

Intrinsic and Apparent Singularities in Flat Differential Systems

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In this paper, we study the singularities of locally flat systems, motivated by the solution, if it exists, of the global motion planning problem for such systems, in the spirit of [5]. More precisely, flat outputs may be only locally defined because of the existence of points where they are singular (a notion that will be made clear later), thus preventing from planning trajectories crossing these points. Such points are of different types. Some of them can be easily ruled out by considering another non singular flat output, defined on an open set intersecting the domain of the former one and well defined at the point in question. However, it might happen that no well-defined flat outputs exist at all at some points. We call these points *intrinsic* singularities and the other ones *apparent*. A rigorous definition of these points is introduced in terms of atlas and charts in the framework of the differential geometry of jets of infinite order and Lie-Bäcklund isomorphisms (see [10, 12]). We then give a criterion allowing to effectively compute intrinsic singularities. Finally, we show how our results apply to global motion planning of the well-known example of non holonomic car.

1 Introduction

Differential flatness became in the past two decades a central concept in non-linear control systems. See [9, 10] and [12] for a thoroughgoing presentation.

Consider a non-linear system on a smooth n -dimensional manifold X given by

$$\dot{x} = f(x, u)$$

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where $x \in X$ is the n -dimensional state vector and $u \in \mathbb{R}^m$ the input or control vector, with $m \leq n$ to avoid trivial situations.

This system is said to be locally (differentially) flat if there exists an m -dimensional vector $y = (y_1, \dots, y_m)$ such that the following statements hold:

- y and its successive derivatives \dot{y}, \ddot{y}, \dots are locally differentially independent;
- y is a locally smooth function of x, u and derivatives of u up to a finite order $\beta = (\beta_1, \dots, \beta_m)$, i.e. $y = \Psi(x, u, \dot{u}, \dots, u^{(\beta)})$, where $u^{(\beta)}$ stands for $(u_1^{(\beta_1)}, \dots, u_m^{(\beta_m)})$ and where $u_i^{(\beta_i)}$ is the β_i th order time derivative of u_i , $i = 1, \dots, m$;
- x and u are locally smooth functions of y and its derivatives up to a finite order $\alpha = (\alpha_1, \dots, \alpha_m)$, i.e. $(x, u) = \Phi(y, \dot{y}, \dots, y^{(\alpha)})$.

Then the vector y is called *flat output*.

The flatness notion is best understood when regarding Φ and Ψ as smooth functions defined on infinite order jet spaces [11, 15, 10, 12]. They are then called *Lie-Bäcklund isomorphisms* and are inverse one of each other. Let us stress that these functions may be defined on suitable open sets that need not cover the whole space and we may want to know where such isomorphisms do not exist at all, a set that may be roughly qualified as *singular*, thus motivating the present work.

This work is indeed a preliminary one. It has been partly inspired by the global motion planning problem of the well-known non-holonomic car (see section 3). This relatively simple example might help the reader to verify that the concepts we introduce, in the relatively arduous context of Lie-Bäcklund isomorphisms, are nevertheless intuitive and well suited to this situation, in particular the notions of *apparent and intrinsic singularities*. The latter notions are introduced thanks to the construction of an *atlas* where *charts* are made of the open sets where flat outputs are non degenerated, in the spirit of [5] where a comparable idea was applied to a quadcopter model. Intrinsic singularities are then defined as points where flat outputs fail to exist, i.e. that are contained in no above defined chart at all. Other types of singularities are called *apparent*, as they can be ruled out by switching to another flat output well defined in the same chart.

Our main result is then the effectively computable characterization of intrinsic singularities. For this purpose, we use the necessary and sufficient conditions for the existence of local flat outputs of [13] (alternative approaches to compute flat outputs may also be found in [4, 3]). They are exploited in the following way.

We firstly consider the above system in the locally equivalent implicit form: $F(x, \dot{x}) = 0$ and assume that F is a meromorphic function. We then introduce the operator τ , the trivial Cartan field on the manifold of global coordinates $(x, \dot{x}, \ddot{x}, \dots)$, given by

$$\tau = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}},$$

thus satisfying the elementary relations $\tau x^{(k)} = x^{(k+1)}$ for all $k \in \mathbb{N}$. The first step of the method of [13] consists of computing the *diagonal* or

Smith-Jacobson decomposition of the polynomial matrix (1.1):

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} \tau \quad (1.1)$$

a matrix lying in the ring of matrices whose entries are polynomials in the operator τ with meromorphic coefficients. Then, we prove that a necessary condition for local flatness is that the matrix $P(F)$ is *hyper-regular* in the corresponding open set (see also [12] where this result was implicit but not completely stated and proven) and show that intrinsic singularities are contained in the intersection of the sets where $P(F)$ is *not* hyper-regular. As a corollary, we deduce that if an equilibrium point is an intrinsic singularity, the system is not first order controllable at this point.

The paper is organized as follows. In section 2, we introduce the basic language of Lie-Bäcklund atlas and charts. Then this leads to a computational approach for calculating intrinsic singularities. In particular, their links with the non-hyper-regularity of the polynomial matrix (1.1) of the variational system is established in Proposition 1 and Theorem 1, and then specialized in Corollary 1 to the case of equilibrium points.

In section 3, we apply our results to the non holonomic car. We build an explicit Lie-Bäcklund atlas for this model, compute explicitly the set of intrinsic singularities and apply the atlas construction to trajectory planning where the route contains several apparent singularities. Finally, conclusions are drawn in section 4.

2 Lie-Bäcklund Atlas, Apparent and Intrinsic Singularities

Recall from the introduction that we consider a controlled dynamical system, given in explicit form by:

$$\dot{x} = f(x, u), \quad (2.1)$$

where x evolves in some n -dimensional manifold X . The control input u lies in \mathbb{R}^m . Then the system can be seen as the zero set of $\dot{x} - f(x, u)$ in $TX \times \mathbb{R}^m$, where TX is the tangent bundle of X .

Converting the system to its implicit form consists in computing its image by the projection π from $TX \times \mathbb{R}^m$ onto TX to get the implicit relation:

$$F(x, \dot{x}) = 0, \quad (2.2)$$

where $F : (x, \dot{x}) \in TX \mapsto \mathbb{R}^{n-m}$ is a meromorphic function, with $m \leq n$.

