Signed tree associahedra

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Spines

T a tree on a signed ground set $\mathsf{V} = \mathsf{V}^- \sqcup \mathsf{V}^+$.

Spine on T = directed and labeled tree S such that

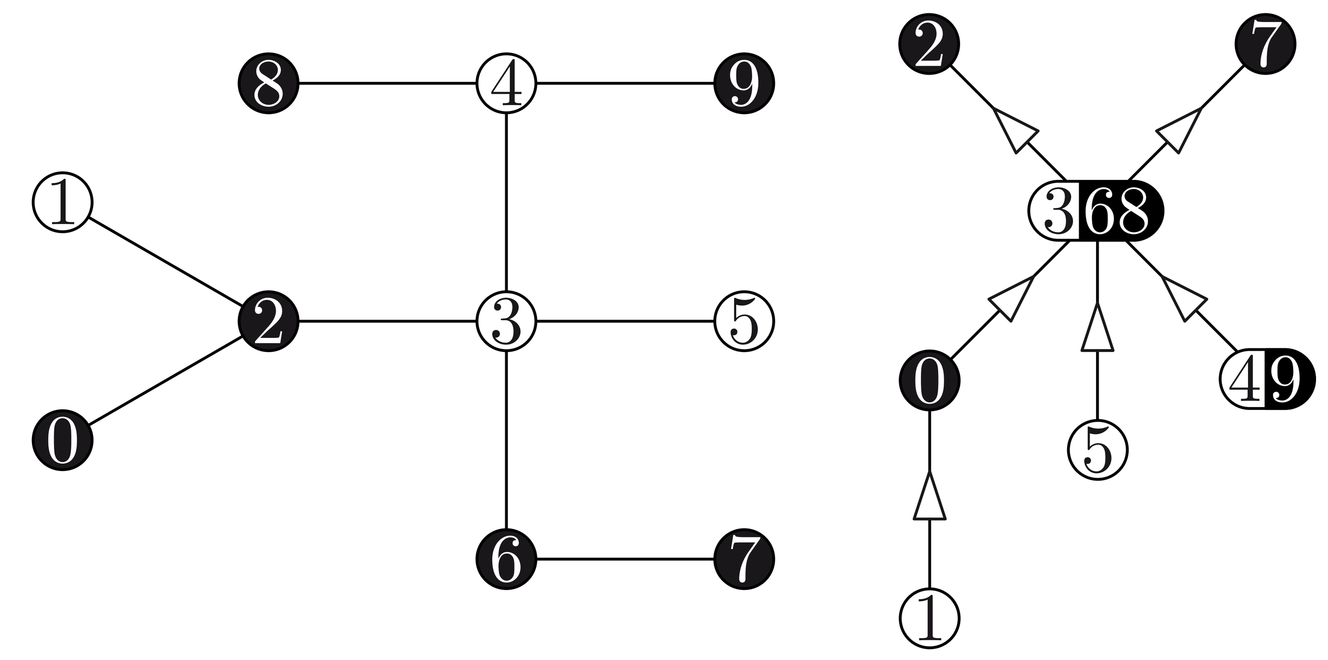
- the labels of the nodes of S form a partition of the signed ground set V,
- at a node labeled by $U = U^- \sqcup U^+$, the source label sets of the incoming arcs are subsets of distinct connected components of $\mathsf{T} \smallsetminus U^-$, and the sink label sets of the outgoing arcs are subsets of distinct connected components of $\mathsf{T} \smallsetminus U^+$.

Spine poset $\mathcal{S}(\mathsf{T})$ = poset of arc contractions on signed spines of T.

Prop. *The spine poset $\mathcal{S}(\mathsf{T})$ is a pure graded poset of rank $|\mathsf{V}|$.*

Signed nested complex = simplicial complex $\mathcal{N}(\mathsf{T}) = \{\mathsf{N}(\mathsf{S}) \mid \mathsf{S} \in \mathcal{S}(\mathsf{T})\}$,
where $\mathsf{N}(\mathsf{S})$ = collection of source sets of S.

