Jacobi's bound and normal forms computations. A historical survey

F. Ollivier, LIX UMRS CNRS–École polytechnique n° 7161
Email `francois.ollivier@lix.polytechnique.fr`.
`http://www.lix.polytechnique.fr/~ollivier/indexEngl.htm`

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This work is dedicated to the memory of Giuseppa Carrà Ferro and Evgeniĭ Vasil'evich Pankratiev.

Abstract

Jacobi is one of the most famous mathematicians of his century. His name is attached to many results in various fields of mathematics and his complete works in seven volumes have been available since the end of the XIXth century and are very often quoted in many papers. It is then surprising that some of his results may have fallen into oblivion, at least in part. We will try to describe some of Jacobi's results on ordinary differential equations and the available, published or unpublished material he left. We will then expose the selective interests of his followers and their own contributions.

There are in fact many interrelated results: a bound on the order of a differential system, a necessary and sufficient condition, given by a determinant, for the bound to be reached, an algorithm to compute the bound in polynomial time, and processes for computing normal forms using as few derivatives as possible.

We give for all of them the form under which they could have been proved or rediscovered, sometimes independently of Jacobi's findings. In conclusion, we give the state of the art and suggest some possible applications of Jacobi's bound to improve some algorithms in differential algebra.

1 Introduction

In two posthumous articles [18, 20], which have been recently translated[19, 21], Jacobi has introduced a bound on the order of a system of n ordinary differential equations in n unknowns. Let $A := (a_{i,j})$ be the matrix such that $a_{i,j}$ is the order of the equation u_i in the unknown function x_j. Let $J = \max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}$. A sum $\sum_{i=1}^{n} a_{i,\sigma(i)}$ is called a *transversal sum* and J is the *maximal transversal sum* . He claims that:

JACOBI'S BOUND. — *The order of the system is bounded by J.*

The bound is still conjectural in the general case.

JACOBI'S ALGORITHM. — *Jacobi gave an algorithm to compute the bound in polynomial time, viz $\mathcal{O}(n^3)$ operations, instead of trying the $n!$ permutations.*

It has been forgotten and rediscovered by Kuhn in 1955 [41], using Egerváry's results (see Schrijver's paper [55] for historical details). The idea is to find a *canon*, i.e. $\lambda \in \mathbf{N}^n$ such that, in the matrix $(a_{i,j} + \lambda_i)$ one can select maximal entries in each column that are located in different rows. Jacobi's algorithm computes the unique canon with a minimal n-uple of integres λ_i that we denote by ℓ. Let $\Lambda = \max_i \ell_i$, $\alpha_i = \Lambda - \ell_i$ and $\beta_j = \max_i a_{i,j} - \alpha_i$. The *truncated jacobian matrix* ∇ is the matrix $\left(\partial u_i / \partial x_j^{(\alpha_i + \beta_j)} \right)$.

THE TRUNCATED DETERMINANT CONDITION. — *Jacobi claims that the order of the system is equal to the bound J iff $|\nabla| \neq 0$.*

This implicitly assumes the *strong bound*, defined with the convention $\mathrm{ord}_{x_j} u_i = -\infty$ if u_i is free of x_j and its derivatives, but he gives no detail about what should be done in such a case.

THE SHORTEST REDUCTION METHOD. — *Jacobi also asserts that it is possible to compute a normal form using only ℓ_i derivatives of equation u_i and that it is impossible to compute one using a smaller number of derivatives.*

This implicitly assumes $|\nabla| \neq 0$; if not, a greater number of derivatives may be required. This is only generically true: for some particular systems, it is possible to differentiate u_i at most $\ell_i - \ell_{i+1}$ times, for $i < n$, assuming $\lambda_1 \geq \cdots \geq \lambda_n$). Jacobi also gives a bound on the order of derivation of the u_i required to compute a resolvent representation, using x_j as a differential primitive element—assuming it is one. *Then, u_i must be differentiated a number of times equal to the maximal transversal sum of the matrix obtained by suppressing the row i and the column j in A.*

The aim of this paper is to describe the content of Jacobi's two papers [18, 20], the genesis of these results, the history of research on the subject and the state of the art. We also describe some related documents from *Jacobis Nachlaß*, kept in the *Archiv der Berlin-Brandenburgische Akademie der Wissenschaften* and give a complete[1] list of references.

2   Unpublished manuscripts

Jacobi himself is possibly the first to have forgotten his own work. According to Koenigsberger [32], his manuscripts on this subject were written around 1836 and were intended to be a part of a forsaken project of a great work on differential equations. Part of it was incorporated in his long paper on the last multiplier [22], but the bound itself was never published in his lifetime. The many versions of the text, containing numerous corrections, suggest that Jacobi was not satisfied of the redaction. However, these manuscripts were clearly intended for publication at the time he wrote them, as it is suggested by the many typographical precisions in German, written in *Kurrentschrift* in the margins.

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[1]For the best of our knowledge... any information about material or sources not mentionned here is welcome.

In his preface to [VD], were the second article [20] was first published, Clebsch asserts that it is a little posterior to Jacobi's lectures at Kőnigsberg university during academic year 1842–43, when Borchardt was his student there.

On page 2231 of manuscript II/23 a), one finds a reference to a paper [23] published in 1834 and his work on normal form computation is mentionned in the second part of the last multiplier paper [22], published in 1845, but one may think that the results were obtained before the redaction of that paper; the first page of the manuscript I/58 a) shows that Jacobi began to correct it in Roma, in december 1843. Assuming that manuscripts II/13 b), II/22, II/23 a), II/23 b) where written at the same time, as many similarities of style and content suggest, it must have been between 1834 and the end of 1843, certainly not later than 1845.

These results are a by-product of his work on the *isoperimetric problem*: *"Let U be a given function of the independent variable t, the dependent ones x, y, z etc. and their derivatives x′, x″, etc., y′, y″, etc., z′, z″, etc. etc. If we propose the problem of determining the functions x, y, z in such a way that the integral*

$$\int U\,dt$$

be maximal or minimal or more generally that the differential of this integral vanishes, it is known that the solution of the problem depends on the integration of the system of differential equations:

$$
0 = \frac{\partial U}{\partial x} - \frac{d\frac{\partial U}{\partial x'}}{dt} + \frac{d^2\frac{\partial U}{\partial x''}}{dt^2} - etc.,
$$

$$
0 = \frac{\partial U}{\partial y} - \frac{d\frac{\partial U}{\partial y'}}{dt} + \frac{d^2\frac{\partial U}{\partial y''}}{dt^2} - etc.,
$$

$$
0 = \frac{\partial U}{\partial z} - \frac{d\frac{\partial U}{\partial z'}}{dt} + \frac{d^2\frac{\partial U}{\partial z''}}{dt^2} - etc.\ etc.
$$

I will call these in the following isoperimetric differential equations ..." (see Jacobi's last multiplier article [22], GW IV p. 495).

