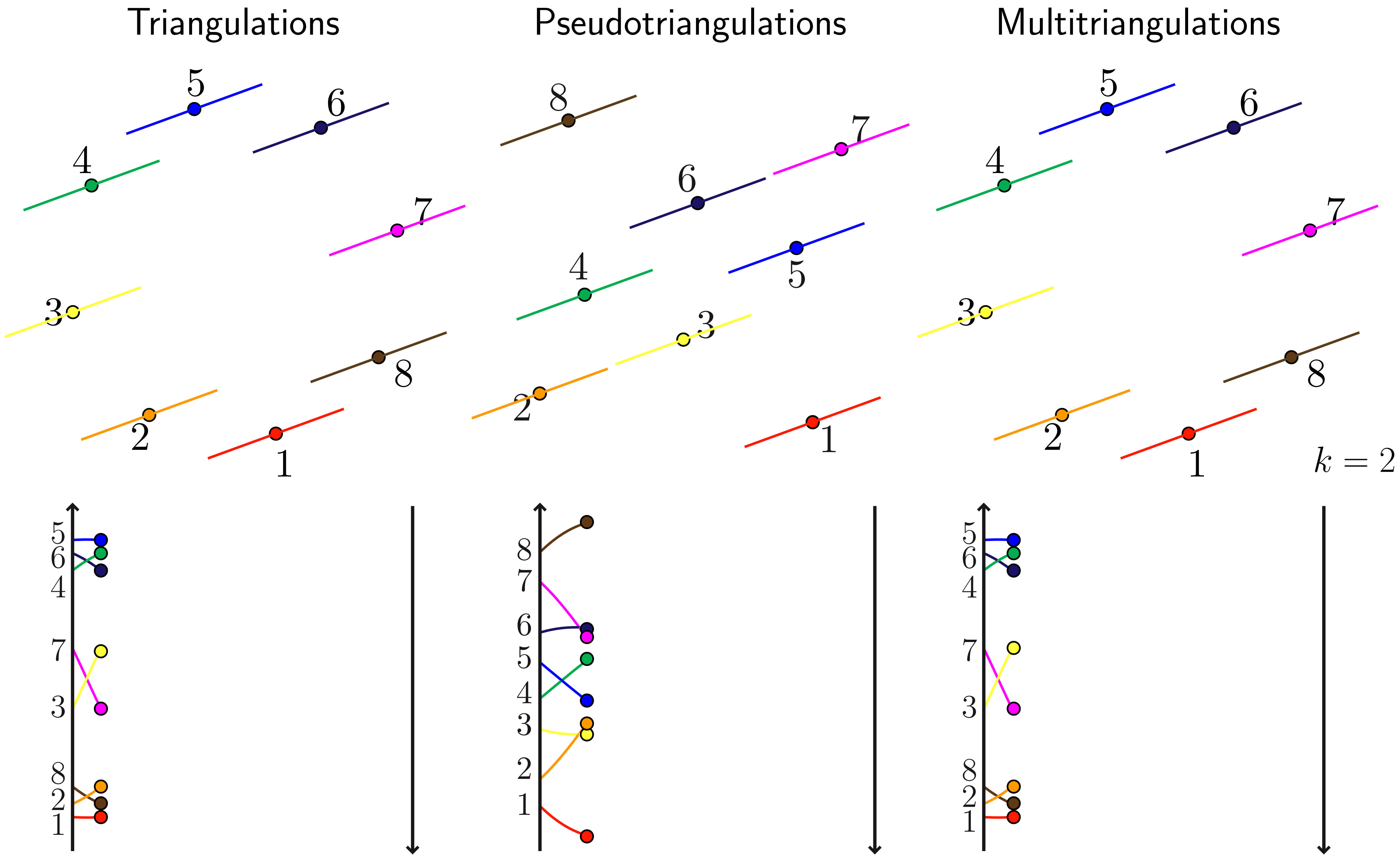
DUALITY

DUALITY



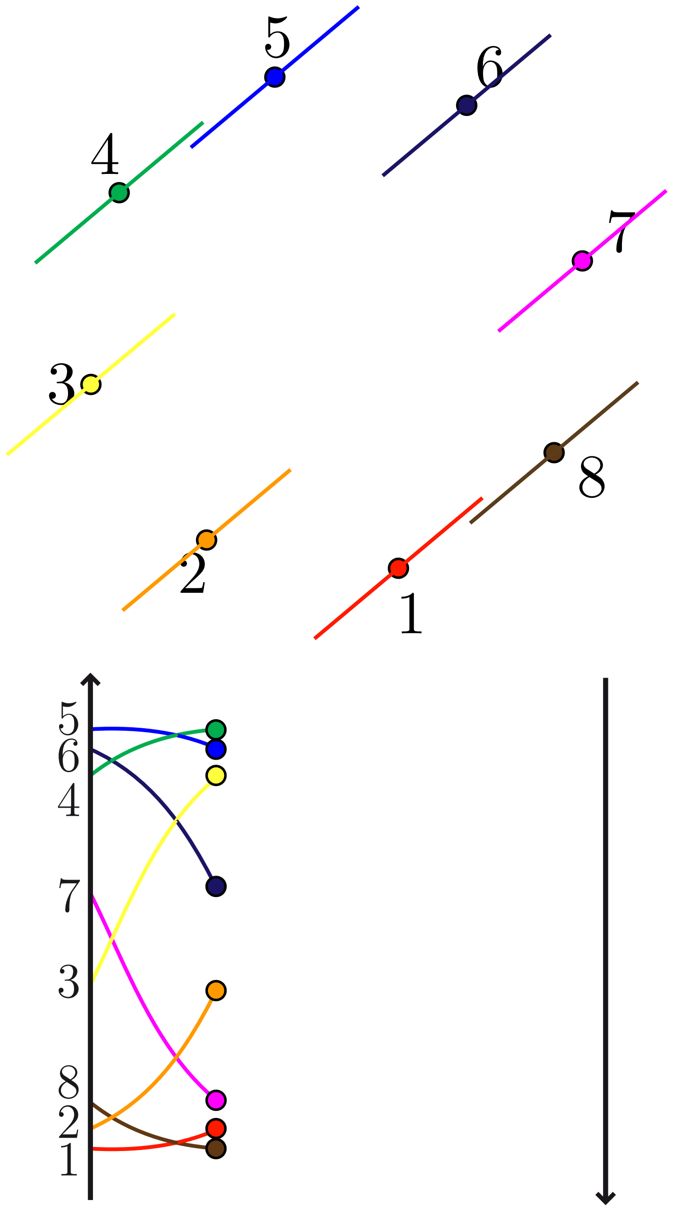
Triangulations
Pseudotriangulations
Multitriangulations
$k = 2$

DUALITY

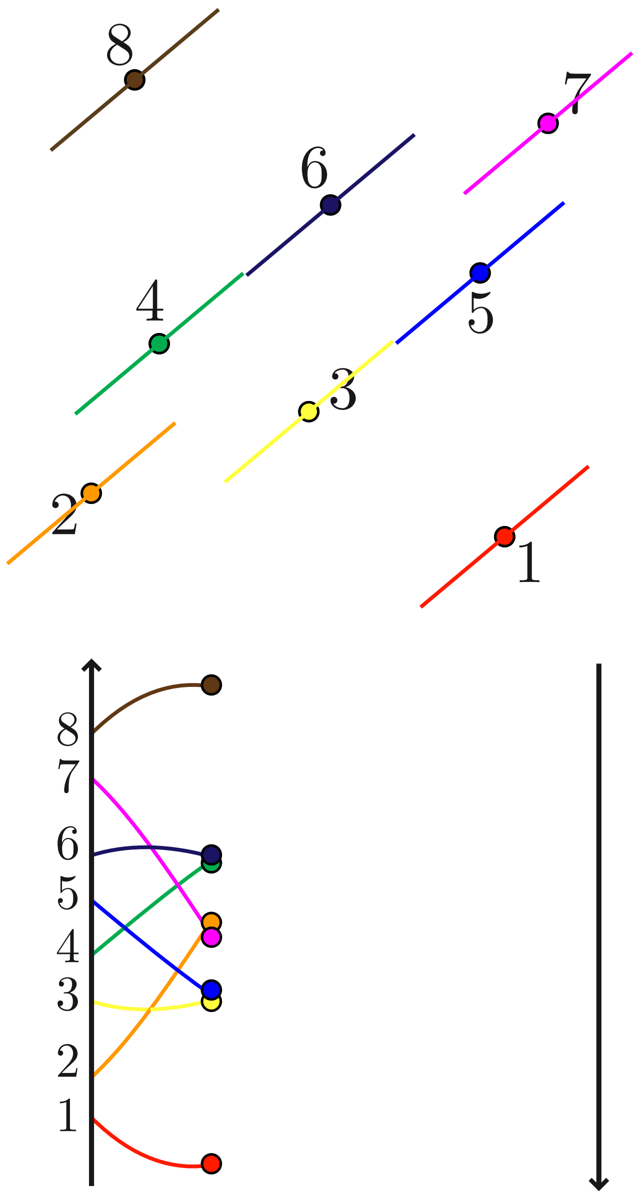

Triangulations
Pseudotriangulations
Multitriangulations
$k = 2$

DUALITY

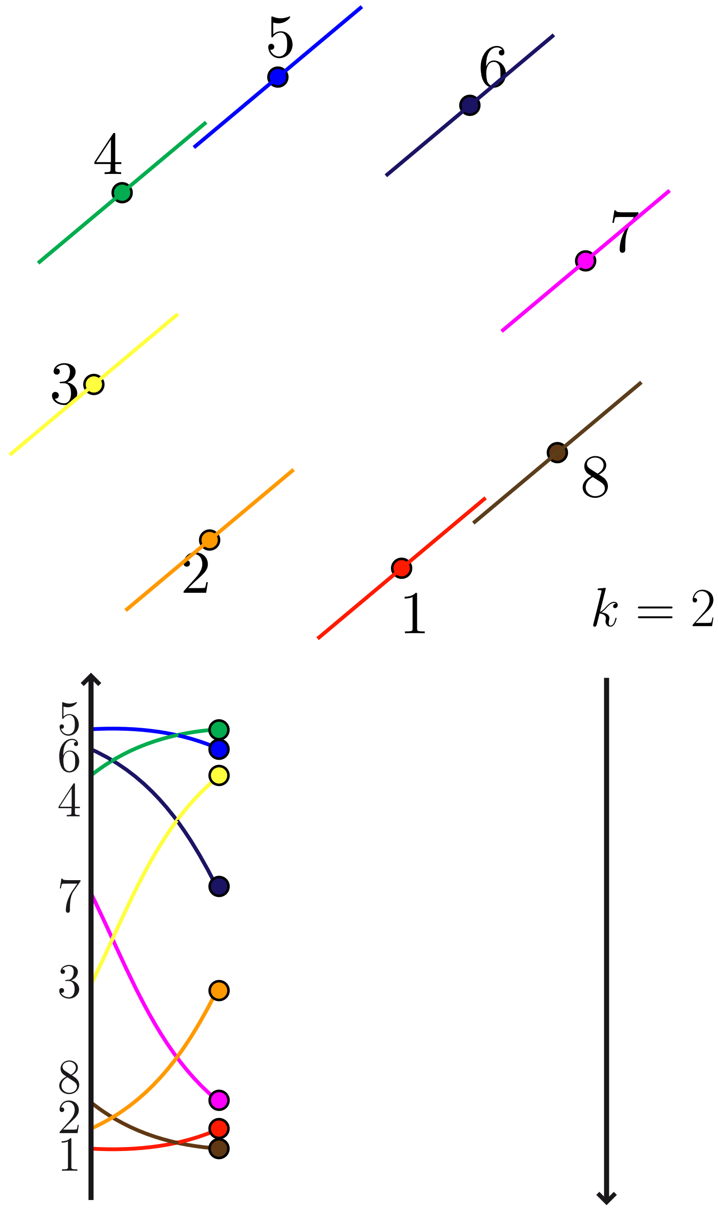
Triangulations



Pseudotriangulations

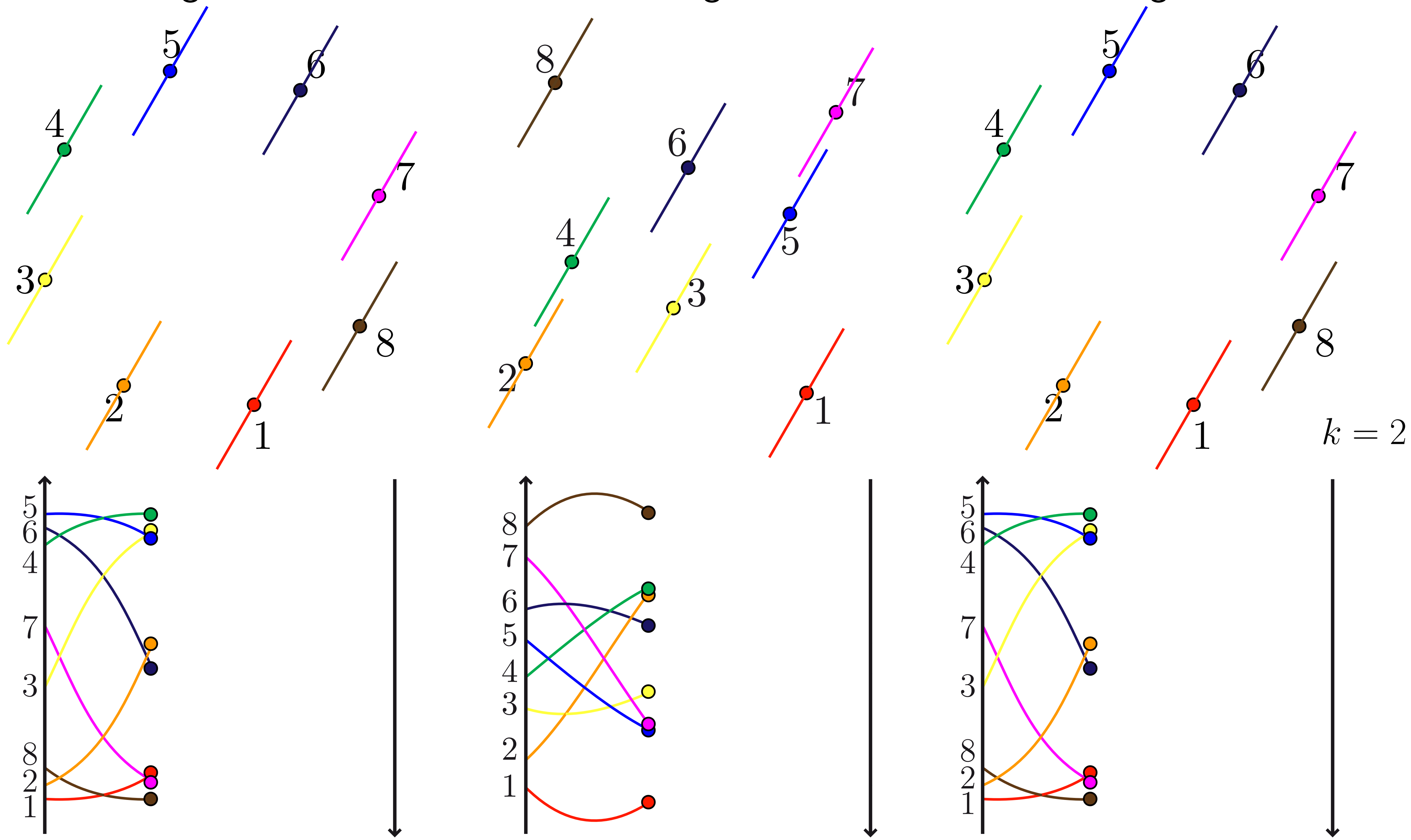


Multitriangulations



DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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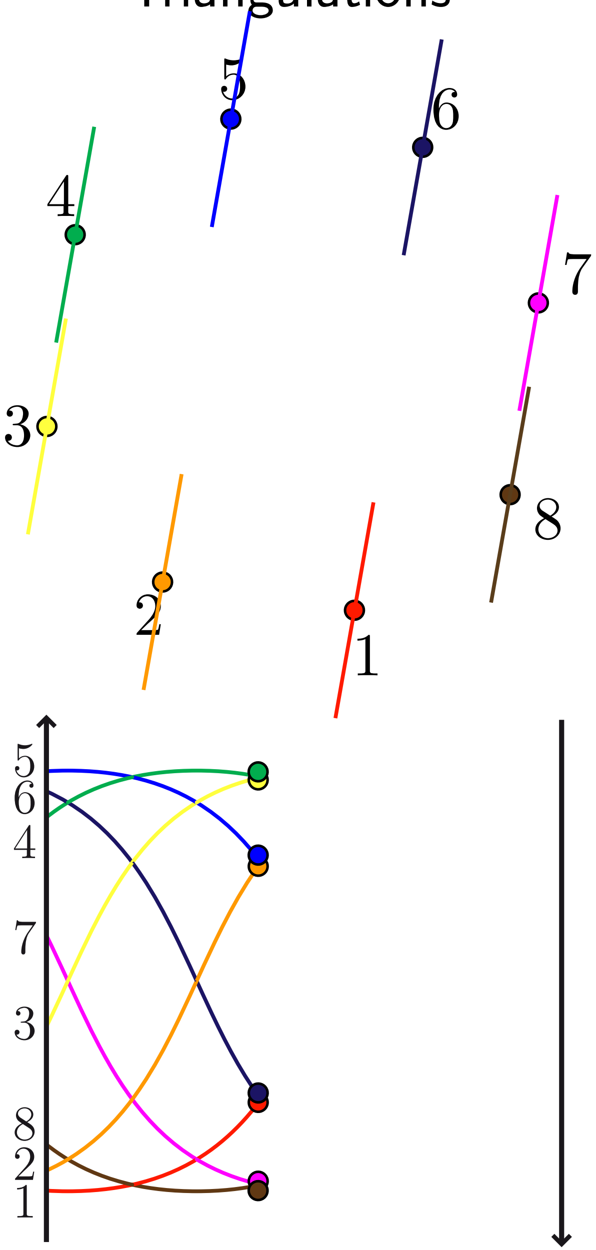
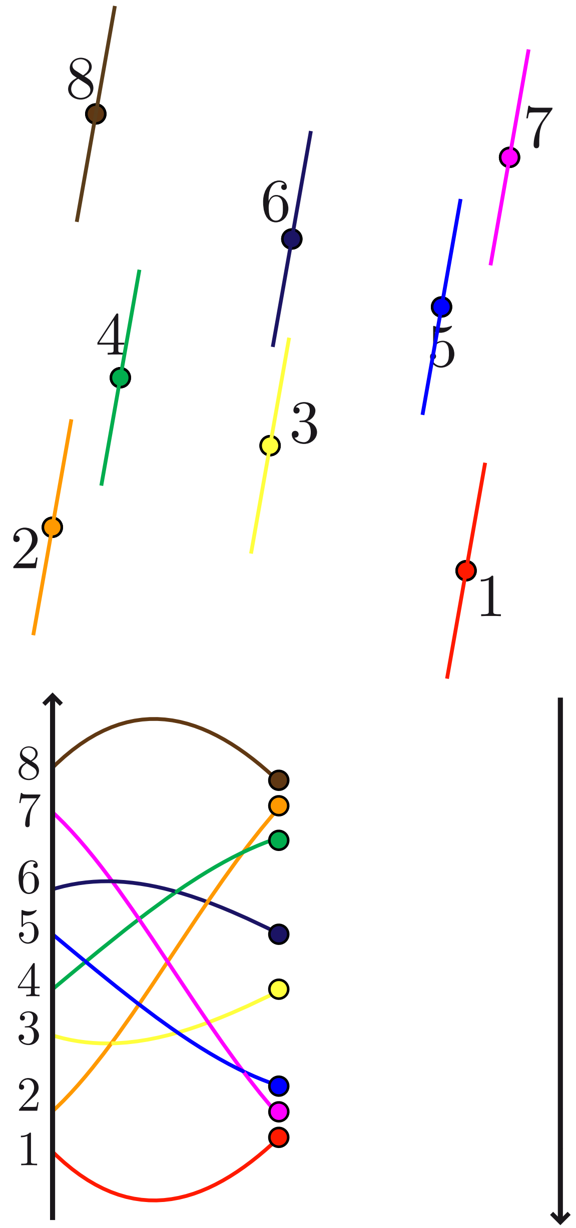
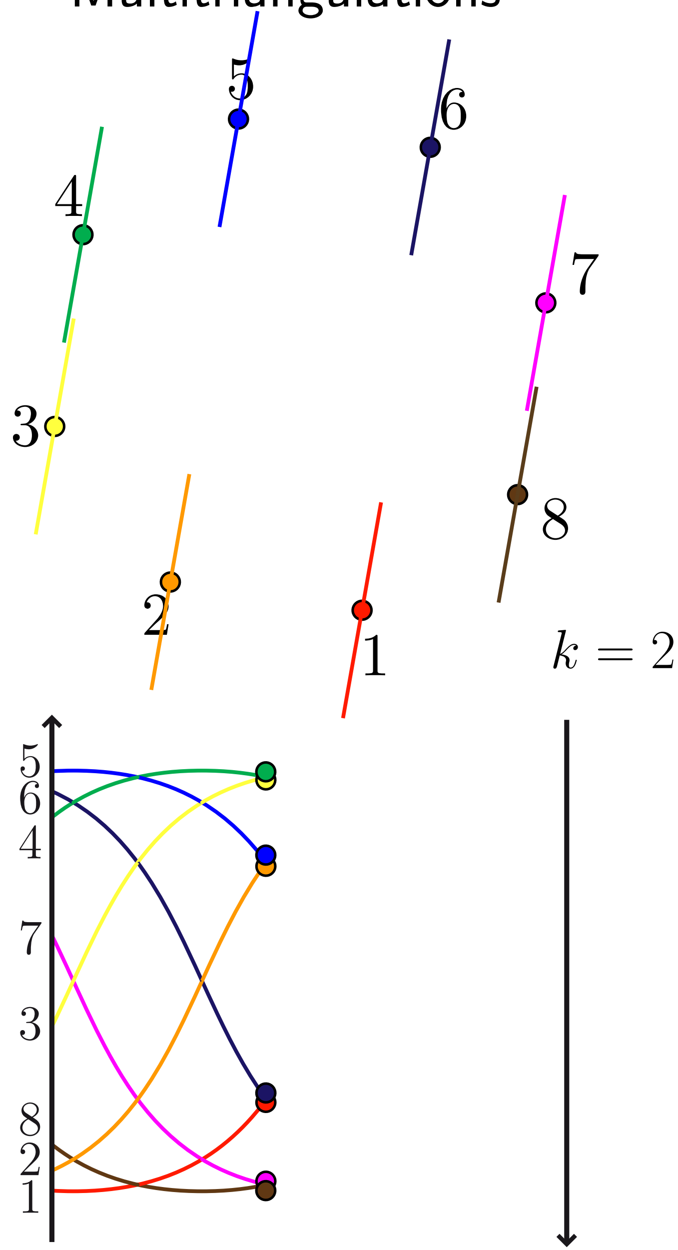
Triangulations



