Acyclic reorientation lattices and their lattice quotients

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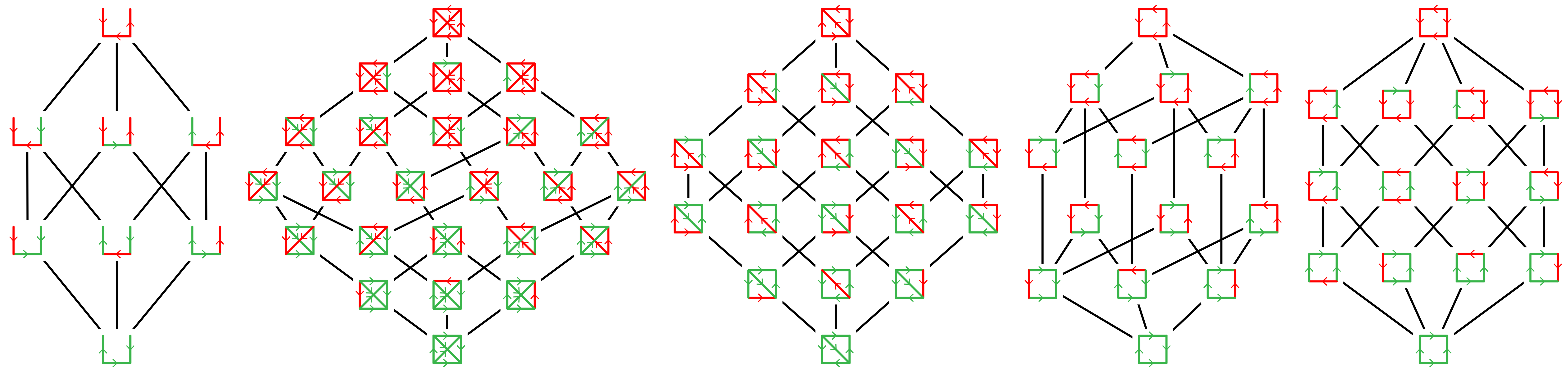
Acyclic reorientation posets

D = directed acyclic graph.
$\mathcal{AR}_D$ = poset of acyclic reorientations of D ordered by inclusion of reversed arcs.

REM. $\min(\mathcal{AR}_D) = D$ and $\max(\mathcal{AR}_D) = \bar{D}$.
$E \mapsto \bar{E}$ is a self-duality of $\mathcal{AR}_D$.
cover relations = reversing an arc in the transitive reduction of E.

EXM. $\mathcal{AR}_{\text{forest}} \simeq$ boolean lattice,
$\mathcal{AR}_{\text{tournament}} \simeq$ weak order.



boolean lattice — weak order lattice — semidistributive lattice — (non semidistr.) lattice — not a lattice

$\mathcal{AR}_D$ lattice $\iff$ D vertebrate

D vertebrate = the transitive reduction of any induced subgraph of D is a forest.

THM. $\mathcal{AR}_D$ is a lattice $\iff$ D is vertebrate.

PROP. If D vertebrate, then $X = \mathrm{bwd}(E)$ for some $E \in \mathcal{AR}_D$
$\iff$ all arcs of D in the transitive closure of X belong to X, and same with $D \smallsetminus X$.

PROP. If D vertebrate,
$\mathrm{bwd}(E \vee F)$ = transitive closure of $\mathrm{bwd}(E) \cup \mathrm{bwd}(F)$,
$\mathrm{fwd}(E \wedge F)$ = transitive closure of $\mathrm{fwd}(E) \cup \mathrm{fwd}(F)$.

$\mathcal{AR}_D$ semidistributive $\iff$ D skeletal

D filled = any directed path joining the endpoints of an arc in D induces a tournament.

D skeletal = vertebrate + filled.

THM. $\mathcal{AR}_D$ semidistributive lattice $\iff$ D is skeletal.