If the highest order derivative of x_i in U is $x_i^{(e_i)}$, the order of x_j in the ith isoperimetric equation is $e_i + e_j$. Then, if the e_i are not all equal to the maximal order $e := \max_i e_i$, we cannot compute a normal form without using *auxiliary equations* obtained by differentiating the ith isoperimetric equation λ_i times with $\lambda_i = e - e_i$. It is also clear that $J = 2\sum_i e_i$ is equal to the order of the system, provided that the Hessian matrix $(\partial^2 U/\partial x_i \partial x_j)$ has full rank. We understand how this example can have inspired the whole theory.

In a letter to his brother Moritz, on Sept. 17th 1836 [26], Jacobi writes about a huge manuscript on mechanics: *"I came accross some very abstract ideas about the treatment of differential equations that appear in mechanical problems, for these differential equations, with their special form, allow some simplifications for the integration, that had not yet been remarked. These considerations will be all the more important, I think, as they extend to the differential equations that appear both in the isoperimetrical problem and the integration of partial differential equations of the first order."* It seems that the remaining manuscripts come from the time of these first investigations, so between 1836 and 1840, rather

than from some later attempt. The tables A–H given in [20] are written on page 2250.a of the manuscript; on the back there is a table of the doubles of prime numbers equal to 1 modulo 8, from 2018 to 20018, a material that could reflect the strong interest of Jacobi in number theory and prime numbers, a short time before the publication of Jacobi's *Canon arithmeticus* [25], a table of discrete logarithms, in 1839. In a letter to the Académie des Sciences de Paris, published by Liouville in 1840 [27], he evocates his work on mechanics. Liouville's commentaries are enthousiastic and suggest that a book will appear soon. But Jacobi writes to his brother in january 1841 [26] that he is embarrassed, as he does not have enough "breath" to achieve his huge project, that should have been entitled *Phoronomie*, the study of physical bodies motion.

But in 1845, Jacobi had clearly still in mind to publish a study on normal forms computation, for he wrote: *"I will expose in another paper the various ways by which this operation may be done, for this question requires many outstanding theorems that necessitate a longer exposition."* [22]

It is quite possible that this project was forgotten because of a change in Jacobi's life—who definitely left Koenigsberg to Berlin after a long trip in Italy—that also opened new contacts and new scientific issues. One may also consider a possible lack of practical examples for such a general method of computing normal forms. The algorithm may have suffered the same absence of contemporary applications. Jacobi was right claiming that the problem of computing the bound was of interest by itself, but the economical questions that strongly motivated the mathematicians of the last century were not yet considered in the middle of the XIXth century. (See section 7.)

3   The publication of the manuscripts

Jacobi's widow gave the manuscripts he left to Dirichlet who began to work for their publication with his friends Borchardt and Joachimsthal. Very few documents remain from their work and the best source seems to be Koenigsberger [32]. The papers were in great disorder. In order to class them, they gave a number to each page. These numbers appear on the envelops were pages that seemed to form a single document or to be related were stored.

Borchardt entrusted Sigismund Cohn[2], who worked on the publication of some others manuscripts of Jacobi, with the documents related to the bound. Cohn indentified (see II/13 a) two sets of manuscripts suitable for publication II/13 b), II/23 b) and worked on a transcription II/13 c) of these sometimes hardly readable texts. After his death in 1861 [24], the work was continued by Borchardt who published the first paper [18] in his journal in 1865. The second [20] was published by Clebsch in the volume *Vorlesungen über Dynamik* [VD] in 1866. This one was quoted by Sofya Kovalevskaya in one of her most famous articles [40] in 1875. The fact that these papers were written in latin did not seem to have been a trouble at that time. Cohn and Borchardt could easily write themselves some paragraphs to fill gaps in the manuscripts and

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[2]Almost nothing is known about him. There was a student of that name at Könisberg university in 1842-43; in 1846 some Sigismundus Cohn defended in Breslau an inaugural dissertation entitled *De medicina talmudica*. According to the *vita* that follows this work in the copy kept at the library of Alliance Israélite Universelle, he was a different man with no interest in mathematics.

Borchardt even tried to rewrite full passages in order to make them clearer. In his biography [32], published in 1904, Koenigsberger did not translate the many quotations in Latin, French and Italian.

Borchardt also wrote some kind of abstracts of the two papers, which show that he fully understood their content and that he did not consider what he published as devoid of rigor. A slightly ironical quotation of Jacobi himself *"Tam quaestiones altioris indaginis poscuntur."*[3] concludes the abstract of a part that clearly did not satisfy Borchardt's standards and was not kept in the published version (see II/13 c).

4 Jacobi's mathematical results

4.1 The manuscripts

The first paper [18] begins with the exposition of the bound and the truncated determinant criterion. Then, the main part of the paper is devoted to a careful explanation of the algorithm with complete proofs. To obtain this paper, Cohn put together two different texts from document II/13 b). The first 6 pages of the manuscript, reproduced in Cohn's transcription, were not kept by Borchardt (see sec. 3). They are related to the different normal forms that a given system could have, with a quite complete description of systems of 2 equations in 2 differential unkowns.

The second [20] begins with a fast exposition of the algorithm, without any proof, followed by the example of a 10×10 matrix. Jacobi then explains how to compute a normal form, using as few derivatives of each variable as possible and how to compute a resolvent representation for some variable x_κ, using again as few derivatives of each equation as possible. This paper reproduces with very few changes a single manuscript II/23 b).

Cohn considered documents II/22, II/23 a) as unusable because they investigate how Jacobi's last multiplier behaves when one changes the order on derivatives and computes a new normal form. Borchardt and him wanted to avoid the multiplier theory, possibly because Jacobi decided not to include this material in his paper on the subject. The § 17 in manuscript II/23 a) fos 2217–2220 corresponds to the same paragraph in the last multiplier paper [22][4], but the manuscript considers the general situation, whereas Jacobi retreated to the linear case in the published version.

Document II/4 is not related to the bound or normal form computation, but rather to the last multiplier theory. It seems an interesting unpublished paper of Jacobi on differential equations, including results such as the linear independence criterion given in Ritt's *Differential algebra* [54] p. 34.

4.2 The algorithm

A square table $(a_{i,j})$ being given, Jacobi calls *transversal maxima* numbers being maximal in their column, that are located in all different rows. The idea of jacobi

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[3]Then these questions require further investigation.
[4]See Crelles 29, Heft 3 221–225 or GW IV 403–407.

for computing $\max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}$ is to look to what he calls a *canon*[5], that is a $n \times n$ square table where one finds a maximal set of n transversal maxima. One starts with a table $(a_{i,j})$ and computes a canon by adding to all the elements of row i some integer ℓ_i. At each step of Jacobi's algorithm, one tries to increase the number of transversal maxima. The integers ℓ_i computed by the algorithm are the smallest ones, meaning that there is no canon derived from the table $a_{i,j}$ such that one of these integers may be smaller.

The algorithm may be sketched as follows. We start with the *preparation process*: we add suitable integers to the rows of the table, so that each row possess a *maximum*, i.e. an element being maximal in its column.