Pseudotriangulations

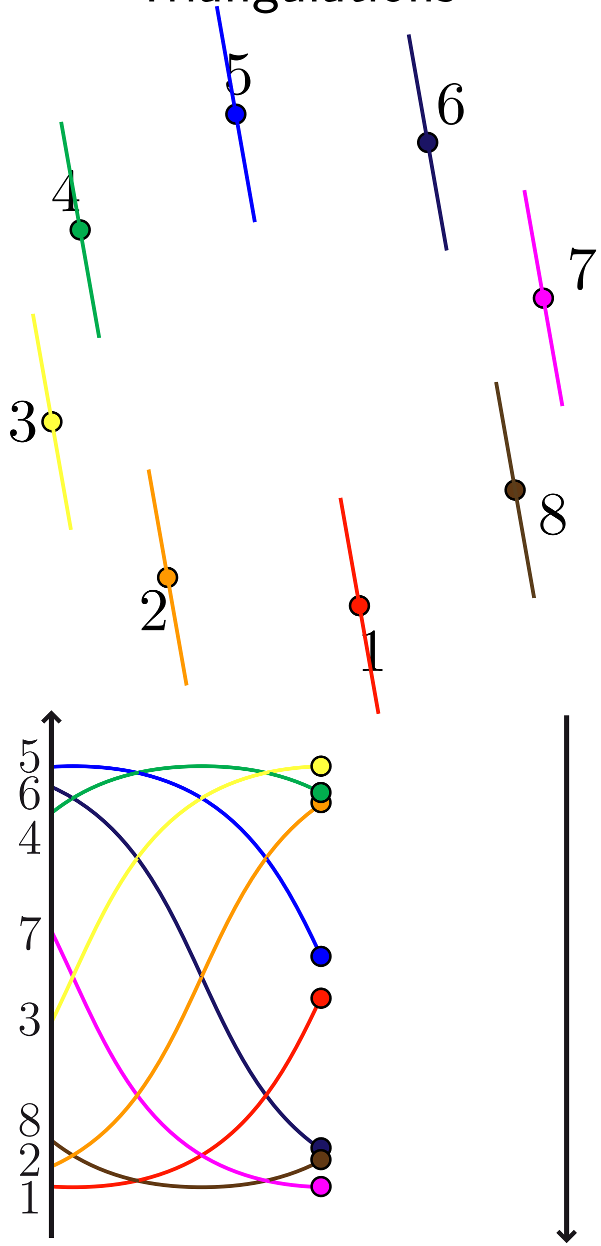


Multitriangulations

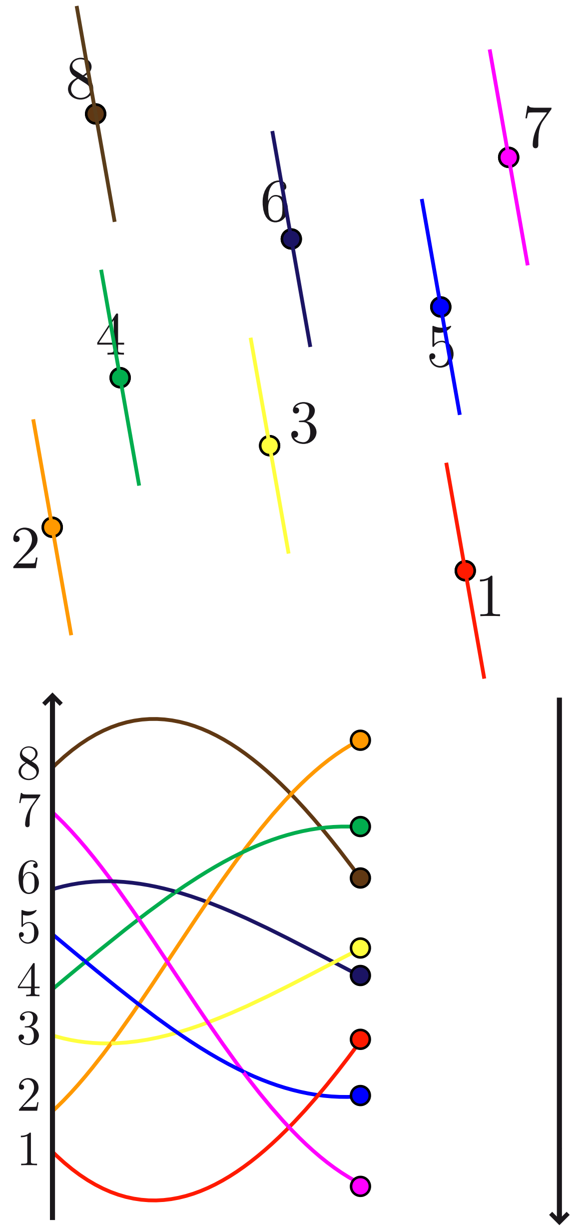


DUALITY

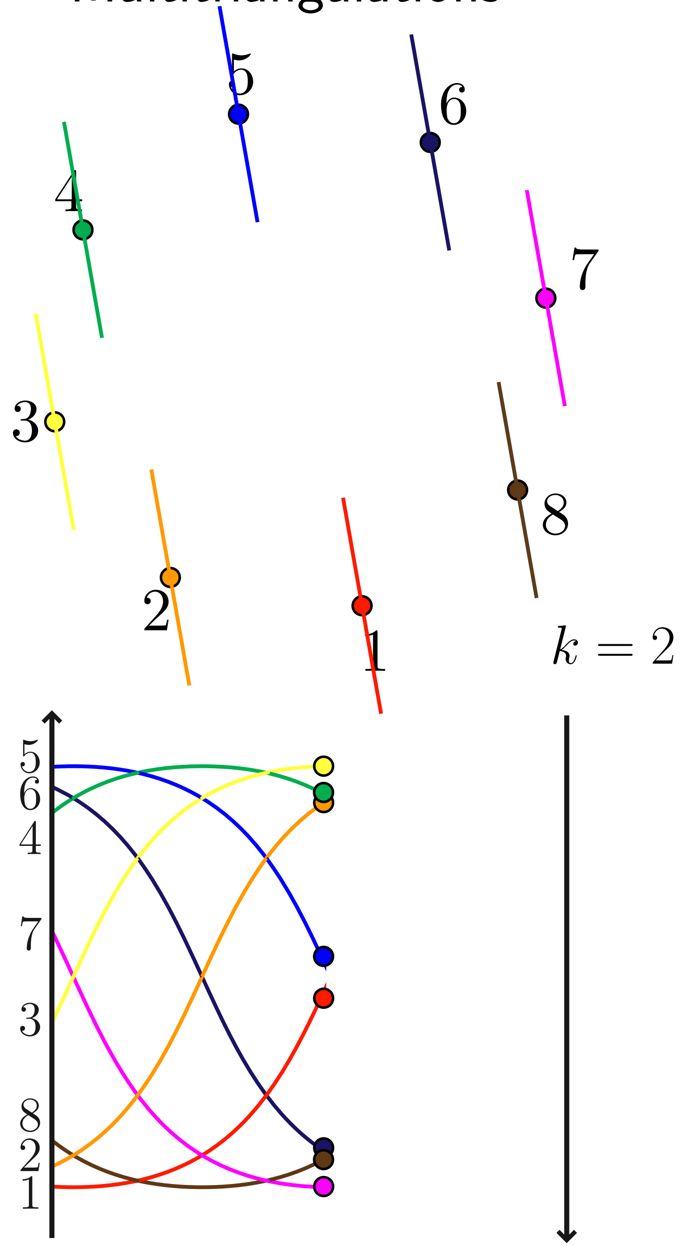
Triangulations



Pseudotriangulations



Multitriangulations



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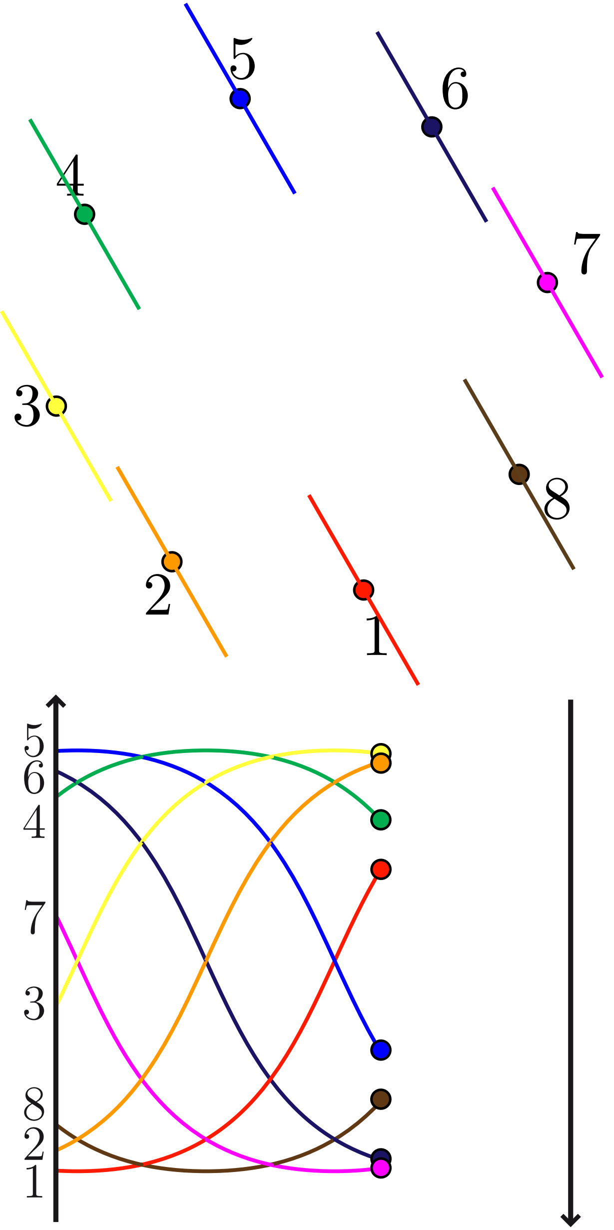
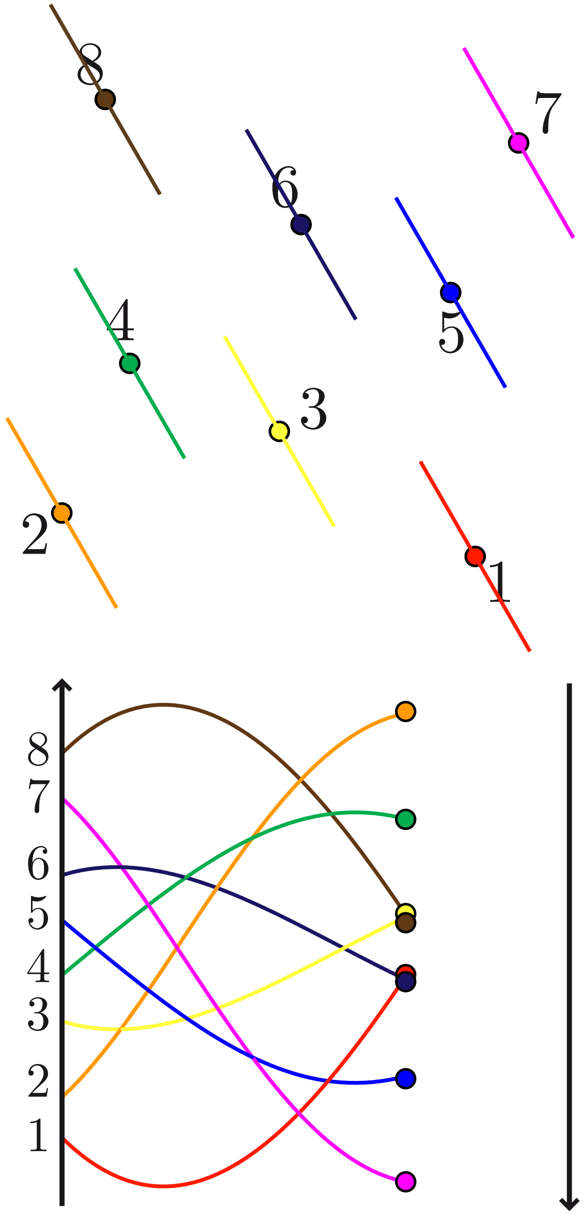
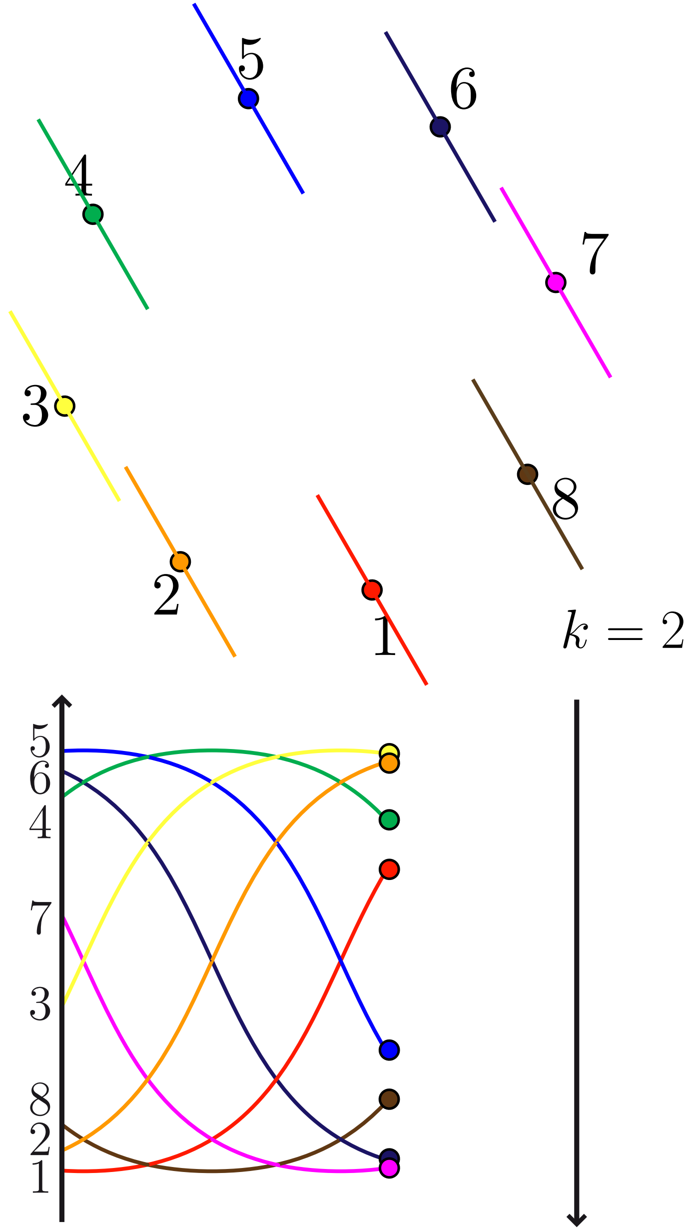
Triangulations



Pseudotriangulations

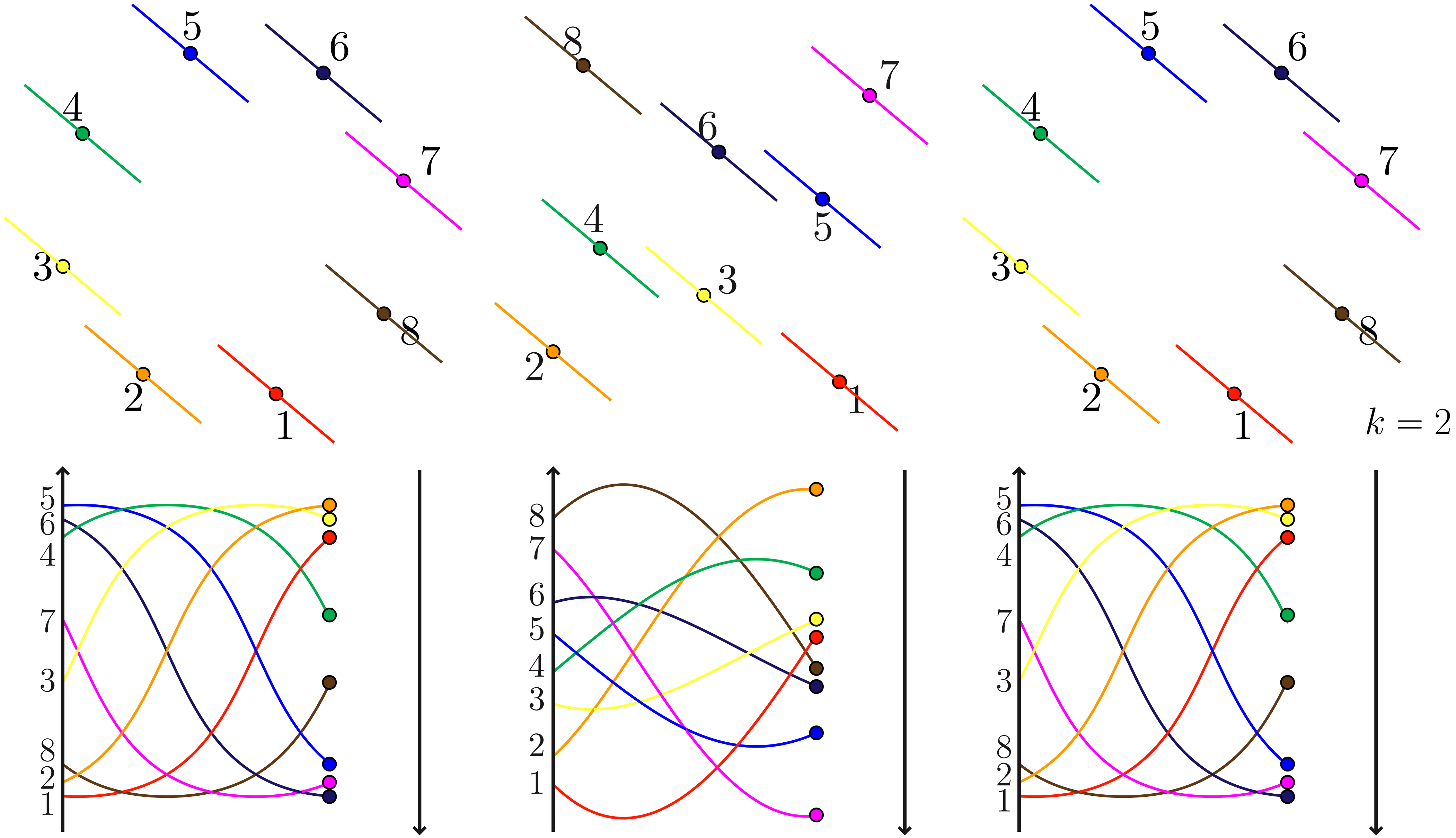


Multitriangulations



DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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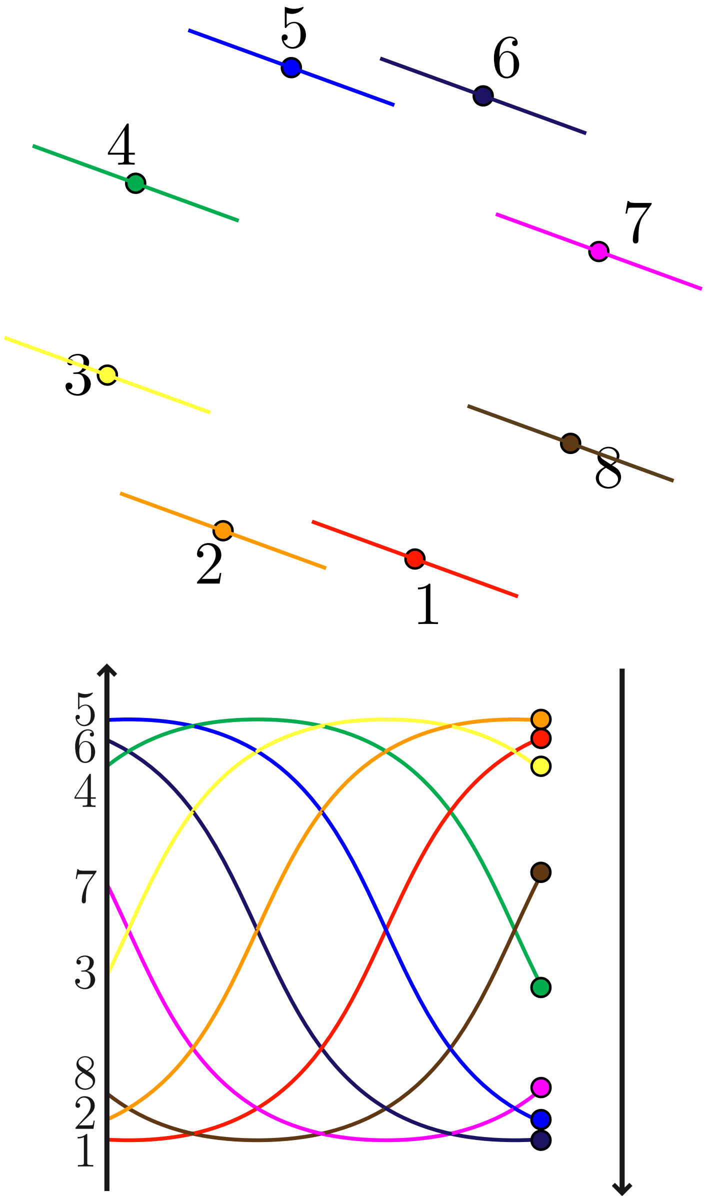
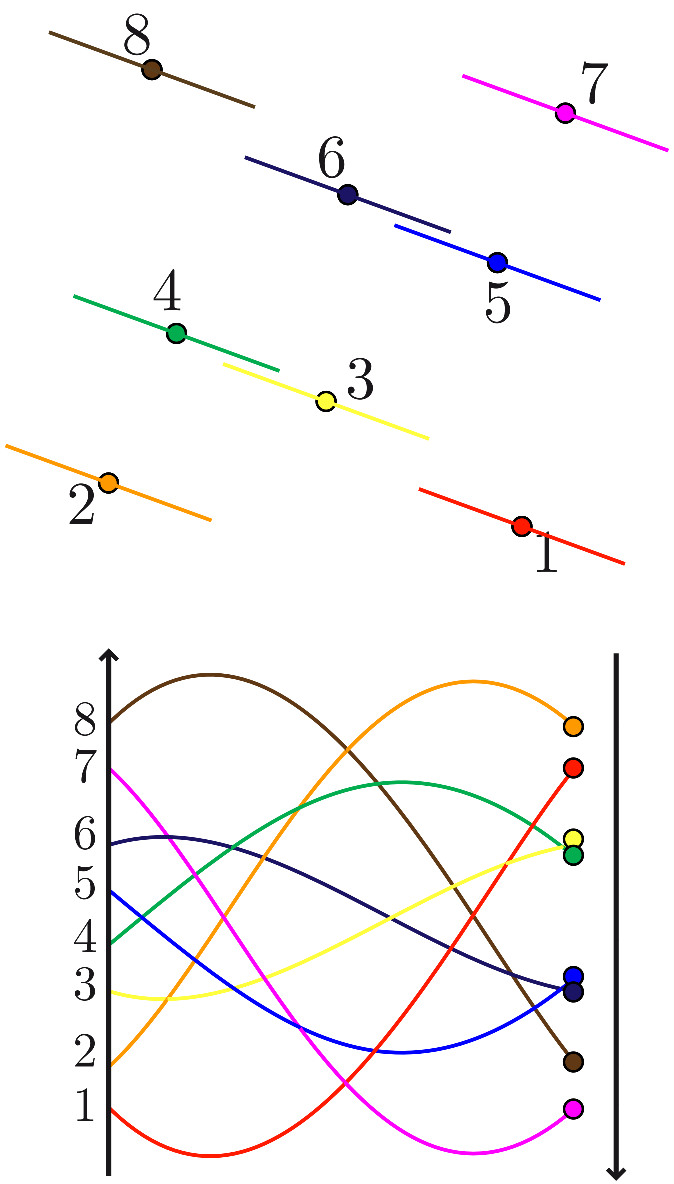
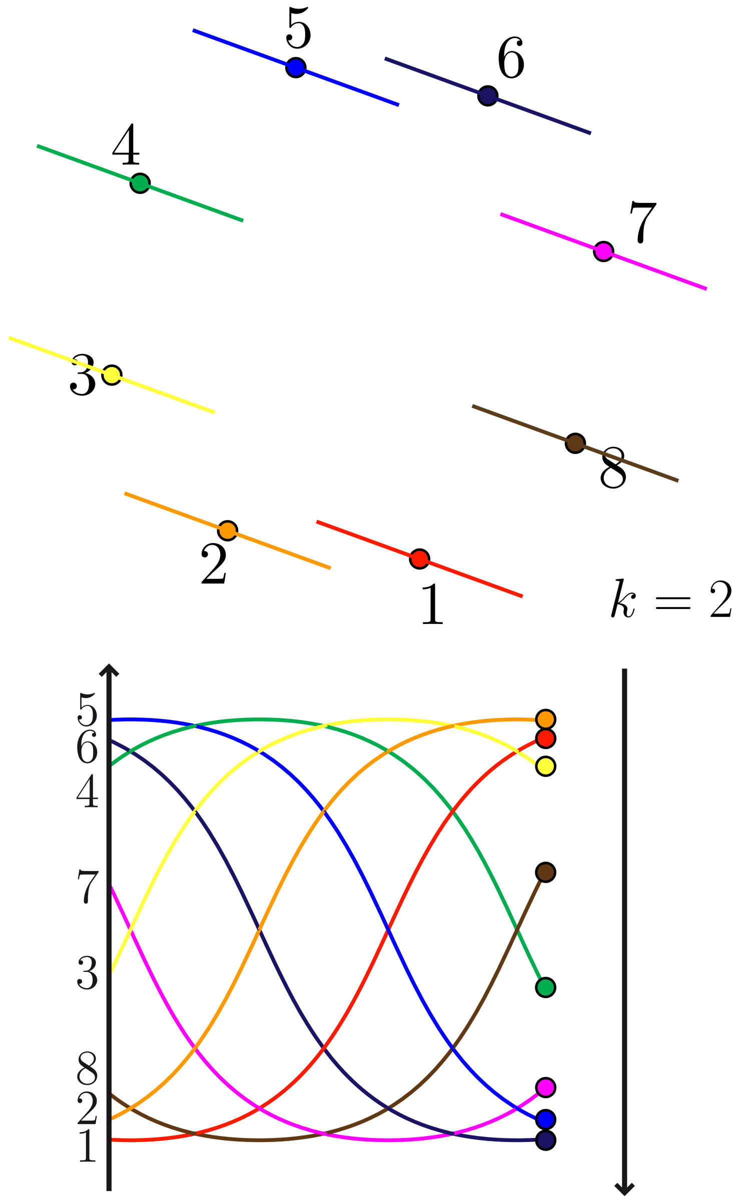
Triangulations



