Deformation cone of hypergraphic polytopes

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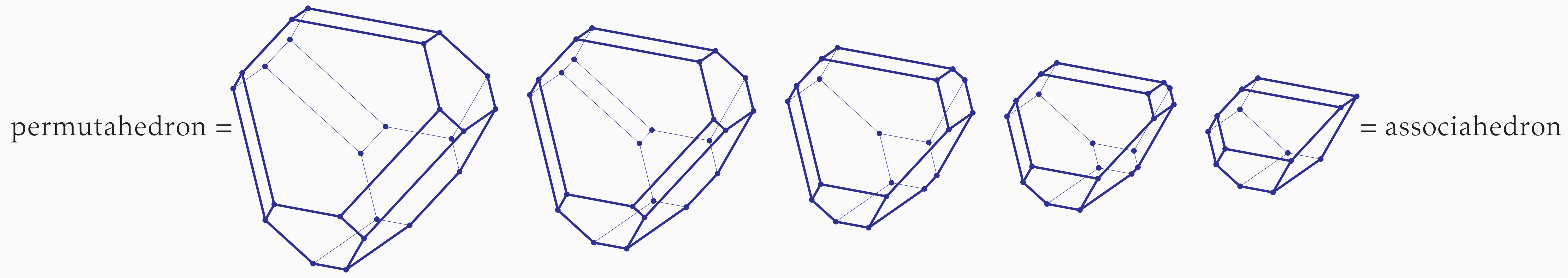
Deformation of polytopes

deformation cone $\mathbb{DC}(\mathsf{P})$ = set of all polytopes whose normal fans coarsen the normal fan of P.

PROP. $\mathbb{DC}(\mathsf{P})$ is a closed convex cone (under dilation and Minkowski sum) and contains a lineality subspace of dimension d (translations).

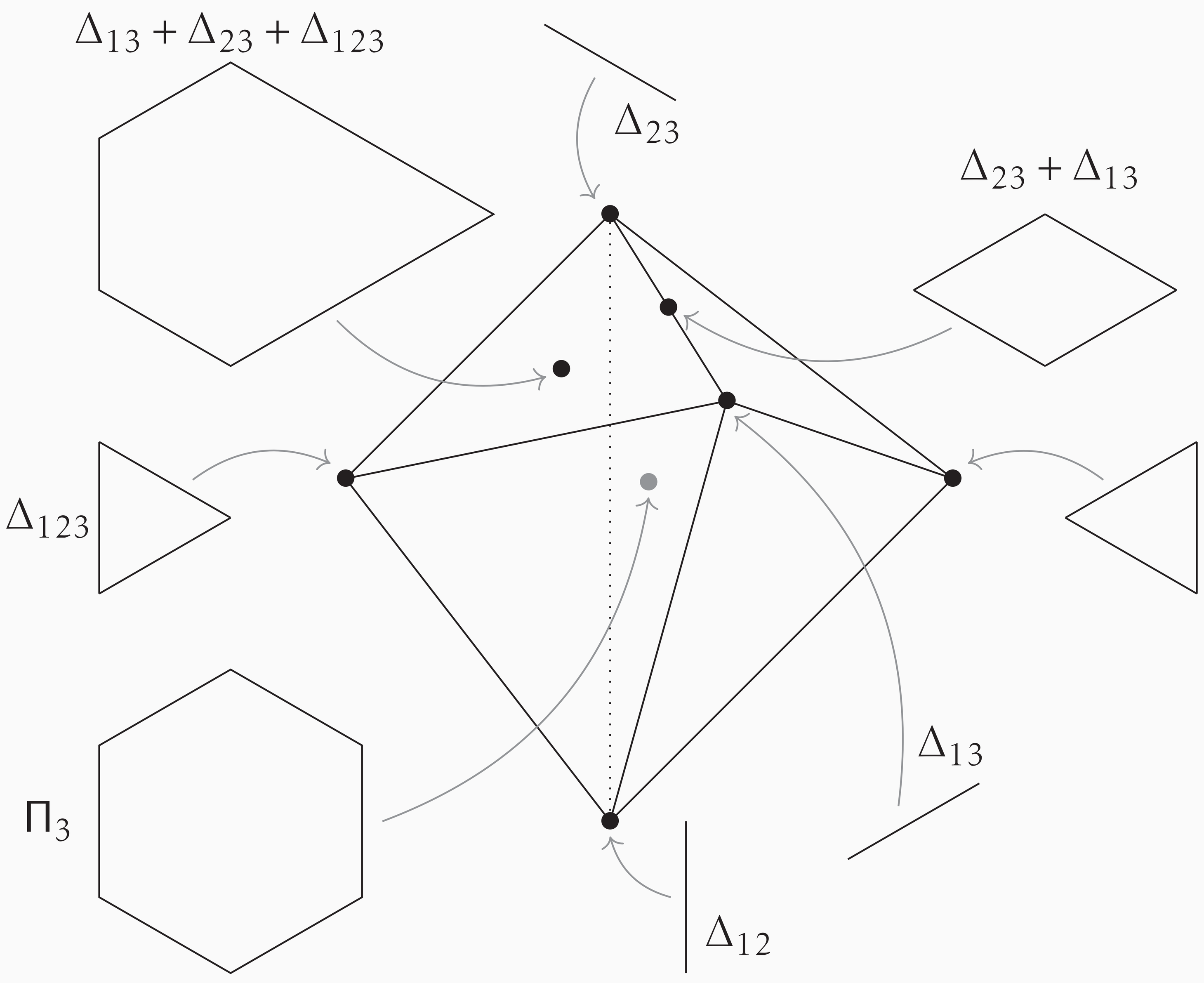
PROP. Q deformation of P $\Rightarrow$ $\mathbb{DC}(\mathsf{Q})$ face of $\mathbb{DC}(\mathsf{P})$.