The second part of the algorithm starts with the *prepared table* the maximum of the first row being the first set of transversal maxima being considered.

A set of transversal maxima being given, we will repeatedly compute a new set containing one more element. We reorganise the rows and columns in the following way: *upper rows* are the rows containing the transversal maxima and *lower rows* the remaining ones; *left columns* are the columns containing the transversal maxima and *right columns* the remaining ones. We denote by asterisks the transversal maxima and the maxima (in their columns) that are located in the upper rows and right columns: maxima with an asterisk are the *stared maxima*. If there is some maximal element in some lower row and right column, we may already add it to the set of transversal maxima.

We say that *there is a path (transitum dari)* from row i to row j if some element of row j is equal to a stared maximum in i, or if there is a path from row i to row i' and from i' to j. Rows of the *first class* are the upper rows containing a stared element in a right column and all the rows to which there is a path from them. If there is a lower row j in the first class, there is a path from an upper row i_0 with a stared maximum α_0 in a right column to an upper row i_1 possessing an element α_1 equal to the transversal maxima of i_0, then a path from row i_1 to row i_2 possessing an element α_2 equal to the transversal maxima of i_1, then a path from row i_2 to row i_3... and at the end a path from an upper row i_{r-1} to the lower row $i_r = j$ containing an element α_r equal to the transversal maxima in i_{r-1}. We may then replace in the set of transversal maxima those located in the rows $i_0, \ldots, i_{r-1}$ by $\alpha_0, \ldots, \alpha_{r-1}$ and get a greater set of transversal maxima by adding α_r. *There is no lower series of the first class iff the set of transversal maxima is maximal.*

If so, we will need to increase some rows. The rows not in the first class, from which there is a path to a lower row form the *third class*. The *second class* contains the remaining rows. We will increase all the elements of the third class rows by the minimal integer such that one of them become equal to a stared element β in a row of the first or second class. The computations of the maxima in each column and of the partition of the rows in classes may be done in $\mathcal{O}(n^2)$ operations.

If the element β belongs to the second class, its rows goes to the third, as well as all the rows from which there is a path to it, so that the change in the classes partition may be computed in $\mathcal{O}(n)$. This may happen at most $\mathcal{O}(n)$

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[5]The choice of this strange word may be related to its use in the title: *Canon arithmeticus*, that according to Schumacher came from a play on words: the computations were done by a *Kanonier Unteroffizier*. See Jacobi's *Briefwechsel* [26] note 2 p. 62.

times before there is no more elements in the second class. If β belongs to the first class, some lower series of the third class will go to the first, and the number of transversal maxima will be increased, as we have seen. So, we need to perform the partition in classes and exhaust the second class, with total cost $\mathcal{O}(n^2)$, at most n times before we get a maximal set of n transversal maxima. The complexity of the whole algorithm is $\mathcal{O}(n^3)$.

Basically, this algorithm is the same as Kuhn's "Hungarian method" (see section 7), but Kuhn also adds constants to the columns, and not only to the rows. The complexity of Jacobi's algorithm, $\mathcal{O}(n^3)$, is the same as the complexity of the variant of Kuhn's method given by Munkres [42].

Jacobi proved further that the ℓ_i are minimal. This result is a consequence of the two following propositions: *i) there is no unchanged row of the third class, i.e. with $\ell_i = 0$; ii) the numbers added to the third class series in the algorithm are the minimal ones that may change the partitions in classes.*

On may remark that this is the only place in the two posthumous papers [18, 20][6] where Jacobi provides complete proofs of his results.

4.3 The shortest reduction in normal form

Jacobi provides [20] a method to compute a normal form, using as few derivatives as possible of the given equations

$$u_1(x_1, \ldots, x_n) = 0, \ldots, u_n(x_1, \ldots, x_n) = 0. \tag{1}$$

His results are only *generically* true; it is easy, as we will see, to make the requested hypotheses explicit. First, he implicitly assumes that the truncated determinant does not vanish. In order to provide rigorous statements, we need to translate his findings within the framework of some formalism that did not exist at this time. We will use here an elementary approach that relies on diffiety theory [50].

DEFINITION 1. — *For short, we say that λ_i is a* canon *for the order matrix $(a_{i,j})$, if $(a_{i,j} + \lambda_i)$ is a canon, i.e. a table possessing a maximal set of transversal maxima.*

Let ℓ_i be the minimal canon of the order matrix $(a_{i,j})$[7], with $a_{i,j} = \mathrm{ord}_{x_j} u_i$, $\Lambda = \max_i \ell_i$, $\alpha_i = \Lambda - \ell_i$ and $\beta_j = \max_i a_{i,j} - \alpha_i$. The truncated jacobian matrix ∇_u is the matrix $\left(\partial u_i / \partial x_j^{(\alpha_i + \beta_j)} \right)$.

We will say that an ordering $<$ on derivatives is a Jacobi ordering *for the system 1 if $k_1 - \beta_1 < k_2 - \beta_2$ implies $x_{i_1}^{(k_1)} < x_{i_2}^{(k_2)}$.*

Jacobi calls an explicit canonical form *a system*

$$x_i^{(e_i)} = f_i(x), \quad 1 \leq i \leq n$$

where the functions f_i only depend on derivatives of x_j smaller than e_j.

Jacobi's shortest reduction method may be expressed in the following way.

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[6]Together with a short passage in the second paper [20] related to resolvent computation and using the same kind of combinatorial arguments. See subsection 4.4.

[7]That we may compute using Jacobi's algorithm, see subsection 4.2.

THEOREM 2. — i) *Assume there exist some functions* $\tilde{X} : t \mapsto \tilde{x}_j(t)$, $[a, b] \mapsto$ **R** *that form a solution of the system u, such that $|\nabla_u| \neq 0$. Then, there exists a normal form of u for a Jacobi ordering, of which $\tilde{x}$ is solution, that may be computed using equation $u_i = 0$ and its derivatives up to the order ℓ_i.*

ii) Assume that ∇_u and all its minors of order $n-1$ that do not contain row μ have full rank, then there is no normal form of u that may be computed using only derivatives of u_μ up to an order strictly less than ℓ_μ.