Exm. $\mathsf{V}^- = \{1,3,4,5\}$ $\mathsf{V}^+ = \{0,2,6,7,8,9\}$



graph G spine S on G

Spine fan

Ambiant space $\mathbb{H} = \left\{ \mathbf{x} \in \mathbb{R}^{\mathsf{V}} \mid \sum_{v \in \mathsf{V}} x_v = \binom{|\mathsf{V}|+1}{2} \right\}$.
Cone $\mathsf{C}(\mathsf{S})$ of a spine $\mathsf{S} = \{\mathbf{x} \in \mathbb{H} \mid x_u \le x_v \text{ for all } u \to v \text{ in } \mathsf{S}\}$.

Theo. *The collection of cones $\mathcal{F}(\mathsf{T}) = \{\mathsf{C}(\mathsf{S}) \mid \mathsf{S} \in \mathcal{S}(\mathsf{T})\}$ defines a complete simplicial fan on $\mathbb{H}$, called the* spine fan *of* T.

The spine fan $\mathcal{F}(\mathsf{T})$ coarsens the braid fan on $\mathbb{H}$. It defines a map κ from linear orders on V to maximal spines on T.

Prop. *The fibers of κ are the classes of* T-congruence *defined by $XuvY \equiv_{\mathsf{T}} XvuY$ iff there is $w \in \mathsf{V}$ in between u and v in* T *and such that $w \in X \cap \mathsf{V}^+$ or $w \in Y \cap \mathsf{V}^-$.*

Signed tree associahedron

Theo. *The spine fan $\mathcal{F}(\mathsf{T})$ is the normal fan of the* signed tree associahedron $\mathsf{Asso}(\mathsf{T})$ *with*

- *a vertex $\mathbf{a}(\mathsf{S}) \in \mathbb{R}^{\mathsf{V}}$ for each maximal $\mathsf{S} \in \mathcal{S}(\mathsf{T})$, with coordinates*

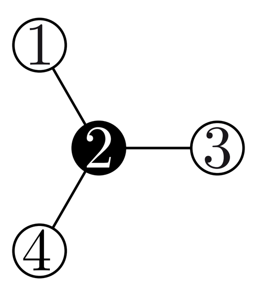
$$\mathbf{a}(\mathsf{S})_v = \begin{cases} \big| \{\pi \in \Pi(\mathsf{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathsf{V}^- \\ |\mathsf{V}| + 1 - \big| \{\pi \in \Pi(\mathsf{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathsf{V}^+ \end{cases}$$

where r_v = unique incoming (outgoing) arc at $v \in \mathsf{V}^-$ ($v \in \mathsf{V}^+$),
$\Pi(\mathsf{S})$ = {(undirected) paths in S*},*

- *a facet for each $B \in \bigcup_{\mathsf{S} \in \mathcal{S}(\mathsf{T})} \mathsf{N}(\mathsf{S})$ defined by the half-space*