Following [12, 13], we embed the state space associated to (2.2) into a diffeity (see [15]), i.e. into the manifold $\mathfrak{X} \triangleq X \times \mathbb{R}_\infty^n$, where we have denoted by \mathbb{R}_∞^n the product of a countably infinite number of copies of \mathbb{R}^n , with coordinates $\bar{x} \triangleq (x, \dot{x}, \ddot{x}, \dots, x^{(k)}, \dots)$,

endowed with the trivial Cartan field: $\tau_{\mathfrak{X}} = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}$. The integral curves

of (2.2) thus belong to the zero set \mathfrak{X}_0 of $\{F, \tau_{\mathfrak{X}}^k F \mid k \in \mathbb{N}\}$ in \mathfrak{X} , i.e.

$$\mathfrak{X}_0 = \{\bar{x} \in \mathfrak{X} \mid \tau_{\mathfrak{X}}^k F(\bar{x}) = 0, \forall k \in \mathbb{N}\}.$$

Therefore, the system trajectories are uniquely defined by the triple $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ that we call *the system* from now on (see [12]). Equivalently, we may also consider that this system is defined in \mathfrak{X}_0 .

Note that there might exist points $\bar{x}_0 = (x_0, \dot{x}_0, \ddot{x}_0, \dots, x_0^{(k)}, \dots) \in \mathfrak{X}_0$ such that the fiber $\pi^{-1}(\bar{x}_0)$ above \bar{x}_0 relative to π is empty, *i.e.* such that $\dot{x}_0 - f(x_0, u) \neq 0$ for all $u \in \mathbb{R}^m$. In that case, \mathfrak{X}_0 should be replaced by a smaller set excluding those points, still denoted by \mathfrak{X}_0 for convenience.

Let us recall the definitions of Lie-Bäcklund equivalence and local flatness for implicit systems ([12, 13]).

For this purpose, we consider two systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ where $\mathfrak{Y} \triangleq Y \times \mathbb{R}_{\infty}^q$, Y being a q -dimensional smooth manifold, with global coordinates $\bar{y} \triangleq (y, \dot{y}, \dots)$ and trivial Cartan field $\tau_{\mathfrak{Y}}$. As before, we note $\mathfrak{Y}_0 = \{\bar{y} \in \mathfrak{Y} \mid \tau_{\mathfrak{Y}}^k G(\bar{y}) = 0, \forall k \in \mathbb{N}\}$.

We say that $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are Lie-Bäcklund equivalent at a pair of points (\bar{x}_0, \bar{y}_0) if, and only if,

- (i) there exist neighborhoods \mathcal{X}_0 of \bar{x}_0 in \mathfrak{X}_0 , and \mathcal{Y}_0 of \bar{y}_0 in \mathfrak{Y}_0 , and a one-to-one mapping $\Phi = (\varphi_0, \varphi_1, \dots)$, meromorphic from \mathcal{Y}_0 to \mathcal{X}_0 , satisfying $\Phi(\bar{y}_0) = \bar{x}_0$ and such that the restrictions of the trivial Cartan fields $\tau_{\mathfrak{Y}}|_{\mathcal{Y}_0}$ and $\tau_{\mathfrak{X}}|_{\mathcal{X}_0}$ are Φ -related, namely $\Phi_* \tau_{\mathfrak{Y}}|_{\mathcal{Y}_0} = \tau_{\mathfrak{X}}|_{\mathcal{X}_0}$;
- (ii) there exists a one-to-one mapping $\Psi = (\psi_0, \psi_1, \dots)$, meromorphic from \mathcal{X}_0 to \mathcal{Y}_0 , such that $\Psi(\bar{x}_0) = \bar{y}_0$ and $\Psi_* \tau_{\mathfrak{X}}|_{\mathcal{X}_0} = \tau_{\mathfrak{Y}}|_{\mathcal{Y}_0}$.

The mappings Φ and Ψ are called *mutually inverse Lie-Bäcklund isomorphisms* at (\bar{x}_0, \bar{y}_0) .

The two systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are called *locally L-B equivalent* if they are L-B equivalent at every pair $(\bar{x}, \Psi(\bar{x})) = (\Phi(\bar{y}), \bar{y})$ of an open dense subset \mathcal{Z} of $\mathfrak{X}_0 \times \mathfrak{Y}_0$, with Φ and Ψ mutually inverse Lie-Bäcklund isomorphisms on \mathcal{Z} .

Finally, the system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is said locally (differentially) flat if, and only if, it is locally equivalent to the trivial system $(\mathbb{R}_{\infty}^m, \tau, 0)$ where τ is the trivial Cartan field on \mathbb{R}_{∞}^m with global coordinates¹ $\bar{y} = (y, \dot{y}, \dots)$, *i.e.* $\tau = \sum_{i=1}^m \sum_{j \geq 0} y_i^{(j+1)} \frac{\partial}{\partial y_i^{(j)}}$, and where 0 indicates that there is no differential equation. In this case, we say that y , or Ψ by extension, is a *local flat output*.

2.1 Lie-Bäcklund Atlas

From now on, we assume that system (2.2) or, equivalently, system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat.

We now introduce the notion of a Lie-Bäcklund atlas for flat systems. It consists of a collection of charts on \mathfrak{X}_0 , that we call *Lie-Bäcklund charts and atlas*, and that will allow us to define a structure of infinite dimensional manifold on a subset of \mathfrak{X}_0 , that can be \mathfrak{X}_0 itself in some cases.

¹The number of components of y must be equal to m (see [10, 12]).

Definition 1. A Lie-Bäcklund chart on \mathfrak{X}_0 is the data of a pair (\mathcal{U}, ψ) where \mathcal{U} is an open set of \mathfrak{X}_0 and $\psi : \mathcal{U} \rightarrow \mathbb{R}_\infty^m$ a local flat output, with local inverse $\varphi : \mathcal{V} \rightarrow \mathcal{U}$ with \mathcal{V} open subset of $\psi(\mathcal{U}) \subset \mathbb{R}_\infty^m$.

Two charts (\mathcal{U}_1, ψ_1) and (\mathcal{U}_2, ψ_2) are said to be *compatible* if, and only if, the mapping

$$\psi_1 \circ \varphi_2 : \psi_2(\varphi_1(\mathcal{V}_1) \cap \varphi_2(\mathcal{V}_2)) \subset \mathbb{R}_\infty^m \rightarrow \psi_1(\varphi_1(\mathcal{V}_1) \cap \varphi_2(\mathcal{V}_2)) \subset \mathbb{R}_\infty^m$$

is a Lie-Bäcklund isomorphism (with the same trivial Cartan field τ associated to both the source and the target) with local inverse $\psi_2 \circ \varphi_1$, as long as $\varphi_1(\mathcal{V}_1) \cap \varphi_2(\mathcal{V}_2) \neq \emptyset$.