PROOF. — i) We may reorder the unknowns x_j and the equations u_i, so that the sequence α_i of definition 1 is increasing and the n principal minors of ∇_u have non vanishing determinants. Let D_k be the determinant of the kth minor. Consider the jacobian matrix of the system $\{u_i^{(k)}|1 \leq i \leq n, \ 0 \leq k \leq \ell_i\}$, with respect to the set of derivatives $E := \{x_i^{(\alpha_i+\beta_i+k)}|1 \leq i \leq n, \ 0 \leq k \leq \ell_i\}$. Its determinant is a product of powers of the D_i and so is not vanishing. We can then use the implicit functions theorem and find, on a suitable open set containing $(\tilde{x}_1(0), \ldots, \tilde{x}_1^{(\alpha_1+\beta_1+\ell_1)}, \ldots, \tilde{x}_1(0), \ldots, \tilde{x}_n^{(\alpha_n+\beta_n+\ell_n)})$, functions expressing the derivatives of E, depending on the derivatives of the set $S := \{x_i^{(k)}|1 \leq i \leq n, \ 0 \leq k < \alpha_i + \beta_i\}$. Hence we get a normal form:

$$x_i^{(\alpha_i+\beta_i)} = f_i(x),$$

of which $\tilde{X}$ is a solution. The order associated to this normal form is a Jacobi ordering.

ii) Assume that there exists a normal form that may be computed using derivatives of u_i up to order λ_i, and that $\lambda_\mu < \ell_\mu$. Let $s := \max_{i=1}^{n} \lambda_i - \ell_i$. As $\nabla_u \neq 0$, we see that there is an element in the normal form depending of some $u_{i_0}^{(\lambda_{i_0})}$ with $\lambda_{i_0} - \ell_{i_0} = s$, and of the form $x_{j_0}^{(\beta_{j_0}+\Lambda+s)} = g(x)$, where $\Lambda = \max_i \ell_i$. Remaining elements in the normal form have left side derivatives $x_j^{(\beta_j+\Lambda+s')}$, with $s' \leq s$, so g does not depend on derivatives $x_j^{(\beta_j+\Lambda+s)}$: this implies that the minor of ∇_u obtained by supressing row i_0 and column j_0 is not of full rank, a contradiction. ∎

Jacobi gave no proof for these statements; the style of the article [20] and of most of his manuscripts on the subject is that of a mathematical cook-book: in the best case, proofs are reduced to a short sketch. Ritt surmised [53] that the bound was suggested to Jacobi by such considerations on normal form computation. This natural assumption is confirmed by Jacobi's claim [18] that his normal form reduction method provides an alternative proof of the bound. We easily see, according to the shape of the normal form above, that the order of u is $\sum_{i=1}^{n} \alpha_i + \beta_i = J$.

Jacobi also claims that there are as many normal forms of this kind as there are permutations σ such that $\sum_{i=1}^{n} a_{i,\sigma(i)} = J$. This is perhaps his single claim that does not stand, even under suitable genericity hypotheses, as shown by the example $x_1'' + x_2'' + x_3'' = 0$, $x_2' = 0$, $x_2 + x_3 = 0$: we only have one possible permutation, but two possible normal forms for shortest reductions: $x_1'' = -x_2'' - x_3''$, $x_2' = 0$, $x_3 = -x_2$ and $x_1'' = -x_2'' - x_3''$, $x_3' = 0$, $x_2 = -x_3$.

If we want to compute a normal form using a different kind of orderings, we may need to differentiate the defining equations a greater number of times. Jacobi provided bounds for the computation of resolvents.

4.4 Resolvent computation

In § 4 of the second article [20], Jacobi investigates the computation of *resolvents*[8], i.e. of normal forms such that one equation of order J, $x_{i_0}^{(J)} = f_{i_0}(x_{i_0})$, only depends on a single indeterminate x_{i_0} and the remaining ones, $x_i = f_i(x_{i_0})$, $i \neq i_0$, express the other variables as functions of x_{i_0} and its derivatives of order lower than J. He writes: *"As mathematicians use to consider such kind of normal forms before others, I will indicate how many times each of the proposed differential equations $u_1 = 0$, $u_2 = 0$, $\ldots u_n = 0$ are to be differentiated in order to make appear auxiliary equations necessary for that reduction."* Jacobi's presentation requires the implicit hypothesis $|\nabla_u| \neq 0$, but his results stand in the more general situation of quasi-regular systems[51].

Let $A := (a_{i,j})$ be defined as in the introduction, Jacobi starts with the new matrix A' defined by adding to each row of A the corresponding numbers ℓ_i defined above. He assumes that a set of transversal maxima is chosen in the canon A'; he denotes transversal maxima with asterisks and calls them *stared terms*, and he underlines the terms being equal to the stared maximum located in their column. He says that row i_1 is *attached (annexa) to row i_0* if it contains a stared term equal to some underlined term in row i_0 or in some row attached to row i_0. The row i_0 is implicitly assumed to be attached to itself. If not all rows are attached to row i_0, he increases all the rows attached to it of the same minimal number that makes new underlined elements to appear and new rows to be attached to row i_0. The process is to be continued until all rows are attached to it. We get a new matrix A''; the last step is to increase all its elements of a same number that makes the stared term in row i_0 become equal to the order J of the system u: we get a new matrix A''', which is obtained by increasing the rows of A by a sequence of numbers h_i, $1 \leq i \leq n$, that will be the researched orders of derivation.

Example 3. — We consider a simple system $x_1'' - x_2' = 0$, $x_2'' - x_3 = 0$ and $x_3' - x_2 = 0$. For this system, the order matrix

$$A = A' = \begin{pmatrix} 2^* & 1 & -\infty \\ -\infty & 2^* & 0 \\ -\infty & 0 & 1^* \end{pmatrix}$$

is already a canon. It is not possible to construct a resolvent representation using x_2 or x_3, but it is possible with x_1. There is no row attached to the first one. Increasing it by 1, row 2 becomes attached to row 1. Then, increasing rows 1 and 2 by 1, row 3 becomes attached to row 1. So, we get the matrix

$$A'' = \begin{pmatrix} 4^* & \underline{3} & -\infty \\ -\infty & 3^* & \underline{1} \\ -\infty & 0 & 1^* \end{pmatrix}.$$

The order J of the system is 5 and we need to increase all terms by 1, so that the stared term of row 1 be made equal to J. We get the new matrix

$$A''' = \begin{pmatrix} 5^* & \underline{4} & -\infty \\ -\infty & 4^* & \underline{2} \\ -\infty & 1 & 2^* \end{pmatrix}.$$

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[8]This word is not used by Jacobi, we borrow it from *Diff. alg.* [54] p. 41

To obtain A''', one has increased row 1 of A by 3, row 2 by 2 and row 3 by 1, so one needs to differentiate the first equation 3 times, the second 2 times and the third 1 time to compute a resolvent representation for x_1: $x_1^{(5)} = x_1''$, $x_2 = x_1^{(4)}$, $x_3 = x_1'''$.

THEOREM 4. — *Assume that $|\nabla_u| \neq 0$ and that x_j is a differential primitive element[9], then a resolvent representation of u using x_{j_0} may be computed using derivatives of equation u_i up to order h_i.*

Jacobi gave no proof for this result; we propose for the reader's convenience the following elementary one, that follows Jacobi's presentation.