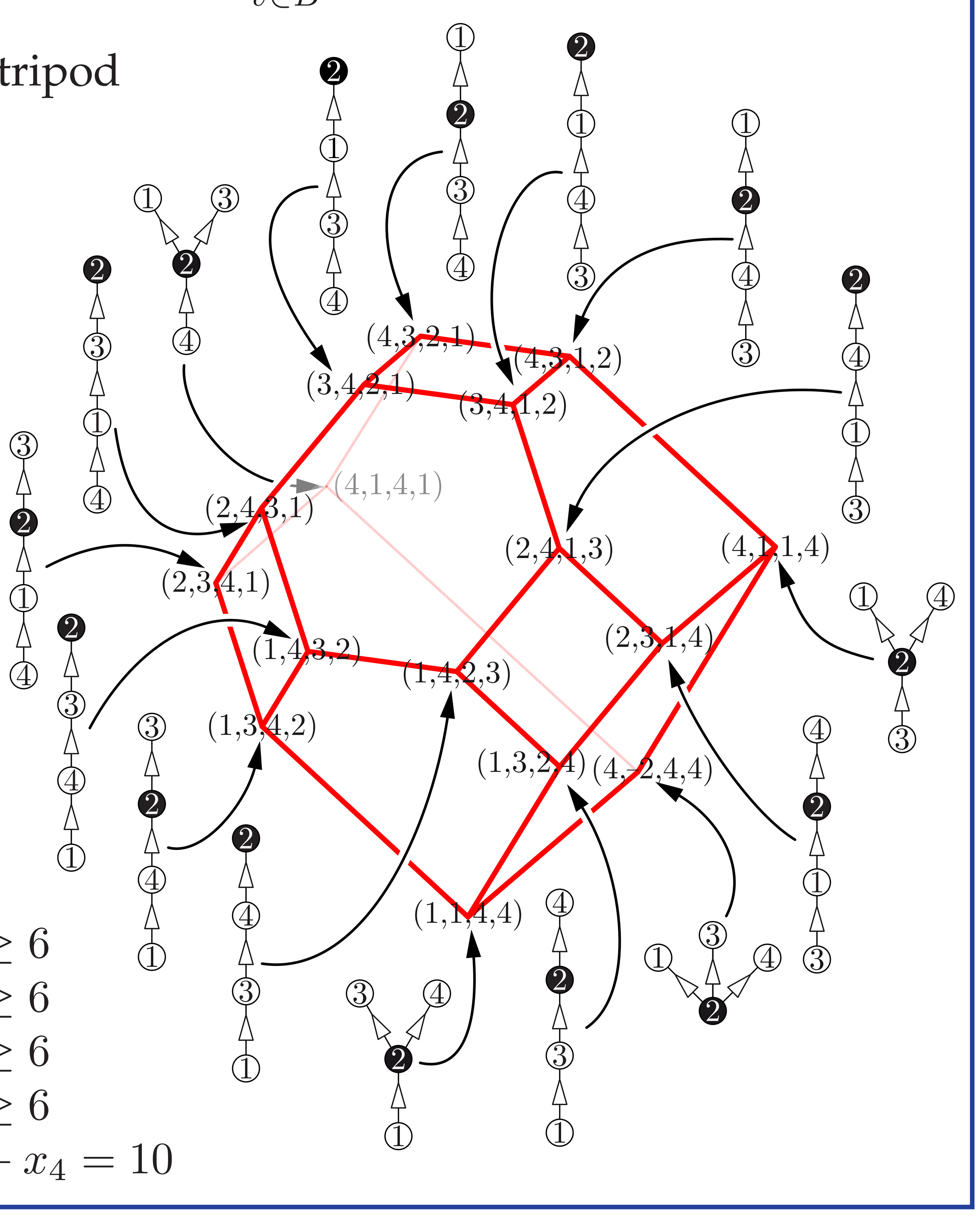
$$\left\{ \mathbf{x} \in \mathbb{R}^{\mathsf{V}} \;\middle|\; \sum_{v \in B} x_v \ge \binom{|B|+1}{2} \right\}.$$