An *atlas* \mathfrak{A} is a collection of compatible charts.

For a given atlas $\mathfrak{A} = (\mathcal{U}_i, \psi_i)_{i \in I}$, let $\mathfrak{U}_{\mathfrak{A}}$ be the union $\mathfrak{U}_{\mathfrak{A}} \triangleq \bigcup_{i \in I} \mathcal{U}_i$.

Here our definition differs from the usual concept of atlas in finite dimensional differential geometry, since, on the one hand, diffeomorphisms are replaced by Lie-Bäcklund isomorphisms and, on the other hand, we do not require that $\mathfrak{U}_{\mathfrak{A}} = \mathfrak{X}_0$. The reason for this difference is precisely related to our objective, i.e. identifying the essential singularities of differentially flat systems. This will become clear in the sequel.

2.2 Apparent and Intrinsic Flatness Singularities

It is clear from what precedes that if we are given two Lie-Bäcklund atlases, their union is again a Lie-Bäcklund atlas. Therefore the union of all open sets that form every atlas is well-defined as well as its complement, which we call the set of intrinsic flatness singularities, as stated in the next definition.

Definition 2. We say that a point in \mathfrak{X}_0 is an intrinsic flatness singularity if it is excluded from all charts of every Lie-Bäcklund atlas. All other singular points are called apparent.

Clearly, this notion does not depend on the choice of atlas and charts. The concrete meaning of this notion is that at points that are intrinsic singularities there is no flat output, i.e. the system is not flat at these points.

On the other hand, points that are apparent singularities are singular for a given set of flat outputs, but well defined points for another set of flat outputs. In the terminology of atlas and charts, an apparent singularity is a point that does not belong to some open set \mathcal{U}_i for some chart (\mathcal{U}_i, ψ_i) but belongs to another chart (\mathcal{U}_j, ψ_j) .

Note, moreover, that obtaining atlases may be very difficult in general situations and a computable criterion to detect intrinsic singularities should be of great help. A simple result in this direction is presented in the following section 2.3.

2.3 Intrinsic Flatness Singularities and Hyper-regularity

The purpose of this section is to give a characterization of the intrinsic singularities. As a by-product we will derive an algorithm to effectively compute them.

Relying on the notations defined at the beginning of the section, we next consider the variational equation, in polynomial form, of system (2.2):

$$P(F)dx = 0, \quad P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \mathfrak{x}} \tau_{\mathfrak{x}} \quad (2.3)$$

where the entries of the $(n-m) \times n$ matrix $P(F)$ are polynomials in $\tau_{\mathfrak{x}}$ with meromorphic functions on \mathfrak{X} as coefficients.

Recall that a square $n \times n$ polynomial matrix is said to be *unimodular* if it is invertible and if its inverse is also a matrix whose entries are polynomials in $\tau_{\mathfrak{x}}$ with meromorphic functions on \mathfrak{X} as coefficients. It is of importance to remark that, according to the fact that the coefficients are meromorphic functions, they are, in general, only locally defined. This local dependence will be omitted unless explicitly needed.

The polynomial matrix $P(F)$ is said *hyper-regular* if, and only if, there exists a unimodular $n \times n$ polynomial matrix U such that

$$P(F)U = \begin{bmatrix} I_{n-m} & 0_{(n-m) \times (m)} \end{bmatrix}. \quad (2.4)$$

We say that $P(F)$ is *hyper-singular* at a given point if, and only if, it is not hyper-regular at this point, i.e. if this point does not belong to any neighborhood where $P(F)$ is hyper-regular or, in other words, if at this point no unimodular matrix U satisfying (2.4) exists.

Let us denote by \mathcal{S}_F the subset of \mathfrak{X}_0 where $P(F)$ is hyper-singular. The following proposition clarifies some previous results of [12, 13] in the context of locally flat systems:

Proposition 1. *If system (2.2) is flat in a given neighborhood V of the point $\mathfrak{x}_0 \in \mathfrak{X}_0$, then $P(F)$ is hyper-regular in V .*

Proof. Assume that system (2.2) is flat in a neighborhood V of the point $\mathfrak{x}_0 \in \mathfrak{X}_0$. Then, denoting as before $\boldsymbol{\eta} \triangleq (y, \dot{y}, \ddot{y}, \dots)$ and $\boldsymbol{\mathfrak{x}} \triangleq (x, \dot{x}, \ddot{x}, \dots)$, by definition, there exists a flat output $\boldsymbol{\eta} = \Psi(\boldsymbol{\mathfrak{x}}) \triangleq (\Psi_0(\boldsymbol{\mathfrak{x}}), \Psi_1(\boldsymbol{\mathfrak{x}}), \Psi_2(\boldsymbol{\mathfrak{x}}), \dots) \in \Psi(V) \subset \mathbb{R}_{\infty}^m$ for all $\boldsymbol{\mathfrak{x}} \in V$ and conversely, $\boldsymbol{\mathfrak{x}} = \Phi(\boldsymbol{\eta}) \triangleq (\Phi_0(\boldsymbol{\eta}), \Phi_1(\boldsymbol{\eta}), \Phi_2(\boldsymbol{\eta}), \dots)$ for all $\boldsymbol{\eta} \in \Psi(V)$ such that $F(\Phi_0(\boldsymbol{\eta}), \Phi_1(\boldsymbol{\eta})) = F(\Phi_0(\boldsymbol{\eta}), \tau\Phi_0(\boldsymbol{\eta})) \equiv 0$.

Taking differentials, we show that dy is a flat output of the variational system. We denote by $P(\Phi_0)$ (resp. $P(\Psi_0)$) the polynomial matrix form with respect to τ (resp. w.r.t. $\tau_{\mathfrak{x}}$) of the Jacobian matrix $d\Phi_0(\boldsymbol{\eta})$ (resp. $d\Psi_0(\boldsymbol{\mathfrak{x}})$) (see [12, 13]).