PROOF. — Using theorem 2, we may assume that the system admits a normal form of the shape $x_i^{(a_{i,i})} = f_i(x)$, that can be obtained using derivatives of u_i up to order ℓ_i at most. Let us denote by w_i the ith equation of this normal form. We may make a more precise evaluation of the order of derivation requested to compute w_{i_0}. If $x_i^{(\alpha_{i_0} + \beta_i)}$ appears in u_{i_0}, then we will need the $(\ell_i - \ell_{i_0})^{\text{th}}$ derivative of w_i to compute w_{i_0}. Then, the row i is attached to the row i_0 in A'. We may recursively prove that, if the $(\ell_i - \ell_{i_0})^{\text{th}}$ derivative of u_i is needed, then the row i is attached to the row i_0 in A'. More precisely, if we need to increase the row i_0 of A' by $s \leq \ell_i - \ell_{i_0}$, so that row i becomes attached to it, we only need to differentiate u_i up to order $\ell_i - \ell_{i_0} - s$ in order to compute w_{i_0}.

As $|\nabla_u| \neq 0$, the order of the system is J and for computing a resolvent, we need the first J derivatives of x_{i_0}; we shall differentiate equation w_{i_0}, which is of order a_{i_0,i_0} in x_{i_0}, up to order $J - a_{i_0,i_0}$. At the beginning of the process and after each step of differentiation, if some $x_j^{a_{j,j}}$, $j \neq i_0$ appears in the right side, we may substitute to it the expression $f_j(x)$, and repeat the process until no such derivative appears[10]. If the row i of A' is attached to row i_0, then we may compute the derivative of order ℓ_{i_0} of w_{i_0} using the derivatives of u_i of order ℓ_i at most, and so we may compute a resolvent using the derivatives of u_i of order at most $J - (a_{i_0,i_0} + \ell_{i_0}) + \ell_i = h_i$. If the row i in A' becomes attached to row i_0 after this last row has been increased of e, then we may compute the derivative of w_{i_0} of order $e + \ell_{i_0}$ using the derivatives of u_i up to order at most ℓ_i. So, we may get the derivative of order $J - a_{i_0,i_0}$ of w_{i_0}, requested to get the resolvent, using derivatives of u_i up to order at most $J - a_{i_0,i_0} - (e + \ell_{i_0}) + \ell_i = h_i$, hence the result. ∎

Remark 5. — It is easily seen that the maximal possible value for h_i is $J - a_{i_0,i_0} + \ell_i - \ell_{i_0} = J - (\beta_{i_0} + \Lambda) + \ell_i$. If the equation u_i has order e_i, the sum of the h_i, that is the number of equations in the system one needs to solve to compute a resolvent, is maximal when e_i is the order of u_i in all the variables: h_i is at most $\sum_{i' \neq i} e_{i'}$ and $\sum_{i=1}^{n} h_i \leq (n-1) \sum_{i=1}^{n} e_i$. On the other hand, assume that, after some reordering, the rows are listed by successive order of "attachment" to row $i_0 = 1$. We have $h_1 = \sum_{i>1} a_{i,i}$ and for $i > 1$, $h_i \geq \sum_{k>i} a_{k,k}$, so that $\sum_{i=1}^{n} h_i \geq \sum_{i>1} (i-1) a_{i,i}$.

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[9]This means that a resolvent exists for that element, see Cluzeau and Hubert [2] for details.

[10]This idea appears in some unpublished manuscripts of Jacobi, e. g. in II/23 a) p. 2217.a: *"et simulac in dextra parte obvenit variabilis $x_{[j]}$ differentiale $[a_{j,j}]^{tum}$, eius e [u] substituo valorem $[f_j]$."* (Mathematical notations between square brackets have been changed to correspond with those used in our proof.) See our translation [21] p. 37.

Remark 6. — In more general situations, that is when $|\nabla_u| = 0$, one may need to differentiate the defining equations a greater number of times, that may also depend on the degree of the equations. But for quasi-regular systems [37, 38, 50], Jacobi's bound for the h_i still stands[51].

Jacobi did not stop at this stage; he has also provided the following elegant version of his result.

THEOREM 7. — *The order h_i up to which one needs to differentiate equation u_i in order to compute a resolvent representation for u using x_{i_0} as a primitive element, is equal to the maximal transversal sum of the $(n-1) \times (n-1)$ matrix obtained by removing from A line i and column i_0.*

Jacobi gives a complete proof of the equality between the order h_i defined by his process and the maximal transversal sum of the above theorem. It relies on the same kind of argument as the proof of his algorithm. We refer to his paper [20] for details[11].

We may remark that a naïve use of this theorem requires $\mathcal{O}(n^4)$ operations to compute the h_i using Jacobi's or Munkres' algorithm [42]. But Jacobi's process, described above, only requires $\mathcal{O}(n^3)$ operations, once the ℓ_i have been computed with the same complexity. Jacobi states first an efficient algorithmic version of his result, before giving an elegant—but less efficient—mathematical version.

These sharp bounds provided by Jacobi for computing normal forms can be used in a straightforward way to improve many algorithms developed in recent years. His results on normal form are especially important for new methods of resolution for algebraic systems relying on the representation of polynomials as *Straight-Line Programs* [12] that begin to be extended to differential systems [7, 8, 9, 10].

4.5 The bound

The work of Jacobi suffers from the lack of a rigorous theory allowing precise definitions of the mathematical objects he considers. However, if one has in mind that his goal is to consider physical situations, that imply implicit hypotheses, his proof of the bound is not so weak as one may think at first sight.

PROOF OF THE BOUND. — The proof relies on three succesive steps. The first is to claim that one can reduce to *linear* equations. This, of course, will not stand for all systems, but for those expressing the laws of mechanics or other problems of physical interest, we can take this for granted.

From a mathematical standpoint, the most general condition under which this may be done is the one given by Johnson [29], expressing that a differential system is in some way "regular"[12]. Under such hypotheses, one may from a system $u_i = 0$ build a new system $\delta u_i = 0$, which is nothing else than what is described by Johnson as "Kähler differentials" [28]. Let $\tilde{X} : t \mapsto \tilde{x}(t)$ be a solution of u. It is a *quasi-regular* solution of u at t_0 if for all $r \in \mathbf{N}$, when substituting to any derivative $x_j^{(k)}$ the value $\tilde{x}_j^{(k)}(t_0)$, the jacobian matrix of the system $u, \ldots, u^{(r)}$ has full rank $n(r+1)$. Then the jet of $\tilde{X}$ at t_0 is a regular

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[11]See our translation [21] p. 62–63.
[12]This was to be developed later by Kondratieva *et al.* [35, 37, 38, 50].

point of the subspace V of $\mathbf{J}^\infty(\mathbf{R}, \mathbf{R}^n)$ defined by u and the order of u, that is also the dimension of V, is equal to the dimension of its tangent space, defined by δu.

It is easily seen that, if $\nabla_u \neq 0$ for $\tilde{X}$, then $\tilde{X}$ is quasi-regular. We say "quasi-regular", for some "singular solutions", according to Ritt's teminology, such as $x = 0$ for the equation $(x')^2 - 4x = 0$, may satisfy the Johnson condition.

Jacobi uses in fact a stronger result, but without proof, claiming that if the system u has order e and if its general solution depends on e arbitrary constants $a_1, \ldots, a_e$, then the set $\partial x / \partial a_i$ is a basis of solutions of the linear system δu. I don't know if it was "well known" at that time.

