SOME THEORETICAL PROBLEMS IN EFFECTIVE DIFFERENTIAL ALGEBRA AND THEIR RELATION TO CONTROL THEORY

Avec le soutien du GDR de Calcul Formel MEDICIS (Mathématiques Effectives, Développements Informatiques, Calcul, Ingénierie et Systèmes) du PRC/GDR Mathématiques et Informatique et du projet européen ESPRIT BRA contract 6846 POSSO (POlynomial Systems SOlving).

François OLLIVIER
Équipe de Calcul Formel, GAGE
École Polytechnique, F-91128 PALAISEAU CEDEX (France)
URA CNRS n° 169
ollivier@ariana.polytechnique.fr

1992

Abstract: Recent works in control theory have shown the interest of differential algebra in this field: it appears to be a convenient language to express some theoretical problems and, may be, a tool to solve them if algorithmic solutions could be found, efficient enough to be implemented on computers. On the other hand, control theory strongly motivates new research on some fundamental and difficult problems of effective differential algebra. We will try through a few examples to illustrate the strong links between some basic problems in those two fields.

First, we will recall how identifiability or observability may be tested using elimination techniques in differential algebra. The essential problem is that of efficiency. We will underline the interest of an effective differential nullstellensatz to prove complexity bounds for an elimination procedure.

If we consider distinguishability, we are led to a difficult problem arising in the formal resolution of general systems. Nothing is known here except for some special cases of structures, such as linear ones, or single output structures.

Before concluding, we will consider the question of initial conditions in the case of quasi-algebraic structures, and show that we are then led to some difficult problems of number theory which cannot be avoided using existing methods.

Keywords: Standard bases, characteristic sets, calculability, complexity, computer algebra, differential algebra, distinguishability, identifiability, effective nullstellensatz, observability.

1. Introduction

The history of differential algebra begins more than a century ago, with the pioneering works of Picard, Vessiot and Riquier. But it is only with Ritt and Kolchin that this theory came to adult age. As the basic methods introduced in those works are almost always effective and could sometimes be easilly translated into program on modern computers, it is quite surprising that the computer algebra community began to pay much attention to them only a short time ago, mostly under the impulse of WU Wentsün (cf. [W]).

In the meanwhile, some control theorists began using differential algebra, which plays an increasing role in the study of non-linear structures (see [F]). This gives a new motivation for algebraists to implement existing algorithms and to try to find convenient solutions to old problems, taking a new interest with the needs of control theory.

We will try to stress here on some computational issues, naturally arising when testing some structural properties, and related to problems of complexity, or of decidability for some problems of differential algebra.

2. Basic differential algebra

A satisfactory overview of the subject would be much too long. We will only try to provide a few concepts to help the courageous naive reader to understand the next sections. The classical book [Ka] provides a very readable introduction.

Differential algebra is a generalisation of commutative algebra. Here the rings to be considered possess a derivation, i.e. an operator δ satisfying $\delta(x+y) = \delta x + \delta y$ and $\delta(xy) = \delta(x)y + x\delta(y)$. In the sequel, we will denote $\delta(x)$ by x' and $\delta^i(x)$ by $x^{(i)}$. E. g., the polynomial ring $k[t]$ is a differential ring with the usual derivation.

To any differential algebra A, one can associate a differential algebra of polynomials in n variables, denoted by $A\{x_1, \ldots, x_n\}$, which is the algebra of algebraic polynomials in $x_1, \ldots, x_n$ and all their formal derivatives $x_1', x_1'', \ldots$ with the unique derivation δ extending the derivation of A and such that $\delta x_i^{(j)} = x_i^{(j+1)}$. E.g. $x''x - (x')^2$ is a differential polynomial in $\mathbf{Q}\{x\}$ ($\mathbf{Q}$ has a unique structure of differential ring defined by $r' = 0 \, \forall r \in \mathbf{Q}$).

A system of algebraic (ordinary) differential equations in n variables over a differential field $\mathcal{F}$ will be a subset of $\mathcal{R} = \mathcal{F}\{x_1, \ldots, x_n\}$. We can of course have a geometrical approach as in commutative algebra: namely differential algebraic geometry will consider the set of solutions of differential algebraic systems in some sufficiently big extension of $\mathcal{F}$, called a universal extension. It may be considered as some kind of differential algebraic closure.