Pseudotriangulations

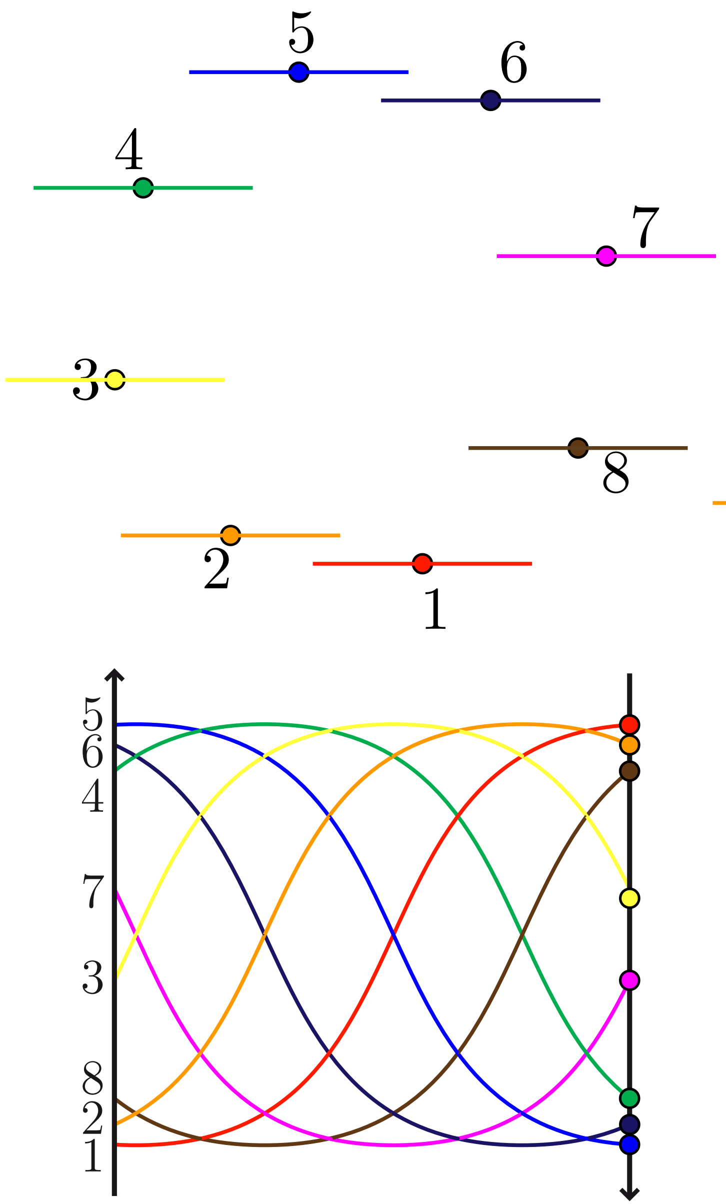


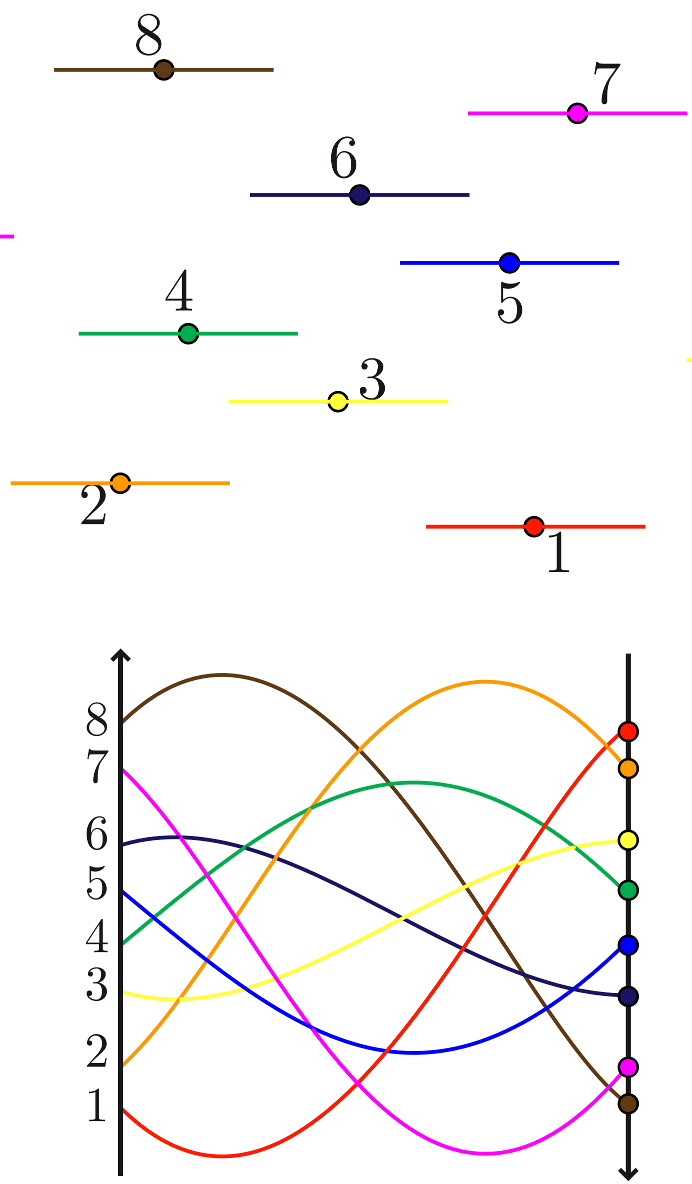
Multitriangulations

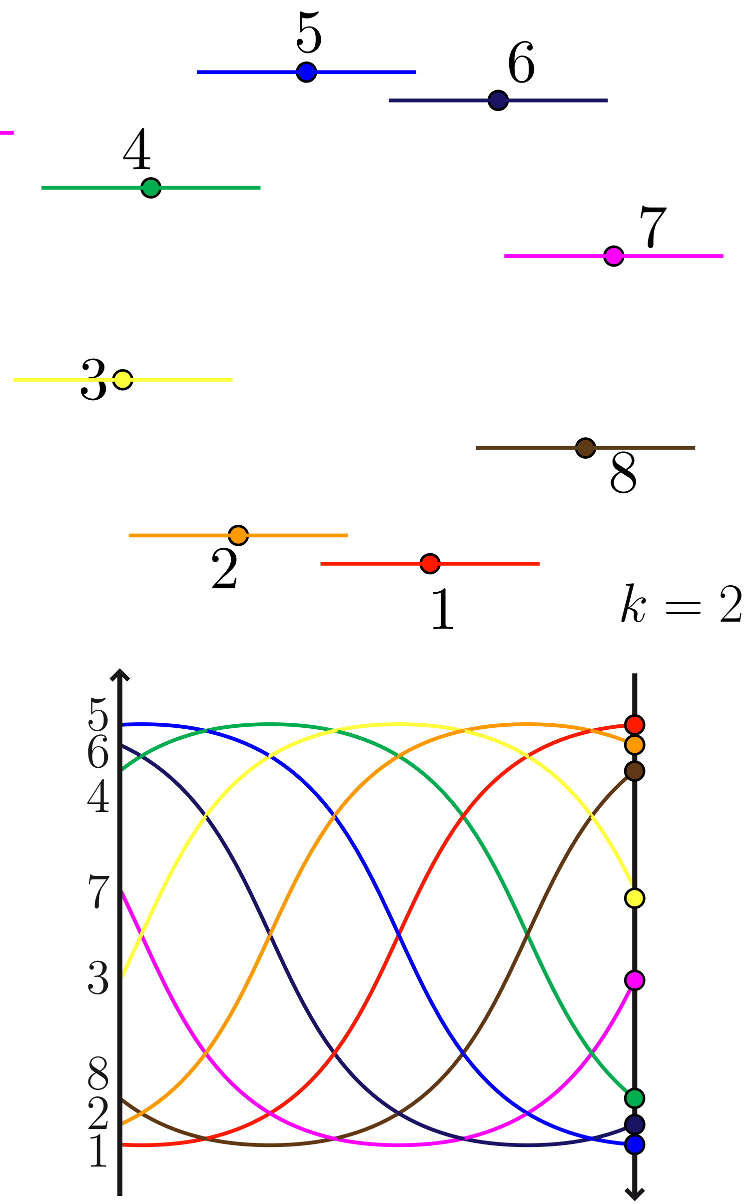


DUALITY

Triangulations	Pseudotriangulations	Multitriangulations







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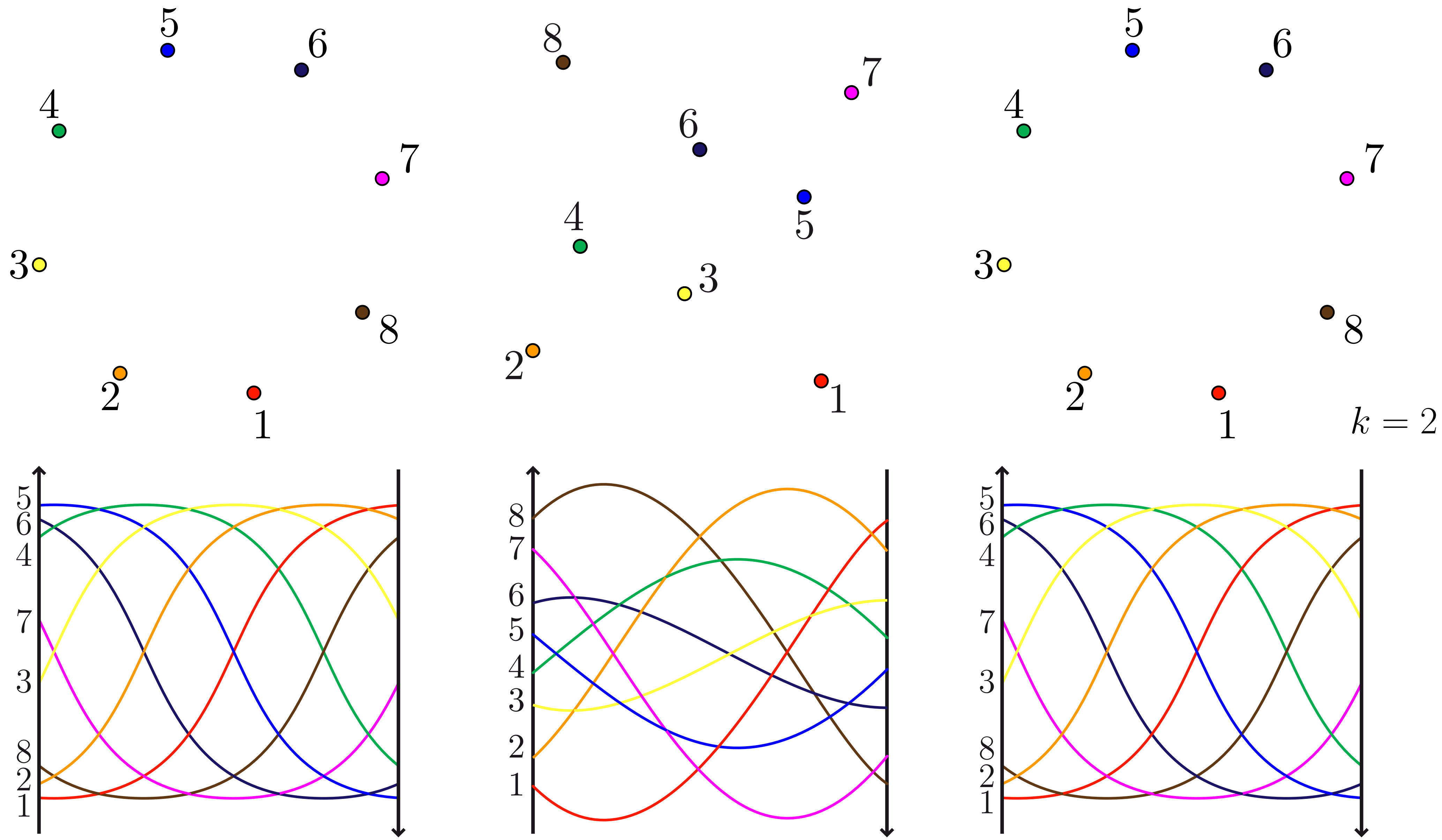
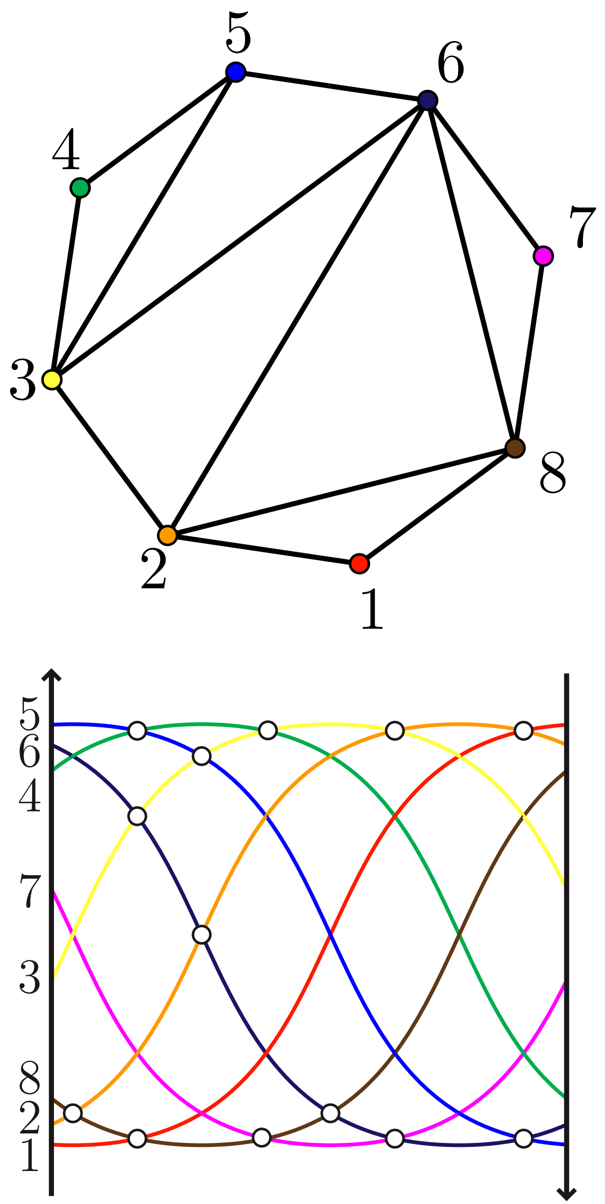
Triangulations | Pseudotriangulations | Multitriangulations