The next step is more surprising, for Jacobi claims, with no justification, that it is enough to consider a linear system with constant coefficients. There is a beginning of proof, striked out by Jacobi on page 2203.a of manuscript II/13 b): *"In exploring the order of a system, as one considers only the highest derivatives in the linear differential equations to which the proposed ones have been reduced, one may assume that the coefficients are constants. For, having differentiated the equations [$\delta u = 0$] many times, in order to obtain new equations"...* This interrupted sentence suggests that the shortest reduction process [20] used to compute normal forms was the basic idea.

In fact, provided that $|\nabla_u|$ does not vanish, we only need to consider the coefficients of the leading derivatives[13] when computing the order: derivatives of the coefficients will only affect smaller derivatives of the variables. If $|\nabla_u|$ vanishes, assume that the equations u_i are sorted by increasing α_i and that i_0 is the smallest integer such that the first i_0 rows of ∇_u are linearly dependent, satifying $\sum_{i=1}^{i_0} c_i l_i = 0$, then we may in δu replace δu_{i_0} by $\sum_{i=1}^{i_0-1} c_i (\delta u_i)^{(\ell_i - \ell_{i_0})}$, as $c_{i_0} \neq 0$, this new system is equivalent to δu and the leading derivatives in δu_{i_0} have been removed: so the new system has a strictly smaller Jacobi number J. This gives an easy proof by induction of the result.

We could conclude from such considerations on normal forms of linear systems, but Jacobi uses a different kind of argument for the last step of his proof. Having reduced his investigations to the case of a linear system with constant coefficients, he looks for solutions of the form $x_j = c_j e^{\lambda t}$. Substituting such an expression in his linear equations, he gets a system of the form $\sum_{j=1}^n P_{i,j}(\lambda) c_j = 0$, $1 \leq i \leq n$ where $\deg_\lambda P_{i,j} = a_{i,j}$. The number of possible values for λ is the degree of the determinant $|P|$, which is at most $\max_{\sigma \in S_n} \sum_{i=1}^n a_{i,\sigma(i)} = J$, with equality if $|\nabla_u| \neq 0$. Jacobi does not consider the case of multiple solutions, etc. but there is no difficulties.

We cannot know precisely how Jacobi could have detailed his demonstration. However, we have shown that we can design a complete proof, using elementary arguments, following the indications he left in his manuscripts, provided that we retreat to the safe ground of regular systems. For an account of the research on singular solutions during the first half of the xix[th] century, see the work of Houtain [16].

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[13]We call here a leading derivative of an equation, a derivative $x_j^{(k)}$ such that $k - \beta_j$ is maximal, i.e. with a maximal Jacobi order.

5 The second part of the XIXth century

The first publication of these two papers[18, 20] in 1865 and 1866 and a new publication in 1890 in the volume V of Jacobi's complete works did not stimulate further research on the subject in Germany. The works of his continuators during the XIXth century are very superficial. They did not seem to have considered the subject could deserve a real mathematical effort.

Nanson in 1876 [49], considers the linear case with constant coefficients, in 2 and 3 variables. He rejects the idea of substituting $x_i = c_i \mathrm{e}^{\lambda t}$ in the equation, in order to obtain the order, claiming that one should first compute the order in order to be sure that a complete solution of that kind could be obtained. He proceeds heuristically, eliminating one variable after the other and bounding at each step the orders in each variable of the equations he gets.

Jordan in 1883 [30] considered the non linear situation with 4 variables. He tried to eliminate x_2, x_3, x_4 in order to compute a resolvent for x_1, using arguments that only work in the most general situation. From heuristic considerations on the number of derivatives to eliminate, he established theorem 7 in four variables: one needs to differentiate u_i a number of times equal to h_i, the maximal transversal sum of the matrix obtained by removing from the order matrix A row i and column 1. So, the order of the resolvent will be $\max_{i=1}^{4} a_{i,1} + h_i = J$.

Nanson, who referred to Boole [1] for systems of order 1 and Cournot [6] for systems of two equations, did not quote Jacobi. Jordan—who did not quote Nanson—did not have a full view of Jacobi's work. He claimed that Jacobi had given an "indirect" proof and that he would give a "direct" one.

The work of Chrystal in 1895 [3] was rigorous, but he only considered the easy linear case with constant coefficients—Jacobi's arguments only worked for all different eigenvalues. Ritt, who gave these references [53] also referred to a paper by Sarminski (*Communications of the University of Warsaw*, 1902) that I was unable to find.

6 Ritt's work

Ritt, who was known to be fluent in many languages, certainly had a better view of Jacobi's results. However, a century after Jacobi wrote them, the style and spirit of mathematics did change. One expects rigorous proofs, but also intrinsic results, attached to geometrical objects. One thinks of varieties (or "components") and not of systems: more precisely, in his article [53], published in 1935, a system means a component. It is remarkable that this change also concerned a mathematician like Ritt, who knew well some very applied style of mathematics[14] and whose activity was dominated by the spirit of "classicism"[15].

One interest of Ritt in this subject was to secure results that could be applied to components intersection and it is not the case for Jacobi's bound (see

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[14]He worked performing computations in the Naval Observatory in Washington during his studies and helped to organise a computation group working for the US artillery during World War I [15].

[15]In Ritt's obituary [45] p. 310, E.R. Lorch writes *"His media are complex function theory of the nineteenth century and differential equations. Much of his work could have been written a half century earlier."*

Diff. Alg. [54] p. 138–144). Having developed a theory that allows to characterize "singular components", he wonders if the bound stands for all components, including those that do not satisfy natural hypotheses of regularity: a difficult question that was certainly not considered by Jacobi.

The less convincing part of Jacobi's "proof" is to go from time varying systems to constant coefficients. Ritt's proof in the linear case [53] solves the problem. But Ritt, who only considers [53, 54] elimination orderings does not prove the necessary and sufficient condition for the bound to be reached, given by the non vanishing of the truncated Jacobian. Such a condition is, as we have seen in 4.5, more easily proved with an orderly ordering for an adapted order defined in this way: $\widetilde{\mathrm{ord}}_{x_j} u_i := \mathrm{ord}_{x_j} u_i - \beta_j$, with β_j defined as in def. 1. Ritt's proof relies on some simplified method for computing a characteristic set in the linear case. He proves the *strong bound*, using the convention $\mathrm{ord}_{x_j} u_i = -\infty$ if x_j and its derivatives do not appear in u_i.

His 1935 paper [53] concludes with a proof of the bound for any component of dimension zero of a system of two polynomial equations A and B in two variables, that is reproduced with a few modifications in *Diff. alg.* [54] p. 136–138. One may remark that in the 1935 article [53], a footnote precises that if one of the variables does not appear in A, the order of A in this variable is 0. In [53], a new footnote claims that we can in fact prove the strong bound. The requested modifications in the proof seem easy, but are not given explicitly. In 1935 [53], Ritt refers to Gourin [13] for the following result: if a zero dimensional differential ideal $\mathcal{I}$ contains a zero dimensional differential ideal $\mathcal{J}$, then $\mathrm{ord}\,\mathcal{I} < \mathrm{ord}\,\mathcal{J}$.