Of course, different systems could have the same variety of solutions, but they will be related in the following way. First, we may define the differential ideal generated by a system Σ. It is the smallest algebraic ideal $\mathcal{I}$ containing Σ and such that $\delta\mathcal{I} \subset \mathcal{I}$. In other words, it is the algebraic ideal generated by $\Sigma, \delta(\Sigma), \delta^2(\Sigma), \ldots$ It is denoted by $[\Sigma]$. Obviously, two systems generating the same ideal define the same variety, but the reciprocal is false. E.g. the ideals $[x]$ and $[x^2]$ define the same variety, containing only 0! We define the radical of an ideal $\mathcal{I}$ to be the ideal $\{a | \exists\, i \in \mathbf{N}\ a^i \in \mathcal{I}\}$ and denote the radical of $[\Sigma]$ by $\{\Sigma\}$.

Proposition 1. — *Two systems Σ and Ξ define the same variety iff $\{\Sigma\} = \{\Xi\}$.* ∎

Special cases of *perfect* ideals—i.e. ideals being their one radical—are prime ideals, which are such that $PQ \in \mathcal{I}$ implies $P \in \mathcal{I}$ or $Q \in \mathcal{I}$. If a radical ideal $\mathcal{I}$ is not prime, we may split the associated variety in two different varieties defined by the systems $\mathcal{I} \cup \{P\}$ and $\mathcal{I} \cup \{Q\}$. Fortunately, there is only a finite number of possible such splitting, due to the following property.

Proposition 2. — *Any radical differential ideal $\mathcal{I}$ is a finite intersection of prime differential ideals*

$$\mathcal{I} = \bigcup_{i=1}^{s} \mathcal{J}_i.$$

∎

This means equivalently that the variety $V(\mathcal{I})$ is a finite union of irreducible varieties $V(\mathcal{J}_i)$. It appears to be a corollary of the following fundamental theorem.

Theorem 3. — *For all radical differential ideal in $\mathcal{F}\{x_1, \ldots, x_n\}$, $\mathcal{F}$ being a differential field, there exist a finite system $P_1, \ldots, P_k$ such that*

$$\mathcal{I} = \{P_1, \ldots, P_k\}.$$

∎

This means that every differential algebraic variety may be defined by a finite system.

Examples. — 4) The situation is not as easy as one may think at first sight. The differential ideal $[x''x - 2(x')^2]$ is not radical. On the other hand, its radical $\mathcal{I}$ is prime, but there exists no finite system Σ such that $\mathcal{I} = [\Sigma]$.

5) Even a variety defined by a single irreducible polynomial may be reducible. The polynomial $P = (x')^2 - 4x$ cannot be factorized, but it defines two different varieties, the first one associated to the prime ideal $[P, x'' - 2]$, and the second to $[x]$. To show it, we may remark that $P' = 2x'(x'' - 2)$ is not prime, each factor inducing one of the two components.

In spite of those examples, one may underline the paralellism with algebraic geometry by defining the dimension. A way to define it in algebraic geometry is to consider the projections on the linear spaces $L(I)\, I \subset [1, n]$ generated by the subset $\{x_i i \in I\}$ of the set of variables. The dimension of V is equal to $\max\{\sharp I | \pi(V) \text{is dense in} L(I)\}$—if $V = \emptyset$, it is taken equal to -1 by convention. The same definition can be used in differential algebra. An equivalent algebraic definition is to consider subsets I such that $\mathcal{I}(V) \cap \mathcal{F}\{x_i | i \in I\} = [0]$.

3. Algorithmic tools

The basic algorithmic tool in differential algebra is the notion of characteristic set. We will avoid a general formal definition and prefer to stick to intuitive ideas. We suppose in the following that the differential ideal $\mathcal{I}$ is prime.

We will consider the set $\Upsilon = \{x_i^{(j)}\}$ of derivatives. We may define on this set a total ordering $<$ such that $\forall\, u\ u < u'$ and $\forall\, (u, v)\ u < v$ implies $u' < v'$. There is two main kinds of such orderings used in practice. First, the *orderly* orderings, being such that $\operatorname{ord} u < \operatorname{ord} v$ implies $u < v$, and the elimination orderings: for a partition of the variables in two sets X_1 and X_2, an ordering such that $x_{i_1} \in X_1$ and $x_{i_2} \in X_2$ implies $x_{i_1}^{(j_1)} < x_{i_2}^{(j_2)} \forall (j_1, j_2) \in \mathbf{N}^2$ is said to eliminate X_2.