THM. If D skeletal, the canonical join representation of an acyclic reorientation E of D is $E = \bigvee_{a \in A} E_a$ where
- A = {arcs of D reversed in the transitive reduction of E},
- an arc is reversed in E_a $\iff$ it is the only arc reversed in E along a path in D joining the endpoints of a.

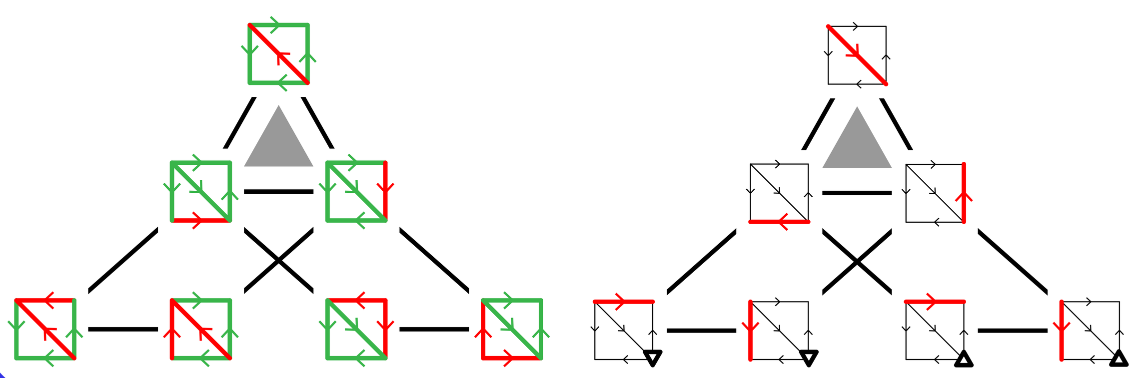
Ropes and rope diagrams

rope of D = quadruple $(u, v, \triangledown, \triangle)$ with
- (u, v) = an arc of D,
- $\triangledown \sqcup \triangle$ = partition of the transitive support of (u, v) minus $\{u, v\}$.

two ropes $(u, v, \triangledown, \triangle)$ and $(u', v', \triangledown', \triangle')$ are crossing if there are $w \neq w'$ such that
- $w \in (\triangledown \cup \{u, v\}) \cap (\triangle' \cup \{u', v'\})$,
- $w' \in (\triangle \cup \{u, v\}) \cap (\triangledown' \cup \{u', v'\})$.

THM. If D is skeletal, then the following correspondences hold:

in $\mathcal{AR}_D$		in D
join irreducibles	$\longleftrightarrow$	ropes,
canonical join representations	$\longleftrightarrow$	non-crossing rope diagrams,
canonical join complex	$\longleftrightarrow$	non-crossing rope complex.



More details?