Since $d\boldsymbol{\eta} = d\Psi(\boldsymbol{\mathfrak{x}})d\boldsymbol{\mathfrak{x}}$ and $d\boldsymbol{\mathfrak{x}} = d\Phi(\boldsymbol{\eta})d\boldsymbol{\eta}$, we get that $dx = P(\Phi_0)dy$, $dy = P(\Psi_0)dx$, $P(F)P(\Phi_0) \equiv 0$ and $P(\Phi_0)$ left-invertible, since $P(\Psi_0)P(\Phi_0) = I_m$, in the same neighborhood.

We next consider the Smith-Jacobson decomposition, or diagonal decomposition [7, Chap. 8], of $P(F)$: there exists an $n \times n$ unimodular matrix U and an $(n-m) \times (n-m)$ diagonal matrix Δ such that $P(F)U = \begin{pmatrix} \Delta & 0 \end{pmatrix}$. Partitioning U into $\begin{pmatrix} U_1 & U_2 \end{pmatrix}$, we indeed get $P(F)U_1 = \Delta$ and $P(F)U_2 = 0$. Thus, by elementary matrix algebra, taking account of the independence of the columns of both U_2 and $P(\Phi_0)$, there exists such a U satisfying $U_2 = P(\Phi_0)$.

Following [8, 13] (see also [1] in a more general context), we introduce the *free* differential module $\mathfrak{K}[dy]$ generated by dy_1, \dots, dy_m over the ring \mathfrak{K} of meromorphic functions from \mathfrak{X}_0 to \mathbb{R} and the differential quotient module $\mathfrak{H} \triangleq \mathfrak{K}[dx]/\mathfrak{K}[P(F)dx]$

where $\mathfrak{K}[P(F)dx]$ is the differential module generated by the rows of $P(F)dx$. Taking an arbitrary non zero element $z = (z_1, \dots, z_m)$ in $\mathfrak{K}[dy]$, and its image $\xi = P(\Phi_0)z$, we immediately get $P(F)\xi = P(F)P(\Phi_0)z = 0$ which proves that ξ is equivalent to zero in \mathfrak{X} . Since $U = \begin{pmatrix} U_1 & P(\Phi_0) \end{pmatrix}$ is unimodular, it admits an inverse $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ and thus $U_1V_1 + P(\Phi_0)V_2 = I_n$. Multiplying on the left by $P(F)$ and on the right by ξ , and using the relation $P(F)P(\Phi_0) = 0$, we get $0 = P(F)\xi = P(F)U_1V_1\xi + P(F)P(\Phi_0)V_2\xi = P(F)U_1V_1\xi$. Consequently, recalling that $P(F)U_1 = \Delta$, we have that $\zeta \triangleq V_1\xi = V_1P(\Phi_0)z$ satisfies $0 = P(F)U_1\zeta = \Delta\zeta$. Consequently, if the entries of the diagonal matrix Δ contain at least one polynomial of degree larger than 0 with respect to τ , say δ_i for some $i = 1, \dots, n - m$, then $\delta_i\zeta_i = 0$, and since $\zeta_i \in \mathfrak{K}[dy]$, we have proven that the non zero component ζ_i is a torsion element of $\mathfrak{K}[dy]$, thus leading to a contradiction with the fact that $\mathfrak{K}[dy]$ is free. Therefore, the entries of the matrix Δ are only elements of \mathfrak{K} and there straightforwardly exists a submatrix U'_1 such that $U' \triangleq \begin{pmatrix} U'_1 & P(\Phi_0) \end{pmatrix}$ is unimodular and satisfies $P(F)U' = \begin{pmatrix} I_{n-m} & 0 \end{pmatrix}$, thus proving that $P(F)$ must be hyper-regular in the considered neighborhood. \square

Remark 1. *The above proof may be summarized by the following diagram of exact sequences:*

$$\begin{array}{ccccccc}
& & \Phi & & & & \\
& & \rightarrow & & F & & \\
0 & \rightarrow & \mathbb{R}_\infty^m & \xrightarrow{\quad} & \mathfrak{X}_0 & \xrightarrow{\quad} & 0 \\
& & \leftarrow & & \leftarrow & & \\
& & \Psi & & & & \\
& & d \downarrow & & d \downarrow & & (2.5) \\
& & d\Phi & & & & \\
0 & \rightarrow & \mathbb{T}\mathbb{R}_\infty^m & \xrightarrow{\quad} & \mathbb{T}\mathfrak{X}_0 & \xrightarrow{P(F)} & 0 \\
& & \leftarrow & & \leftarrow & & \\
& & d\Psi & & & &
\end{array}$$

Since $\mathbb{T}\mathbb{R}_\infty^m$, is isomorphic to the free differential module $\mathfrak{K}[dy]$, then $\mathbb{T}\mathfrak{X}_0$, that can also be seen as a differential module, must also be free. In other words, the kernel of $P(F)$ must be equal to the image of $\mathbb{T}\mathbb{R}_\infty^m$ by the one-to-one linear map $d\Phi$, thus sending a basis of $\mathbb{T}\mathbb{R}_\infty^m$ (flat outputs) to a basis of $\mathbb{T}\mathfrak{X}_0$.

Remark 2. *Due to the Smith-Jacobson decomposition, the hyper-regularity property gives a practical algorithm to compute \mathcal{S}_F (see [2] and the car example in section 3.3 below). The hyper-singular set is then deduced by complementarity.*

According to Proposition 1, it is clear that on \mathcal{S}_F , the system cannot be flat. We thus have the following straightforward result:

Theorem 1. *The set \mathcal{S}_F contains the set of flatness intrinsic singularities of the system.*

In fact (see [8, 12]), \mathcal{S}_F corresponds to the points where the system is no more F-controllable, i.e. controllable in the sense of free modules, and therefore non flat (see [6, 9, 10, 12]). Note that, at equilibrium points, F-controllability boils down to first order controllability, or controllability of the tangent linear system.

Corollary 1. *The set made of equilibrium points that are not first order controllable contains the set of equilibrium points which are flatness intrinsic singularities of the system.*

As a consequence of this result, in order to identify the intrinsic singularities, it suffices to restrict our computation of the list of charts for which the local Lie-Bäcklund isomorphisms are ill-defined, only to those intersecting \mathcal{S}_F .

3 Applications: Route Planning For the Non Holonomic Car

In this section, we show how the theoretical analysis that we carried out above comes into concrete terms in a specific example.