Denote the biggest variable of a differential polynomial P by u_P, then we may define a partial ordering on $\mathcal{R}$ by $P < Q$ if $u_P < u_Q$ or if $u_P = u_Q$ and $\deg_{u_P}(P) < \deg_{u_P}(Q)$: we sort polynomials first according to their main derivatives, and then according to their degrees in those derivatives. By convention 0 will be the biggest polynomial, and non zero polynomials of degree 0 will be smaller than any non zero polynomial of positive degree.

Definition 1. — *A polynomial P is said to be reduced with respect to a polynomial Q if u_P is not a proper derivative of u_Q, and if $\deg_{u_Q}(P) < \deg_{u_Q}(Q)$.*

We will then present an abstract construction of a characteristic set of $\mathcal{I}$. First we pick up in $\mathcal{I}$ a minimal polynomial P_1. Then, we consider the subset E_2 of polynomials in $\mathcal{I}$ reduced with respect to P_1. If $E_2 = \emptyset$ we stop. If not, we continue and pick up P_2 being minimal in E_2—we may remark that $P_1 < P_2$ so that P_1 is reduced with respect to P_2. Then E_3 will be the subset of $\mathcal{I}$ of polynomials reduced with respect to P_1 and P_2... This process will stop because if P and Q are reduced, one with respect to the other, their main derivatives are derivatives of two different variables, so that the length of such a chain is at most the number n of variables.

We call the result of that process a characteristic set $\mathcal{A} = \{P_1, \ldots, P_k\}$ for $\mathcal{I}$.

The main interest is that characteristic sets, even if they are not unique can characterize a prime ideal. Indeed, there is a reduction process, based on pseudo-euclidean reduction (you may multiply Q by the leading coefficient of P in order to divide it) which allows to secure a rest $R_{\mathcal{A}}(Q)$ being reduced with respect to the members of $\mathcal{A}$. It then provides a membership test for $\mathcal{I}$.

PROPOSITION 2. — *If $\mathcal{A}$ is a char. set of $\mathcal{I}$ and $\mathcal{I}$ is prime, then $Q \in \mathcal{I}$ iff $R_{\mathcal{A}}(Q) = 0$.* ∎

Moreover, we have an (unfortunately non effective) algebraic characterization of $\mathcal{I}$, knowing a char. set $\mathcal{A}$. Denote by I_P the leading coefficient of P considered as a one variable polynomial in u_P, and by S_P $I_{P'}$, i.e. $\frac{\partial P}{\partial u_P}$. Let $H_{\mathcal{A}} = \prod_{P \in \mathcal{A}} I_P S_P$, then

$$\mathcal{I} = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} := \{Q | \exists\, i \in \mathbf{N}\ QH_{\mathcal{A}}^{i} \in \mathcal{I}\}.$$

Let $X = X_1 \cup X_2$ be a partition of the set of variables, then if $\mathcal{A}$ is a char. set of $\mathcal{I}$ for a ordering eliminating X_1, then $\mathcal{A} \cap \mathcal{F}\{X_2\}$ is a char. set of $\mathcal{I} \cap \mathcal{F}\{X_2\}$.

We also can determine easilly most useful information about $\mathcal{I}$ once we know a char. set. Let X_1 be the set of variables such that their derivatives do not appear as leading derivatives in $\mathcal{A}$. Then $\mathcal{A}$ is also a char set for an ordering eliminating X_1, which shows that $\mathcal{I} \cap \mathcal{F}\{X \setminus X_1\} = [0]$. Indeed, it may be proved that $\dim \mathcal{I} = \sharp X_1$.

Ritt has introduced in [R] an algorithm allowing to compute for any finite set of algebraic differential equations Σ a decomposition of $\{\Sigma\}$ into a finite intersection $\bigcap_{i=1}^{s} \mathcal{J}_i$, each $\mathcal{J}_i$ being defined by a char. set $\mathcal{A}_i$.

This methods suffers two main shortcomings. First, we need to be able to factorize a polynomial in $\mathcal{F}[x]$, which imposes a restriction on the ground field $\mathcal{F}$. Even if this is possible on "usual" fields such as $\mathbf{Q}$ or $\bar{\mathbf{Q}}$, factorization are still very expensive. The second drawback is that we are in general unable to check that two prime ideals $\mathcal{I}$ and $\mathcal{J}$, given by char. sets are included one in the other. Only equality can be tested. So the decomposition given by Ritt's algorithm may contain imbeded components.

