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Flat difference systems. A definition, some properties and open problems

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Introduction

Differential flatness [5, 6], introduced by Fliess, Lévine, Martin and Rouchon, is a mathematical notion which goes back to Monge's problem [12, 9, 20] and which proved to be a very fruitfull tool for control theory. The aim of this poster is to provide an similar definition in difference algebra and to investigate their properties, by analogy with the differential case. Such an approach is known to work in many cases, but the difference case could sometimes hide subtle difficulties.

Our two main results are analogs of the criterion of Charlet, Lévine and Marino, for difference dimension 1, and of the necessary condition for flatness expressed by Rouchon's criterion. We show however that there is no difference analog of the Lüroth–Ritt theorem. We also show that every controllable linear difference system is flat.

To avoid any misunderstanding, it should be noticed that this work has no claim to direct applicability. It should be considered as a first theoretical step for the sudy of non-linear *mixed systems* of differential-difference equations.

1 Difference algebra

We refer to Cohn's classical book [4] or to Levine [11] for an introduction to difference algebra, a theory first introduced by Ritt [15, 16, 17]. For computational issues, one may refer to [10, 8] and the references therein, or to [2, 14] in the linear case.

We will be only concerned here with ordinary *inversive* difference fields, that is a field $\mathscr{F}$ equipped with an automorphism τ, with inverse τ^{-1}; τ is call a *transforming operator*, $\tau^j a$ is called a *transform* of a. We denote by $\mathscr{F}\{x_1,\ldots,x_n\}$ the algebra of inversive difference polynomials, that is the algebra $\mathscr{F}[\tau^j x_i | i\in[1,n],\ j\in\mathbf{Z}]$, with the action of τ and τ^{-1} extended in an obvious way. We denote by $\mathscr{F}\langle x\rangle$ the fraction field of $\mathscr{F}\{x\}$; this is the *transformally transcendental* extension generated by $x_1,\ldots,x_n$.

By an inversive difference ideal, we mean an ideal $\mathscr{I}$ such that $\tau\mathscr{I}=\tau^{-1}\mathscr{I}=\mathscr{I}$. A morphism ϕ of difference rings $A\mapsto B$ is a ring morphism such that $\tau\circ\phi=\phi\circ\tau$. Let $\mathscr{F}\subset\mathscr{G}$, it is said that $\mathscr{G}/\mathscr{F}$ is a difference field extension if the injection of $\mathscr{F}$ in $\mathscr{G}$ is a morphism of difference rings. If $X\subset\mathscr{G}$, we denote by $\mathscr{F}\langle X\rangle$ the smallest difference field extension of $\mathscr{F}$ that contains X.

Let $\mathscr{P}$ be a prime inversive difference ideal of $\mathscr{F}\{x_1,\ldots,x_n\}$, we associate to it a difference field extension $\mathscr{G}/\mathscr{F}$, where $\mathscr{G}$ is the fraction field of the difference domain $\mathscr{F}\{x\}/\mathscr{P}$. Difference transcendence bases and difference dimension are defined in an obvious way.

We refer to [8] for a precise definition of characteristic sets for difference ideal. In the context of control theory, we call a differential system a characteristic set of a prime differential ideal [18] and a difference system a characteristic set of a prime difference ideal of $\mathscr{F}\{x_1,\ldots,x_n\}$, assuming $\mathscr{F}$ to be of characteristic 0.

2 Controlable and flat difference extension

2.1 Controlability

A differential system $\mathscr{A}$, defining a differential field extension $\mathscr{G}/\mathscr{F}$, with derivations denoted by δ, is said to be *controllable* if, assuming $\mathscr{A}$ to be of order 1 and $x_1,\ldots,x_m$ to be a trancendance basis of the extension, the Lie algebra generated by δ and the partial derivatives $\partial/\partial x_i'$, for $1\le i\le m$ contains all partial derivatives $\partial/\partial x_i$, for $1\le i\le n$. See [7]. The notion does not depend of the chosen coordinates, characteristic set or transcendence basis. Controlability implies that all elements of $\mathscr{G}$ that are differentially algebraic over $\mathscr{F}$ must belong to $\mathscr{F}$, but the converse is known to the false, as shown by the example: $x_1'=x_1x_2'$. Every trajectory is such that $x_1e^{-x_2}=\mathrm{Cste}$, which contradicts controllability, but $x_1e^{-x_2}\notin\mathscr{G}$.

For difference systems, we propose the following definition.

Definition 1. — *Assume $y_1,\ldots,y_m$ to be a* transformal *trancendance basis of the extension $\mathscr{G}/\mathscr{F}$ and $x_1,\ldots,x_k$ to be an* algebraic *transcendence basis of $\mathscr{G}$ over $\mathscr{F}\langle y\rangle$. The extension is said to be* controllable *if the Lie algebra generated by $\tau^j\partial/\partial y_i\tau^{-j}$, $1\le i\le m$, $j\in\mathbf{Z}$ contains every partial derivation $\partial/\partial x_i$, $1\le i\le k$.*

Proposition 2. — *The extension $\mathscr{G}/\mathscr{F}$ is controllable iff there is no element in $\mathscr{G}\setminus\mathscr{F}$ which is transformally algebraic over $\mathscr{F}$.*

It is easily seen that such elements form a basis of solution of the partial differential system associated with the Lie algebra of the definition and this is the main idea of the proof.