3.1 Car Model

The car (kinematic) model is made of the following set of explicit differential equations:

$$\begin{cases} \dot{x} &= u \cos \theta \\ \dot{y} &= u \sin \theta \\ \dot{\theta} &= \frac{u}{l} \tan \phi \end{cases} \quad (3.1)$$

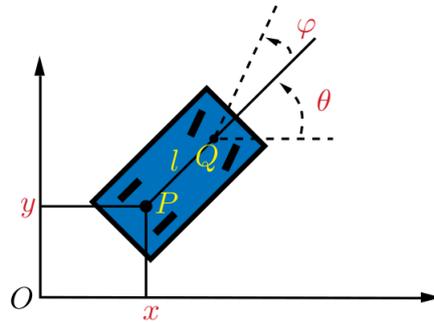


Figure 3.1: *Car Model: the state vector is made of rear axle's center coordinates (x, y) and of the angle θ between the car's axis and the x -axis. The commands are the speed u and the angle ϕ between the wheels' axis and the car's axis. The length l is the distance between the two axles.*

See figure 3.1 for details about the notations. Under this explicit form, the system evolves in the manifold $\mathfrak{X}_1 = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1$ where the variables are (x, y, θ, u, ϕ) . For the sake of clarity, we shall write $\mathfrak{X}_{11} = \mathbb{R}^2 \times \mathbb{S}^1$ for the space of state variables (x, y, θ) and $\mathfrak{X}_{12} = \mathbb{R} \times \mathbb{S}^1$ for the space of control variables (u, ϕ) . The tangent bundle of \mathfrak{X}_{11}

will be denoted $\mathbb{T}\mathfrak{X}_{11}$. With these notations, the system can be seen as the zero set in $\mathbb{T}\mathfrak{X}_{11} \times \mathfrak{X}_{12}$ of the following function:

$$\mathfrak{F}(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u, \phi) = \begin{pmatrix} \dot{x} - u \cos \theta \\ \dot{y} - u \sin \theta \\ \dot{\theta} - \frac{u}{l} \tan \phi \end{pmatrix}$$

Similarly to the general case, considered in section 2 and again following [12, 13], we shall consider the local implicit representation of the system, obtained by eliminating the controls. In this context, the dynamics (3.1) are locally equivalent to the following implicit differential equation:

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \dot{x} \sin \theta - \dot{y} \cos \theta = 0. \quad (3.2)$$

Then the system can be locally seen as the zero set of F in $\mathbb{T}\mathfrak{X}_{11}$.

In order to have a global equivalence, let us consider the canonical projection $\pi : \mathbb{T}\mathfrak{X}_{11} \times \mathfrak{X}_{12} \rightarrow \mathbb{T}\mathfrak{X}_{11}$. The zero sets of \mathfrak{F} and F are denoted $Z(\mathfrak{F})$ and $Z(F)$. It is clear that $\pi(Z(\mathfrak{F})) \subset Z(F)$.

Proposition 2. *We have*

$$\pi(Z(\mathfrak{F})) = Z(F) \setminus \mathfrak{Z} \quad (3.3)$$

with the notation $\mathfrak{Z} \triangleq \{(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) \in \mathbb{T}\mathfrak{X}_{11} \mid \dot{x} = \dot{y} = 0, \dot{\theta} \neq 0\}$.

Proof. For an arbitrary point $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) \in \pi(Z(\mathfrak{F}))$, it is straightforward to verify that it satisfies $F = 0$ and that $\dot{x} = 0$ and $\dot{y} = 0$ implies $\dot{\theta} = 0$, which proves that $\pi(Z(\mathfrak{F})) \subset Z(F) \setminus \mathfrak{Z}$.

Conversely, we consider a point $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) \in Z(F) \setminus \mathfrak{Z}$. We can always define $u = \sqrt{\dot{x}^2 + \dot{y}^2}$, and compute $\phi = \arctan(\frac{\dot{y}}{\dot{x}})$ as long as $u \neq 0$, i.e. $\dot{x} \neq 0$ or $\dot{y} \neq 0$. But, if $u = 0$, i.e. $\dot{x} = 0$ and $\dot{y} = 0$, since the case $\dot{\theta} \neq 0$ is excluded, this point belongs to $\pi(Z(\mathfrak{F}))$, hence (3.3). \square

Again following [12, 13], we embed the state space associated to (3.2) into a diffeity (see [15]), i.e. into the manifold $\mathfrak{X} = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}_{\infty}^3$, endowed with the trivial Cartan field: $\tau_{\mathfrak{X}} = \sum_{i=1}^3 \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}$, where we set $x_1 = x$, $x_2 = y$ and $x_3 = \theta$. The system is now defined in \mathfrak{X} as the zero set \mathfrak{X}_0 of $\{F, \tau_{\mathfrak{X}}^k F \mid k \in \mathbb{N}\}$.

3.2 Lie-Bäcklund Atlas for the Car Model

Now we shall define an atlas on \mathfrak{X}_0 . We simply enumerate the charts, as in [5] in the context of quadcopters. Each chart is defined on an open set associated to a local Lie-Bäcklund isomorphism ψ_i from \mathfrak{X}_0 to \mathbb{R}_{∞}^2 with local inverse denoted by $\phi_i : \mathbb{R}_{\infty}^2 \rightarrow \mathfrak{X}_0$. For simplicity's sake, we only define ϕ_i by its three first components. The other ones are deduced by differentiation, i.e. by applying $\tau_{\mathfrak{X}}$ to them an arbitrary number of times. A similar abuse of notation holds for the definition of ψ_i . A point in \mathfrak{X}_0 will be denoted \mathfrak{x} .

1. Over $U_1 = \{\dot{x} \neq 0\}$, we take $y_1 = (x, y) = \psi_1(\mathbf{r})$ and the inverse Lie-Bäcklund transform is given by:

$$\phi_1 = \begin{pmatrix} x \\ y \\ \tan^{-1}(\frac{\dot{y}}{\dot{x}}) \end{pmatrix}$$

2. Over $U_2 = \{\dot{y} \neq 0\}$, we take $y_2 = (x, y) = \psi_2(\mathbf{r})$ and the inverse Lie-Bäcklund transform is given by:

$$\phi_2 = \begin{pmatrix} x \\ y \\ \cotan^{-1}(\frac{\dot{x}}{\dot{y}}) \end{pmatrix}$$

3. Over $U_3 = \{\dot{\theta} \neq 0\}$, we take $y_3 = (\theta, x \sin \theta - y \cos \theta) = \psi_3(\mathbf{r})$. Here for the sake of simplicity, we shall denote (z_1, z_2) the components of y_3 . In that case the inverse Lie-Bäcklund transform is given by:

$$\phi_3 = \begin{pmatrix} \frac{\dot{z}_2}{z_1} \cos z_1 + z_2 \sin z_1 \\ \frac{\dot{z}_2}{z_1} \sin z_1 - z_2 \cos z_1 \\ z_1 \end{pmatrix}$$

4. Finally note that the above charts do not contain the set $V = \mathfrak{X}_0 \setminus (\bigcup_{i=1}^3 U_i) = \{\dot{x} = \dot{y} = \dot{\theta} = 0\}$, which corresponds to the set of equilibrium points of the system.

