Testing regularity and freeness of D-modules

Applications to local differential flatness

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Abstract

This paper describes an algorithm to test the freeness of a module over the ring $\mathcal{D} := \mathfrak{k}[\![t]\!][\mathrm{d}/\mathrm{d}t]$. The freeness of the linearized system provides a necessary condition for local differential flatness in control theory, that we prove to be sufficient for flat systems with 1 control but not in higher differential dimension.

Flatness implicitely assumes regularity. We introduce a notion of $\mathcal{D}$-regularity for $\mathcal{D}$-module, which is the analogue for the linearized system and generalizes the classical notion of regular linear system. We give a criterion, showing that $\mathcal{D}$-modules without torsion elements are $\mathcal{D}$-regular and that a torsion module $\mathcal{M}$ is regular iff $\bigcap_{r\in\mathbf{N}} t^r\mathcal{M} = 0$.

1 Introduction

The main motivation of this paper is to provide a necessary condition for local differential flatness. Differential flatness was introduced by Fliess *et al.* [8, 10, 19] at the beginning of the 90's and proved to be a powerfull tool for trajectory planning in control theory. Although flatness is not generic, flat systems are ubiquitous in all branches of engineering, chemistry etc.

Flatness means that a system is, on a dense open set, locally isomorphic to a controllable linear system, *viz.* associated to a free module. This implies two possible difficulties. First, even if a system is flat everywhere, one may have to use different charts when moving in its configuration space as illustrated by a quadcopter in Chang and Eun [4]. Second, there may exist a singular place where the system is not flat. Specific precautions must be taken to avoid this singular locus, or cross it when the regular set is not connected. This justifies a systematic study of singularities of flat systems, initiated in [14].

Although there is no general algorithm to test flatness, physical intuition often easily provides suitable changes of coordinates, defined by so-called "flat outputs". Proving that no such flat outputs exist is a harder task but there is an easy necessary condition for local flatness: the linearized system must be associated to a free module.

Testing freeness over the ring $\tilde{\mathcal{D}} := \mathfrak{k}(\!(t)\!)[\mathrm{d}/\mathrm{d}t]$ reduces to computing a Jacobson normal form [12, 18, 23]. But here, the local nature of our computations implies to work in $\mathcal{D} := \mathfrak{k}[\![t]\!][\mathrm{d}/\mathrm{d}t]$, which is harder, as then Jacobson normal forms do not exist in the genaral case. An algorithm seems to be missing in the literature.

Section 2 provides a few basic definitions and properties of flat systems and their intrinsic singularities, a notion that implicitely assumes some kind of geometric regularity of the point, such as those defined by Johnson [13]. We investigate an alternative definition he suggests, using characteristic sets.

In section 3, we prove that the linearized system of a flat point must be free, and that this is a necessary and sufficient condition for a flat system of differential dimension 1. We exhibit an example of a non flat point in differential dimension 2, for which the linearized system defines nevertheless a free module.

Section 4 is devoted to the wider class of "regular" modules. This notion is the linear analog of the characteristic set regularity of section 2. It generalizes in arbitrary differential dimension the classical notion of regular linear system. We show that a torsion $\mathcal{D}$-module $\mathcal{M}$, *i.e.* a $\mathcal{D}$-module of differential dimmension 0 is regular iff $\bigcup_{r\in\mathbf{N}} t^r\mathcal{M} = 0$ and that a $\mathcal{D}$-module without torsion elements is regular. These two properties provide a complete regularity criterion.

The last section 5 provides a freeness algorithm for a regular system.

Notations. — We will denote the partial derivation $\partial/\partial x$ by ∂_x. The total or Cartan derivation $\mathrm{d}/\mathrm{d}t$ will be τ, $\mathfrak{k}[\![t]\!][\tau]$ will be denoted by $\mathcal{D}$ and $\mathfrak{k}(\!(t)\!)[\tau]$ by $\tilde{\mathcal{D}}$. The $\mathfrak{k}$-vector space generated by Σ will be $\langle\Sigma\rangle$, the $\mathfrak{k}[\![t]\!]$-module by (Σ) and the $\mathcal{D}$ or $\tilde{\mathcal{D}}$-modules by $[\Sigma]$ according to the context.

2 Differentially flat systems

The notion of flatness is closely related to Monge's problem [20]. See also Hilbert [11] or Cartan [2]. In some informal setting, flat systems are systems of ordinary differential equations in n variable x_i, such that there exist m differentially independent functions Z_j of the x_i and their derivatives with the following property: *the general solution of the system is parametrized by the Z_j and* *their derivatives, i.e. we have* $x_i = X_i(Z, \ldots, Z^{(r)})$. These functions Z are called *flat outputs*.

Example 1. *The system* $\theta' = (\sin(\theta)x' - \cos(\theta)y')/L$ *describes a simplified model of a car, where* θ *is the angle of the car axis with a reference direction and* (x, y) *the coordinate of a point of the car, at distance L or the rear axis. The coordinates of the point* $(z_1, z_2) = (x - \cos(\theta)L, y - \sin(\theta]L)$*, which is on the rear axis are flat outputs and we see that:* $\theta = \arctan(z_2'/z_1') \pm k\pi$*, so that the whole trajectory may be reconstructed knowing* (z_1, z_2)*, provided that* $z_1' \neq 0$*. We may also use* $\theta = \mathrm{arccot}(z_1'/z_2') \pm k\pi$*, provided* $z_2' \neq 0$*. We see that this parametrization is only local, and that we may have to change charts, as the car turns around a traffic circle.*

Assuming we have a special car for mathematicians, with a single front wheel that may turn in all direction, we may even deal with $z_1' = z_2' = 0$*, provided that* $\theta' \neq 0$*. If we take* $\zeta_1 = \theta$ *and* $\zeta_2 = \sin(\theta)z_1 - \cos(\theta)z_2$*, we have:* $\zeta_2' = \theta'(-\sin(\theta)z_1 + \cos(\theta)z_2)$*, so that we can recompute* (z_1, z_2) *and from them* (x_1, x_2) *if* $\theta' \neq 0$*.*

From a theoretical standpoint, flatness may be introduced in the two complementary formalisms of differential algebra (see Ritt [24] and Kolchin [15]) and diffiety theory (see Vinogradov *et al.* [17] or Zharinov [26]). Diffiety theory is usefull as flat outputs are not always algebraic, but most systems encountered in practice are algebraic. The most confortable compromize is to consider algebraic diffieties, *i.e.* denumerable subvarieties of the jet space defined by an algebraic differential system.