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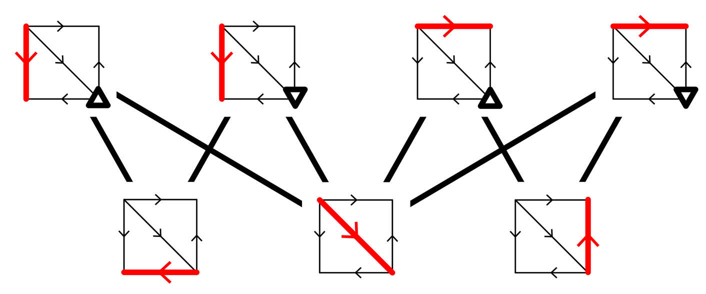
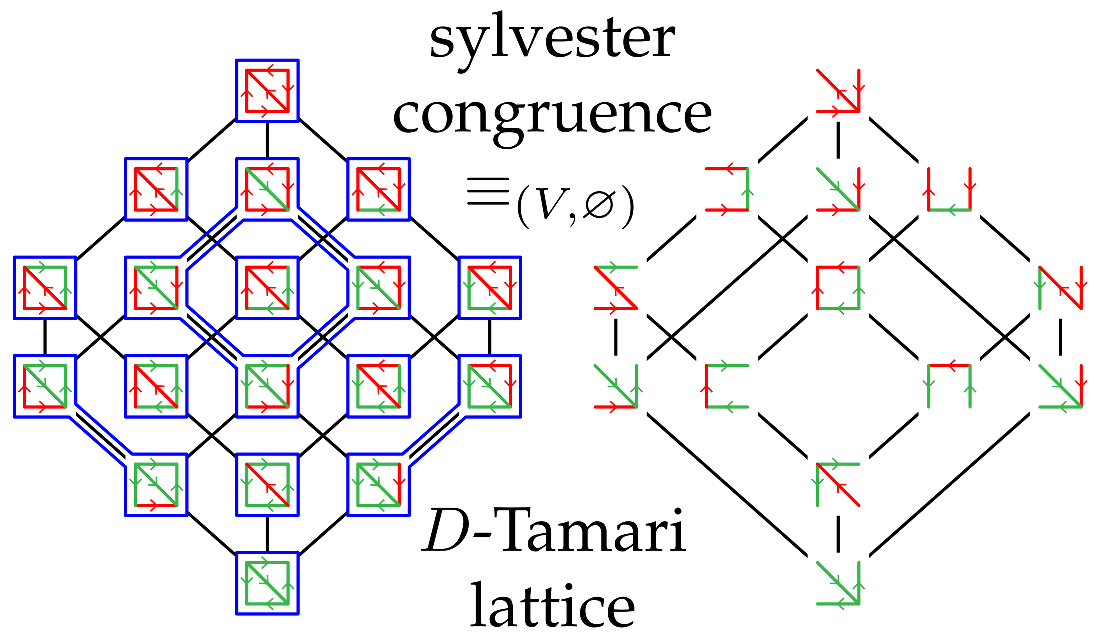
Congruences

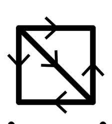
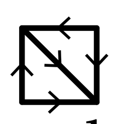
$(u, v, \triangledown, \triangle)$ subrope of $(u', v', \triangledown', \triangle')$ if
- $u, v \in \{u', v'\} \cup \triangledown' \cup \triangle'$,
- $\triangledown \subseteq \triangledown'$ and $\triangle \subseteq \triangle'$.



THM. If D skeletal, then lattice congruences of $\mathcal{AR}_D$ $\longleftrightarrow$ lower ideals of subrope order.

EXM. $(\mho, \Omega)$ = arbitrary subsets of V.
$\mathbb{I}_{(\mho,\Omega)}$ = lower ideal of ropes $(u, v, \triangledown, \triangle)$ such that $\triangledown \subseteq \mho$ and $\triangle \subseteq \Omega$.
coherent congruence $\equiv_{(\mho,\Omega)}$ = congruence with subrope ideal $\mathbb{I}_{(\mho,\Omega)}$.