An important argument is not explicitly stated in the proof. Ritt reduces the situation to the case of two polynomials, A and E, were E depends only on x_1. He claims that E must effectively depend on that variable. For this, he needs to use the fact that a single equation in two variables cannot define a component of dimension 0, which is true by *Diff. alg.* [54] chap. III § 1 p. 57.

In *Diff. alg.* [54] p. 139–144, Ritt also proved an important result. He considers irreducible differential polynomials in two variables, and investigates the order of the intersection of their *general solutions*. He first proves that Jacobi's bound stands if A and B have order not greater than unity. Then, he exhibits a family of polynomials of order $r > 3$ in x_1 and x_2, the general solution of which intersects the manifold of x_1 in an irreducible manifold of order $2r - 3$. So, the bound cannot stand for manifolds, but just for systems.

7   The assignment problem

In 1944, the R.A.F. tried to optimize the reaffectation of soldiers of disbanded units [55]. No practical solution could be used before the end of World War II, but this initiated the first research on the problem. It was then considered to optimize the affectation of n workers to n tasks, $a_{i,j}$ representing the productivity of worker i if affected to task j. One looks for a maximum, with the constraint that two different workers must be given two different tasks. The Monge problem [48] may be considered as a first, continuous, example of such problems (how to transport earth from a given area to some other with the least amount of carriage).

It is not the place to give much details about the discovery of the Hungarian method by Harold Kuhn in 1955 [55]. Anyway, it may be of interest to consider the situation from the standpoint of the transmission of mathematical results. Jacobi's algorithm was sleeping in papers written in a dead language, with titles that cannot be related to the assignment problem. It also seems that the mathematical community was not always of a great help for the practitioner who wanted to solve his optimization problem in a short time. It is amazing that trying $n!$ possibilities may have been considered as an acceptable solution, provided that their number is finite, in the middle of the xx^{th} century (see Schrijver [55] p. 8). For Jacobi, trying $n!$ solution was not a solution at all. He claimed indeed to look for *a solution*, whereas we would rather say that we are looking for an efficient one. The efficiency issue was at that time—very strongly—implicit.

One may also notice that rediscovering Jacobi's method took more than 10 years, from 1944 to 1955, and that prominent mathematicians such as John von Neuman considered the problem. It could have been much longer if Kuhn did not translate from Hungarian Egerváry's paper [11] that allowed him to conclude. Inspired by Kőnig, a pioneer of graph theory[39], Egerváry considered the problem as a weighted variant of the maximal matching problem, but he did not give a polynomial time algorithm [31]. Possibly, Egerváry could have contributed to the question himself[16], but it seems that the research was concentrated in the eastern part of the world, mostly in the United States. It was also strongly motivated by economical and organizational issues and one may guess that it did not facilitate collaborations with eastern scientists during the cold war. Egerváry heard of Kuhn's algorithm as late as in 1957 and went back to such matters with two papers on the transportation problem. Tragic circumstances interrupted his research[17].

Richard Cohn is the first, for the best of my knowledge, to have made a link between Jacobi's work and the assignment problem [5], but the information did not spread into the optimization community before 2005.

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[16]Kőnig commited suicide in october 1944, a few days before Budapest Jews were forced into the ghetto and their deportation began.

[17]He killed himself in November 1958. This is often attributed to "bureaucratic persecutions", which may suggest "communist repression". The facts are quite different, according to the testimony of one of his former students, Pál Rózsa. Egerváry enjoyed very good working conditions as a member of the Hungarian Academy of Science. He was awarded twice the Kossuth price in 1949 and 1953, and also the Order of Work by the Government in 1956.

As a consequence of the failed Revolution of 1956, some liberal changes appeared in Hungarian universities. Contracts with industrial firms were made possible, in order to improve the situation of the professors and their staffs. They were not prepared to accountancy and the procedure was not precisely regulated. During a general revision of book-keeping in 1958, irregularities were found and exemplary measures were taken against four department, including the one Egervary was heading at the Budapest University of Technology. He was forced to retire and his deputy professor arrested. Freed from all charges after some months and having been dismissed from the university, he could find a research job in some electric factory. The other dismissed professors could also continue their careers in research institutions. The significance of the irregularities had just been exagerated.

In that period, innocents had been sent to jail and Egerváry might have feared such a fate. The feeling of Prof. Rózsa is that Egerváry was a "gentleman of the old school" who could not stand being interrogated by the police and accused of neglecting his duty because he just signed the financial documents, letting his deputy professor do an administrative work for which he had no taste.

8 The second part of the xxth century

The second part of the xxth century is dominated by the work of Richard Cohn and his students.

8.1 Greenspan's bound

Greenspan proved a different bound, in the framework of difference algebra [14] in 1959. It is easily translated in differential algebra: let $r_j = \max_{i=1}^{n} \mathrm{ord}_{x_j} u_i$ and η_i be the greatest integer such that $\mathrm{ord}_{x_j} u_i^{(\eta_i)} \leq r_j$, $1 \leq j \leq n$, Greenspan's bound is $G := \sum_{i=1}^{n} r_j - \max_{i=1}^{n} \eta_i$. It was proved by Cohn in 1980 [4] that the order of any zero dimensional component of an arbitrary differential system is bounded by G. One may remark that this result implies Jacobi's bound in two variables, and that it may be proved using an adapted version of Ritt's proof.

8.2 Lando's bound

Barbara Lando proved in 1970 [43] the *"weak bound"* for order one differential systems. The *"weak bound"* means that if x_j and its derivatives do not appear in u_i, we use the convention $\mathrm{ord}_{x_j} u_i = 0$. This result was translated in difference algebra [44] in 1972. *Proving the strong bound for order one systems would imply the strong bound for any system, for the strong bound is compatible with the classical reduction of a system to order one equations.* Lando's proof uses results on matrices of zeros and ones that are very close to some theorems of Kőnig and Egerváry.

B. Lando also proved that any order matrix A is the order matrix of a system for which the bound is reached.

8.3 Tomasovic and PDE systems

There was a attempt to generalize the bound to partial differential systems [56], due to Tomasovic in 1976. Consider a system of n partial differential equations in n variables and m derivatives. Let $\mathcal{P}$ be a component of of $\{u\}$ and

$$\omega_{\mathcal{P}}(r) = \sum_{i=0}^{m} a_i \binom{r+i}{i}$$

be the Hilbert polynomial of $\mathcal{P}$. If the dimension of $\mathcal{P}$ is 0, then $a_m = 0$. The *Jacobi conjecture* of Tomasovic states that $a_{m-1} \leq J$, where J is defined as above, according to the order matrix of the PDE system. This conjecture had already been presented by his thesis adviser, Kolchin, in 1966 [33]. Tomasovic proved it for linear systems and for $n \leq 2$. A proof in the linear case was also given by Kondratieva *et al.* [36].