Other methods could be used in some cases. If $\mathcal{I}$ is already given by a char set—say for an orderly ordering—, we may compute a new char set—e.g. for an elimination ordering, **without any factorization** (cf [O1]). We may also try to compute a *differential standard basis*, which will provide much more information, but such a basis is in general infinite. However, we have a completion procedure, using only field operations, converging to the basis and we can obtain partial informations if we know how to stop it at some convenient order (cf. [C] and [O2]).

Another method, introduced by Seidenberg (see [D]), allows one to perform elimination by giving a description of the projection by equations and inequations—in general it is a semi-algebraic set, e.g. the projection of $V(xy-1)$ on the x axis is defined by $x \neq 0$. In this sense it is more precise than Ritt's method which only determines the adherence but on the other hand, it is unable to determine the dimension or even to check whether a solution do exist.

4. The use of elimination

4.1. Representation of data

This question is a central one when one wants to manipulate algebraic objects. Sometimes, mostly for finite objects such as polynomials, the different possible codings are equivalent, even if some of them could be better for a given computation. But for infinite objects, it may often be difficult, or even impossible to go from one representation to another. In the special case of differential ideals, it is impossible to describe them in all cases using a system of generators, because they are not always finitely generated. Provided they are radical, we may at least give a finite system Σ such that $\mathcal{I} = \{\Sigma\}$. Such a system was called by Rit a *basis*. In particular, this also applies to prime ideals. But they can also be defined by a characteristic set.

We may compute a char. set $\mathcal{A}$, Σ being given, *provided we know a priori that the ideal is prime*, but no algorithm is known to recover a basis knowing $\mathcal{A}$. We will see that that the answer to some problems will depend greatly on the chosen representation of prime ideals.

We can give the following, now quite classical abstract definition of an algebraic structure or parametric model. It is a prime ideal of

$$\mathcal{F}\{u_1, \ldots, u_m, x_1, \ldots, x_n, y_1, \ldots, y_p, \theta_1, \ldots, \theta_q\},$$

where the u_i denote the inputs, the x_i the state, the y_i the output, and the θ_i are the parameters. We impose the following natural conditions:

(i) the dimension of $\mathcal{I}$ is m, and $\mathcal{I} \cap \mathcal{F}\{U\} = [0]$

(ii) $\mathcal{I} \cap \mathcal{F}\{\Theta\} = [\theta_i' = 0 | i = 1, \ldots, q]$, which means that the parameters are constants.

Off course, to check those two properties, one needs to use Ritt's method which may be very difficult. Often, one will have to deal with structures defined by equations of the following type:

$$X' = F(X, U, \Theta)$$
$$Y = G(X, \Theta),$$

where X stands for $(x_1, \ldots, x_m)$, F for $(f_1, \ldots, f_m)$, etc. and the f_i and g_i are algebraic functions respectively defined by the equations $P_i(x_i', X, U, \Theta)$, $Q_i(y_i, X, \Theta)\ldots$ Such a systems defines the wanted ideal—provided we complete it with $\Theta' = 0$, because it is a characteristic set for an orderly ordering. We may notice that the associated ideal is not prime in general, but we still can perform most computations in this situation using the method given in [L]. We shall need to factorize to check primality. The main drawback is that it is not always a basis. More general cases of algebraic structure could easilly be reduced to this definition.

4.2. State elimination

Suppose that we have a structure, being defined as above. The problem of state elimination obviously reduces to the elimination of the variables X, which may be done in many different ways. If we need a very precise set theoretic description of the projection, one may use Seidenberg's method as Diop did (see [D1]). It may be faster to determine only its Zariski adherence using Ritt's method. If we assume the representation of the structure by a char. set as above, the method introduced in [O1] after Lazard's work [L] also applies, and allows to avoid factorization.

4.3. Identifiability

Identifiability may also be reduced to an elimination problem, using the following definition.

DEFINITION 1. — *A structure is locally (resp. globally) identifiable if there exist algebraic (resp. rational) functions h_i such that $\theta_i = h_i(U, Y)$.*

This is easily tested using the computation of a char. set of the ideal defining the structure, using a ordering eliminating the X and then the Θ. Then, one only has to check that the char sets contains polynomials $R_i(\theta_i, U, Y)$, algebraic in θ_i. Global identifiability correspond to the case where R_i is of degree 1 in θ_1.

Such a method has been introduced by Glad et Ljung in [GL].

4.4. Distinguishability

We will provide a differential algebraic definition of distinguishability and show that this problem can be possibly solved by elimination, or becomes much more difficult according to the chosen representation of the structure.