Some deformations of the *standard permutohedron* $\Pi_n = \mathrm{conv}\{(\sigma(i))_{i\in[\![1,n]\!]}\ ;\ \sigma \in \mathcal{S}_n\}$.

Deformed permutahedra and $\mathbb{DC}(\Pi_n)$

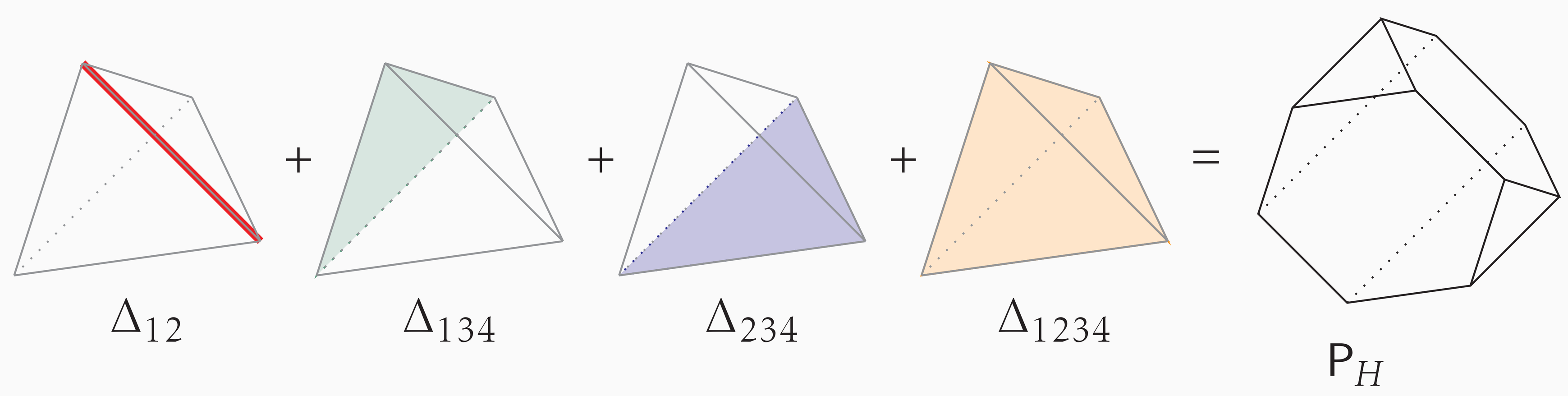
$\mathbb{DC}(\Pi_n)$ = deformed permutahedra = cone of submodular functions
$\dim \mathbb{DC}(\Pi_n) = 2^n - n - 1$; number of facets of $\mathbb{DC}(\Pi_n) = 2^{n-2}\binom{n}{2}$.
Remains to understand since the 70s: rays, faces...