Tomasovic's results remained unpublished, due to his untimely death. His dissertation contains interesting material that requires further examination.

8.4 Order and dimension

In 1983, Cohn proved that the bound would imply the *"dimension conjecture"*: *every component defined by a system of r equations has differential codimension*

at most r. Tomasovic proved in the chap. 6 of his thesis [56] that the dimension conjecture is equivalent to this one: *If a system has a component of differential dimension* 0, *then its Jacobi number* J *is not* $-\infty$.

We have seen in section 6 that Ritt's proof in two variables requires the dimension conjecture for $r = 1$. More precisely, Cohn's proof shows that proving the bound for a system of n equations implies the dimension conjecture for a system of equations in $r < n$ variables. Furthermore, Cohn proved that the weak bound and the dual of "Bézout's bound"[18] also imply the dimension conjecture.

It is known that the intersection of two manifolds of differential codimensions r and s may contain components of codimension greater that $r + s$ (see *Diff. alg.* [54] p. 133) and, as we have already seen above in section6, Jacobi's bound may only be expected to stand for systems and not for manifolds. If the examples given by Ritt in these two cases are clearly distinct, we may remark that they are both closely related to the structure of the singular place of the manifold. We still need to understand better such paradoxical behaviours and their possible connections.

8.5 Kondratieva's proof

As we have already seen in subsection 4.5, the case where Jacobi's linearization argument works corresponds to the regularity hypothesis defined by Johnson in order to prove Janet's conjecture [29] in 1978: *the differentials* $du_i^{(k)}$, $1 \leq i \leq n$, $k \in \mathbf{N}$, *are linearly independent.* Kondratieva *et al.* call such systems *independent systems*; they were able in 1982 [35] to prove the strong bound, using first linearization as in Jacobi's approach [18], then Ritt's proof for the linearized system du. However, this result received little attention, the paper being written in Russian and difficult to find. A new proof, also valid for independent partial differential systems has been given in 2008 by the same authors [37, 38].

8.6 Other works

In 1960, Jacobi's strong bound was rediscovered independently by Volevich [57] for arbitrary linear systems, assuming that the truncated determinant does not vanish. One may also mention the works of Magnus, [46, 47] who refers to Chrystal and Jacobi.

9 Beginning of the XXIth century

Hrushovski in 2004 [17] proposed a proof for the bound in difference algebra. His method is completely different from those used so far in this field, but it does not seem possible, or at least not easy, to deduce from this result a proof in the differential case.

See our article [50] for a proof of the truncated jacobian condition in the framework of diffiety theory, assuming the regularity condition of Johnson. A generalization to underdetermined systems is also considered. Kondratieva *et al.*

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[18]It is defined as $\sum_{i=1}^{n} \max_j \operatorname{ord}_{x_j} u_i$.

[37, 38] provided a proof of Jacobi's bound for independent partial differential systems.

In 2001, Pryce [52] rediscovered Jacobi's shortest reduction method in order to provide an efficient method of computing power series solutions of implicit differential algebraic systems.

10 Conclusion

One says that Jacobi once told to a student who wanted to read all the mathematical literature before starting his research: *"Where would you be if your father before marrying your mother had wanted to see all the girls of the world?"* So, we may hope that he would have forgiven us for having forgotten some of his results.

We see nevertheless that a closer look to the past may be fruitfull. Contemporary mathematicians who turn to computer algebra will find some common spirit with the approach of these times, when a great familiary with hand computation produced a still unformal but deep attention to efficiency.

Some of the results presented here, beyond their intrinsic mathematical interest, could help producing improved bounds of complexity or designing new algorithms. E.g., the shortest reduction leads to the choice of an ordering for which the computation of a characteristic set may be easier.

Warning

The bibliography is divided in four parts. The first, *Manuscripts* contains primary material, mostly Jacobi's manuscript, comming from *Jacobis Nachlaß, Archiv der Berlin-Brandenburgische Akademie der Wissenschaften*. Documents are denoted

by their archive index, e.g. [I/58 a)]. The second, *Complete works* contains the book were Jacobi's work were published. They are denoted by [GW ...], [VD], ... The third *Crelle Journal* contains the issues were the quoted papers of Jacobi were published. Their are denoted by [Crelle...] The fourth and last contains the remaining material. They are denoted by numbers.

References

Manuscripts

[I/58 a)] *Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi.* Copyist work with corrections by Jacobi und Crelle, with the date *Roma d. 27 Dec. 1843* written by Jacobi on the first page. It corresponds to the first part of the article [22].

[II/4] *De particulari (speciali) determinantibus genere, quibus aequationum differentialium linearium integratio inuititur.* Pages 747–761. 30 p. Manuscript written by Jacobi with pencil notes by Borchardt and reference to a publication in Crelle Bd 22, S. 319–359.

[II/10] *Papiere zu meiner un Cohn's Abschrift von Jacobis Abhandlungen über Reduction simultaner Differentialgleichungen in die canonische Form gehörig, die aber ungeeignnet sind an Clebsch zur Ansicht gesendt zu werden.* Papers from Borchardt's hand with computations and notes related to Jacobi's papers.

[II/13 a)] Letter from Sigismund Cohn probably to C.W. Borchardt. Hirschberg, 25.8.1859., 3 p.

[II/13 b)] *De ordine systematis aequationum differentialium canonici variisque formis quas inducere potest.* Manuscript by Jacobi. Pages: 2186–96, 2200–2206. 35 p. The basis of Cohn's transcription II/13 c). Two different fragment: pages 2206, 2005, 2004–2200 (§ 21–23) and pages 2186–2189, 2196, 2195, 2191–2194, 2190 (§ 19).

[II/13 c)] Cohn [Sigismund]; Transcription from p. 2205, 2206, 2204, 2203, 2202, 2201, 2200, 2187, 2188, 2189, 2196, 2195, 2191, 2192, 2193, 2194, with an abstract by Borchardt. 39 S. The basis of the first article [18].

[II/22] Jacobi: *De reductione simplicissima systematis aequationum differentialium ad formam canonicam.* manuscript by Jacobi, pages: 2182–2213 and 2207–2213. Two fragments 2182, 2183, 2212, 22313 (§ 16); 2184, 2185 (§ 18), the remaining pages are computations and notes in German.

[II/23 a)] *Reductione simultaner Differentialgleichungen in ihre canonische Form und Multiplicator derselben.*, manuscript by Jacobi, pages: 2214–2237. Fives different fragments: 2214–2216; 2217–2220 (§ 17) (see our translation of this part [21] 36–43); 2221–2225 (§ 17); 2226–2229; 2230–2232, 2235, 2237, 2236, 2238 (numbered from 1 to 13).

[II/23 b)] *De aequationum differentialium systemate non normali ad formam normalem revocando*, manuscript by Jacobi p. 2238, 2239–2241, 2242–2251. 25 p. Envelop by Borchardt. The basis of the second article [20].

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Complete works

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