Exm. For the tripod



$\mathsf{V}^- = \{1,3,4\}$
$\mathsf{V}^+ = \{2\}$

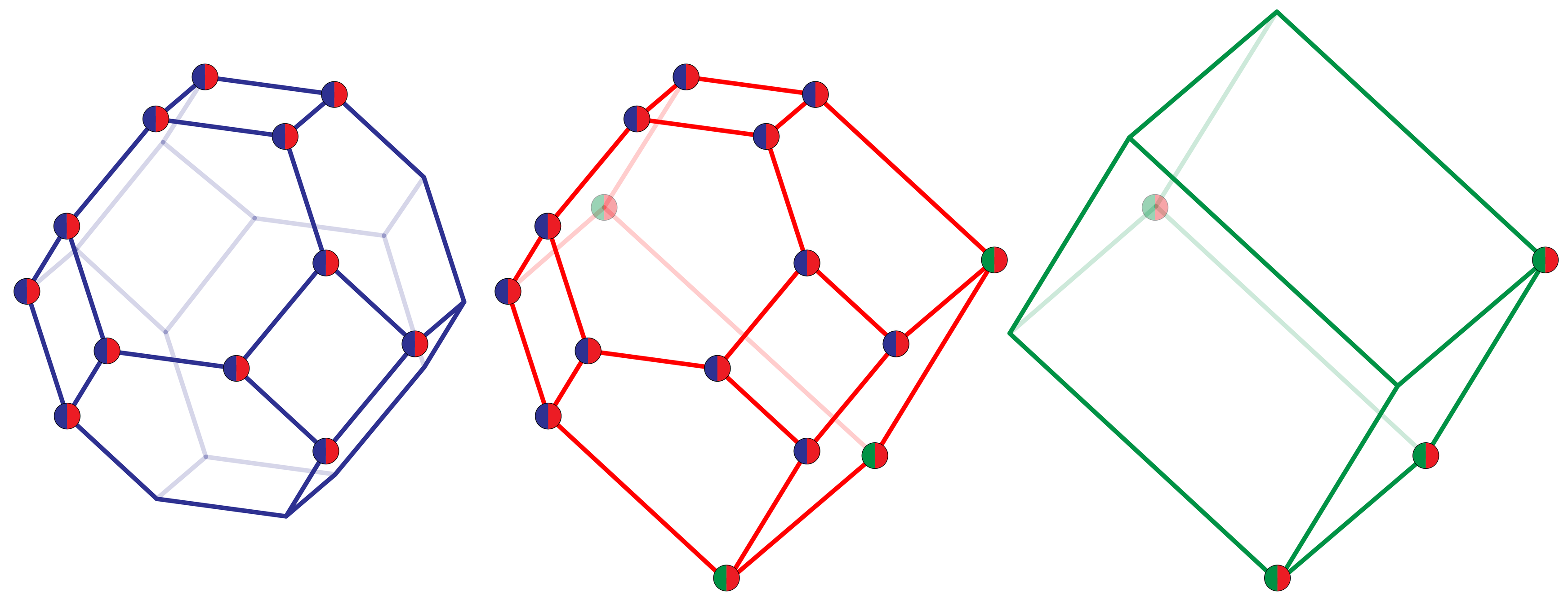
half-spaces:
$x_1 \ge 1$
$x_2 \ge 1$
$x_4 \ge 1$
$x_1 + x_4 \ge 3$
$x_1 + x_3 \ge 3$
$x_3 + x_4 \ge 3$
$x_1 + x_2 + x_3 \ge 6$
$x_1 + x_2 + x_4 \ge 6$
$x_1 + x_3 + x_4 \ge 6$
$x_2 + x_3 + x_4 \ge 6$
$x_1 + x_2 + x_3 + x_4 = 10$



Some properties

Prop. *The signed tree associahedron* $\mathsf{Asso}(\mathsf{T})$ *is sandwiched between the permutahedron* $\mathsf{Perm}(\mathsf{V})$ *and the parallelepiped* $\mathsf{Para}(\mathsf{T})$*:*

$$\sum_{u \ne v \in \mathsf{V}} [e_u, e_v] = \mathsf{Perm}(\mathsf{T}) \subset \mathsf{Asso}(\mathsf{T}) \subset \mathsf{Para}(\mathsf{T}) = \sum_{uv \in \mathsf{T}} \pi_{uv}[e_u, e_v]$$



Common vertices of

- $\mathsf{Asso}(\mathsf{T})$ *and* $\mathsf{Para}(\mathsf{T}) \equiv$ *orientations of* T *which are spines on* T,
- $\mathsf{Asso}(\mathsf{T})$ *and* $\mathsf{Perm}(\mathsf{T}) \equiv$ *linear orders on* V *which are spines on* T,