$3D$ affine section of $\mathbb{DC}(\Pi_3)$ ($4D$-cone with 5 rays).

Hypergraphic polytopes P_H

Hypergraph on V = collection H of $U \subseteq V$ such that $|U| \geq 2$.
Hypergraphic polytope $\mathsf{P}_H = \sum_{U\in H} \Delta_U$ with $\Delta_U := \mathrm{conv}\{\boldsymbol{e}_u \mid u \in U\}$.



Hypergraphic polytope for $H = \{12, 134, 234, 1234\}$.

Hypergraphic fan coarsens braid fan (normal fan of Π_n) $\Rightarrow \mathsf{P}_H \in \mathbb{DC}(\Pi_n)$ and $\mathbb{DC}(\mathsf{P}_H)$ is a face of $\mathbb{DC}(\Pi_n)$. We study $\mathbb{DC}(\mathsf{P}_H)$.

Wall-crossing inequalities give a redundant description of $\mathbb{DC}(\mathsf{P}_H)$.

THM. The deformation cone $\mathbb{DC}(\mathsf{P}_H)$ is isomorphic to the set of polytopes $\left\{x \in \mathbb{R}^V \;\middle|\; \sum_{u\in U} x_u - \sum_{v\notin U} x_v \leq \boldsymbol{h}_U \text{ for all } U \subseteq V\right\}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{2^V}$ defined by the following **redundant** description:

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for each $S \subseteq V$ and each $\{u,v\} \subseteq V \smallsetminus S$ such that $U \notin H$ for any $\{u,v\} \subseteq U \subseteq V \smallsetminus S$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for each $\{u,v\} \subseteq U \in H$ and $S \subseteq V \smallsetminus U$.

Dimension of $\mathbb{DC}(\mathsf{P}_H)$

$K \subseteq V$ induced clique =
$\forall u, v \in K, \exists U \in H,\ \{u,v\} \subseteq U \subseteq K$.

THM. Span $(\mathbb{DC}(\mathsf{P}_H))$ **independent** eqns:

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$; $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for $\varnothing \neq S \subseteq V$ with $V \backslash S$ not an induced clique, $U \notin H$ for any $\{u,v\} \subseteq U \subseteq V \backslash S$.

CORO. The simplices Δ_K for the induced cliques $K \neq \varnothing$ of H form a linear basis of the vector space spanned by $\mathbb{DC}(\mathsf{P}_H)$.