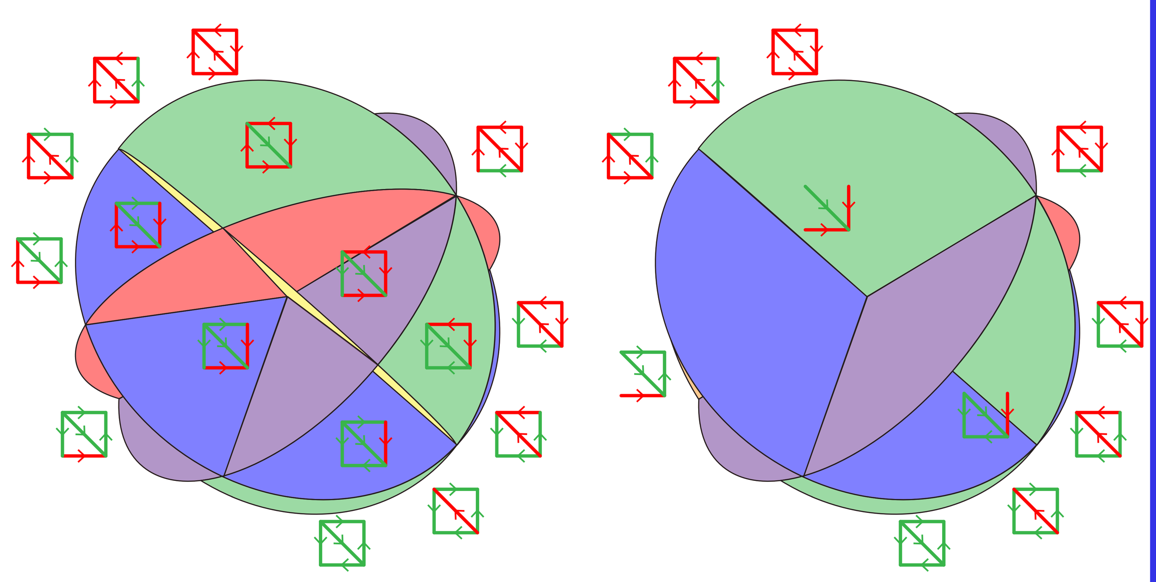


CONJ. D has no induced  or  $\iff$ the D-Tamari lattice is regular.

Quotientopes

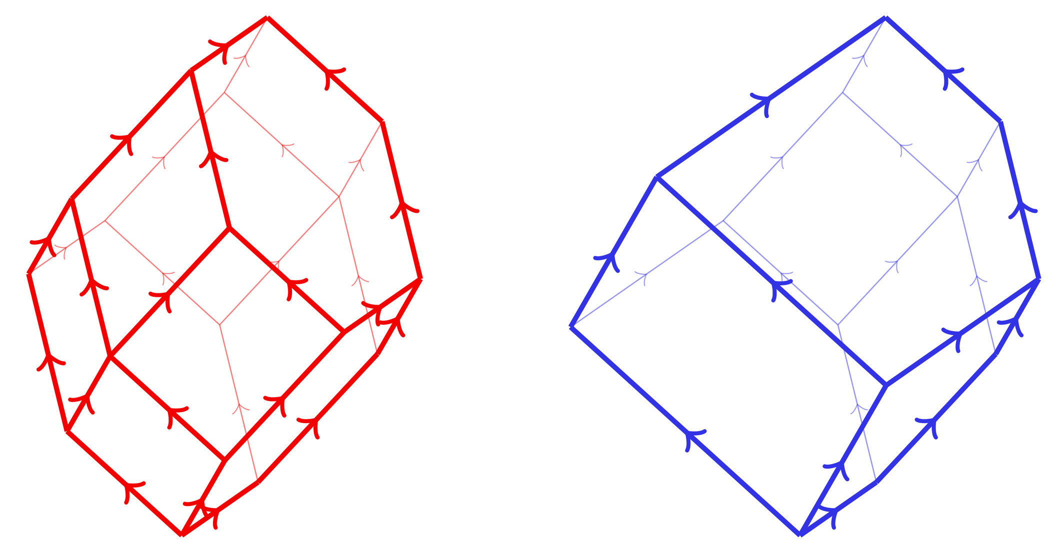
graphical fan $\mathcal{F}_D$ fan of the hyp. arr. $x_u = x_v$ for $(u, v) \in D$

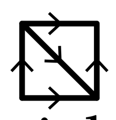
quotient fan $\mathcal{F}_\equiv$ chambers obtained by glueing classes of chambers of $\mathcal{F}_D$



graphical zonotope $\mathcal{Z}_D$ Minkowski sum of $[\mathbf{e}_u, \mathbf{e}_v]$ for $(u, v) \in D$

quotientope $\mathcal{Q}_\equiv$ Minkowski sum of shard polytopes



CONJ. D has no induced  $\iff$ all Cambrian associahedra of D have isomorphic face lattices.

CONJ. $\mathcal{F}_{\equiv_{(\mho,\Omega)}}$ is always removahedral.