We consider *systems*, defined by ordinary explicit differential equations.

$$x_i' = f_i(x, u, t), \tag{1}$$

where the x_i, $1 = 1, \ldots, n$ are state function, the u_j, j=1, ..., n are the controls, t stands for the time and the f_i are rational or algebraic functions. In a more general setting, one may consider implicit equations $P_i(x, t) = 0$, $i = 1, \ldots, n - m$, with some extra natural assumption, the most obvious one being that they form a characteristic set of some prime differential ideal $\mathcal{P}$ for some ordering on the derivatives. From this abstract standpoint, we associate to the system a *prime differential ideal* $\mathcal{P}$ that defines an intrinsic differential field extension, say $\mathcal{F}/\mathbf{R}(t)$. The number of controls m is the *differential transcendence degree* of the extension $\mathcal{F}/\mathbf{R}(t)$ or the *differential dimension* of the prime differential ideal $\mathcal{P}$. The car example can be reduced to the algebraic case by replacing θ with $\tan(\theta/2)$.

Definition 2. *A differential extension* $\mathcal{F}/K$ *is flat is the algebraic closure of* $\mathcal{F}$ *is isomorphic to the algebraic closure of a differentially transcendental extension* $K\langle z_1, \ldots, z_m\rangle$*. The* z_j *are* flat outputs *of the flat system.*

Accordingly, we may associate to the system a *diffiety*, that is a denumerable variety, equipped this the coarsest topology that makes projections on each coordinate continuous, and a *Cartan derivation* acting on its ring of functions, that are $\mathcal{C}^\infty$ functions depending on a finite number of coordinates. For example, a diffiety can be associated to the system (1) on the open set V of

$\mathbf{R}^{n+m+1}$ where the functions f_i are defined. Assuming that the derivatives of the control can take any values, they belong to $\left(\mathbf{R}^{\mathbf{N}}\right)^m$, so that the diffiety is $V \times \left(\mathbf{R}^{\mathbf{N}}\right)^m$ equipped with the Cartan derivation

$$\tau := \partial_t + \sum_{i=1}^{n} f_i(x,u,t)\partial_{x_i} + + \sum_{i=1}^{m}\sum_{k\in\mathbf{N}} u_j^{(k+1)}\partial_{u_j^{(k)}}. \tag{2}$$

The jet space $J^\infty(\mathbf{R},\mathbf{R}^n)$, that we denote $\mathbf{J}^n$ has a natural structure of diffiety and *finite type diffieties* are those that may be seen as the regular points of a subvariety of $\mathbf{J}^n$ defined by a differential system. If the system is algebraic, we speak of an *algebraic diffiety*. In the sequel, we will always assume our diffieties to be of finite type and freely consider them as included in $\mathbf{J}^n$. We will also assume that the differential system defining the diffiety is algebraic and defined by a prime differential ideal $\mathcal{P} \subset \mathbf{R}[t]\{x\}$. Using such an implicit definition one may have to consider different charts to cover the whole diffiety. The classical diffiety structure assumes the regularity of its points, according to the following definition.

Definition 3. *A point ξ of an algebraic diffiety $V \subset \mathbf{J}^n$ defined by a prime differential ideal $\mathcal{P} \subset \mathbf{R}[t][x]$ is* regular *if there exists a diffiety $V_2 \subset \mathbf{J}^{n_2}$ defined by a prime differential ideal $\mathcal{Q} \subset \mathbf{R}[t][y]$, and a local isomorphism of diffieties ϕ from a neighbourhood of ξ to a neighbourhood of $\zeta = \phi(xi)$, such that $\mathcal{Q}$ admits for some ordering a characteristic set $\mathcal{A}$ with all the separants S_A, $A \in \mathcal{A}$ not vanishing at the point $\zeta = \phi(\xi)$.*

We see that it is possible to use the implicit function theorem at each regular point in order to define an explicit differential system. In practice, the existence of such an explicit system may be taken for granted. But from a theoretical standpoint, testing if a point is regular is a difficult task. In fact, testing if a point ξ where the polynomials of a characteristic set vanish but also some separant is a zero of $\mathcal{P}$ is equivalent to Ritt's problem. Johnson has given an alternative definition of regularity, as an intrinsic property of the local ring $(\mathbf{R}[t]\{x\}/\mathcal{P})_\xi$ (see [13]).

We may now provide the definition of flatness in the context of diffieties.

Definition 4. *A diffiety V is flat if it contains a dense open set W such that any point $\xi \in W$ admits a neighboroud which is isomorphic to an open space of $\mathbf{J}^m$. The base function coordinates of $\mathbf{J}^m$ are expressed as functions $Z_j \in \mathcal{O}(V)$ that are* flat outputs*.*

The following example shows that some algebraic diffiety may be flat even if the associated differential extension is not. Flatness in the algebraic meaning requires the existence of *algebraic* flat outputs, which is stronger.

Example 5. *The system $x_1' = u$, $x_2' = x_2 u$ is algebraic. It is flat if the framework of diffiety theory, but not of differential algebra. Indeed, a flat output is $z := e^{x_1}x_2$, and it is functionally unique: all flat outputs are functions of this z (see th. 11 below). So, no flat output is algebraic.*

3 Local flatness criteria

We have seen with the car (example 1) that indeed some flat parametrization related to some choice of a flat output may fail to be defined outside a non-trivial open set. As we will see, it may be proved that actually the car is not flat on the closed set $x' = y' = \theta' = 0$. This justify the following definition.

Definition 6. *A point ξ in a flat diffiety V is singular for the flat output z if $z = Z(x)$ is not defined at ξ or $Z(x) = z$ does not define a local isomorphism with $\mathbf{J}^m$ at ξ,* i.e. *it is not locally equivalent to $x_i = X_i(z)$, for $z = Z(\xi)$.*

A point ξ is flat if it admits a neighborhood isomorphic to some open subset of $\mathbf{J}^m$ and flat singular (or not flat) if not, that is to say if all flat outputs are singular at ξ.

Flat singularity does not mean that one cannot recompute the value of the state x knowing the value of a flat output Z at $x = \xi$; the inverse function may fail to be regular enough, as shown by the next example.

Example 7. *Considering the car example, we can also use time-varying flat outputs such as $z_1 = x - L\cos(\theta) - t\sin(\theta)$ and $z_2 = y - L\sin(\theta) + t\cos(\theta)$. Easy computations whow that when $(z_1')^2 + (z_2')^2 \neq 1$, the value of θ is not unique (modulo $k\pi$), but we have 2 locally unique regular solutions: $\theta = \arctan(z_2'/z_1') \pm \arccos(1/\sqrt{(z_1')^2 + (z_2')^2})$ For $(z_1')^2 + (z_2')^2 = 1$, we have a unique but singular double solution for the value θ.*

We need now some technical definition in order to introduce a necessary local flatness condition.