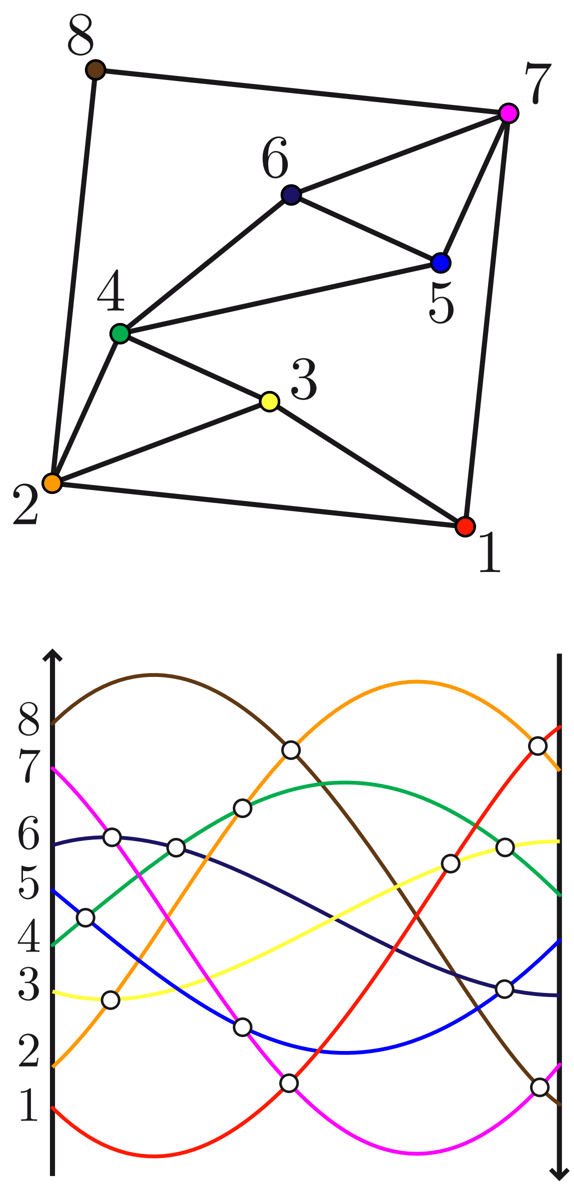


DUALITY

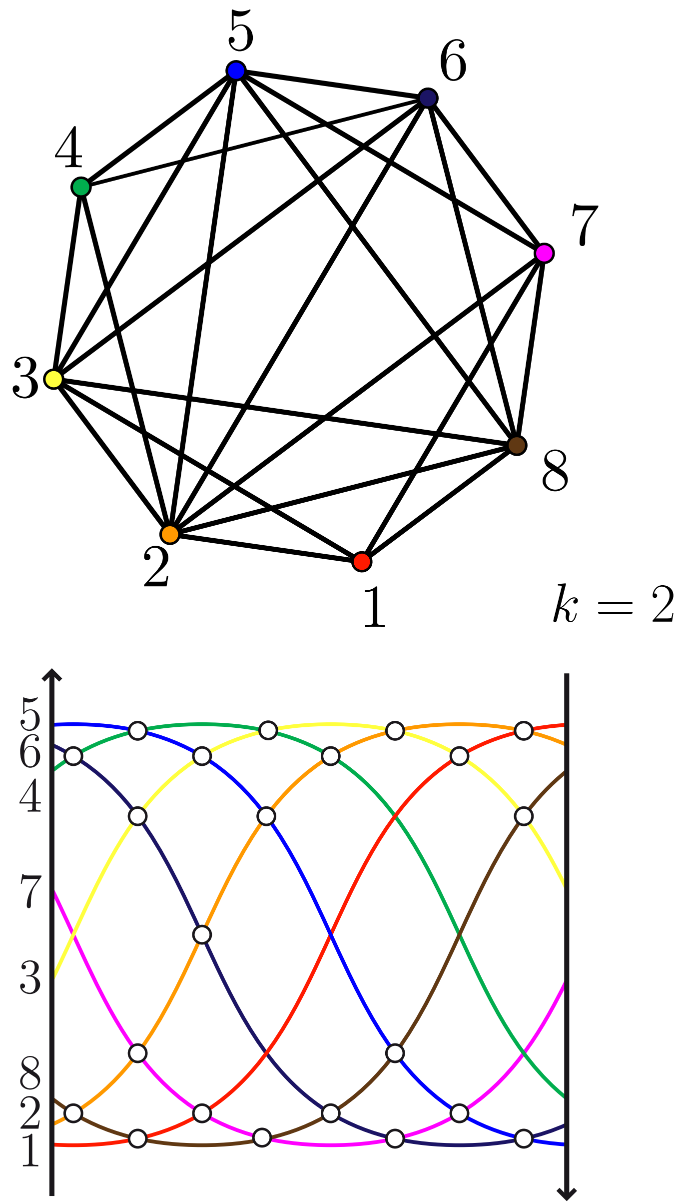
Triangulations



Pseudotriangulations

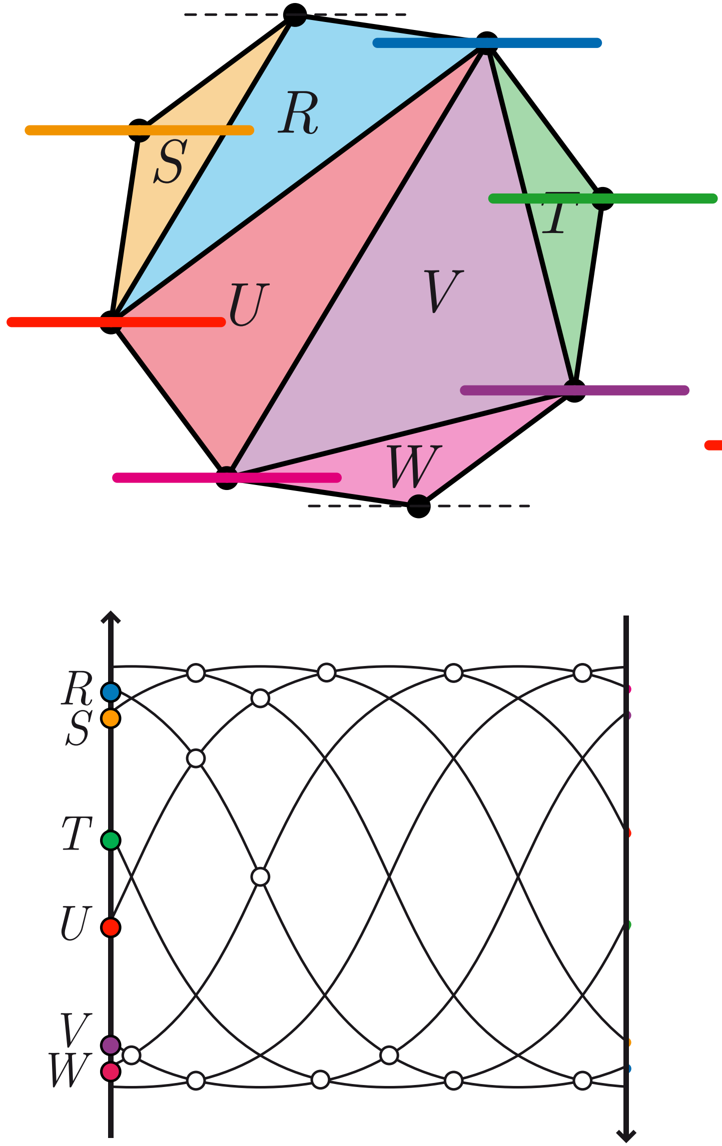


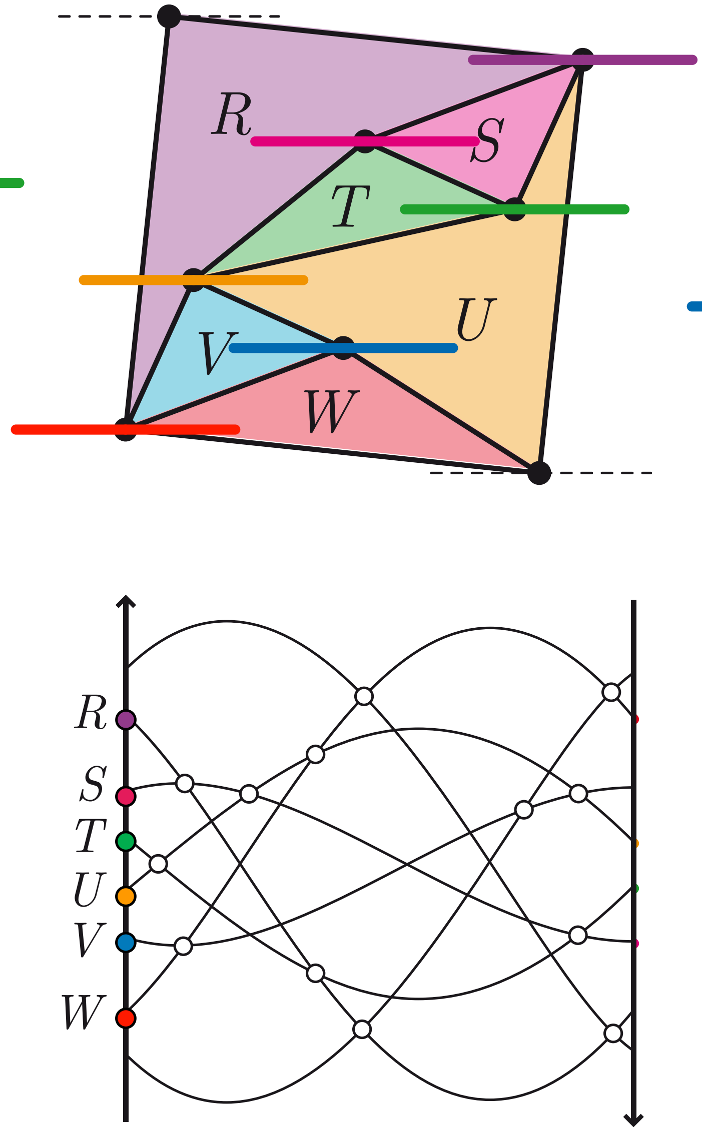
Multitriangulations

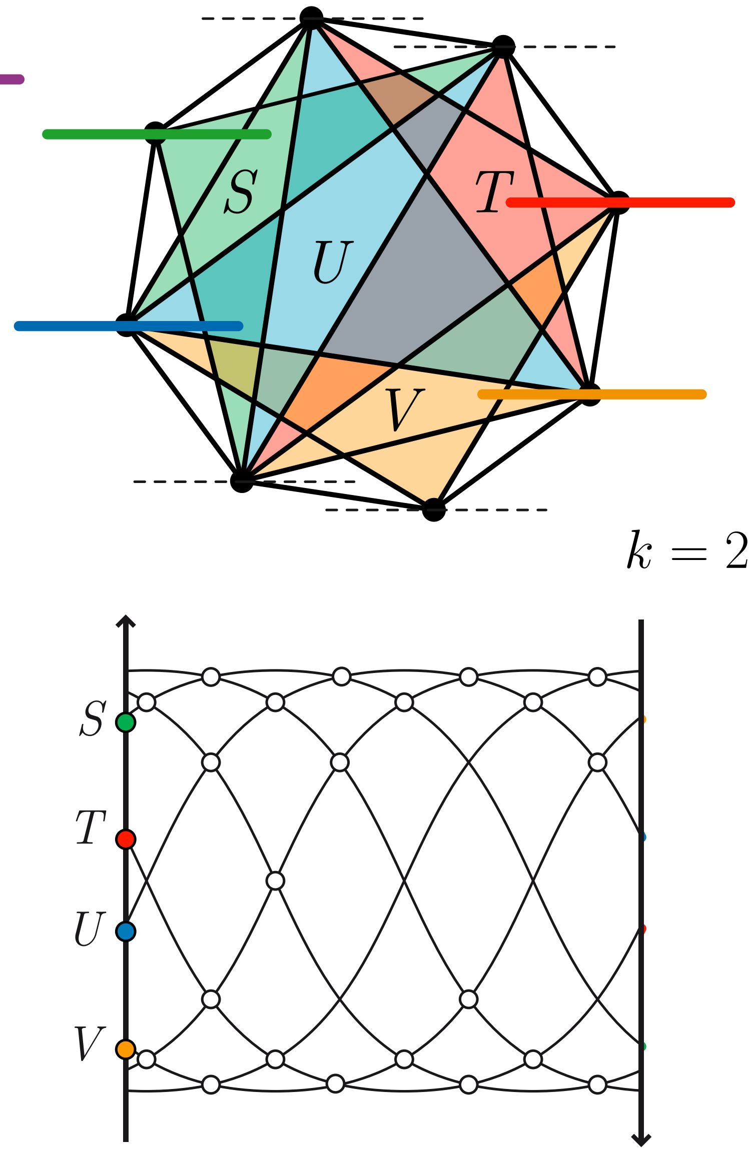


DUALITY

Triangulations | Pseudotriangulations | Multitriangulations







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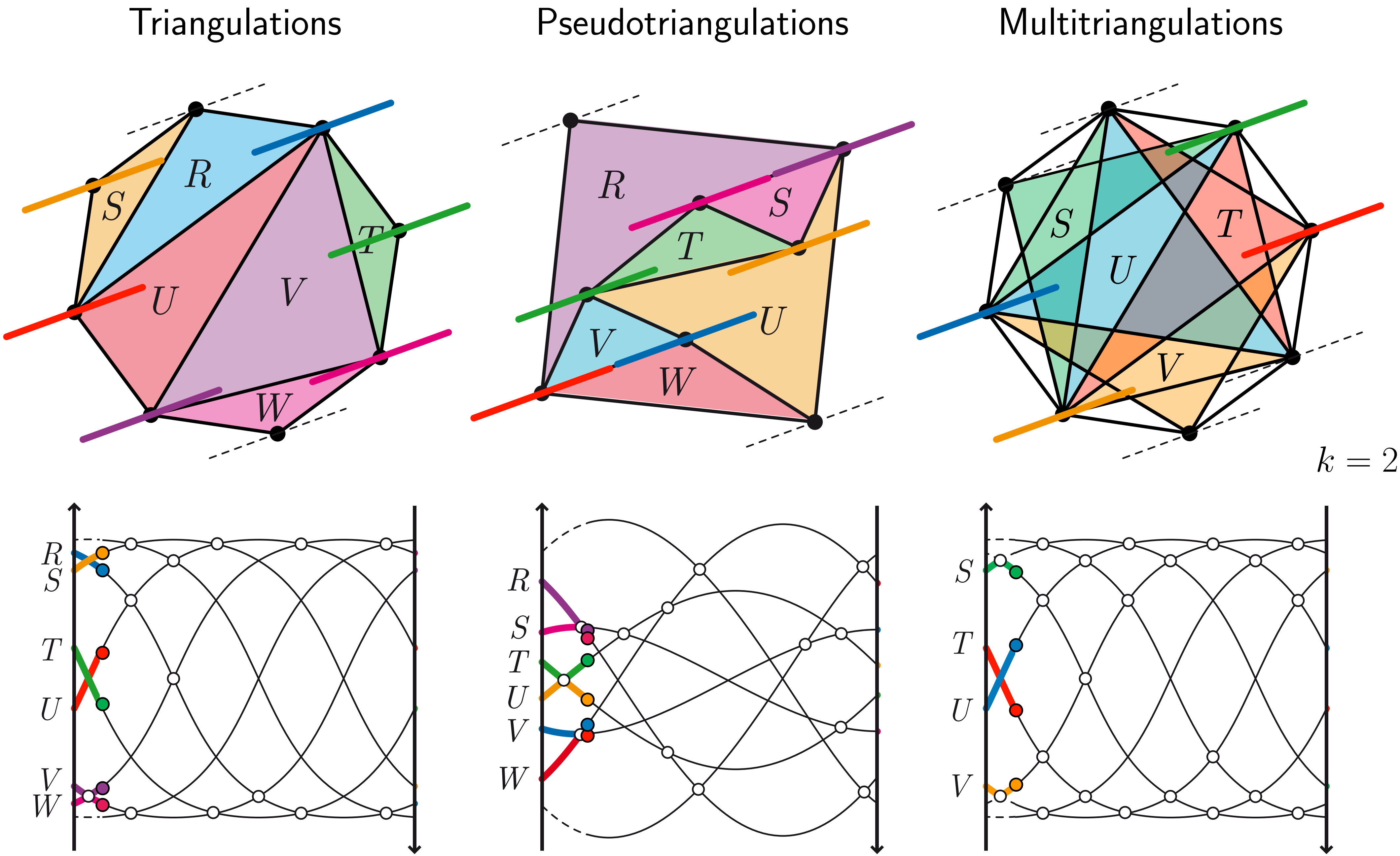
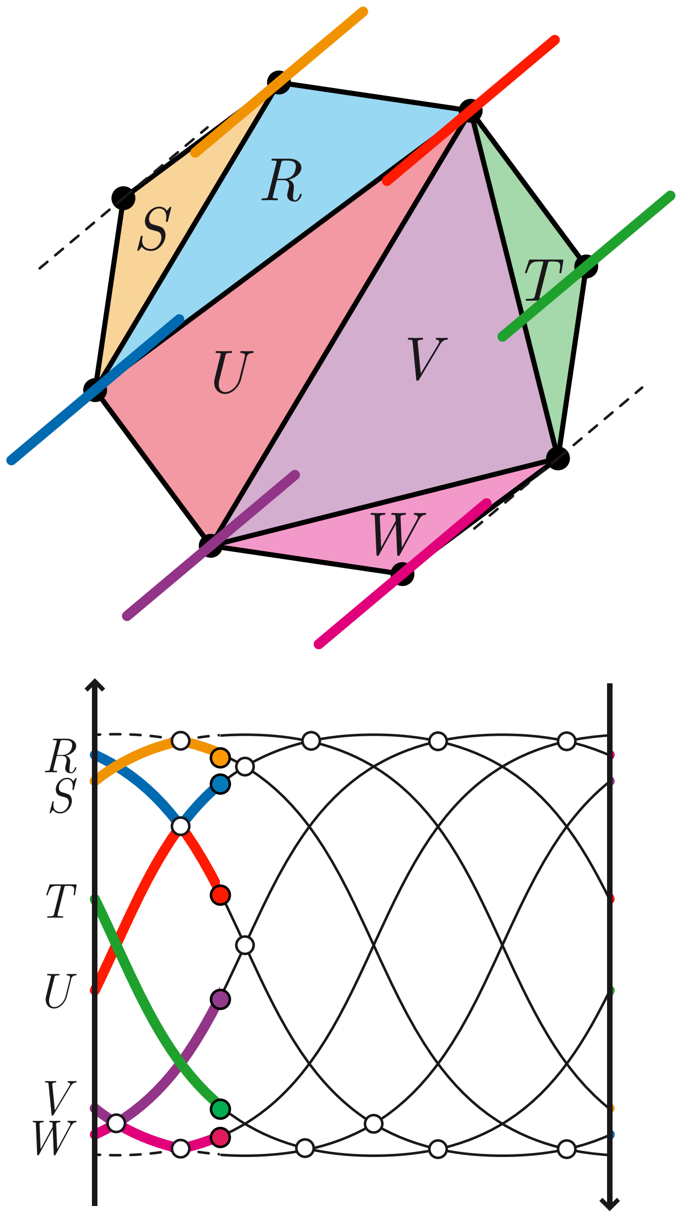
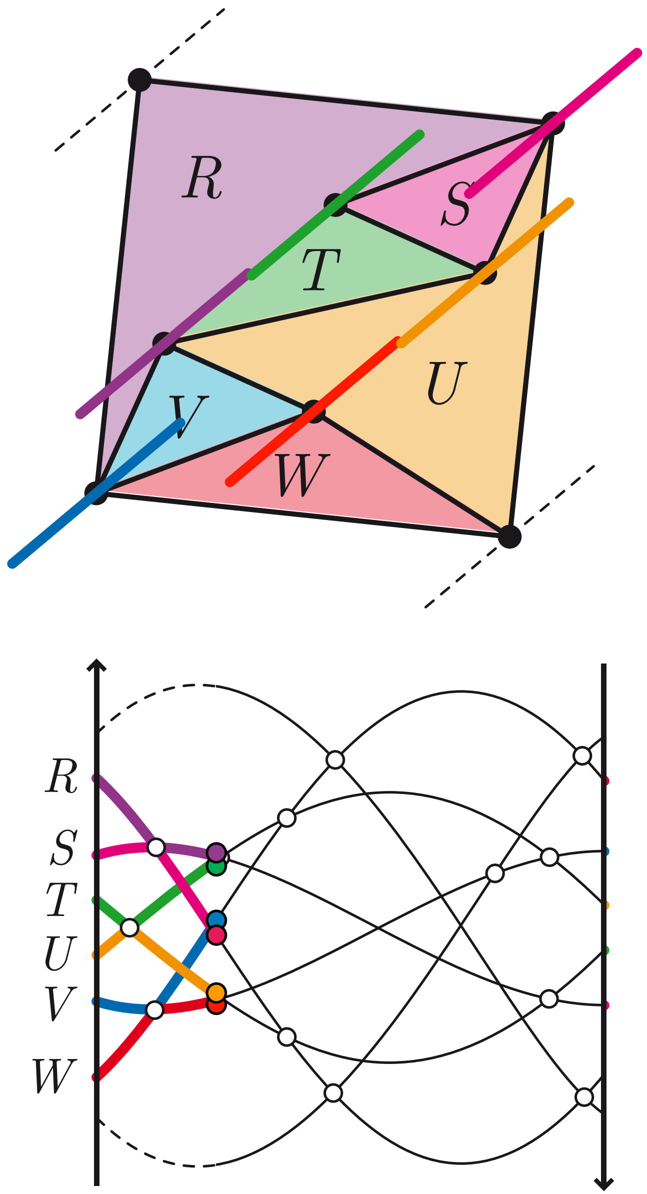

Triangulations
Pseudotriangulations
Multitriangulations
R
S
T
U
V
W
$k = 2$

DUALITY

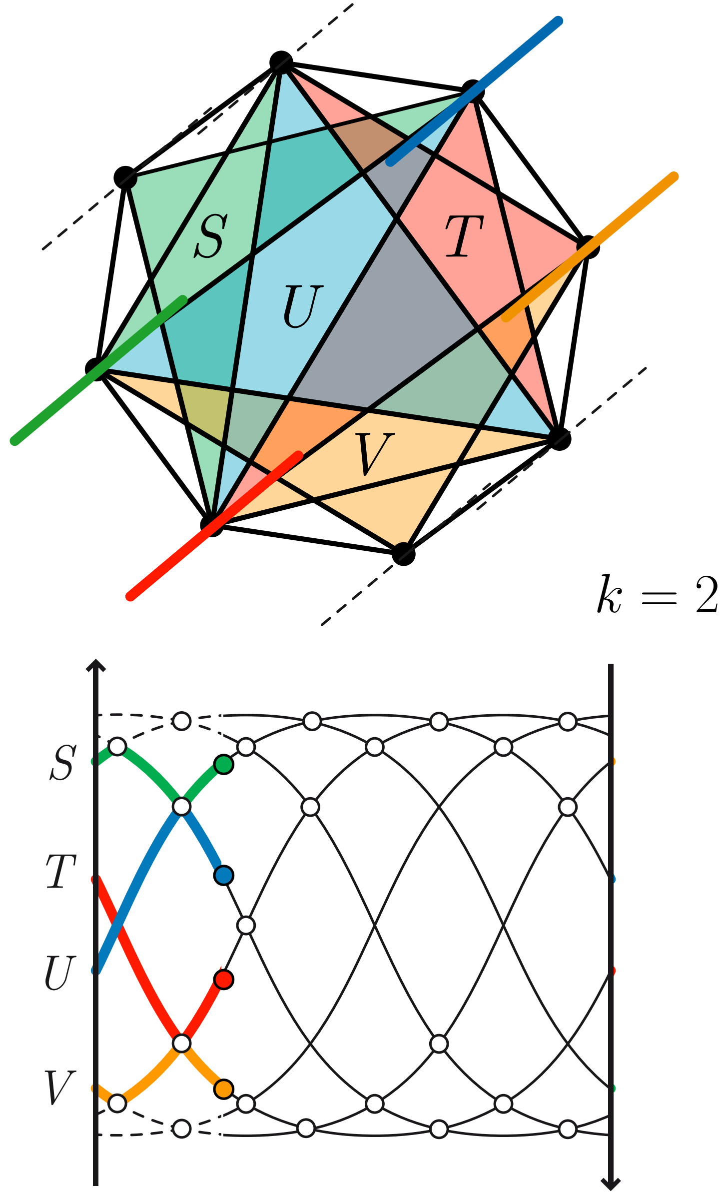
Triangulations



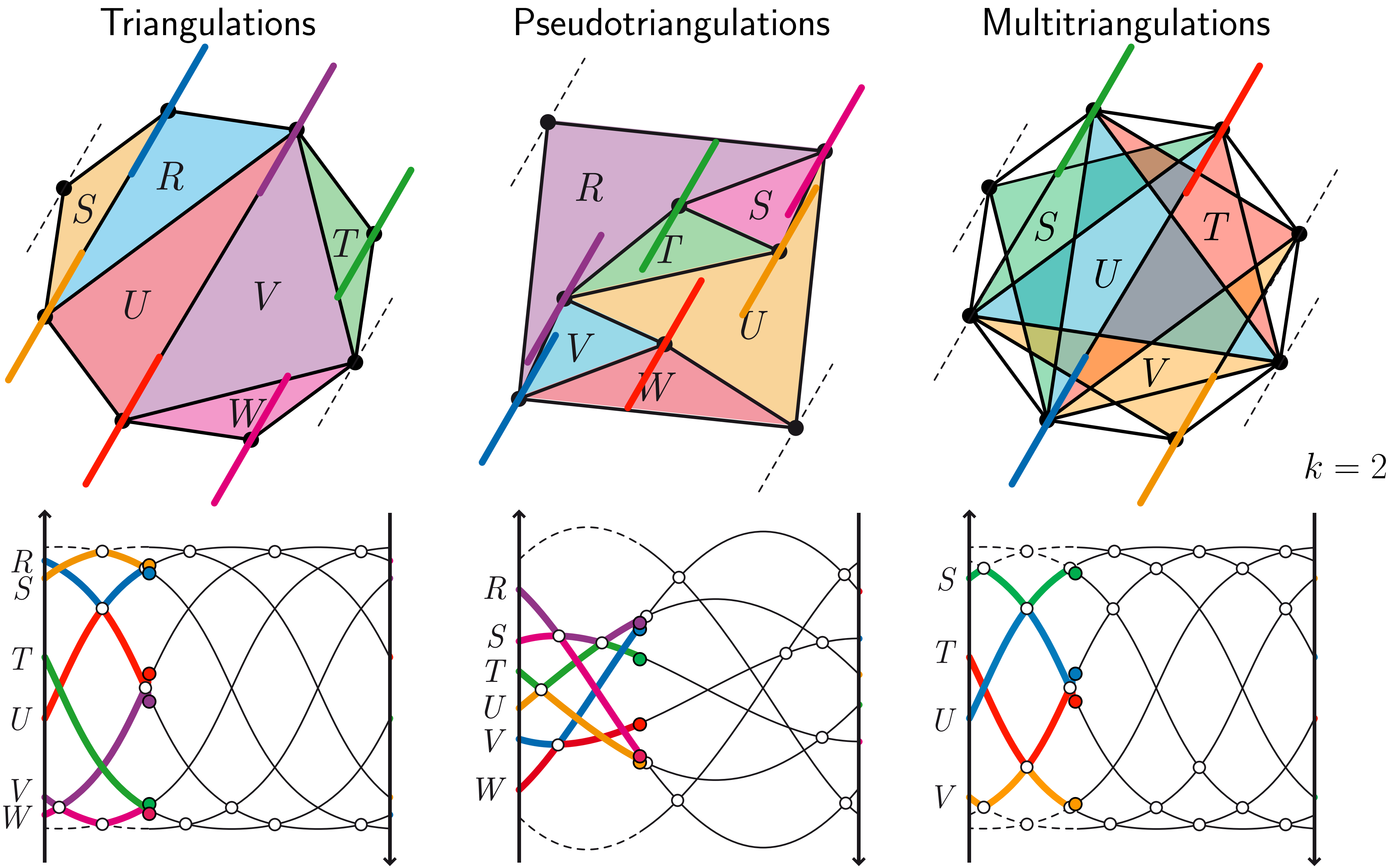
Pseudotriangulations



Multitriangulations



DUALITY



Triangulations
Pseudotriangulations
Multitriangulations
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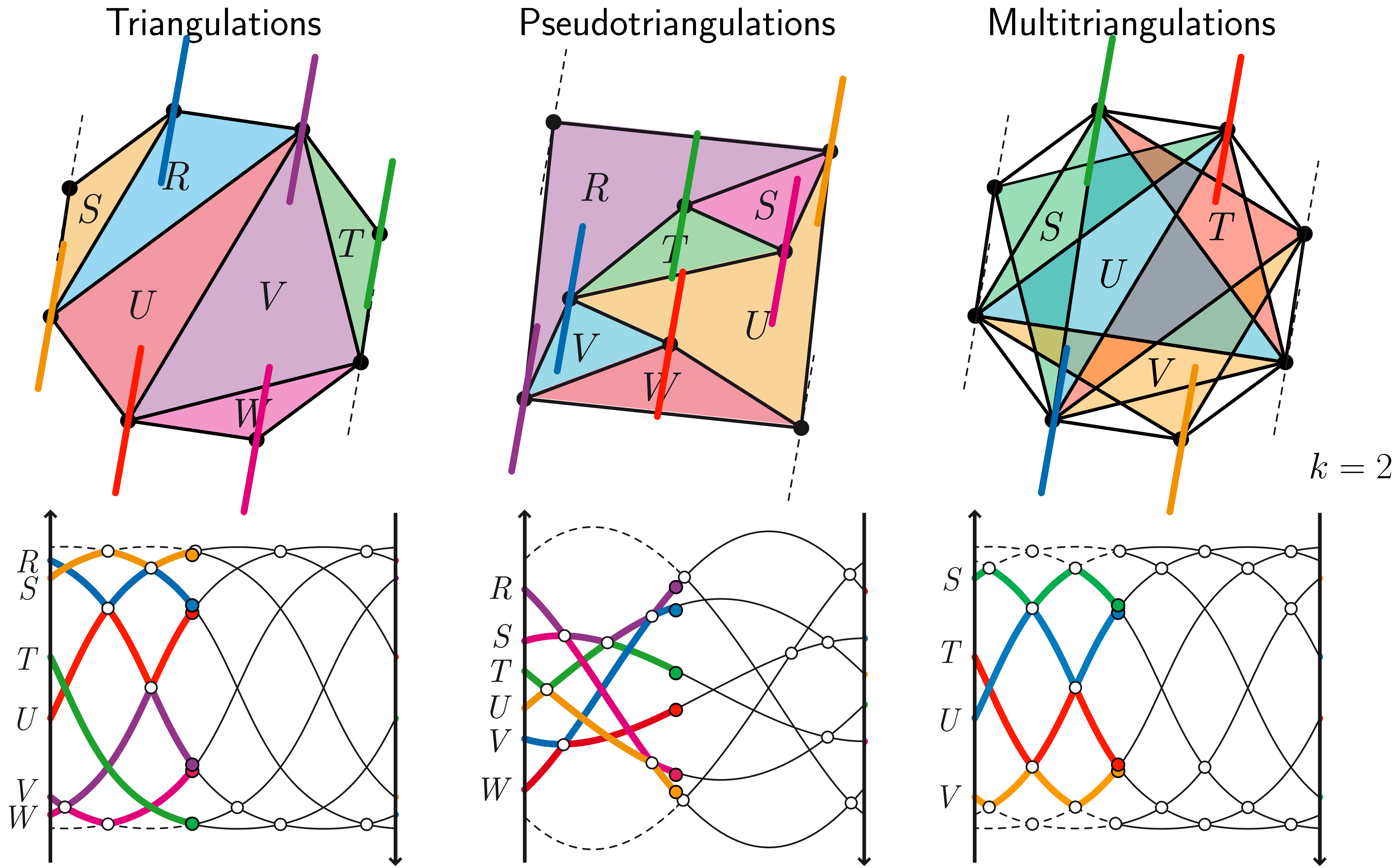
k = 2

DUALITY


Triangulations
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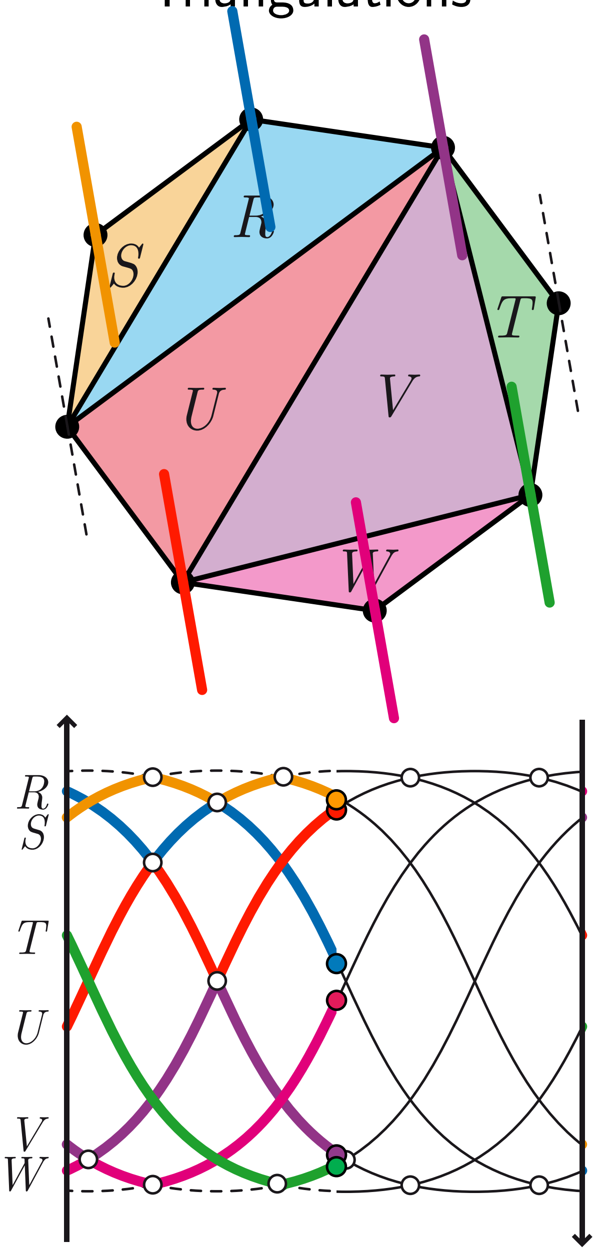
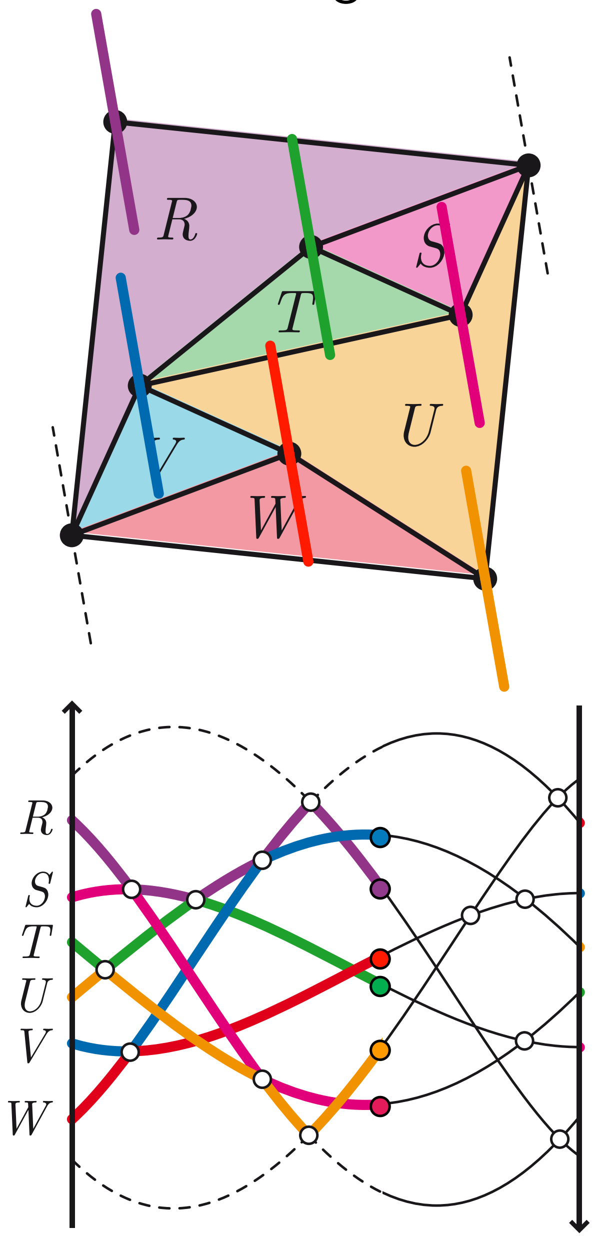
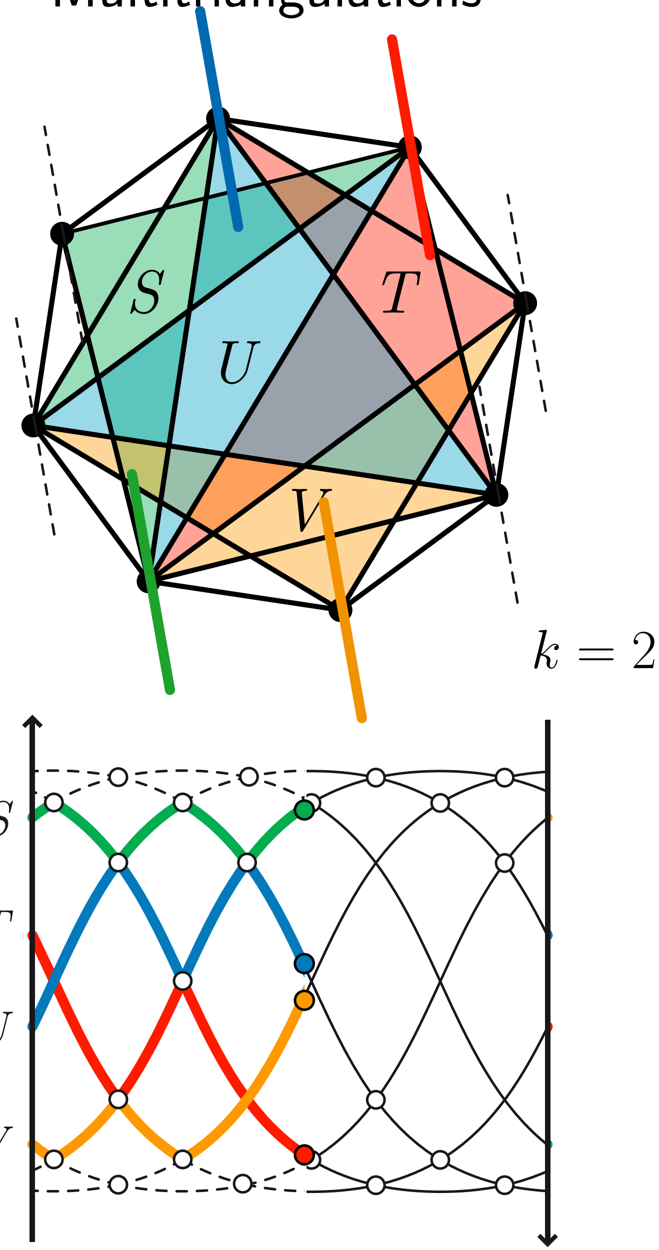
Triangulations