DEFINITION 1. — *Let S and S' be two d. a. structures defined by ideals $\mathcal{I}$ and I'. We suppose that the two structure are comparable, meaning that they have the same inputs U and outputs Y. But we take different sets of variables X and X', Θ and Θ' for their states and parameters. Then, S is distinguishable from S' if $\mathcal{I}' \cap \mathcal{F}\{U, Y\} \subset \mathcal{I} \cap \mathcal{F}\{U, Y\}$, or equivalently if $(\mathcal{I} + \mathcal{I}') \cap \mathcal{F}\{U, \Theta\} \neq [\theta_i' = 1]$.*

.

We can already underline some limitations of this definition: we only consider the generic situation, and we disregard real problems (effective methods in real algebraic geometry seems for the moment out of reach of actual computations). But at least, this is a sufficient distinguishability condition, if we compare it with more precise ones.

Then, if the two structures are given by two bases Σ and Σ' such that $\mathcal{I} = \{\Sigma\}$ and $\mathcal{I}' = \{\Sigma'\}$, we may check this by applying Ritt's method, for an ordering eliminating all variables except $U \cup \Theta$. The ideal $\mathcal{I} + \mathcal{I}'$ is not prime in general, but we only have to check that $\mathcal{J}_i \cap \mathcal{F}\{U, \Theta\}$ is not $[0]$ for all the components.

If we assume that the structures are given by some char. sets, we have no algorithm to conclude. Indeed, no method is known to check whether to irreducible diff. algebraic varieties are included one in the other, except in the one variable situation. As we assume that the inputs and parameters are generic, we may reduce to this case for single output structures. The use of the *low power theorem* (see [R]) allows then to conclude.

We may notice however, that for linear systems, char. sets are always generating the associated ideals, so that our difficulties are purely non-linear.

4.5. Complexity

The complexity of the elimination techniques used above also depends of the presentation of data. The complexity of Ritt's method is unknown in the general case, corresponding to structures defined by bases. But, if we assume that they are already given by char. sets, we may avoid factorizing, which means a great gain of practical efficiency. We would even conjecture a "good"—meaning simple exponential—complexity in such a situation. The reason is that Ritt's analog of Bézout's theorem allows to bound the order of derivatives involved during the computation.

In order to secure complexity bounds in the general case, one would certainly need an "effective differential nullstellensatz", meaning a bound on the order of intermediate computations necessary to check whether a system admits solutions. In the algebraic case, such a bound exists for the degree and has already allowed to provide a complexity bound for the first step of Ritt's method for pure algebraic equations (see [GM]).

5. The difficulty of working with initial conditions

5.1. More general systems

It would sometimes be necessary to complete the data of a structure with a priori known initial conditions. We could also wish to deal with structures involving transcendental functions. Provided they are themselves solution of some differential equations, we may reduce to differential algebraic structures, but the use of initial conditions becomes then absolutely necessary.

Example 1. — If we consider the pendulum equation:

$$x'' = -\theta_1 \sin(x) + \theta_2 u,$$

we can reduce it to an algebraic structure

$$\begin{aligned} x_1'' &= -\theta_1 x_2 + \theta_2 u \\ x_2' &= x_1 x_3 \\ x_3^2 &= -x_2^2 + 1, \end{aligned}$$

where x_1 stands for x, x_2 for $\sin(x)$ and x_3 for $\cos(x)$. But we must complete it with the initial conditions $x_2(0) = \sin(x(0))$ and $x_3(0) = \cos(x(0))$. Except in such simple cases as $x(0) = 0$, we shall need to introduce transcendental constants.

5.2. Identifiability with initial conditions

It has been shown in [O3] how to treat the problem of identifiability for general algebraic structure with initial conditions in a computable field of constants. The same method could be extended for structure involving arbitrary compositions of transcendental functions defined by differential equations, provided we have an oracle to decide equality in the extended field of constants. Indeed, the essential point is that we need to test whether a given polynomial in X, U, Y, Θ is identically 0, according to initial conditions. Other methods for this have already been proposed (see [S]).

It may be noticed that the general problem of constants is undecidable in a language defined using exp, log, 0, 1, $+$, $\times$ and $|x|$. If we suppress $|x|$, nothing is known, but the problem would be solved if Schanuel's conjecture could be proved. Roughly speaking, the conjecture claims that no new relation can arise besides those one can derive using classical formulas. At least it provide a satisfactory heuristic test.