Definition 8. *Let ξ be a point in a diffiety V, defined as a subvariety of $\mathbf{J}^n$ by a prime differential ideal $\mathcal{P} \subset \mathbf{R}(t)\{x\}$. By the data of ξ we mean that of the value of successive derivatives $x_i^{(k)} = \xi_i^{(k)}$ and a time value $t = \tau$. Let P be a differential polynomial $P \in \mathbf{R}[t]\{x\}$, we denote by $P[\xi]$ the result of substituting $\sum_{k\in\mathbf{N}} \xi_i^{(k)}(t-\tau)^k/k! \in \mathbf{R}[\![t]\!]$ to x_i in P. We define $\mathcal{M}_\xi V$ as the tangent module to V at ξ defined by the tangent linear system $\delta_\xi P$, $P \in \mathcal{P}$, where:*

$$\delta_\xi P := \sum_{i=1}^{n} \sum_{k\in\mathbf{N}} \frac{\partial P}{\partial x_i^{(k)}}[\xi]\delta x_i^{(k)}. \tag{3}$$

Denoting by $[\Sigma]$ the $\mathbf{R}[\![t]\!][\mathrm{d}/\mathrm{d}t]$-module generated in the free $\mathbf{R}[\![t]\!][\mathrm{d}/\mathrm{d}t]$-module $[\delta x_i]$, $\mathcal{M}_\xi := [\delta x]/[\delta_\xi P | P \in \mathcal{P}]$.

From now on, we will alway take $\tau = 0$ to simplify notations.

We see that if $\mathcal{A}$ is a characteristic set of $\mathcal{P}$ such that the separants of $\mathcal{A}$ do not vanish at ξ, then $[\delta_\xi P | P \in \mathcal{P}] = [\delta_\xi A | A \in \mathcal{A}]$, allowing to consider a finite set of generators. We don't need here any assumption of convergence on our power series, but we will have to restrict to series that belong to an effective $\mathfrak{k}$-algebra, where $\mathfrak{k} \subset \mathbf{R}$ is an effective field.

Theorem 9. *The diffiety V defined by the prime differential ideal $\mathcal{P}$ regular at point ξ is such that the functions Z_i, $i = 1, \ldots, m$ are flat outputs regular at ξ iff $\mathcal{M}_\xi$ is a free ring and δZ a basis.*

Proof. This condition is necessary. Indeed, if Z is a flat output then it is differentially functionnally independent and so $\delta_\xi Z$ is free in $\mathcal{M}_\xi$. Moreover, $x_i = X_i(z)$, so that $\delta_\xi x_i = \delta_\xi X_i(z)$ and $\delta_\xi Z$ is a basis of $\mathcal{M}_\xi$, which is a free module.

Reciprocally, as ξ is regular we may assume, up to a change of variables, that $\mathcal{P}$ admits a characteristic set $\mathcal{A}$ whose separants do not vanish at ξ. If $\delta_\xi Z$ is a basis of $\mathcal{M}_\xi$, the unique component W of the system $\mathcal{A}$, $Z(x) = z$ (where the z_i are arbitrary functions) that contains $(\xi, Z(\xi))$ is quasi-regular at according the definition given in Ollivier and Sadik [21] in the diffiety case and Kondratieva *et al.* [16] in the algebraic case. This means that the order of the component W is that of the linearized system $\delta_\xi \mathcal{A}$, $\delta_\xi Z = \delta_\xi z$, hence 0: so there exist functions X_i defined on some neighborhood of ξ such that $x_i = X_i(Z(x))$. □

Remark 10. *We see that we are faced in the theorem above and its proof with at least three different notions of "singularity" or "regularity": regular points in some algebraic diffiety, including in differential dimension* 0, i.e. *in finite algebraic dimension, the case of classical singularity theory for algebraic varieties; quasi-regular systems, for which the notion characterizes the possibility to reduce the study of a component to that of its linearized system at a point; and flat regular points which are regular points in the first meaning in the neighborhood of which flat outputs do exist, meaning that this neighborhood is isomorphic to an open subset of $\mathbf{J}^m$.*

According to th. 9, the freeness of the module $\mathcal{M}_\xi$ is a necessary condition for ξ to be flat. The sections 4 and 5 will be devoted to an algorithmic test for this property. Before foccussing on this topic, one may remark that freeness of $\mathcal{M}_\xi$ is a generic condition, encountered for most point of most diffieties of positive differential dimension (see *e.g.* Fliess *et al.* [9]) even if flatness is on the contrary non generic (see Rouchon [25]). So, one may suspect this condition not to be sufficient, even for a flat system. We first prove that this is nevertheless a sufficient condition for flatness at some point ξ in differential dimension 1, that is for systems with 1 control. We will need a classical characterization of flatness in that case, that may be traced back to Cartan [2], in a different setting. We refer to Charlet *et al.* [5] for details.

Theorem 11. *Let $x_i' = f_i(x, u, t)$, $i = 1, \ldots, n$ be a system describing a diffiety V of differential dimension* 1*, that is with one control u. Denote by τ the Cartan derivation and $\partial_u := \partial/\partial_u$. We also use the notation $\hat{\tau}\partial_u := [\tau, \partial_u]$ and $\hat{\tau}^{r+1}\partial_u := [\tau, \hat{\tau}^r \partial_u]$. We define the Lie algebra $\mathcal{L}_r$ generated by $\partial_u, \ldots, \hat{\tau}^r\partial_u$.*

i) The diffiety V is flat iff $\dim \mathcal{L}_r = r + 1$, $r \leq n + 1$ *and a flat output is a functions of the x_i that is a constant for the derivations in $\mathcal{L}_n$.*

ii) All flat outputs are functionally dependent.

iii) A point ξ of a flat diffiety V of differential dimension 1 *is flat iff $\mathcal{M}_\xi$ is a free module.*

Proof. i) The condition means that a flat output Z does not depend on u and its derivatives, and moreover $Z, \ldots, Z^{(n)}$ do not depend on u. Assume that Z depends on $u^{(k)}$, then by [21], the system Σ defined by the system equations and $Z(x,u) = z$ is quasi-regular and has order $k+n$ at any point ξ where $\partial_{u^{(k)}} Z \neq 0$. So we cannot express the value of the x using Z and its derivatives. Assume now that $Z(x)$ is such that $Z^{(r)}$ depends on u for $r < n+1$. Then Z, $\ldots, Z^{(r-1)}$ provide at most $r-1$ independent functions of the x and the order of Σ is at least $n-r+1 > 0$, which is again impossible.