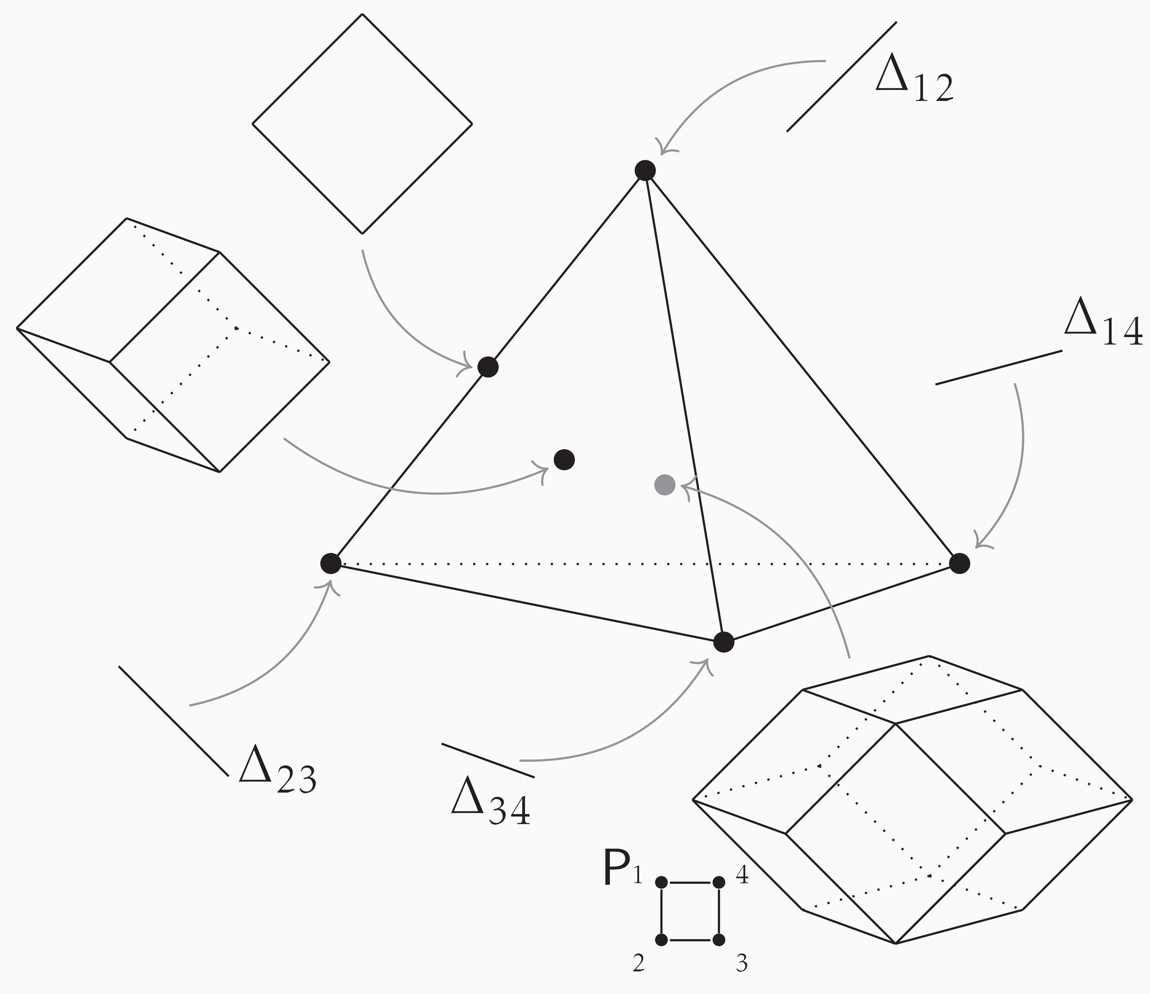
CORO. $\dim \mathbb{DC}(\mathsf{P}_H)$ = number induced cliques.

We have an irredundant description of $\mathbb{DC}(\mathsf{P}_H)$ for two classes of hypergraphs:

- Graphical zonotopes,
- Nestohedra.

Graphical zonotopes

Graphical zonotope = P_G when $H = G$ a graph ($\forall U \in H,\ |U| = 2$)