5.3. Observability with initial conditions

If we suppose generic initial conditions, we may disregard them and treat observability again using simple elimination. This relies on the following definition given by Diop and Fliess (cf. [DF]).

Definition 1. — *A structure S defined by the ideal $\mathcal{I}$ is observable, if for each x_i there exist an polynomial $R_i(x_i, U, Y, \Theta)$ in $\mathcal{I}$, being algebraic in x_i.*

The computation of a char. set for an ordering eliminating the X will provide a satisfactory solution. But if we consider the system:

$$\begin{aligned} x' &= u \\ y &= \sin(x). \end{aligned}$$

we may reduce to the diff. alg. equations:

$$\begin{aligned} x' &= ux \\ (y')^2 &= -(uy)^2 + u^2, \end{aligned}$$

being not observable according to the above definition. But of course, one needs to complete it with initial conditions to obtain an equivalent system. So one may have to deal with more complex criteria. The X being algebraic over the field extension $\mathcal{G} = \mathcal{F}\langle U, Y, \theta\rangle$ is certainly a sufficient condition of observability. On the other hand, a necessary condition is that they are differentialy algebraic over $\mathcal{G}$. The dubious cases should be solved using initial conditions, but as above, one will get in great trouble with transcendental constants.

6. Conclusion

We have faced here two types of difficulties. The first are related to the intrinsic complexity of any formal resolution of systems of differential equations. The quasi-equivalence between the problems of differential algebra and the corresponding problem in controls make us think that we cannot escape those difficulties. Other troubles are more strongly related to the way one poses the problem. And perhaps this must depend on the concrete example one wishes to consider and cannot be solved from a pure abstract standpoint.

We would like at least to escape problems of number theory which seem at first sight an artefact, even if they naturally occur in the situation described above.

On the other hand, one could expect that most interesting systems in control share some algebraic properties making their formal resolution easier. E.g. it is shown in [FLMR] that *differential flatness* allows to reduce to purely algebraic problems after an algebraic change of coordinates. But to test such a property seems a real challenge for algebraist. So control theory will probably continue to motivate in the future new researchs in differential algebra.

7. References

[C] G. CARRA'-FERRO *Gröbner bases and differential ideals,* Proc. of AAECC5, Menorca, Spain, 129–140, Springer Verlag, June 1987.

[DF] S. DIOP and M. FLIESS, *On non-linear observability,* proceedings of ECC'91, Grenoble, France, 170–176, 152–157, HERMES, 1991.

[F] M. FLIESS, *Automatique et corps différentiels,* Forum Math. **1**, 227–238, de Gruyter, 1989.

[FLMR] M FLIESS, J. LÉVINE, P. MARTIN AND P. ROUCHON, *Sur les systèmes non-linéaires différentiellement plats,* preprint, 1991.

[GL] S.T. GLAD and L. LJUNG, *Parametrization of nonlinear model structure as linear regressions,* Proc. of IFAC World Congress, Tallin, August 1990. [GM] G. GALLO and B. MISHRA, *Efficient algorithms and bounds for Wu-Ritt characteristic sets,* proceedings of MEGA'90, Castiglioncello, Progress in Math. **94**, Birkhäuser, 1991.

[K] I. KAPLANSKY, *Differential Algebra,* Hermann, Paris, 1957.

[L] D. Lazard *A New Method for Solving algebraic systems of positive dimension,* prépublication du LITP, Paris VII, n° 89-77, invited conference to AAECC'7, Toulouse, 1989.

[O1] F. OLLIVIER, *Le problème de l'identifiabilité structurelle globale : étude théorique, méthodes effectives et bornes de complexité,* PHD Thesis, École Polytechnique, 1990.

[O2] F. OLLIVIER *Standard Bases of differential ideals,* Proc. of AAECC'8, Tôkyô, Lecture Notes in Computer Science, Springer, 1990.

[O2] F. OLLIVIER, *Generalized standard bases and application to control theory*, proceedings of ECC'91, Grenoble, France, 170–176, HERMES, 1991.

[R] J.F. RITT, *Differential Equations from the Algebraic Standpoint,* Amer. Math. Soc. Colloq. Publ., vol. 14, A.M.S., New-York, 1932.

[S] J. SHACKELL, *A differential-equation approach to functional equivalence,* proc. of ISSAC'89, 7–10, ACM Press, 1989.

[W] WU W.-T. *A Zero Structure Theorem for Polynomial Equation Solving and its applications* Proc. of ISSAC'88, Roma, Italy, Springer Verlag, 1988.