Pseudotriangulations



Multitriangulations



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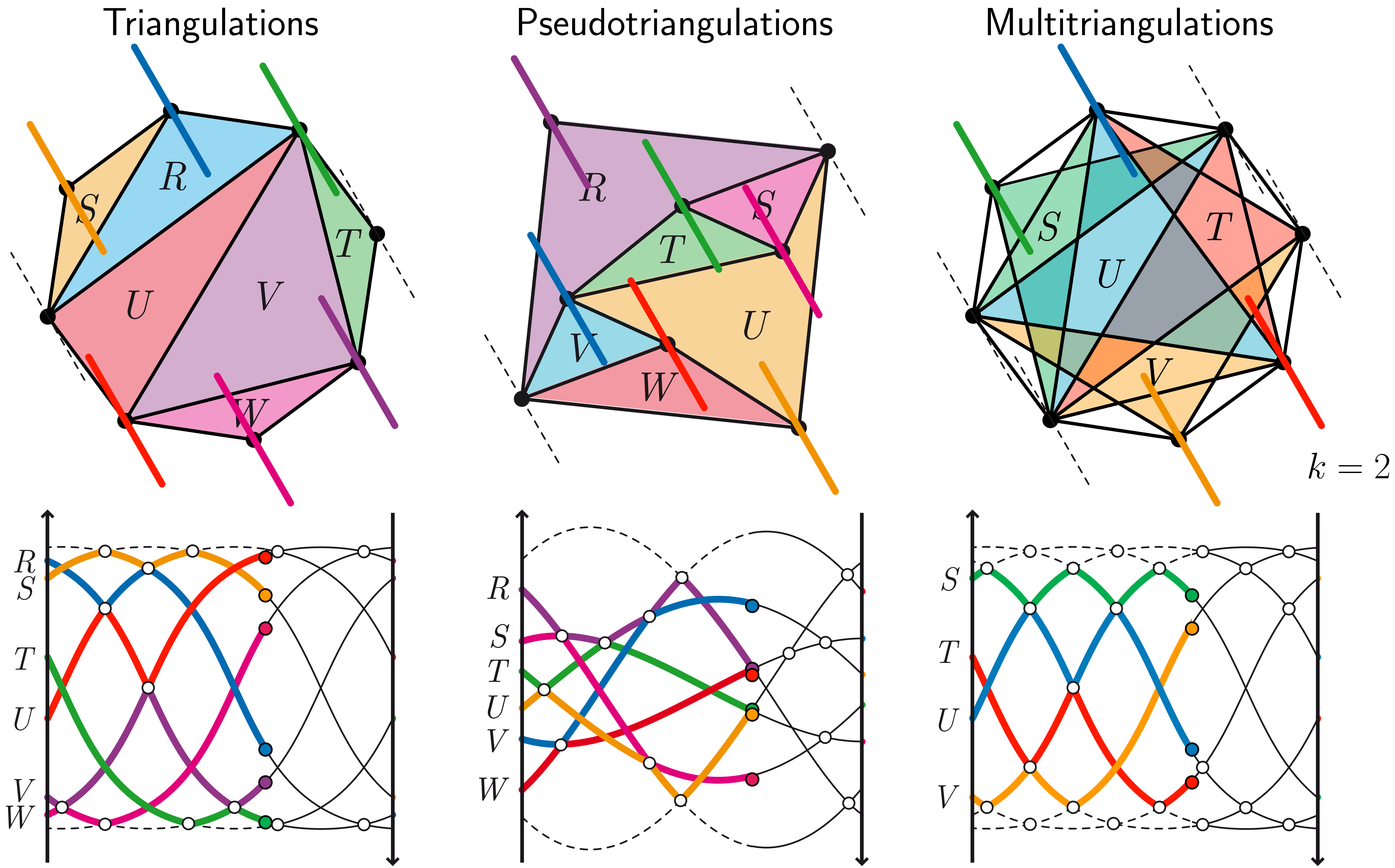

Triangulations
Pseudotriangulations
Multitriangulations
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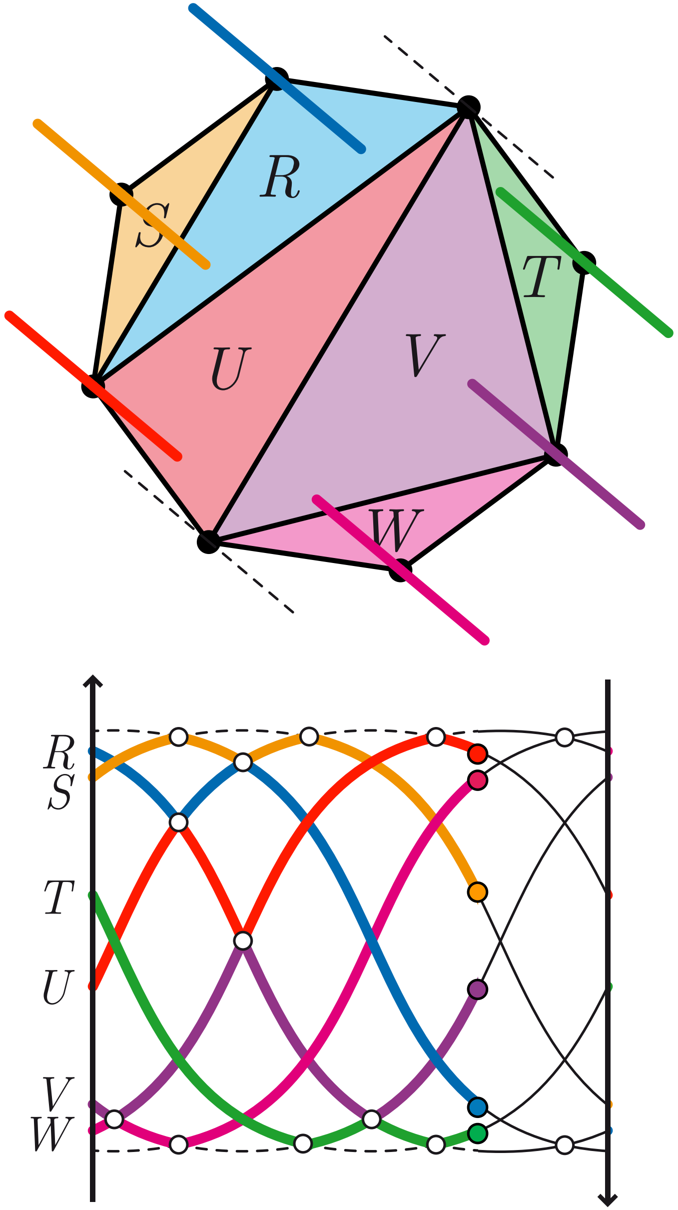
DUALITY

Triangulations



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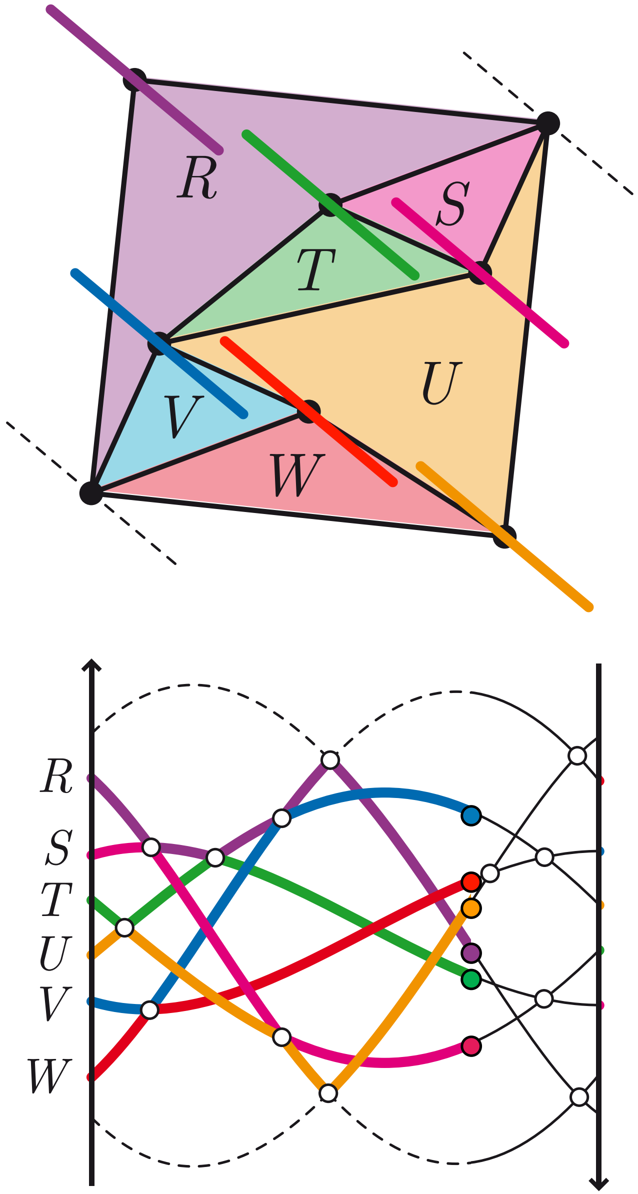
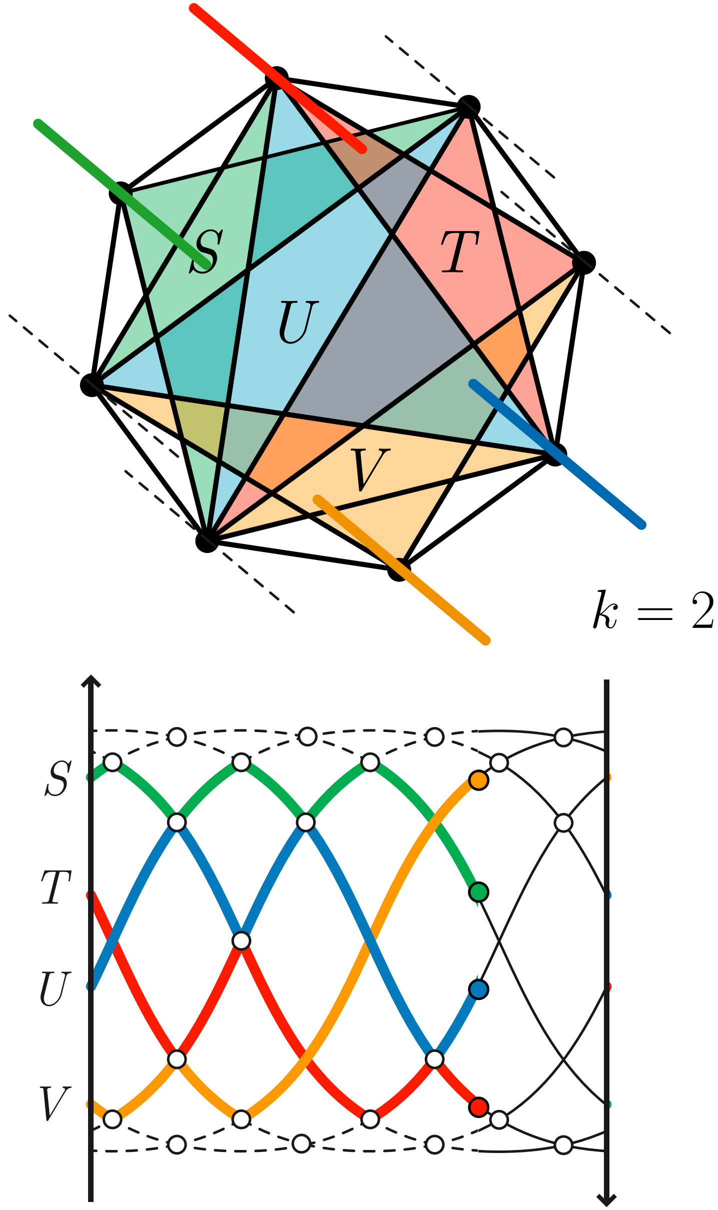


Multitriangulations



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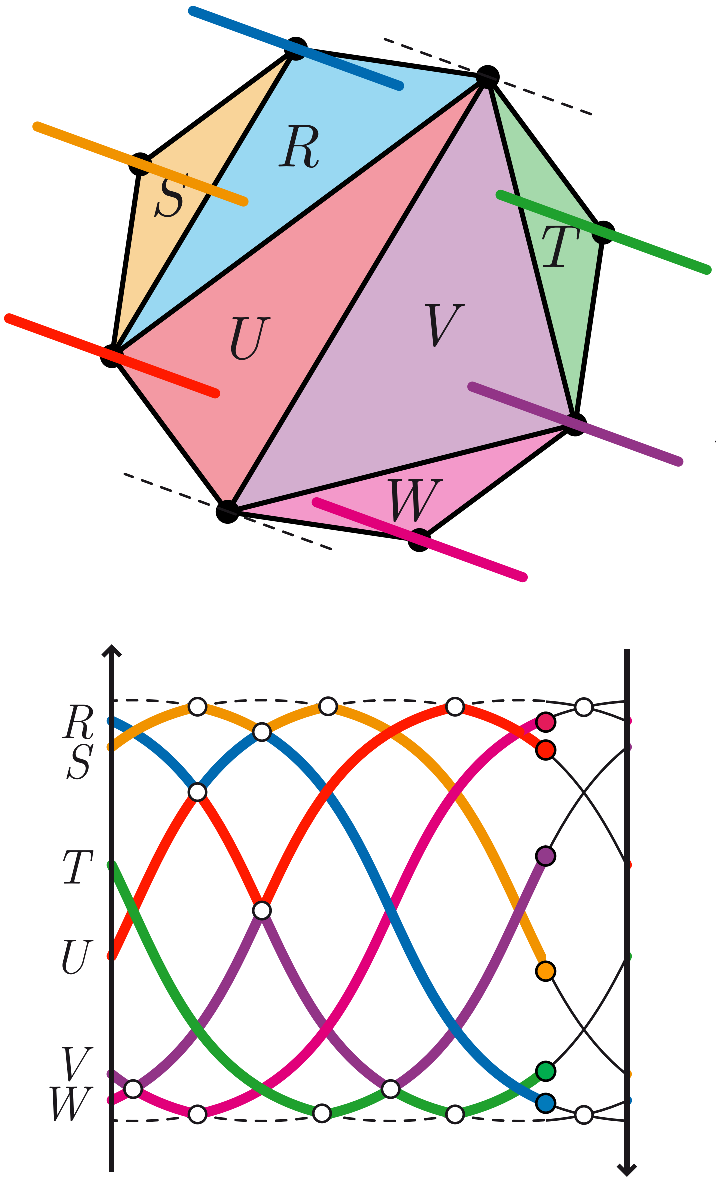
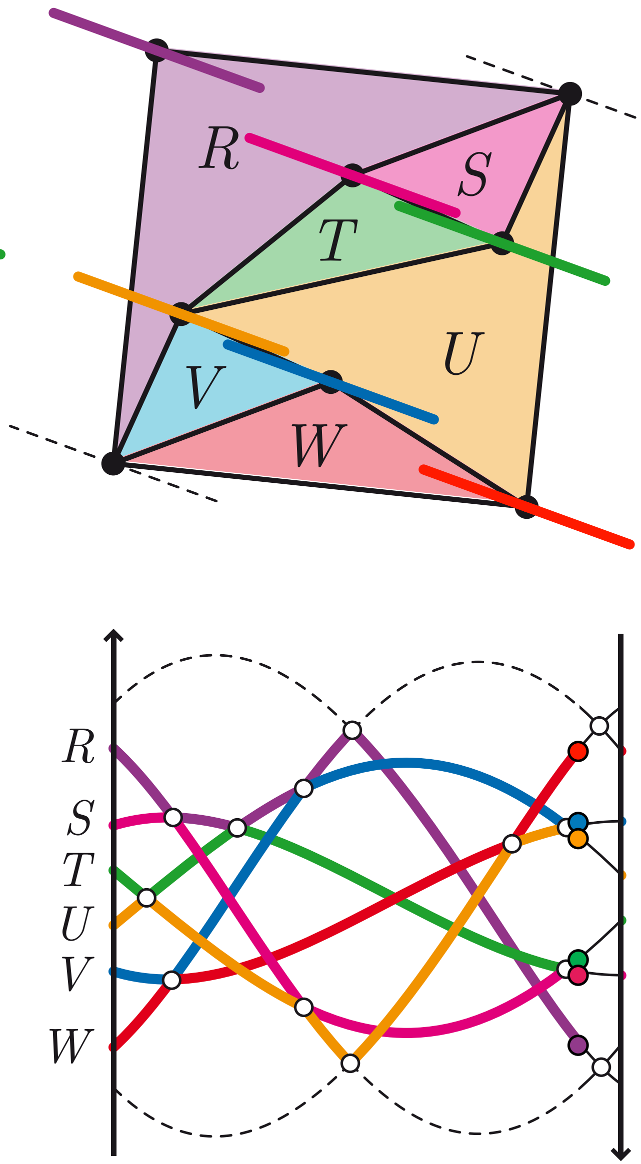
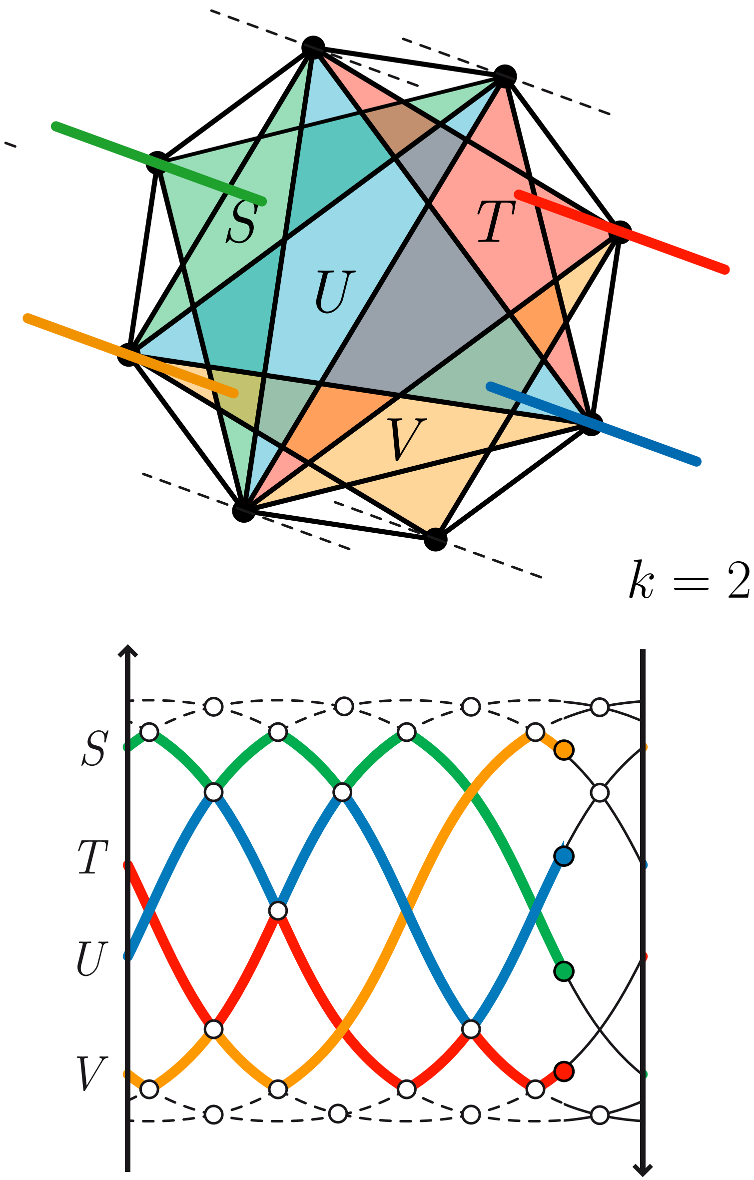
Triangulations