Remark 3. *The case $(\dot{x} = \dot{y} = 0)$ and $(\dot{\theta} \neq 0)$ exists in the context of the implicit system. However as discussed in section 3.1, in the explicit model this situation does not have a physical meaning since then $u = 0$ and the car is at rest, preventing θ from evolving.*

One can check that for all i, j , $Im(\phi_i) \subset \mathfrak{X}_0$ and that the $\psi_j \circ \phi_i$'s satisfy the compatibility definition of section 2.1 on \mathbb{R}_∞^2 . Therefore we have indeed defined an atlas of $\bigcup_{i=1}^3 U_i = \mathfrak{X}_0 \setminus \{\dot{x} = \dot{y} = \dot{\theta} = 0\}$. Among other things, this allows us to conclude that the car dynamics is globally controllable provided one avoids the singular set V , as illustrated in section 3. Note that at this level, we are not able to conclude that the set $\{\dot{x} = \dot{y} = \dot{\theta} = 0\}$ is an intrinsic flatness singularity since, according to definition 2 above, we still have to prove that no other atlas can contain this set, hence the importance of the next section based on the results of 2.3.

3.3 Flat Outputs and Intrinsic Flatness Singularities of the Car Example

One first considers the differential of the implicit equation:

$$dF = d\dot{x} \sin \theta + \dot{x} \cos \theta d\theta - d\dot{y} \cos \theta + \dot{y} \sin \theta d\theta = (\dot{x} \cos \theta + \dot{y} \sin \theta) d\theta + \sin \theta d\dot{x} - \cos \theta d\dot{y}$$

Note that, if z is an arbitrary variable of the system, $d\dot{z} = d(\tau_{\mathfrak{X}}z) = \tau_{\mathfrak{X}}dz$ since the exterior derivative d commutes with the Cartan field $\tau_{\mathfrak{X}}$. Thus we get the following expression of the matrix $P(F)$:

$$P(F) = \begin{bmatrix} (\sin \theta)\tau_{\mathfrak{X}} & -(\cos \theta)\tau_{\mathfrak{X}} & \dot{x} \cos \theta + \dot{y} \sin \theta \end{bmatrix}$$

thus satisfying $P(F) \begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} = 0$ for all $dx, dy, d\theta$ that are differentials of the variables x, y, θ satisfying system (3.2).

Now in the context of the car system given by (3.2), we are in a position to prove the following:

Proposition 3. *The intrinsic singular set of system (3.2), given by $\{\dot{x} = \dot{y} = \dot{\theta} = 0\}$, is equal to \mathcal{S}_F .*

Proof. We compute the set where $P(F)$ is not hyper-regular. Let us define

$$A = \dot{x} \cos \theta + \dot{y} \sin \theta.$$

After a single permutation, $P(F)$ becomes $[A, (\sin \theta)\tau_{\mathfrak{X}}, -(\cos \theta)\tau_{\mathfrak{X}}]$. Then the first column of U , say u_1 is $u_1 = [1/A, 0, 0]^t$ (the superscript t denotes the transposition operator). The second one u_2 is given by $[P_0, P_1, P_2]^t$ where P_0, P_1, P_2 are polynomials of $\tau_{\mathfrak{X}}$ with $\deg(P_0) = 1 + \max_{i=1,2} \deg(P_i)$, such that $AP_0 + (\sin \theta)\tau_{\mathfrak{X}}P_1 - (\cos \theta)\tau_{\mathfrak{X}}P_2 = 0$, or $P_0 = -\frac{1}{A}((\sin \theta)\tau_{\mathfrak{X}}P_1 - (\cos \theta)\tau_{\mathfrak{X}}P_2)$. The third column u_3 is obtained in the same way: $u_3 = [P'_0, P'_1, P'_2]^t$ with $P'_0 = -\frac{1}{A}((\sin \theta)\tau_{\mathfrak{X}}P'_1 - (\cos \theta)\tau_{\mathfrak{X}}P'_2)$ and P'_1, P'_2 such that the matrix $\begin{bmatrix} P_1 & P'_1 \\ P_2 & P'_2 \end{bmatrix}$ is unimodular. Therefore every decomposition exhibits at least one singularity defined by the vanishing of A . Moreover, it is readily seen that the following 0 degree choice $P_1 = \sin \theta, P_2 = -\cos \theta, P'_1 = \cos \theta, P'_2 = \sin \theta$ is such that

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{A} & -\frac{1}{A}\tau_{\mathfrak{X}} & \frac{\dot{\theta}}{A} \\ 0 & \sin \theta & \cos \theta \\ 0 & -\cos \theta & \sin \theta \end{bmatrix}$$

is singular if, and only if, $A = 0$. We thus conclude that $P(F)$ is hyper-regular if and only if $A \neq 0$.

Finally, the equation $A = \dot{x} \cos \theta + \dot{y} \sin \theta = 0$, combined with $F = \dot{x} \sin \theta - \dot{y} \cos \theta = 0$ leads to $\dot{x} = \dot{y} = 0$, and thus to $\dot{x} = \dot{y} = \dot{\theta} = 0$ using the explicit model. We therefore have shown that $\mathcal{S}_F = \{\dot{x} = \dot{y} = \dot{\theta} = 0\}$, in other words that the only obstruction to the hyper-regularity of $P(F)$ is a flat output singularity, hence intrinsic according to Theorem 1. \square

3.4 Route Planning

Now we shall show how the atlas we have built can be used to control the car over a route along which there are several apparent singularities.

Imagine we want the car to travel along the route depicted in figure 3.2.