Reciprocally, one may show that there exists a non trivial (actually depending on some variable x_i) function $Z(x)$ that is a constant for the derivations of $\mathcal{L}_n$. Without loss of generality, we may assume that the derivations ∂_{x_1} and $\hat{\tau}^r \partial_u$ are independent at ξ and all point in some neighborhood W of ξ. Our PDE system is classically solved in W using the method of characteristics. We may impose that $Z(x_1, \xi_2, \ldots, \xi_n, t) = g(x_1)$, for any function g.

For any point $(x,t) \in W$, let δ be the field such that $\delta \in \mathcal{L}_n$ (depending on t) and $\delta_j = x_i - \xi_i$, $i = 2, \ldots, n$. We may integrate this field with initial conditions $X(t,0) = x$, then $X_i(t,1) = \xi_i$, $i = 2, \ldots, n$ and we have $Z(x,t) = g(X_1(1))$. The fact that it is a flat output is a consequence of iii).

ii) The solution $Z_{\mathrm{Id}}(x,t)$ and the solution Z_g are by construction such that $Z_g = g(Z_{\mathrm{Id}})$. So that all solutions are functionally dependent.

iii) We know by th. 9 that, if ξ is flat, then $\mathcal{M}_\xi$ is free. So, we just have to prove the reciprocal. We may apply i) to the linear system defining $\mathcal{M}$, so that denoting by τ the derivation on $\mathcal{M}$, δZ is a non trivial zero of the operators $\partial_{\delta u}$, $\hat{\tau}^r \partial_{\delta u}$, $r = 0, \ldots, n$, which characterizes a basis of $\mathcal{M}$ if $\mathcal{M}$ is free. A solution of this system is unique up to multiplication, so $\delta Z_{\mathrm{Id}} \notin \mathcal{M}$ is a basis.

This means that the determinant of the Jacobian matrix of functions $Z, \ldots,$ $Z^{(n+1)}$ with respect to the x_i and is non zero, so that we can locally express the x_i and u as functions of $Z, \ldots, Z^{(n+1)}$ and ξ is flat. □

We may complete this presentation with an example of flat system with 2 controls that satisfy the freeness condition at a non flat point.

Example 12. *We consider the diffiety V defined by*

$$x_1' = x_1 u_1 + u_2^2 \quad x_2' = u_2. \tag{4}$$

It is trivially flat with flat output x and y, with $u_1 = [x_1' - (x_2')^2]/x_1$, where $x_1 \neq 0$, but these flat outputs are singular when $x_1 = 0$, as well as any flat outputs of order 0 in the x_i. Consider the case where one flat output Z_1 is of order 0 in the x_i. We may take it as a new coordinate function and take for $\{i_0, j_0\} = \{1,2\}$, $\partial_{x_{i_0}} Z_1(\xi) \neq 0$, so $x_{i_0} = H(Z_1, x_{j_0})$.

Then, by substitution, $Z_2 = F(x_{j_0}, u_{j_0}, Z_1)$, with F of order 0 in x_{j_0}. We cannot compute the x_i unless F does not depend on u_{j_0} and its derivatives and $\partial_{x_{j_0}} F \neq 0$ at ξ. This case is thus equivalent to the choice of x_1 and x_2, which is singular.

Let us show that V is not flat when $\xi_1 = 0$. Assume that there are regular flat outputs $Z_i(x)$, $i = 1,2$, at some point ξ with $\xi_1 = 0$. We have a parametrization

$x_i = X_i(z_1, z_2)$, $u_i = U_i(z_1, z_2)$. Assume that, for some $i = 1, 2$, $e :=$ $\max(\mathrm{ord}_{z_i} X_1, \mathrm{ord}_{z_i} X_2)$. Denoting by $X_1(Z(\xi), z_i^{(k)}, \ldots, z_i^{(e)})$ the value of X_i when all its argument are the $Z_i^{(k)}(\xi)$, except the $z_{i^{(\ell)}}$, $k \le \ell \le e$, $i = 1, 2$, we will prove by recurrence that $X_1(Z(\xi), z_i^{(k)}, \ldots, z_i^{(e)}) = X_1(\xi) = 0$ and that $\partial_{z_i^{(k+1)}} X_i$ is divisibble by X_1. This stands for $k > e$.

Assume the assertion stands for $k + 1$. The terms $(\partial_{z_i^{(k)}} X_j z_i^{(k+1)})^j$, $j = 1, 2$ are the only non zero terms in $(z_i^{(k+1)})^j$ in $X_1' - (X_2')^2 =$ $U_1 X_1$. For all $r \in \mathbf{N}$ $\partial^r_{z_i^{(k+1)}} X_1(Z(\xi), z_i^{(k)}, \ldots, z_i^{(e)}) = 0$ according to our hypothesis. When $X_1 = 0$, we have $\partial^r_{z_i^{(k+1)}} X_1 U_1 = 0$ and so $\partial^r_{z_i^{(k+1)}} (X_1' - (X_2')^2) = 0$. This implies that $\partial_{z_i} X_j = 0$, $j = 1, 2$, so that these functions are divisible by X_1.

This implies $\partial_{z_i^{(k)}} X_1 = X_1 H$. Solving this ordinary equation with respect to $z_i^{(k)}$ is a classical way gives:

$$X_1(Z(\xi), z_i^{(k)}, \ldots, z_i^{(e)}), = \xi_1 \exp\left(\int_{Z_i^{(k)}(\xi)}^{z_i^{(k)})} H(z) \mathrm{d} z_i^{(k)}\right) = 0.$$

So $X_1 = \xi_1 = 0$, which is impossible.

This example shows the limitations of our freeness criterion but also the existence of other possible methods to test local flatness, awaiting further investigations.

4 D-Regular modules

In this section and the next one, we will freely consider linear systems defining modules as a special case of differential algebraic system and extend notions such as the differential dimension in an obvious way.

We need to consider in this section a notion of $\mathcal{D}$-regularity for modules, that is the analog of the geometric regularity for a point of an algebraic diffiety (def. 3). This notion generalizes the usual notion of regularity for linear operators or systems. See *e.g.* Chen *et al.* [6] and the refereces therein.

We are first concerned with modules $\mathcal{M}_\xi$, associated to a regular point of a diffiety V defined by a prime ideal $\mathcal{P}$. In our setting, regularity is expressed in a more trivial way in the framework of characteristic sets (def. 3). We can translate the definition for modules in the following way. In the sequel, we recall that we denote by $\mathcal{D}$ the ring of operators $\mathfrak{k}[\![t]\!][\tau]$ and by $\tilde{\mathcal{D}}$ the ring $\mathfrak{k}(\!(t)\!)[\tau]$, where $\mathfrak{k}$ is a field of constants with $\mathbf{Q} \subset \mathfrak{k} \subset \mathbf{R}$ and τ is a derivation with $\tau t = 1$. The canonical bases of $\mathcal{D}^n$ will be denoted by x_i or y_i.