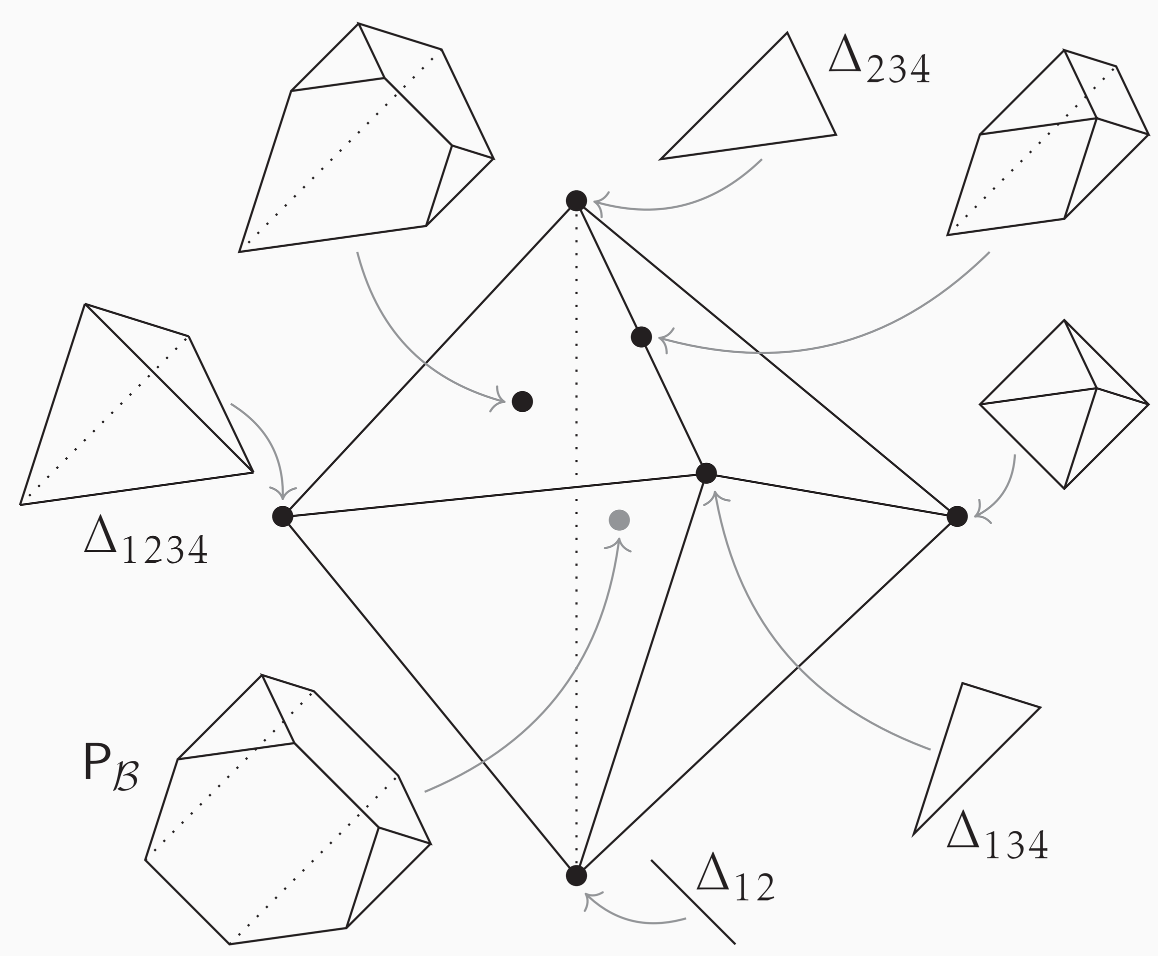
THM. **Irredundant** description $\mathbb{DC}(\mathsf{P}_G)$:

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\smallsetminus\{u\}} + \boldsymbol{h}_{S\smallsetminus\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\smallsetminus\{u,v\}}$ for each $\varnothing \neq S \subseteq V$ and any $\{u,v\} \in \binom{S}{2} \smallsetminus E$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for each $\{u,v\} \in E$ and $S \subseteq N(u) \cap N(v)$.

CORO. $\mathbb{DC}(\mathsf{P}_G)$ simplicial $\Leftrightarrow$ G has no triangle (clique of size 3).

Nestohedra

Nestohedron = $\mathsf{P}_\mathcal{B}$ when $H = \mathcal{B}$ a building set ($\forall U_1, U_2 \in H,\ U_1 \cap U_2 \neq \varnothing \Rightarrow U_1 \cup U_2 \in H$)
U elementary = max $W \subsetneq U$, $W \in H$, disjoint.



THM. **Irredundant** description $\mathbb{DC}(\mathsf{P}_\mathcal{B})$:

- $\sum_{K\in\overline{\kappa}(\mathcal{B})} \boldsymbol{h}_K = 0$ ($\overline{\kappa}(\mathcal{B})$: max $U \in H$ and $\varnothing$),
- $\sum_{B\in\mu(P)} \boldsymbol{h}_B \geq \boldsymbol{h}_P$ for elementary $P \in H$,
- $\boldsymbol{h}_A + \boldsymbol{h}_B + \sum_{K\in\kappa(P\smallsetminus(A\cup B))} \boldsymbol{h}_K \geq \boldsymbol{h}_P + \sum_{K\in\kappa(A\cap B)} \boldsymbol{h}_K$
  P not elementary, $A \neq B$ maximal in P.

CORO. $\mathbb{DC}(\mathsf{P}_\mathcal{B})$ simplicial $\Leftrightarrow$ all U with ≥ 3 distinct maximal subblocks are elementary.

Want more details?

arXiv:2109.09200
arXiv:2111.12422

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