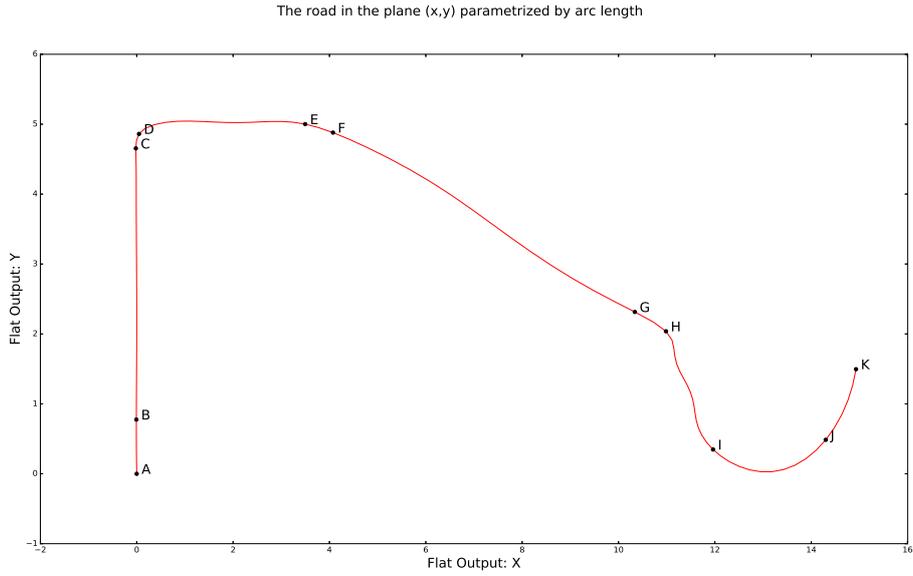


Figure 3.2: *Planned car route, parametrized by arc length.*

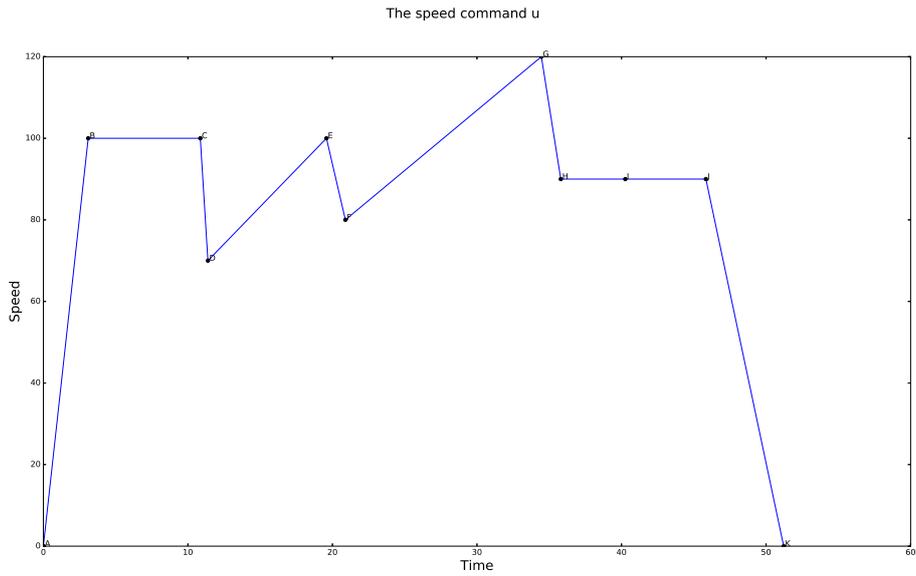


Figure 3.3: *The speed corresponding to the route depicted in figure 3.2*

This route has been defined in several steps. First points were chosen in \mathbb{R}^2 . Two univariate splines were fitted for respectively the x and y coordinates, parametrized by a pseudo time variable. Then the curve in \mathbb{R}^2 has been re-parametrized by arc length on an interval $[0, L]$, with unit speed. These steps have been performed in order to allow the

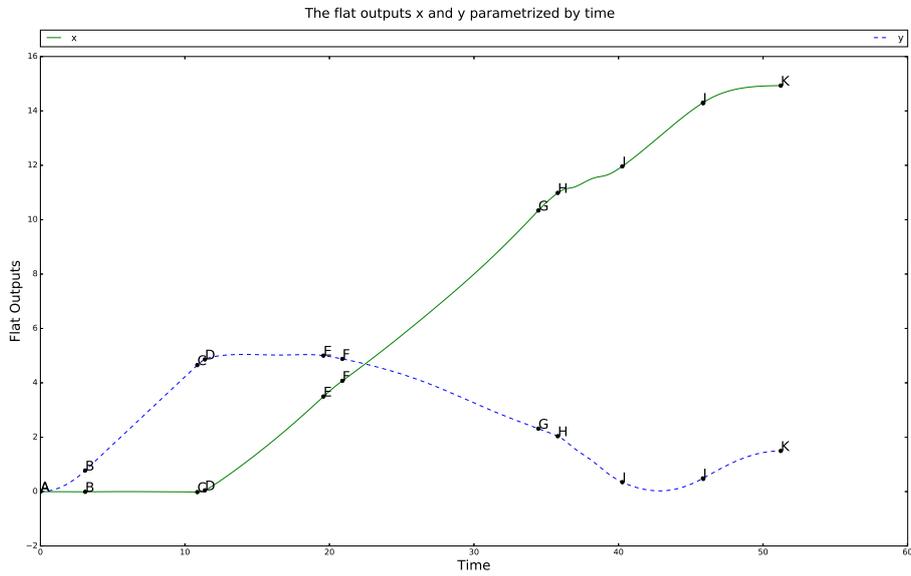
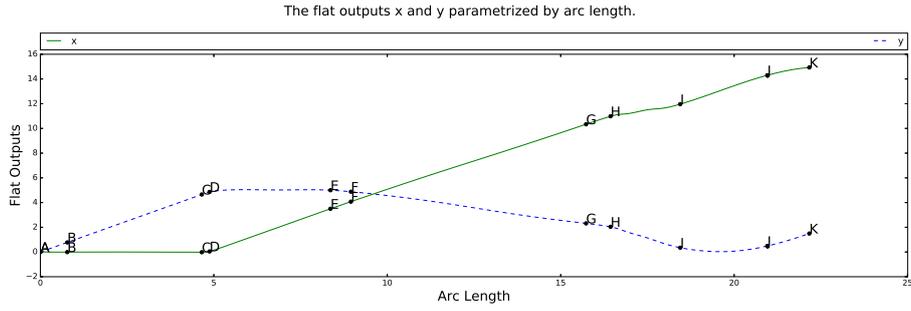


Figure 3.4: *The flat outputs parametrized first by arc length and then by time corresponding to the route depicted in figure 3.2*

design of a speed profile, *i.e.* the control or the energy, over time without restrictions, as follows.