To insure computability, we just need to require that $\mathfrak{k}$ is an effective field and that the power series coefficients in our series belong to an effective subring of $\mathfrak{k}[\![t]\!]$. *E.G.* this is the case for rational or algebraic series or regular solutions of systems of ODE (see Péladan-Germa [22]).

We can rely on Gröbner bases for $\mathcal{D}$-modules for testing submodule membership. On this topic, see, *e.g.*, Castro and Granger [3] and the references therein.

Let $s(t) = \sum_{k=r}^{\infty} c_k t^k$ with $c_r \neq 0$ we will use the valuation $\mathrm{v}s := r$. We denote by y_i the canonical basis on $\mathcal{D}^n$. We will also use notations such as $\partial_{x^{(e)}}$ which have an obvious meaning in the module context.

Definition 13. *A finite type $\mathcal{D}$-module $\mathcal{M}$ is said to be regular if there exists a free $\mathcal{D}$-module $M \subset \mathcal{D}^n$ such that $\mathcal{M}$ is isomorphic to $\mathcal{D}^n/M$ and M admits, for some admissible ordering on the derivatives of the y, a basis G such that the leading term of all $g_i \in G$ is $c_i(t)y_i^{(k_i)}$ with invertible leading coefficients c_j, that is $\mathrm{v}c_j = 0$. We call this a* regular basis*.*

Proposition 14. *If an algebraic diffiety V is regular at point ξ, then the module $\mathcal{M}_\xi$ is regular.*

Proof. One only has to consider a regular representation of V, with a characteristic set $\mathcal{A}$ with non vanishing separants. Using the same order for the derivatives of x_i and of $y_i := \delta x_i$, the set $\delta_\xi A$, $A \in \mathcal{A}$ gives a regular basis of M, as the coefficient of the main derivative of $\delta_\xi A$ has coefficient j S_A, which is invertible as S_A does not vanish at ξ $\square$

Considering a differential local algebra R with maximal ideal $\mathfrak{m}$, Johnson [13] proves in his setting that, if the algebra is regular, then $\bigcap_{r\in\mathbf{N}} \mathfrak{m}^r = 0$. We have then a module analog for Johnson proposition.

Theorem 15. *If $\mathcal{M} = \mathcal{D}^n/M$ is a regular module, then $\bigcup_{r\in\mathbf{N}} t^r\mathcal{M} = 0$.*

Proof. i) Let G be a regular Gröbner basis of M as in def. 13, so with leading term $\tau^{e_i}x_i$. This means that the derivatives $x_i^{(k)}$, for $1 \le i \le n-m$, $0 \le k < e_i$ and for $n-m < i \le n$, $k \in \mathbf{N}$ form a basis of the $\mathfrak{k}[\![t]\!]$-module $\mathcal{M}$. Let any non zero element of $\mathcal{M}$ be

$$w := \sum_{i=1}^{n-m} \sum_{k=0}^{e_{i-1}} s_{i,k}(t) x_i^{(k)},$$

where the e_i, for $n-m+1 \le i \le n$ can take arbitrary finite values. Then $\mathrm{v}w := \min_{i=1}^{n-m} \min_{k=0}^{e_{i-1}} \mathrm{v}s_{i,k}(t)$ is such that $w \in t^\alpha\mathcal{M}$ and $w \notin t^{\alpha+1}\mathcal{M}$. So $\bigcup_{r\in_N} t^r\mathcal{M} = 0$. $\square$

Obviously, if $\mathcal{M} = \mathcal{D}^n/M$ is a free module, then $w \in M$ implies that $w/t^{\mathrm{v}w} \in M$.

Lemma 16. *Let $\mathcal{M} = \mathcal{D}^n/M$ be a regular module, then for $w \in \mathcal{M}$, $tw = 0$ implies then $w = 0$.*

Proof. As in the proof of th. 15, M is a free module, with a basis G such that for all g_i, $\mathrm{v}g_i = 0$. If $tw = 0 \in \mathcal{M}$, then $tw = \sum_{i=1}^{n-m} \sum_{k\in\mathbf{N}} a_{i,k} g_i^{(k)}$, with $\min_{i=1}^{n-m} \min_{k\in\mathbf{N}} \mathrm{v}a_{i,k} > 0$, so that $w = \sum_{i=1}^{n-m} \sum_{k\in\mathbf{N}} (a_{i,k}/t) g_i = 0$. $\square$

This will prove to be usefull, as in the case where $M = \delta_\xi \mathcal{P}$, where $\mathcal{P}$ is a prime differential ideal, we may not know a set of generators for $M = [\delta\mathcal{P}]$ if some separant vanish at ξ.

Definition 17. *We define $M : t^\infty$ to be the set $\{m | \exists r \in \mathbf{N}\, t^r m \in M\}$.*

Proposition 18. *For any $\mathcal{D}$-module M, $M : t^\infty$ is a $\mathcal{D}$-module.*

Proof. We only have to check that, if $t^{r_1} w_1 \in M$ and $t^{r_2} w_2 \in M$, then:
i) $t^{\max(r_1,r_2)}(w_1 + w_2) \in M$;
ii) $t^r w \in M$ implies $(t^{r+1}w)' - (r+1)t^r w = t^{r+1}w' \in M$. □

As $\mathfrak{k}(\!(t)\!)$ is a field, the situation is easy: every submodule M of $\tilde{\mathcal{D}} := \mathfrak{k}(\!(t)\!)[\tau]$ is free because A is an euclidean domain and all Gröbner bases of a $\tilde{\mathcal{D}}$-module are bases in the usual algebraic meaning. So, we may associate to any module $M \subset \mathcal{D}^n$ some Gröbner basis G of the $\tilde{\mathcal{D}}$-module it generates. We assume first that we are interested in $M : t^\infty$ and will look for a set of generators of this module. We may further assume by multiplying each g_i by a suitable power of t that $g_i \in \mathcal{D}^n$ and $g_i \notin t\mathcal{D}^n$. In the sequel, we will assume that we have chosen some elimination ordering $y_1 \gg y_2 \gg \cdots \gg y_n$ and that the leading derivative of g_i is a derivative of the main variable y_i.

Definition 19. *Let $w \in M$ have main variable y_{i_0}, with $w = \sum_{i=i_0}^{n} \sum_{k=0}^{\alpha_i} c_{i,k}(t) y_i^{(k)}$ with $c_{i,\alpha_i} \neq 0$. We define the slanted weight of w to be $\varpi\, w := \max_{k=0}^{\alpha_{i_0}}(k - \mathrm{v}\, c_{i_0,k}(t))$. If $\mathrm{v}w = 0$, we will denote by $\kappa_{(t)}w$ the local head $\max\{x_i^{(k)} | \mathrm{v}c_{i,k}(t) = 0\}$ and the head $\kappa\, w := x_{i_0}^{\alpha_{i_0}}$*

If $w = c\kappa_{(t)}w + R$, the reduction of w_2 by w $\mathrm{red}(w, w_2)$ is obtained by replacing in w_2 $\kappa_{(t)}w$ by $-R/c$.