Pseudotriangulations



Multitriangulations



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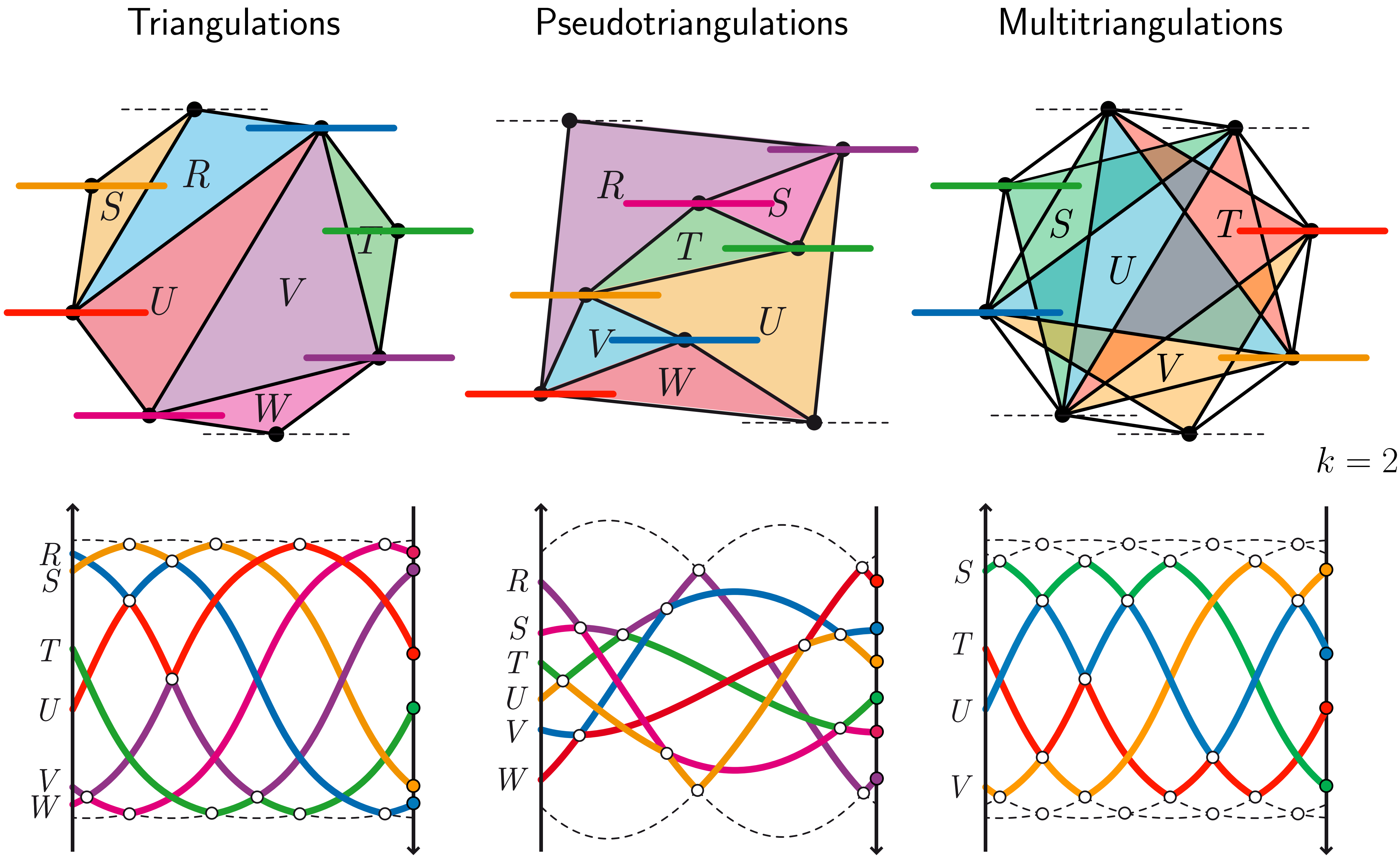

Triangulations
Pseudotriangulations
Multitriangulations
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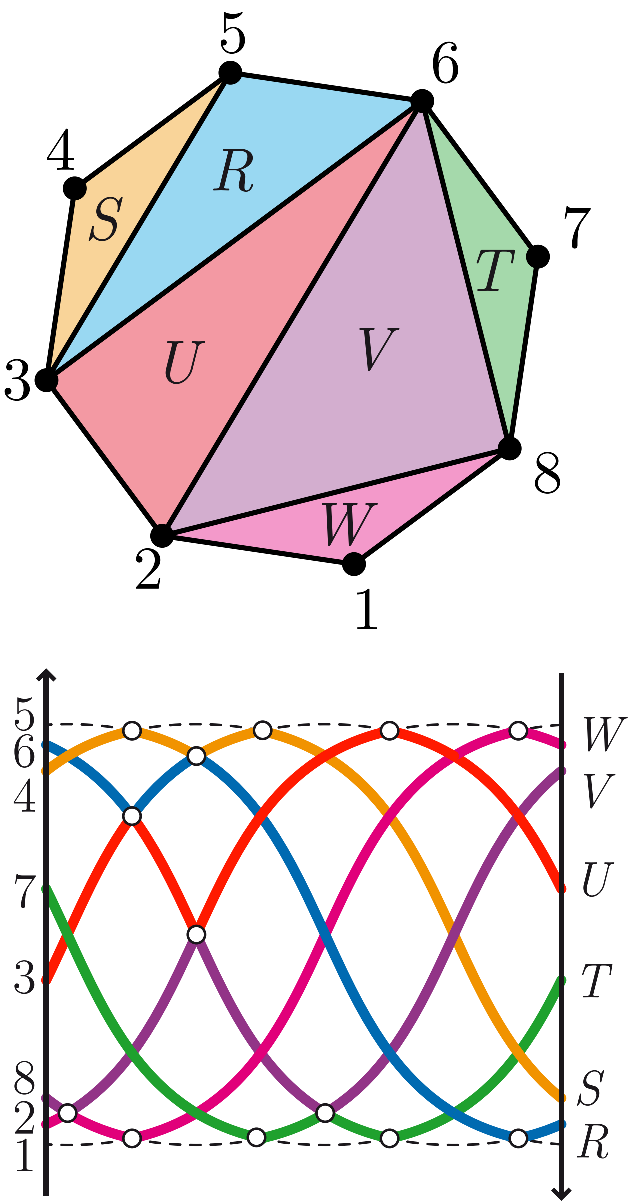
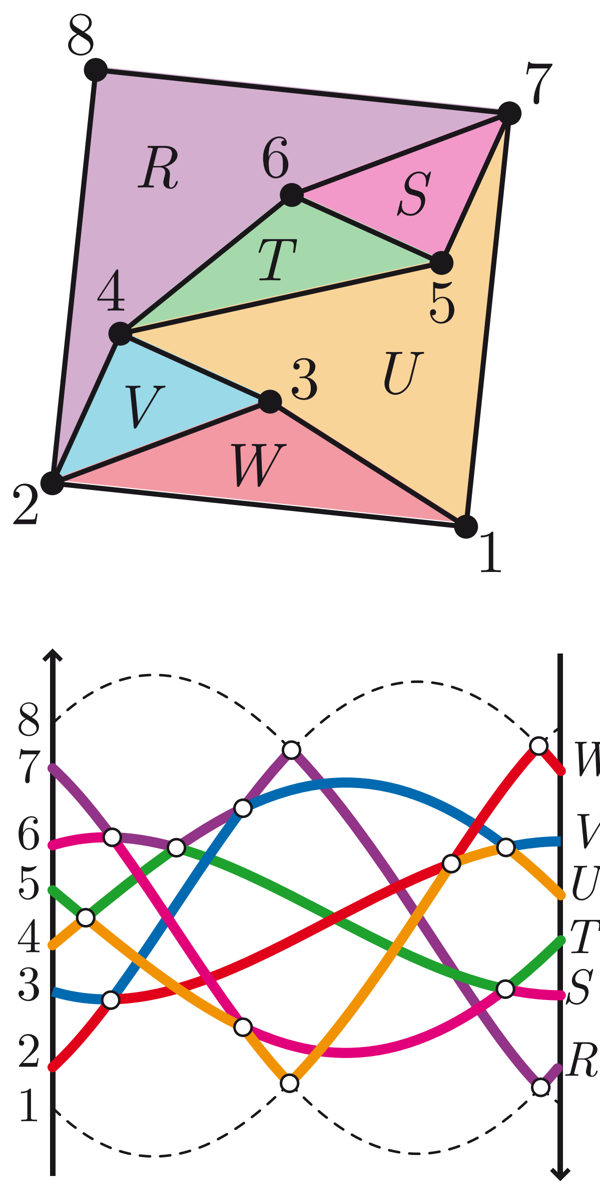
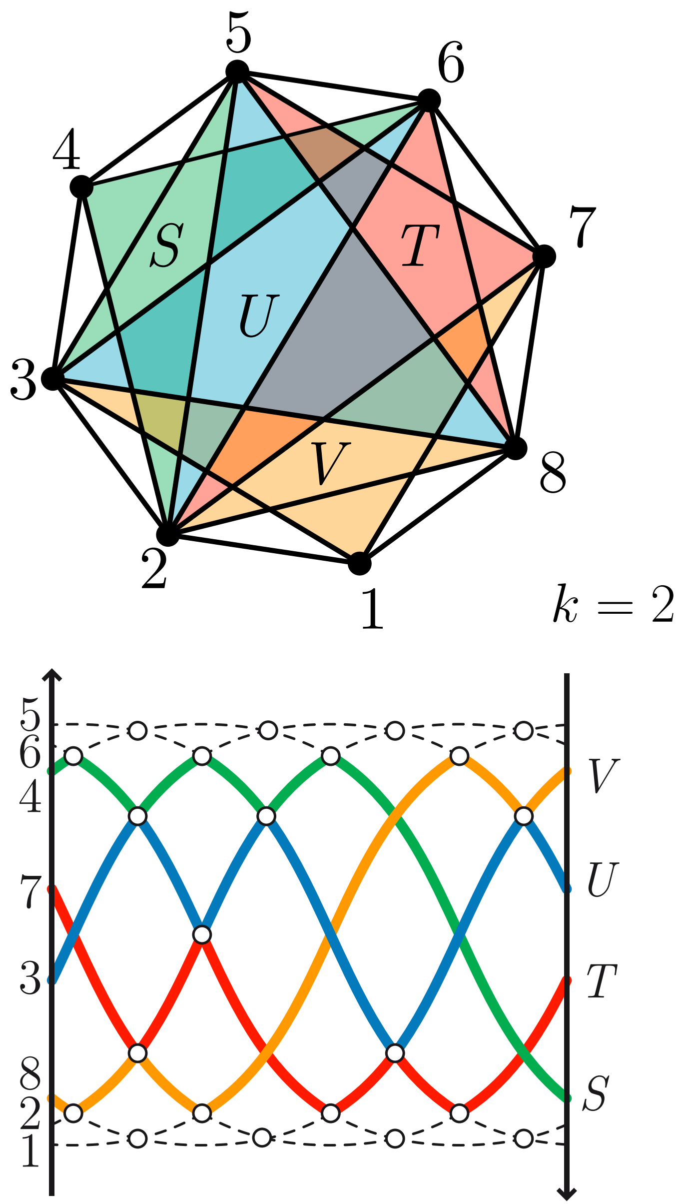
DUALITY

Triangulations	Pseudotriangulations	Multitriangulations



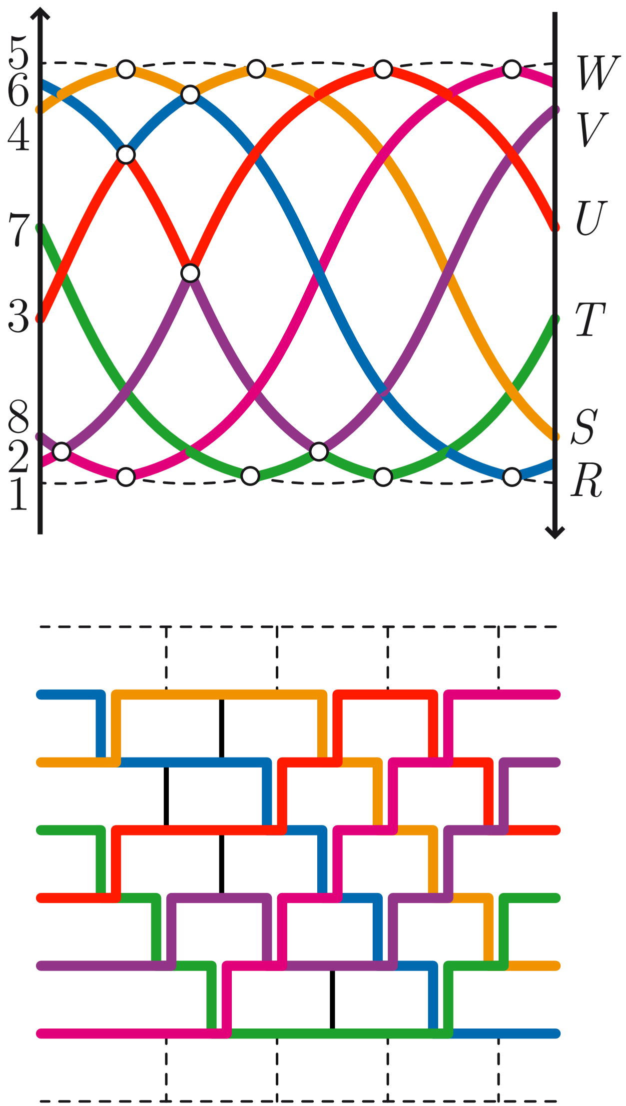




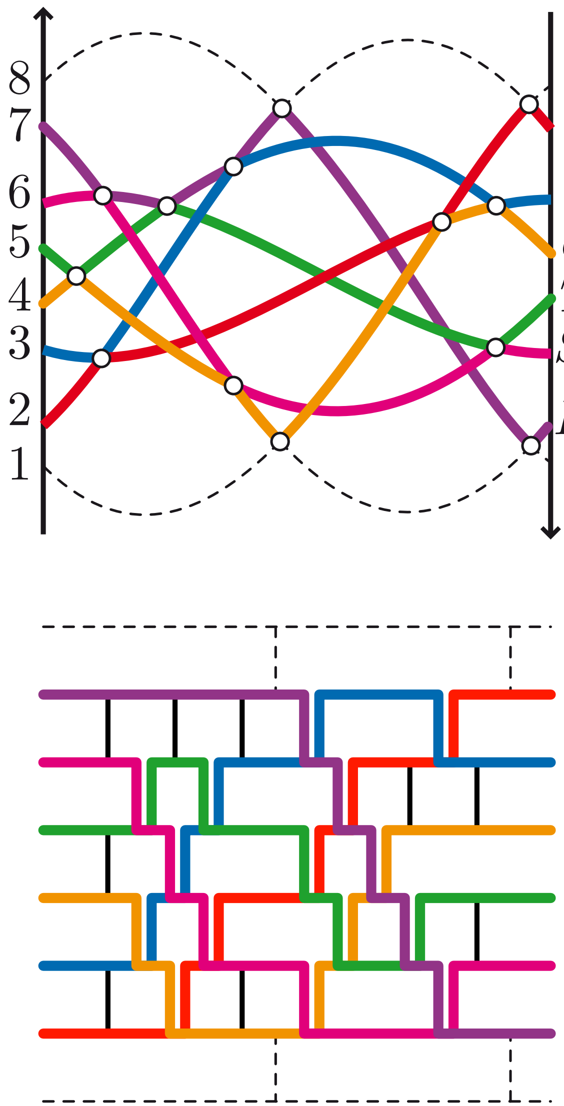
VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2010$^+$.

DUALITY

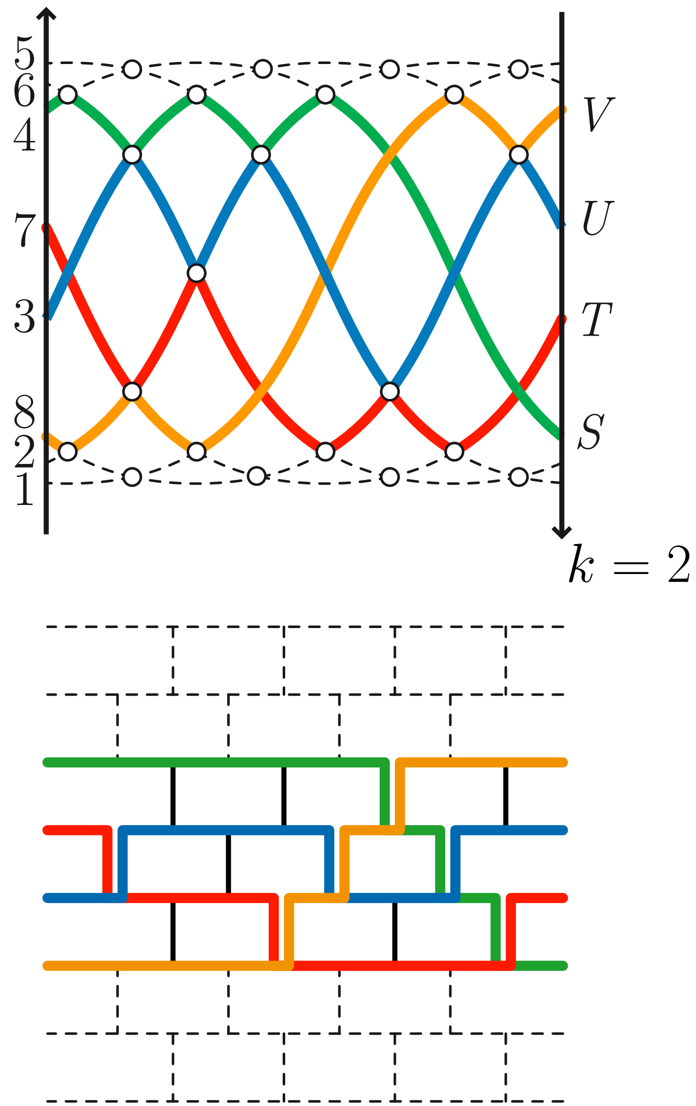
Triangulations



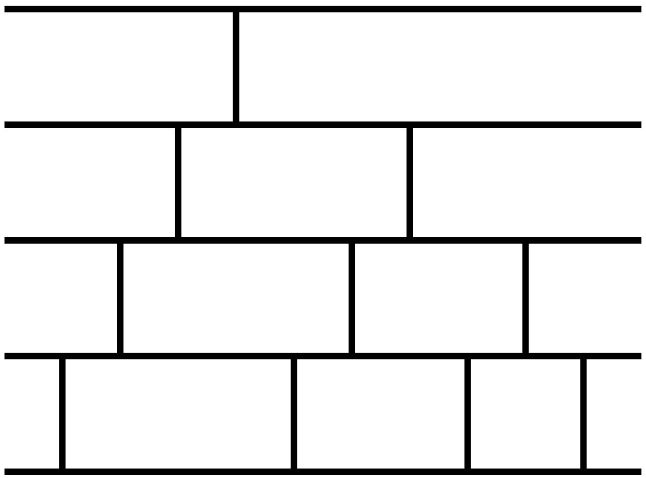
Pseudotriangulations



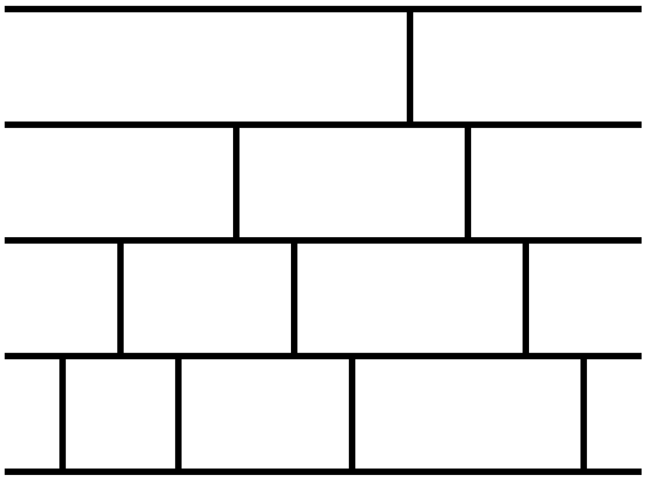
Multitriangulations



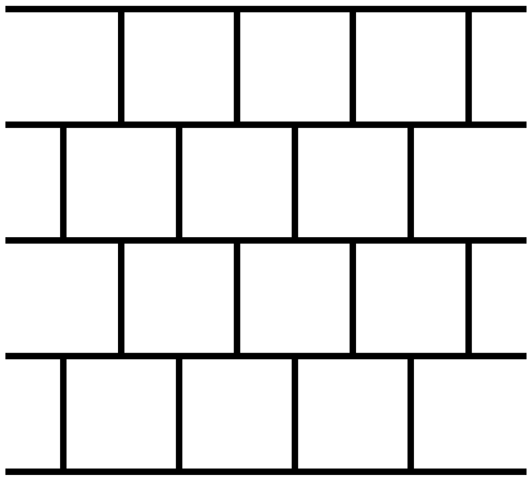
SORTING NETWORKS



Bubble sort



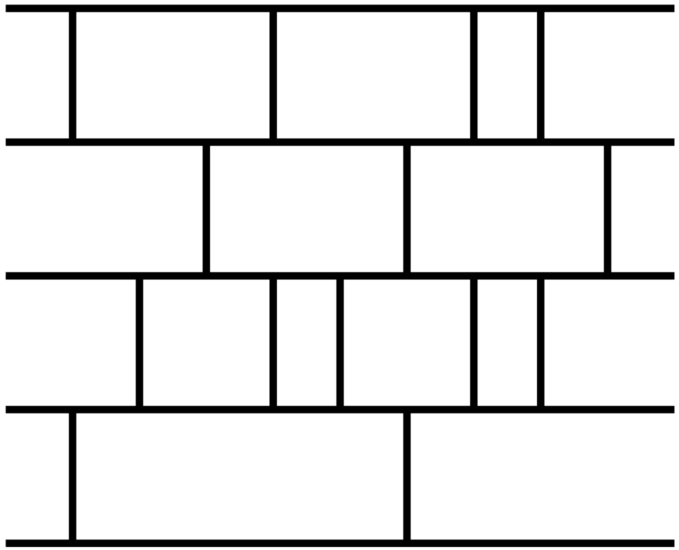
Insertion sort



Even-odd sorting

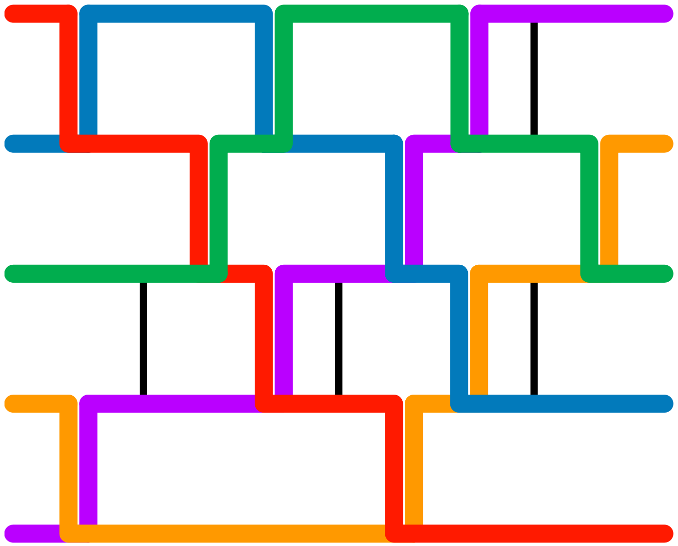
D. Knuth. The art of Computer Programming (Vol. 3, Sorting and Searching). 1997.

NETWORKS & PSEUDOLINE ARRANGEMENTS



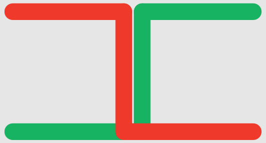
network $\mathcal{N}$ $= n$ horizontal levels and m vertical commutators.
bricks of $\mathcal{N}$ $=$ bounded cells.

NETWORKS & PSEUDOLINE ARRANGEMENTS



network $\mathcal{N}$ $=$ n horizontal levels and m vertical commutators.
bricks of $\mathcal{N}$ $=$ bounded cells.

pseudoline $=$ x-monotone path which starts at a level l and ends at the level $n+1-l$.

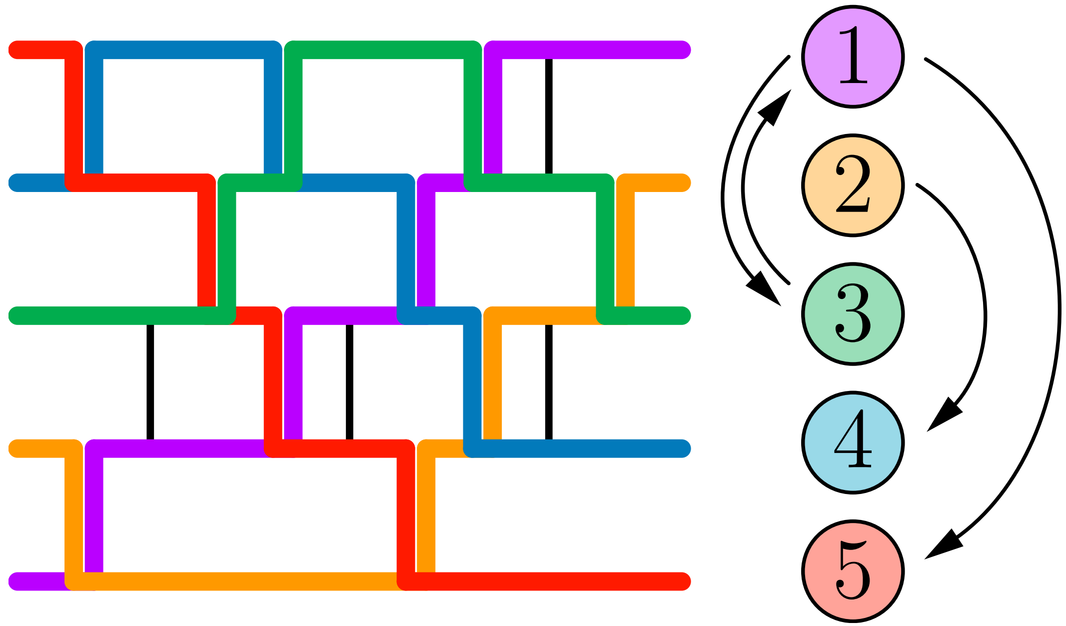
crossing $=$  contact $=$ 

pseudoline arrangement (with contacts) $=$ n pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, eventually some contacts, and no other intersection.