$\Rightarrow$ *no common vertex of the three polytopes except if* T = *signed path.*

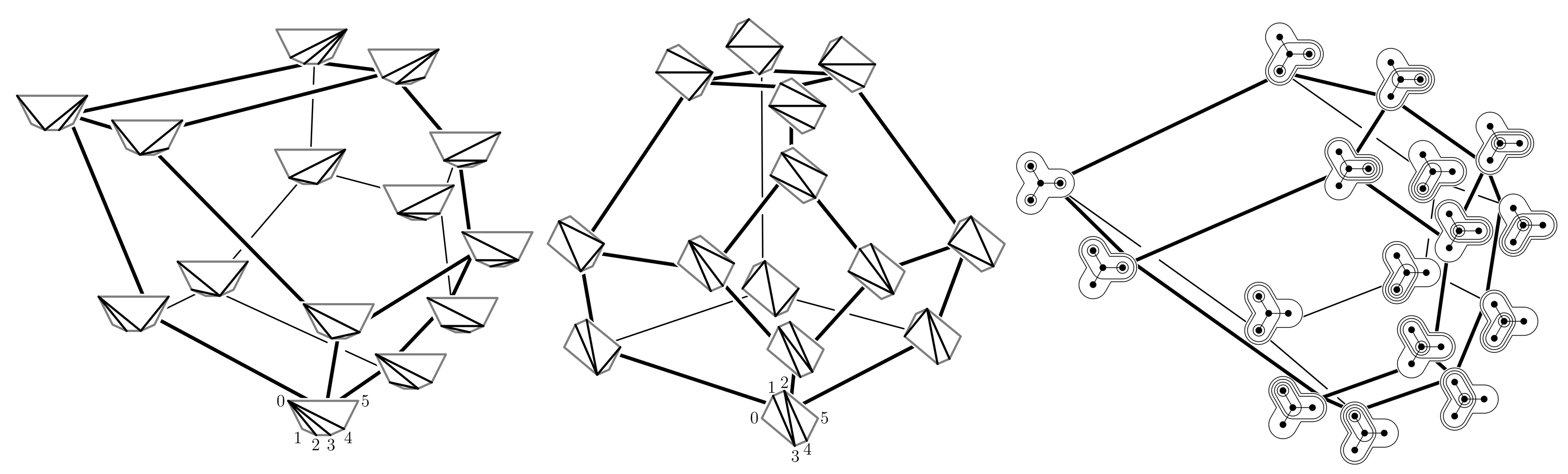
Prop. $\mathsf{Asso}(\mathsf{T})$ *and* $\mathsf{Asso}(\mathsf{T}')$ *isometric* $\iff$ T *and* T′ *isomorphic or anti-isomorphic up to the signs of their leaves, i.e. there is a bijection $\theta : \mathsf{V} \to \mathsf{V}'$ st. $\forall u, v \in \mathsf{V}$*

- *$u - v$ edge in* T $\iff$ *$\theta(u) - \theta(v)$ edge in* T′,
- *if u is not a leaf of* T*, the signs of u and $\theta(u)$ coincide (resp. differ).*

Examples

For a signed path P, Asso(P) is the classical associahedron
faces ⟷ dissections ⟷ Schröder trees,
vertices ⟷ triangulations ⟷ binary trees.

Loday, Realization of the Stasheff polytope, 2004
Hohlweg & Lange, Realizations of the associahedron and cyclohedron, 2007



For an unsigned tree T, Asso(T) is the T-associahedron
facets ⟷ *tubes* = connected induced subgraphs of T,
faces ⟷ *tubings* = collections of tubes which are pairwise nested, or disjoint and non-adjacent.

Carr & Devadoss, Coxeter complexes and graph associahedra, 2006

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