To define successive reduction $\mathrm{red}(S, w)$, we reduce in sequence by the elements $\mu \in S$, sorted according to $\kappa_{(t)}\mu$.

We define $P_w(r)$ to be the constant coefficient of $x_i^{(\varpi\, w + r)}$ in $w^{(r)}$.

Our reduction strategy can make appear higher derivatives, but with coefficients of strictly positive valuation. This has non consequence for the next algorithm and is the basis of th. 23.

Obviously, $P_w(r)$ is a polynomial in r for $r \geq 0$: $P_w = \sum_{k - \mathrm{v}c_{i_0,k} = \varpi\, w} c_{i_0,k}(0)\binom{\mathrm{v}c_{i_0,k}}{r}$[1]. So, it admits only a finite number of zeros. We need some preparation algorithm.

Algorithm 20. ***Input:*** *A Gröbner basis G.*
Let $S : \emptyset$; $\Gamma := \emptyset$.
Loop 1. *Consider the g_i with increasing leading variable: $i = n - m + 1, \ldots, 1$.*
Let $w = g$.

[1]We borrow this idea from Denef and Lipshitz [7]. See also Barkatou *et al.* [1].

Loop 2. *If P_w has no non negative integer zero, then:*
$\Gamma := \Gamma \cup \{w\}$;
Iterate loop 1 *and consider* g_{i-1}.
Loop 3. *Consider in increasing order the non negative integer zeros* d_ℓ *of* P_w
Reduce $w^{(d_\ell)}$ *using* S*, the derivatives of* Γ *and* $w, w', \ldots, w^{(d_\ell - 1)}$ *to get* w_2.
If $\mathrm{v} w_2 > 0$ *then:* $S := S \cup \{w, w', \ldots, w^{(d_\ell - 1)}\}$
Loop 4. Until $\mathrm{v} w_2 = 0$ ***repeat***
$w_2 := w_2 / t^{\mathrm{v} m_2}$;
$w_2 := \mathrm{red}[\Gamma, S, w, w', \ldots, w^{(d_\ell - 1)}$
Iterate loop 2 *with* $w := w_2$.
Iterate loop 1 *and consider* g_{i-1}.
Return Γ *and* S.

Example 21. *Consider the module generated by $m := tx' - px$. The polynomial P_m is $d - p$, which vanishes for $d = p$. Indeed $m^{(p)} = tx^{(p+1)}$, of positive valuation. Then we take $\gamma = x^{(p+1)}$.*

Theorem 22. *The set $\Gamma \cup S$ generates $M : t^\infty$.*

Proof. We only have to remark that S and the derivatives of Γ form a basis of the $\mathfrak{k}[\![t]\!]$-module $M : t^\infty$, as their twisted heads are all different and with coefficient of valuation 0. $\square$

If M is defined by a set of generators Σ, proving that $\mathcal{M}$ is a $\mathfrak{k}[\![t]\!]$-module without torsion amounts to proving that $\Gamma \cup S \subset M$, which may be done by computing a Gröbner basis of $[\Sigma]$.

Assume that M is torsion, that is of differential dimension $m = 0$. Then the next theorem characterizes regular modules.

Theorem 23. *Let M be a module of differential dimension $m = 0$ and $\mathcal{M} = \mathcal{D}^n / M$.*
i) The module $\mathcal{M}$ is regular iff for all $\gamma \in \Gamma$, $\kappa_{(t)}\, \gamma = \kappa\, \gamma$.
ii) The module $\mathcal{M}$ is regular iff $\bigcup_{r \in \mathbf{N}} t^r \mathcal{M} = 0$.
iii) Any module $\mathcal{M}_2 \subset \mathcal{M}$ is regular.

Proof. i) Let us prove first that the condition is necessary. The proof relies on the reduction process. Let $\Gamma_2 := \{\gamma \in \Gamma | \kappa\gamma > \kappa_{(t)}\gamma\}$. We also denote by Γ^* the set of all the derivatives of Γ. If $\Gamma_2 \neq \emptyset$, then for $\gamma \in \Gamma_2$, we may iterate the reduction process by Γ^* r times and get $\mathrm{red}^r(Gamma^*, \kappa_{(t)}\gamma = t^r R_{r,1} + R_{r,2}$, where $R_{r,1} \neq 0$ depends only of derivatives of the $\kappa_{(t)}\gamma$, $\gamma \in \Gamma_2$ and $R_{r,2}$ of derivatives of the y_i smaller that the $\kappa_{(t)}\gamma$, $\gamma \in \Gamma$.

Let N_s be the $\mathfrak{k}[\![t]\!]$-module generated by the $\kappa_{(t)}\gamma - R_{r,2}$ for $r \geq s$. Let B_s be a Gröbner basis of the $\mathfrak{k}[\![t]\!]$-module N_s, using the same order on derivatives. For all s, B_s contains a element m with head $\kappa_{(t)}\gamma$, so $\bigcap_{s \in \mathbf{N}} t^s \mathcal{D}^n / M \supset \bigcap_{s \in \mathbf{N}} N_s \neq 0$.

To prove that the condition is sufficient, we have to consider the set S produced by algorithm 20 and build regular coordinates. For this, we consider the derivatives of the x_i lower than $\kappa\gamma_i$. Let E be the set of such derivatives that are not main derivatives of elements of S and F the set of the remaining derivatives. We have a partition $E = \bigcup_{i=1}^{n} D_i \cup \bigcup_\ell E_\ell$, with $E_\ell = \{x_{j_\ell}^{(k)} | p_\ell \leq k \leq q_\ell\}$ such that $x_{j_\ell}^{(p_\ell - 1)} \notin E$ and $x_{j_\ell}^{(q_\ell+1)} \notin E$ and $D_i = \{x_i^{(k)} | \hat{p}_i \leq k \leq \hat{q}_i\}$ with $x_i^{(\hat{p}_i - 1)} \notin E$ and $x_i^{(\hat{q}_i+1)} = \kappa\gamma_i$. We then introduce new variables $y_\ell = x_j^{(p_\ell)}$ and $z_i = x_i^{\hat{p}_i}$. Using the relations in S we can express all derivatives in F as functions H depending on derivatives in E, and rewrite the relations γ to get expressions of $\kappa\gamma$ as functions J of derivatives in E too. So that we have the regular system $x_{j_\ell}^{(q_\ell - p_\ell + 1)} = H_\ell(y)$, $z_i^{(\hat{q}_\ell - \hat{p}_\ell + 1)} = J_i(z, y)$, which is equivalent to $\Gamma \cup S$.