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

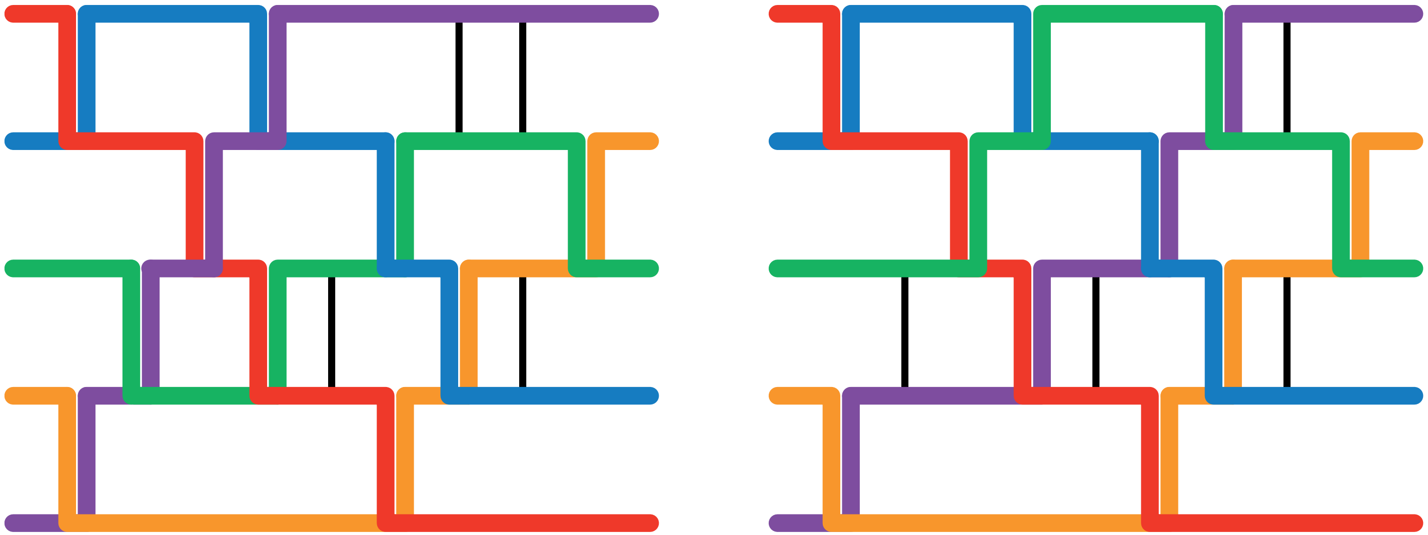
Contact graph $\Lambda^{\#}$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.



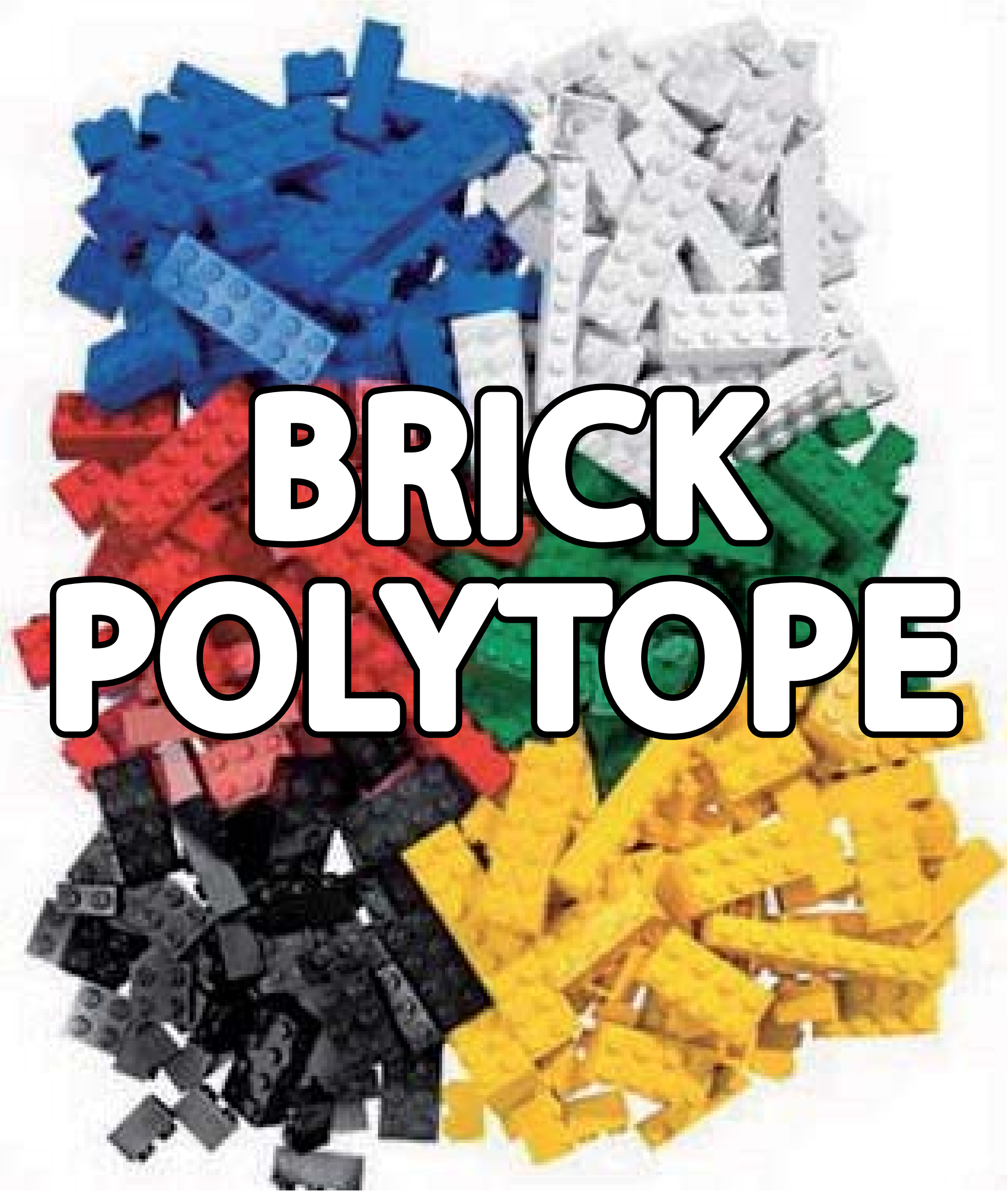
FLIPS

flip $=$ exchange a contact with the corresponding crossing.



THEOREM. Let $\mathcal{N}$ be a sorting network with n levels and m commutators. The graph of flips $G(\mathcal{N})$ is $\left(m - \binom{n}{2}\right)$-regular and connected.

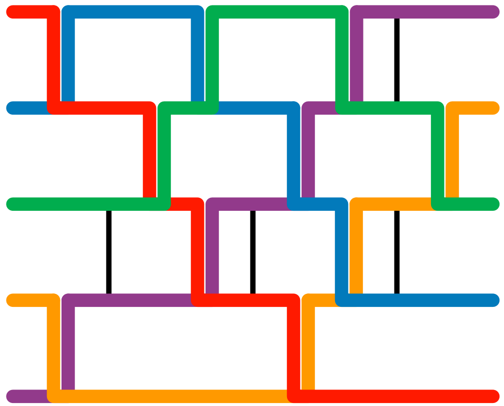
QUESTION. Is $G(\mathcal{N})$ the graph of a simple $\left(m - \binom{n}{2}\right)$-dimensional polytope?


BRICK
POLYTOPE

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$.

$\omega(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ.



Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$.

BRICK POLYTOPE

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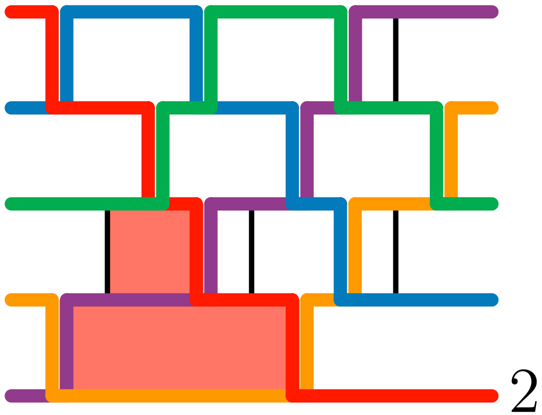

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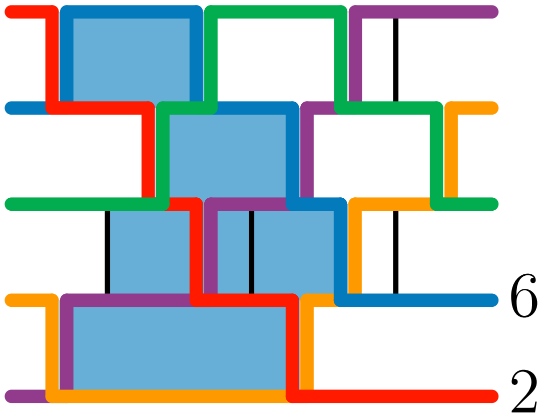

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.

BRICK POLYTOPE

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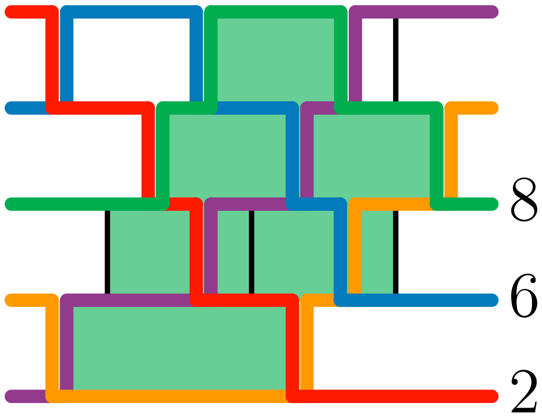

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.

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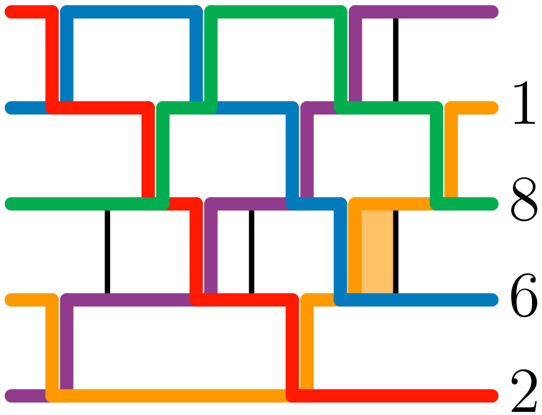

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N} \quad \longmapsto \quad$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$.

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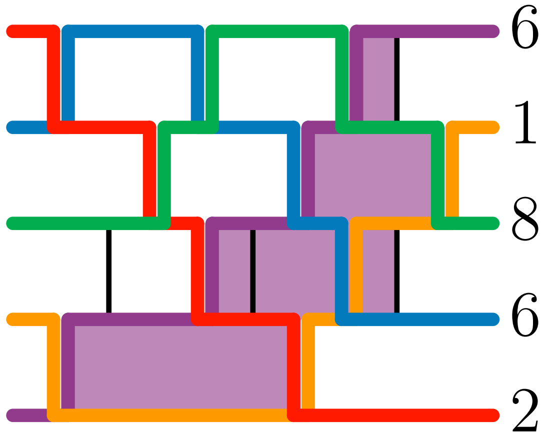

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$.

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$.
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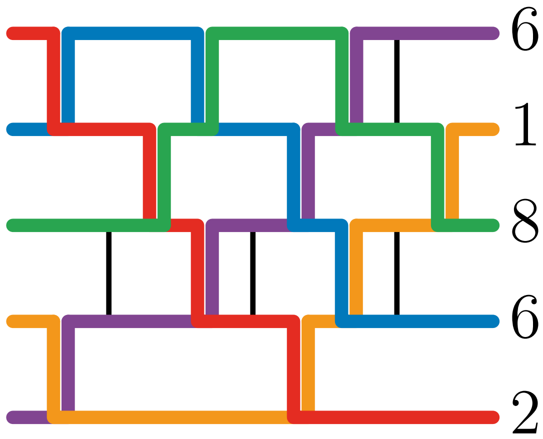


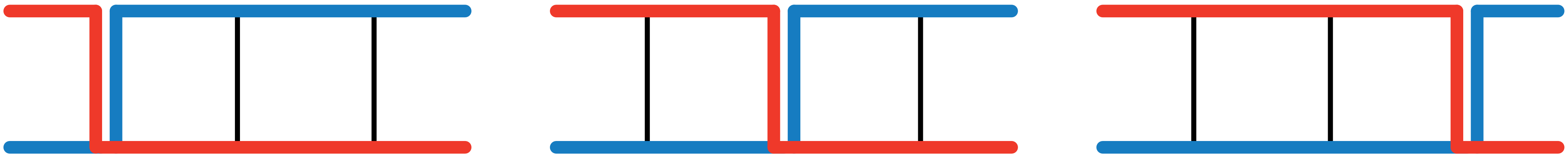
Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$.

REMARK. The brick polytope is not full-dimensional:

$$\Omega(\mathcal{N}) \subset \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; \sum_{i=1}^{n} x_i = \sum_{b \text{ brick of } \mathcal{N}} \text{depth}(b) \right\}.$$

EXAMPLE: 2-LEVELS NETWORKS

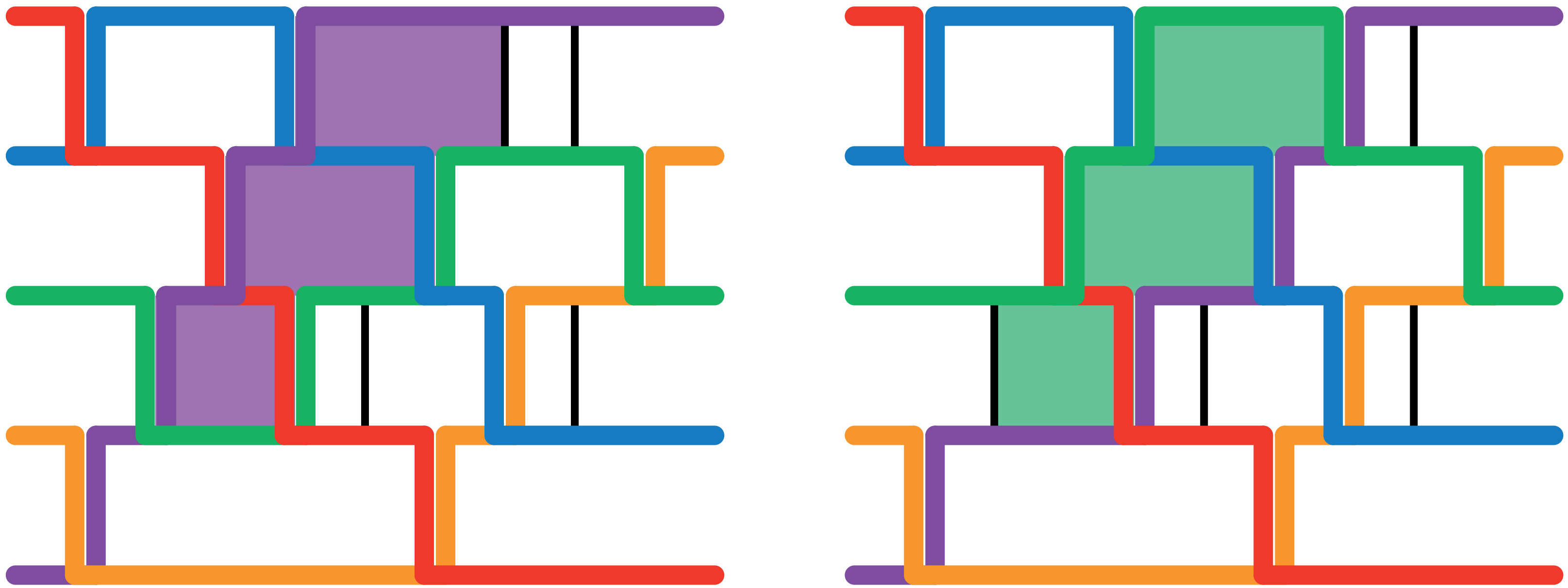
$\mathcal{X}_m$ = network with two levels and m commutators.



Graph of flips $G(\mathcal{X}_m)$ = complete graph K_m.

Brick polytope $\Omega(\mathcal{X}_m) = \text{conv}\left\{ \begin{pmatrix} m-i \\ i-1 \end{pmatrix} \,\middle|\, i \in [m] \right\} = \left[\begin{pmatrix} m-1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ m-1 \end{pmatrix} \right]$.

BRICK VECTORS AND FLIPS



REMARK. If Λ and Λ' are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their ith and jth pseudolines, then $\omega(\Lambda) - \omega(\Lambda') \in \mathbb{N}_{>0}\,(e_j - e_i)$.

INCIDENCE CONE OF A DIRECTED MULTIGRAPH

G directed (multi)graph $\longmapsto$ Incidence configuration $I(G) = \{e_j - e_i \mid (i,j) \in G\}$,
$\longmapsto$ Incidence cone $C(G)$ = cone generated by $I(G)$.

REMARK.
independant sets in $I(G)$ $\longleftrightarrow$ forests in G,
spanning sets of $\langle\, \mathbb{1} \mid x \,\rangle = 0$ $\longleftrightarrow$ connected spanning subgraphs of G,
basis of $\langle\, \mathbb{1} \mid x \,\rangle = 0$ $\longleftrightarrow$ spanning trees of G,
circuits in $I(G)$ $\longleftrightarrow$ simple cycles in G,
cocircuits in $I(G)$ $\longleftrightarrow$ minimal cuts in G,
and signs correspond to the orientations of the edges.

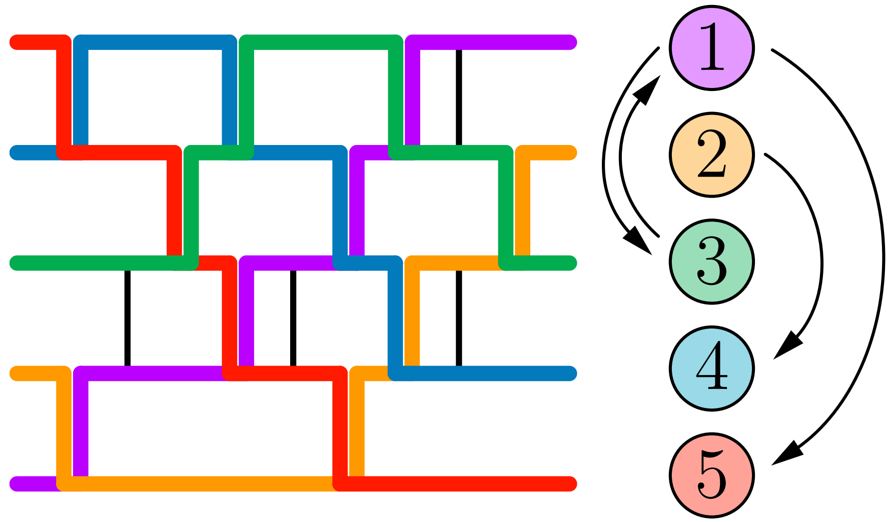
REMARK. H subgraph of G. Then $I(H)$ forms a k-face of $C(G)$ $\iff$ H has $n-k$ connected components and G/H is acyclic. In particular:

$C(G)$ is pointed $\longleftrightarrow$ G is acyclic,
facets of $C(G)$ $\longleftrightarrow$ complements of the minimal directed cuts of G.

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

Contact graph $\Lambda^\#$ of a pseudoline arrangement Λ =

- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.



THEOREM. The cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$ is the incidence cone $C(\Lambda^\#) = \text{cone}\left\{e_j - e_i \;\middle|\; (i,j) \in \Lambda^\#\right\}$ of the contact graph of Λ.

COMBINATORIAL DESCRIPTION

THEOREM. The cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$ is the incidence cone $C(\Lambda^{\#})$ of the contact graph of Λ:

$$\text{cone}\,\{\omega(\Lambda') - \omega(\Lambda) \mid \Lambda' \text{ supported by } \mathcal{N}\} = \text{cone}\left\{e_j - e_i \,\middle|\, (i,j) \in \Lambda^{\#}\right\}.$$

VERTICES OF $\Omega(\mathcal{N})$

The brick vector $\omega(\Lambda)$ is a vertex of $\Omega(\mathcal{N}) \iff$ the contact graph $\Lambda^{\#}$ is acyclic.

GRAPH OF $\Omega(\mathcal{N})$

The graph of the brick polytope is a subgraph of $G(\mathcal{N})$ whose vertices are the pseudoline arrangements with acyclic contact graphs.

FACETS OF $\Omega(\mathcal{N})$

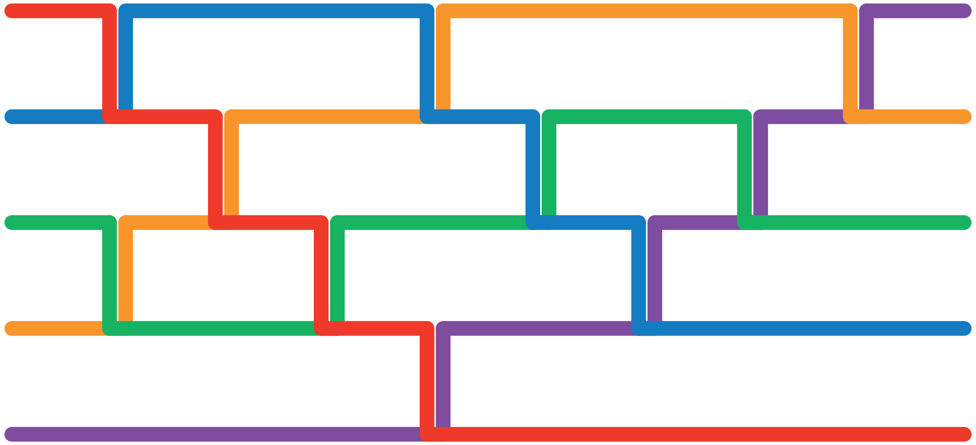
The facets of $\Omega(\mathcal{N})$ correspond to the minimal directed cuts of the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$.

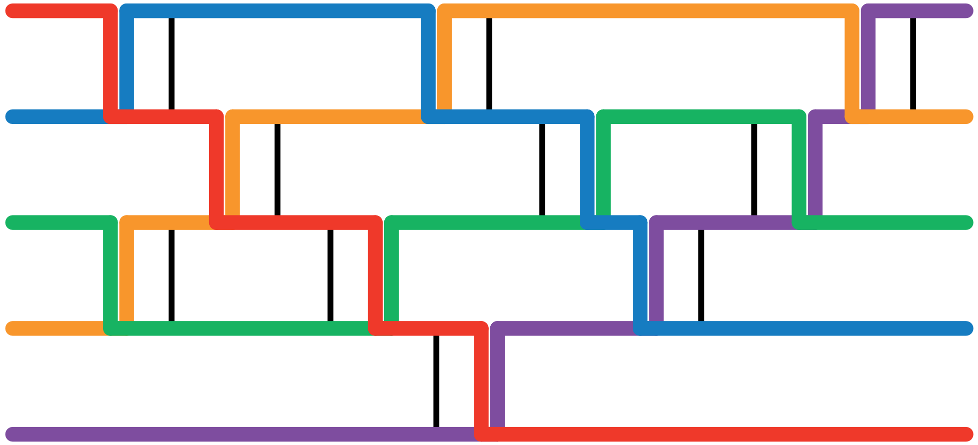
ASSOCIAHEDRA & PERMUTAHEDRA

DUPLICATED NETWORKS: PERMUTAHEDRA

Reduced network $=$ network with n levels and $\binom{n}{2}$ commutators.
It supports only one pseudoline arrangement.

Duplicated network Π $=$ network with n levels and $2\binom{n}{2}$ commutators obtained by duplicating each commutator of a reduced network.