We constructed a function $\sigma : [0, T] \rightarrow [0, L]$ defined up to a time T that is increasing and everywhere differentiable. The function σ has the required derivative and its values are determined so that the speed segments are mapped to the arc length segments shown in figure 3.2. The function σ has been chosen piecewise quadratic. At the extreme points, σ satisfies $\dot{\sigma}(0) = \dot{\sigma}(T) = 0$.

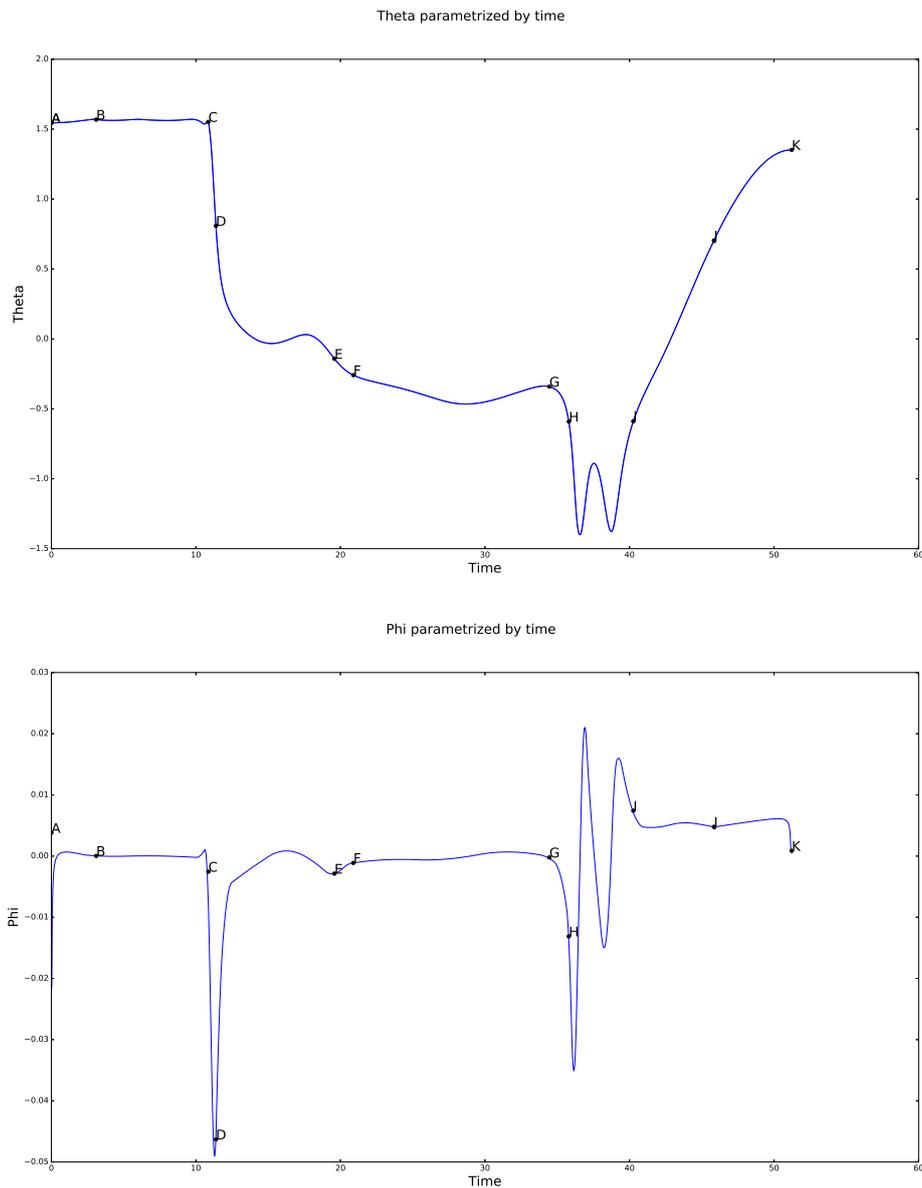


Figure 3.5: *The angles θ and ϕ parametrized by time corresponding to the route depicted in figure 3.2. For the computation of ϕ , the car length has been chosen equal to $l = 2m$.*

The speed profile of the car is shown in figure 3.3.

The flat outputs and the variables θ and ϕ , are then computed using this new parametrization.

The computation of θ precisely requires to take profit of the previously built atlas: on segments $[A, B]$ and $[B, C]$, the derivative of x vanishes and one must use the second chart defined on U_2 with $\theta = \cotan^{-1}(\frac{\dot{x}}{\dot{y}})$. On segment $[D, E]$, the derivative of y nearly

vanishes and one must use the first chart defined on U_1 , such that $\theta = \tan^{-1}(\frac{\dot{y}}{\dot{x}})$. The third chart must be used only on the arc \widehat{HK} .

For the computation of ϕ , we exclude the end points where the speed vanishes and thus where ϕ is not defined. See figure 3.5. Those points are indeed intrinsic singularities and we show that we can approach them as closed as we want but exactly stopping on them with a prescribed orientation is impossible.

4 Concluding Remarks

In this paper, the concepts of intrinsic and apparent flatness singularities have been defined. These notions are of paramount importance for trajectory planning through apparent singularities and avoiding intrinsic singularities.

We have also exhibited a characterization of intrinsic singularities, which yields an effective algorithm for their computation. The approach relies on the hyper-regularity of the matrix $P(F)$ defined in the process of flat output computation.

This analysis is exemplified by the non holonomic car motion planning. In this context, we have explicitly constructed an atlas of flat outputs, which allows resolving all apparent singularities. The remaining singularity corresponding to the set of equilibria is intrinsic and cannot be resolved.

We have proven that this intrinsic singularity is the only one that necessarily appears in the process of computing flat outputs. More precisely this singularity corresponds to the points where $P(F)$ is not hyper-regular, a property shared by all possible flat outputs of this system.

We also have demonstrated the practical interest of introducing a Lie-Bäcklund atlas by explicitly parametrizing a complex trajectory passing through all possible charts.

Note finally that this approach may be used to compute the largest possible controllable set of a system.

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