ii) We have just proved above that $\kappa_{(t)}\,\gamma = \kappa\,\gamma$, $\gamma \in \Gamma$ is a regularity condition and that it is also equivalent to $\bigcup_{r \in \mathbf{N}} t^r \mathcal{M} = 0$.

iii) It is a straightforward consequence of ii). □

Example 24. *We continue example 21. In this case, we have a new coordinate $y = x^{(p)}$ and the regular system $y' = 0$. We also have $x^{(k)} = k!/p!t^{p-k}y$, so that the two systems are equivalent.*

Torsion elements of $\mathcal{M}$ are also torsion elements of $\tilde{\mathcal{M}} = \tilde{\mathcal{D}}^n/\tilde{\mathcal{D}}M$, which are easilly computed, as $\tilde{\mathcal{D}}$ is an Euclidean ring. We recall that starting with generators $x_i' = f_i(x, u, t)$, these torsion elements are such that their derivatives cannot depend on the "controls" u, so that they are functions T that satisfy $\hat{\tau}^r \partial_{u_\ell}$, for all $\ell = 1, \ldots, m$ and $r \in \mathbf{N}$. It is easilly seen that the dimension, as a $\mathfrak{k}(\!(t)\!)$-vector space, of the Lie algebra $\mathcal{L}$ that these derivations generate is at most $n + m$, and $n + m$ iff $\tilde{\mathcal{M}}$ has no torsion elements, and so $\mathcal{M}$.

This gives a fast and easy criterion.

Now, we may consider the case where $\mathcal{M}$ has no torsion elements.

Theorem 25. *A module $\mathcal{M}$ without torsion elements is regular.*

Proof. The $\tilde{\mathcal{D}}$-module $\tilde{\mathcal{D}}\mathcal{M}$ is free and admits a basis z. All generators x_i of $\mathcal{M}$ can then be expressed as $x_i = X_i(z)$. Let d be the maximal degree of the numerators in the X_i and e their maximal order in the z_j. Then, we may write $z_j = t^{d+e}\tilde{z}$, so that we have $x_i = \tilde{X}_i(\tilde{z}_i)$, where the denominators in t have disapeared.

So, $\mathcal{M}$ is a submodule of the $\mathcal{D}^m$-module $[\tilde{z}]$. Let G be a Gröbner basis of $\mathcal{M}$ for an ordering such that $t^{\alpha_1}\tilde{z}_{i_1}^{(k_1)} < t^{\alpha_2}\tilde{z}_{i_2}^{(k_2)}$ if $i_2 > i_1$ or $i_2 = i_1$ and $k_2 > k_1$ or $i_2 = i_1$, $k_1 = k_2$ and $\alpha_2 > \alpha_1$.

We order the $g_i \in G$ by decreasing leading term κg_i and use an ordering on their derivatives such that $g_i < g_j$ if $\kappa g_i < \kappa g_j$ or $\kappa g_i = \kappa g_j$ and $i < j$.

With such an ordering, it is known that the reduction to 0 of the S-elements associated to all the suitable couples $(g_i, g_j) \in G^2$ produces a Gröbner basis of the module of relations between the g_i, that will have the shape $g_i^{(k_i)} - L_i(g)$, so that it is regular. Indeed, the S elements are of the form $g_i^{(k_i)} - t^{\alpha_{i-1}} g_{i-1}$, with $g_i^{(k_i)} > t^{\alpha_{i-1}} g_{i-1}$, according to our conventions. □

5 Testing freeness for regular modules

We assume here that $\mathcal{M} = \mathcal{D}^n/M$ is regular and that we dispose of an explicit regular Gröbner basis $G \subset \mathcal{D}^{n+m}$, that we may assume to have the form:

$$g_i := x_i' - \sum_{j=1}^{n} c_{i,j} x_j + \sum_{\ell=1}^{m} c_{i,\ell} u_\ell. \tag{5}$$

Definition 26. *We may represent an element v of $\mathcal{M}$ as*

$$\sum_{i=1}^{n} c_i(t) x_i + \sum_{k=0}^{k_0} \sum_{j=1}^{m} c_{k,j}(t) u_j^{(k)}.$$

We will denote in this section by κv the non zero sum $\sum_{j=1}^{m} c_{k_0,j}(t) u_j^{(k_0)}$.

We will also use $\mathrm{ord}_u v := k_0$ with $\mathrm{ord}_u v = -\infty$ if v is free of the u_ℓ and their derivatives and denote by C_v the m-uple $(c_{k_0,1}, \ldots, c_{k_0,m}) \in \mathfrak{k}[\![t]\!]$.

The following lemma plays a key role in our approach.

Lemma 27. *i) With the above notations, if $\mathcal{M}$ is free and for $\ell = 1, \ldots, r$ and $i = 1, \ldots, n$, $c_{i,\ell}(0) = 0$, then u_ℓ, $\ell = 1, \ldots, r$ can be completed to form a basis B that does not depend of strict derivatives of the u_ℓ and contains at least one element that does not depend of the u_ℓ.*

Proof. First, we prove that we may find a basis with $\mathrm{ord}_u B \le 0$.

If $\mathrm{ord}_u B_i = -\infty$ for $i = s, \ldots, m$, let μ_i be the smallest integer such that $\mathrm{ord}_u B_i^{(\mu_i)} \ge 0$. We denote $\tilde{B}_i = B_i^{(\mu_i)}$. If $\mathrm{ord}_u B_i \ge 0$, then $\tilde{B}_i := B_i$ and $\mu_i := 0$.

Assume the B_i are ordered by decreasing $\mathrm{ord}_u B_i$, and then increasing μ_i. Let then B be such that the m-uple $\mu_B := (\mathrm{ord}_u B_1, \mathrm{ord}_u B_{s-1}, -\mu_s, \ldots, -\mu_m)$ is minimal for the lexicographic ordering.

If B is a basis of $\mathcal{M}$, then $\dim_{\mathfrak{k}(\!(t)\!)} \mathcal{M}/[B] = 0$. Now, the $m \times m$ matrix C with line i equal to $C_{\tilde{B}_i}$ has a determinant which is not identically 0[2], then $\dim_{\mathfrak{k}(\!(t)\!)} \mathcal{M}/[B] \le n + \sum_{i=1}^{s-1} \mathrm{ord}_u \tilde{B}_i - \sum_{i=s}^{m} \mu_i)$. Indeed, up to a permutation, we may assume that all the principal minors of C do not vanish, and then we get a basis by taking u_i and its derivatives up to $\mathrm{ord}_u - 1$ that we may complete with at most $n - \sum_i \mu_i$ functions of the x_i.