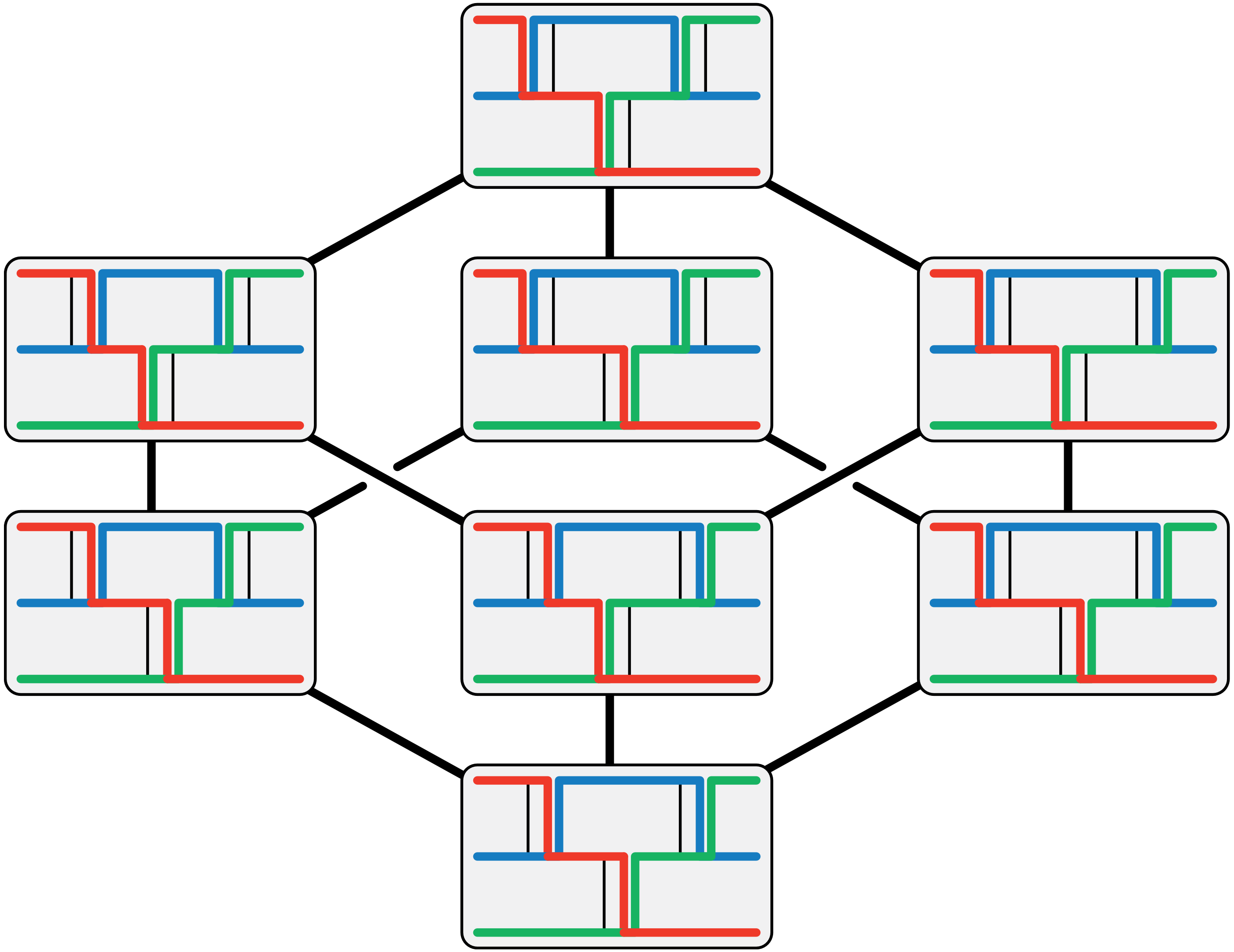




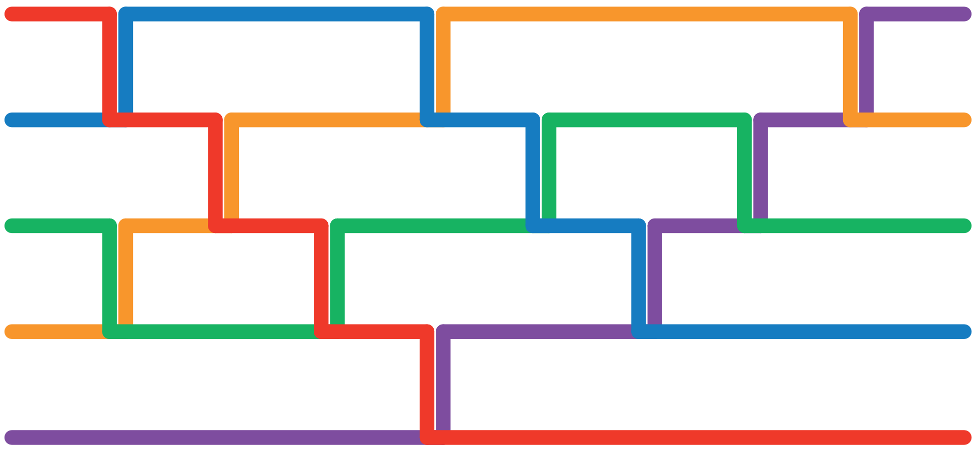
Any pseudoline arrangement supported by Π has one contact and one crossing among each pair of duplicated commutators.

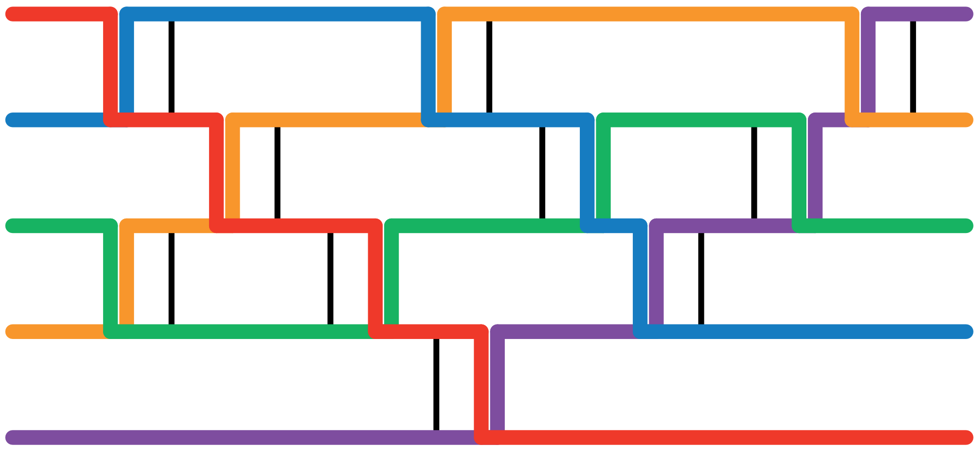
DUPLICATED NETWORKS: PERMUTAHEDRA

Graph of flips $G(\Pi) = \binom{n}{2}$-dimensional cube.



DUPLICATED NETWORKS: PERMUTAHEDRA





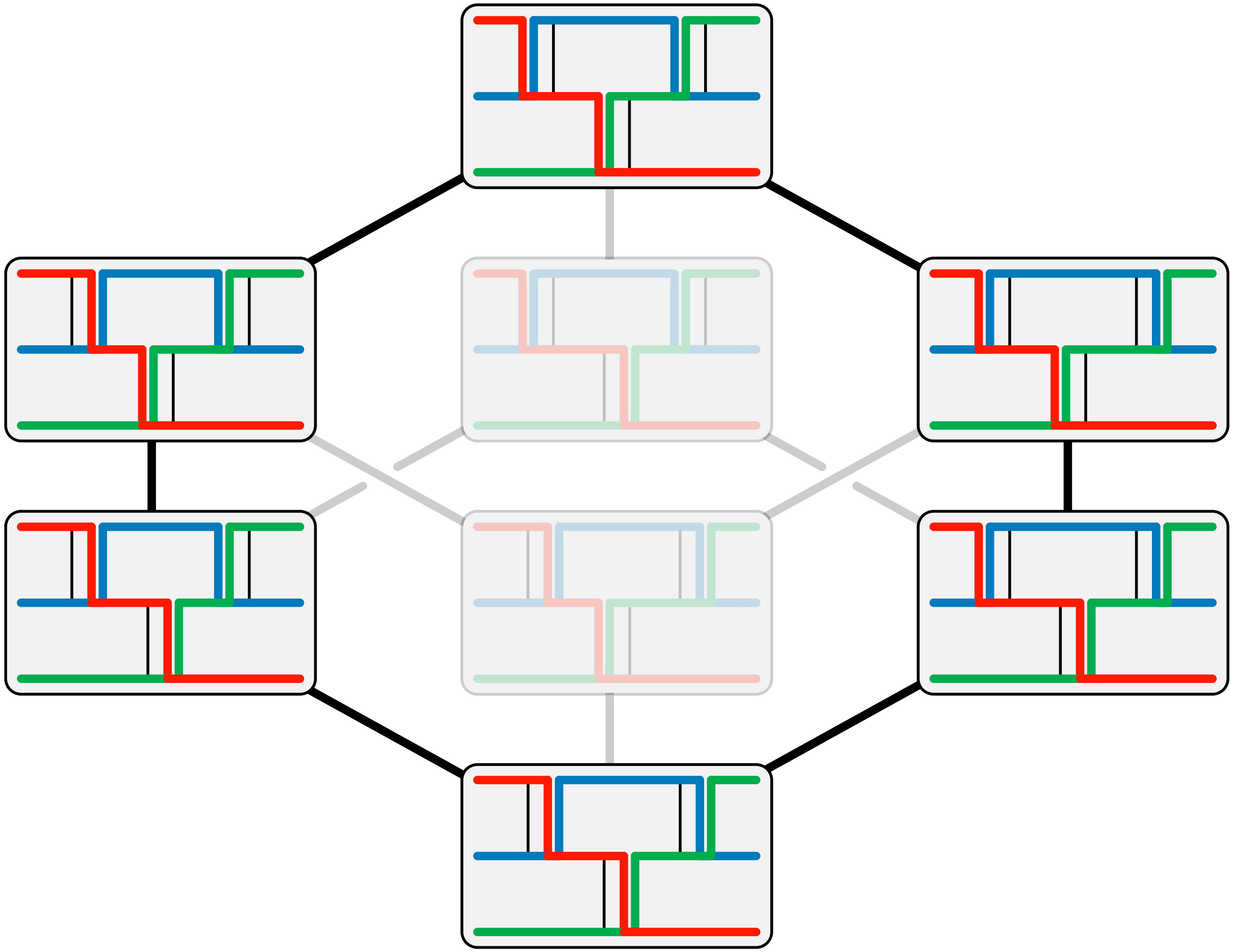
Any pseudoline arrangement supported by Π has one contact and one crossing among each pair of duplicated commutators. $\Longrightarrow$ The contact graph $\Lambda^{\#}$ is a tournament.

Vertices of $\Omega(\Pi)$	$\Longleftrightarrow$	acyclic tournaments	$\Longleftrightarrow$	permutations of $[n]$,
Facets of $\Omega(\Pi)$	$\Longleftrightarrow$	cuts in a tournament	$\Longleftrightarrow$	ordered bipartitions of $[n]$.

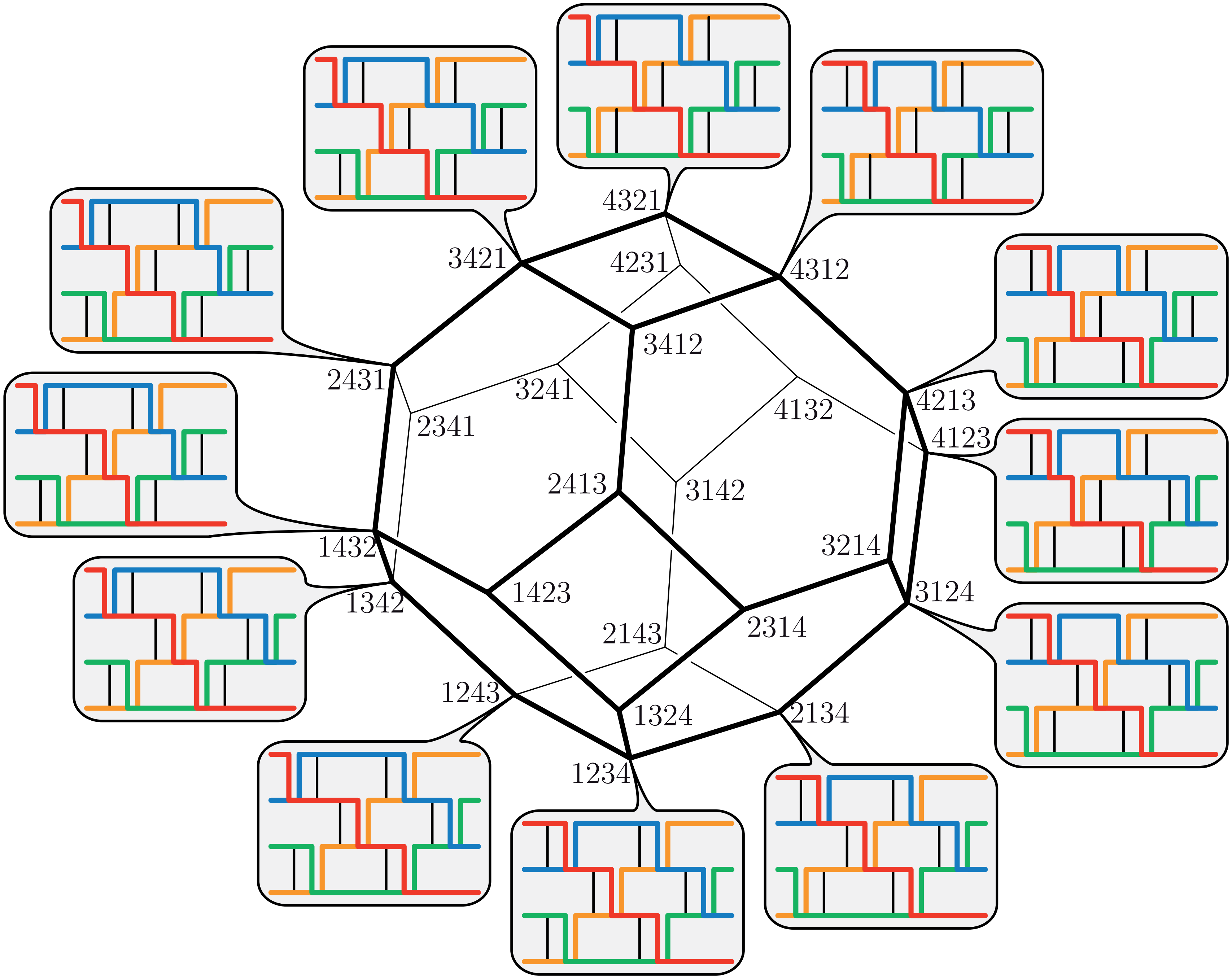
Brick polytope $\Omega(\Pi) =$ permutahedron.

DUPLICATED NETWORKS: PERMUTAHEDRA

Brick polytope $\Omega(\Pi)$ = permutahedron.

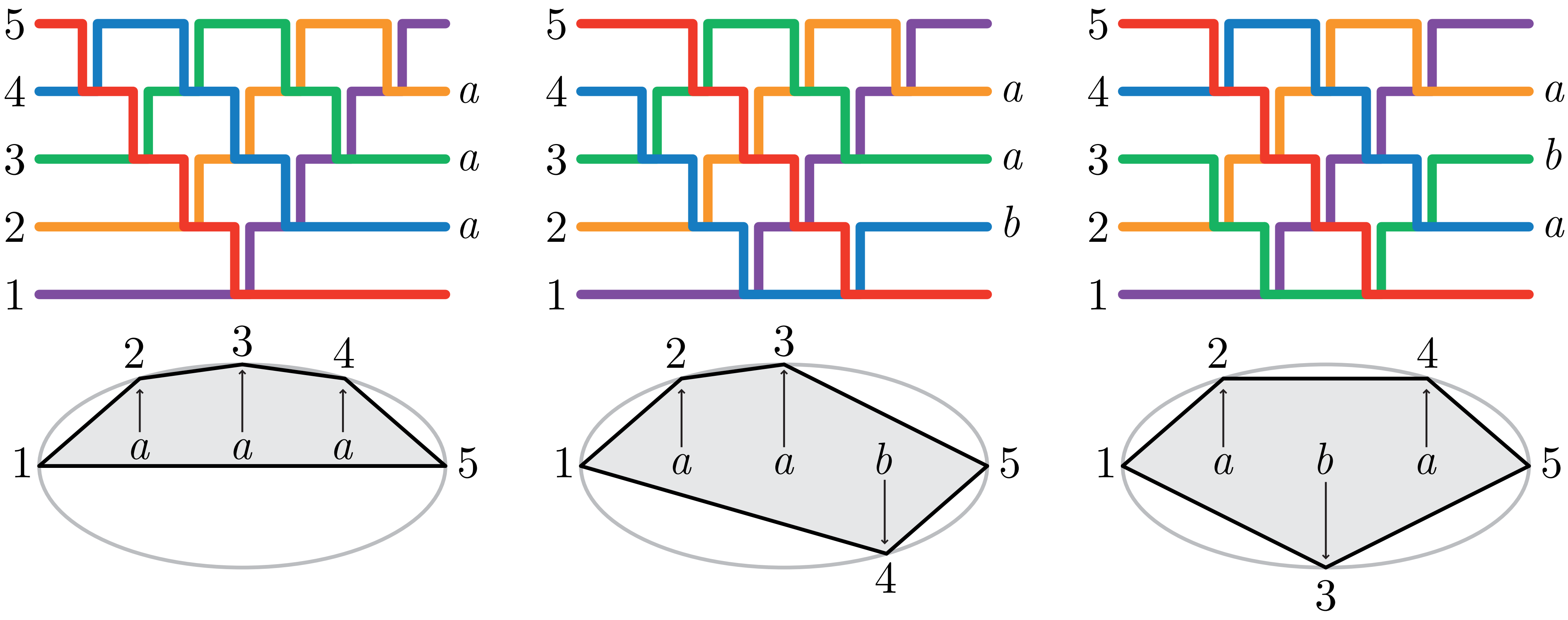


DUPLICATED NETWORKS: PERMUTAHEDRA



ALTERNATING NETWORKS: ASSOCIAHEDRA

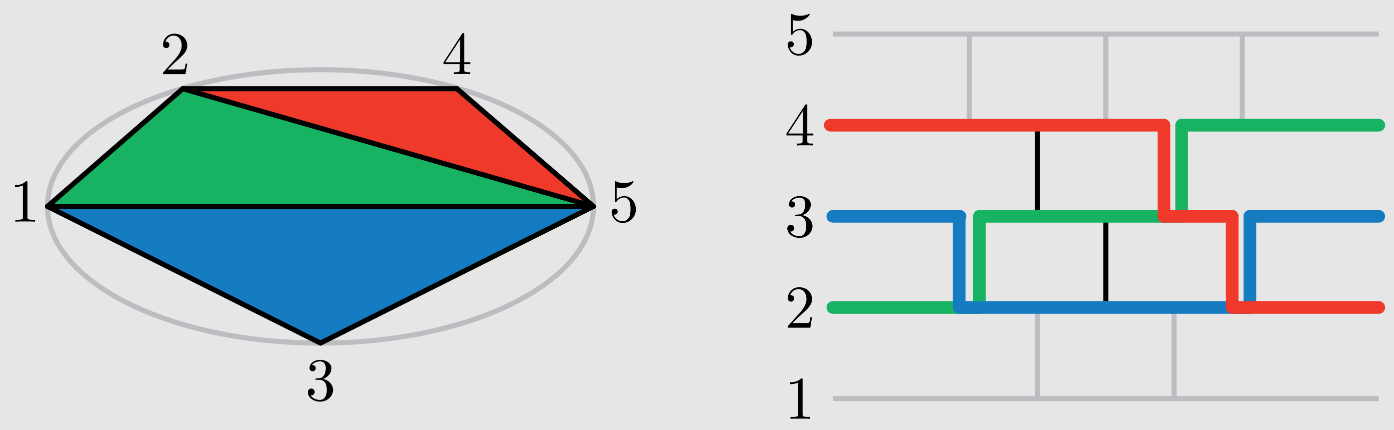
For $x \in \{a,b\}^{n-2}$, we define a reduced alternating network $\mathcal{N}_x$ and a polygon $\mathcal{P}_x$.



$\mathcal{N}_x$ is the dual pseudoline arrangement of the polygon $\mathcal{P}_x$.

ALTERNATING NETWORKS: ASSOCIAHEDRA

THEOREM. There is a duality between the pseudoline arrangements supported by $\mathcal{N}_x^1$ and the triangulations of the polygon $\mathcal{P}_x$.



T triangulation of $\mathcal{P}_x$ $\longleftrightarrow$ T^* pseudoline arrangement supported by $\mathcal{N}_x^1$

Δ triangle of T $\longleftrightarrow$ Δ^* pseudoline of T^*

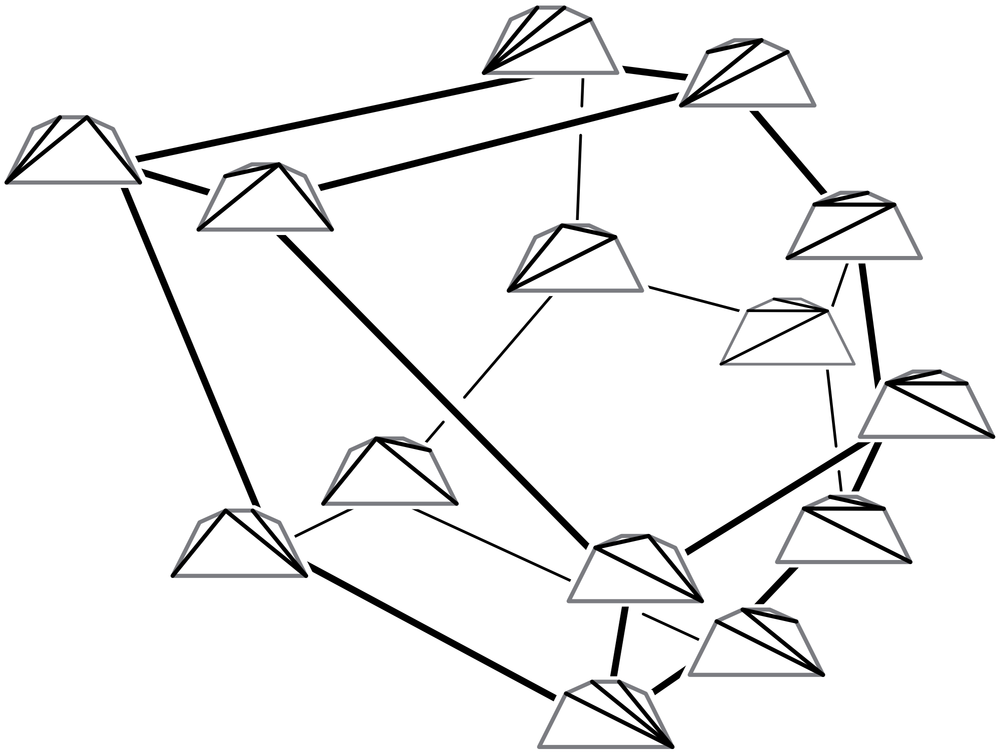
e common edge of Δ and Δ' $\longleftrightarrow$ e^* contact between Δ^* and Δ'^*

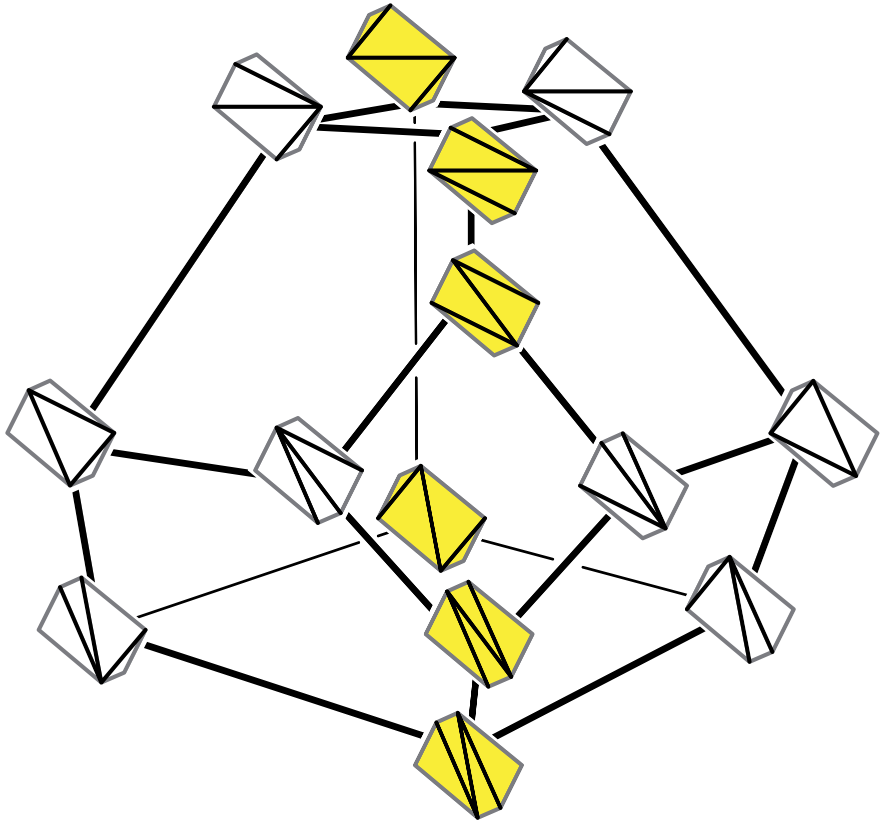
f common bissector of Δ and Δ' $\longleftrightarrow$ f^* crossing between Δ^* and Δ'^*

COROLLARY. (i) The graph of flips $G(\mathcal{N}_x^1)$ is (isomorphic to) the graph of flips $G(\mathcal{P}_x)$.
(ii) The contact graph $(T^*)^{\#}$ is (isomorphic to) the dual binary tree of T.

HOHLWEG & LANGE'S ASSOCIAHEDRA

THEOREM. For any word $x \in \{a,b\}^{n-2}$, the simplicial complex of crossing-free sets of internal diagonals of the convex n-gon $\mathcal{P}_x$ is (isomorphic to) the boundary complex of the polar of the brick polytope $\Omega(\mathcal{N}_x^1)$.





REMARK. Up to translation, we obtain Hohlweg & Lange's associahedra.

C. Hohlweg & C. Lange, Realizations of the associahedron and cyclohedron, 2007.

THANK YOU