[2]This is a special case of jacobi's *truncated determinant* (see [21]).

So $|C| = 0$. Let $\sum_{i=i_0}^m a_i C_{\tilde{B}_i} = 0$ be a relation with $a_{i_0} \neq 0$ and i_0 minimal. If $\mathrm{v} a_{i_0} = 0$, then a_i is intertible and we can replace B_{i_0} with $B_{i_0} - \sum_{i>i_0} a_i / a_{i_0} \tilde{B}_i^{(\mathrm{ord}_u \tilde{B}_{i_0} - \mathrm{ord}_u \tilde{B}_i - \mu_{i_0} + \mu_i)}$. Then $\mathrm{ord}_u B_{i_0}$ will decrease or μ_{i_0} increase, contradicting the minimality of the m-uple μ_B.

Let us prove that the case $\mathrm{v} a_{i_0} > 0$ is impossible. Let then G_2 be the Gröbner basis of the module generated by G (5), the B_i, $i \neq i_0$ and $\hat{B}_{i_0} := a_{i_0} B_{i_0} - \sum_{i>i_0} a_i / \tilde{B}_i^{(\mathrm{ord}_u \tilde{B}_{i_0} - \mathrm{ord}_u \tilde{B}_i - \mu_{i_0} + \mu_i)}$. Now, the module of relations between the B_i and $\hat{B}_{i_0}$ is generated by $\hat{B}_{i_0} - a_{i_0} B_{i_0} + \sum_{i>i_0} a_i / \tilde{B}_i^{(\mathrm{ord}_u \tilde{B}_{i_0} - \mathrm{ord}_u \tilde{B}_i - \mu_{i_0} + \mu_i)}$. This means that computing a Gröbner basis of $[G, B]$ adding $\hat{B}_{i_0}$ to G_2 is reduced to computing a single S-expression, between $t^{\mathrm{v} a_{i_0}} \mathrm{red}(G, \hat{B}_{i_0})$ and some member of G, that must be reduced to 0, so that $\{\mathrm{red}(G, \hat{B}_{i_0})\} \cup G_2$ is a Gröner basis of $[G, B]$, which is impossible, as such a basis must be of the form $\{x_1, \ldots, x_n, u_1, \ldots, u_m\}$.

So, we must have $\mathrm{ord}_u B_i = 0$, $i = 1, \ldots, s-1$ and $\sum_{i=s}^m \mu_i = n$. We may then reduce the B_i, $i = 1, \ldots, s-1$ using the B_i, $i = s, \ldots, m$ and their derivatives up to order $\mu_i - 1$, that generate $(x_i | 1 \leq i \leq n)$, in order to reduce to the case where the B_i, $i = 1, \ldots, s-1$ only depend on the u_j.

We have our result if for all $1 \leq j \leq r$, $u_j \in (B_i | 1 \leq i \leq s)$. Assume this is not the case. We may take $B_i = u_i$, $i = 1, \ldots, p < r$, $B_{p+i} = u_{r+i}$, $i = 1, \ldots, s-p$. Then $\tilde{B}_i$, $i = s+1, \ldots, m$ may be reduced to depend only of u_i, for $i = s-p+1, \ldots, m$ and $i = p+1, \ldots, r$. The values of the u_i for this last set of indexes cannot be expressed from B and their derivatives, as they appear with coefficient of strictly positive valuation. This contradiction concludes the proof. □

This lemma suggests an algorithm, with successive steps of reduction of the state dimension n. In the following algorithm, we assume that the number r of inputs u_i that appear with non invertible coefficient in maximal, in other words the matrix $(c_{i,\ell} | i = 1, \ldots, n,\ \ell = r+1, \ldots, m)$ has full rank $m - r$. If not, we can easilly reduce to this case by a change of coordinates.

Algorithm 28 (Contraction)**.** ***Input:*** *a reg. system G.*
*If $r = m$, **exit:** "not free".*
Reorder the u_i, so that $[\tau, \partial_{u_i}]$, $i = m-k+1, \ldots, m$ are independent over $\mathfrak{k}(\!(t)\!)$ with k maximal.
*If $k = 0$, **exit:** "not free (torsion elements)" .*
Compute a maximal basis of solutions y_ℓ, linear in the x_i of $[\tau, \partial_{u_i}] Y(x, t) = 0$, $i = m-k+1, \ldots, m$.
Complete them with some x_i to get a basis of (x). Call them v_j, $j = 1, \ldots k$.
Compute a new system G_2: $y_i - L_i(y, u_1, \ldots, u_{r+k}, v)$.

Compute a new set of controls w with maximal number r_2 of control appearing with non invertible coefficient.

Return: *The system $G_3(y, w)$ obtained at the last step.*

The main algorithm consist in repeated application of this process.

Algorithm 29. ***Input:*** *a regular basis G.*

until $n = 0$ ***repeat:***

$G := Contraction(G)$

Return: *The controls u of the last system.*

The algorithm stops when the state dimension n is 0 and then returns the controls u that form a basis. If $r = m$ the contraction is impossible, according to lemma 27 and one exits the process, as well as in the case where the state dimension is strictly positive and stationary, meaning that torsion elements exists.

Example 30. *The module defined by $x' = tu$ is not free.*

Example 31. *The module defined by $x_1' = tu_1 + x_2$ and $x_2' = u_2$ is nfree, as we can take u_1 as a flat output, completed with x_1.*

In the case when the module is not free, we may conclude that our control systems is not flat in the neighbourhood of a given point. This is important to notice, but does not mean that we cannot use some kind of non bijective parametrization.

Example 32. *We go back to the car (example 1) at a non flat point where $z_1' = z_2' = \theta' = 0$ at $t = 0$.*

We would like to use the flat outputs: $\zeta_1 = \theta$ and $\zeta_2 = \sin(\theta)z_1 - \cos(\theta)z_2$. We may pose $\zeta_1 = t\lambda' - \lambda$. Then, we have $\zeta_1' = t\lambda''$ and $\lambda'' \neq 0$ when $\zeta_1'' \neq 0$. So,

$$\zeta_2'/\lambda'' = -\sin(\theta)z_1 + \cos(\theta)z_2$$

and the values of z_1 and z_2, and so the whole state of the system, can be computed at the non flat point.

6 Conclusion

We have provided an algorithm to test the freeness of a $\mathcal{D}$-module, which produces an effective necessary condition for local differential flatness. An example shows that this condition is not sufficient in differential dimension greater than 1. Another example has shown that the situation at a non flat point is not lost and that a suitable parametrization, although not bijective, can achieve motion planning.

These remarks provide new directions for further investigations.

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