A straightforward consequence of this proposition is that the definition does not depend of the chosen difference transcendence basis. We may also express the condition of the proposition in the following form. Assume that the difference system is reduced to order one, as we may, using a suitable change of variables, so that $y_1,\ldots,y_m$ form a difference transcendance basis of $\mathscr{G}/\mathscr{F}$, and that $\{\tau^j y_i|1\le i\le m,\ i\in\mathbf{Z}\}\cup\{y_i|m<i\le n\}$ is an algebraic transcendance basis of $\mathscr{G}/\mathscr{F}$. Then, denoting by $\overline{\mathscr{F}\langle A\rangle}$ the algebraic closure in $\mathscr{G}$, the condition of the proposition is equivalent to

$$\bigcap_{j\in\mathbf{Z}}\overline{\mathscr{F}\langle\tau^j x_1,\ldots,\tau^j x_n\rangle}=\overline{\mathscr{F}}.$$

2.2 Flat difference extension

Our definition exactly reproduces the differential one, so that there is little ambiguity about the way one could translate flatness in the difference case.

Definition 3. — *We say that a difference extension $\mathscr{G}/\mathscr{F}$ is flat if $\overline{\mathscr{G}}$ is isomorphic to the algebraic closure of some transformally transcendental extension $\overline{\mathscr{F}\langle z_1,\ldots,z_m\rangle}$*

The elements $z_1,\ldots,z_m$ are said to be a linearizing output.

Example 4. — The system $\tau x_1=x_1+(\tau x_2-x_2)^2$ is not flat as we will prove below. This system is an analog of the non flat differential system $x_1'=x_2'^2$.

Example 5. — The system $(\tau x_1-x_1)(\tau x_2-x_2)=(\tau x_3-x_3)$ is flat. A flat output is x_1, $(x_1-\tau^{-1}x_1)x_2-x_3$. This is an analog of the differential flat system $x_1'x_2'=x_3'$ of Rouchon.

3 Analogue of the Charlet, Lévine and Marino criterion

In the case of an extension of differential dimension one, there is a necessary and sufficient condition of flatness due to Charlet, Lévine and Marino [3]. We propose here a difference analog.

Theorem 6. — *Let $\mathscr{G}/\mathscr{F}$ be a difference extension of difference dimension* 1, *using the coordinates y, as defined above, $\mathscr{G}/\mathscr{F}$ is flat iff it is controllable and*

$$\bigcap_{j=0}^{n-2}\overline{\mathscr{F}\langle\tau^j x_1,\ldots,\tau^j x_n\rangle}\neq\overline{\mathscr{F}}.$$

In such a case, this intersection is equal to $\overline{\mathscr{F}\langle z\rangle}$, for some $z\in\mathscr{G}$, which is a linearizing output.

We follow the same scheme of proof as in the differential case. First, an easy recurrence shows that if the system is controllable, $\bigcap_{j=0}^{k}\overline{\mathscr{F}\langle\tau^j x_1,\ldots,\tau^j x_n\rangle}$ is of algebraic transcendental dimension at most $n-k$ over $\mathscr{F}$, so if $\bigcap_{j=0}^{n-2}\overline{\mathscr{F}\langle\tau^j x_1,\ldots,\tau^j x_n\rangle}\neq\overline{\mathscr{F}}$, it must be of algebraic transcendental dimension 1 and a transcendental element z will be a linearizing output.

If the extension is flat, it is obviously controllable and we can chose a linearizing output that actually depends of $x_1,\ldots,x_n$. Then, it cannot depend of transforms of these elements of order k non zero. If it were the case, $\tau Z,\ldots,\tau^j Z$ would respectively depend of elements of order $|k|+1,\ldots,|k|+j,\ldots$, so that it would be impossible to express the x_i as function of transforms of z.

So $z=Z(x_1,\ldots,x_n)$. Now, the x_i must be expressed as functions of $Z,\tau Z,\ldots,\tau^{n-2}Z$. Using the same kind of arguments, one shows that these transforms can only depend of $x_1,\ldots,x_n$. We easily conclude then that z must belong to that field intersection.

Using this criterion, it is easilly seen the the system of example 4 is non flat.

We may notice that there is no difference analog of the Lüroth – Ritt theorem, as shown by considering the subfield $\mathscr{F}\langle x\tau x, x^2\rangle$ of $\mathscr{F}\langle x\rangle$.

4 An analog of Rouchon's criterion

In [19], Rouchon gave a necessary flatness condition for differential systems. We provide here the following analog.

Theorem 7. — *Let $\mathscr{G}/\mathscr{F}$ be a flat difference extension, then, using coordinates $x_1,\ldots,x_n$, as defined above, such that $x_1,\ldots,x_m$ is a difference transcendance basis, there exist two derivations expressed by finite sums $d_1=\sum_{i=1}^m\sum_{j>0}c_{1,i,j}(x)\partial/\partial\tau^j x_i$ and $d_2=\sum_{i=1}^m\sum_{j<0}c_{2,i,j}(x)\partial/\partial\tau^j x_i$, where the $c_i(x)$ are algebraic transformal functions, such that $d_1,\tau^{-1}\circ d_1\circ\tau,\ldots,d_2$, and $\tau\circ d_2\circ\tau^{-1},\ldots$ commute.*

Assume the extension is flat and let $z_1,\ldots,z_m$ be a linearizing output. We proceed as in the proof given in [13] for the differential case, and take for d_1 and d_2 the derivatives with respect to the highest and lowest transforms of some z_i, the coordinate functions $y_1,\ldots,y_n$ actually depend on. The result is then straightforward.

We see however that deducing an effective criterion is not as easy. The best we can do in the general case is to reduce to some system of differential equation and then to test the existence of solutions by computing a characteristic set. In some simple cases such as the example 4, they can be performed by hand.

5 Linear systems

The linear theory is as easy as in the differential case.

Theorem 8. — *A linear difference system is flat iff it is controllable.*

To show it, we may again use coordinates $x_1,\ldots,x_n$ as defined above. Considering a generic $d_1=\sum_{i=1}^m c_i\partial/\partial\tau x_i$, where the c_i are constants, a basis of solutions of the equation $d_1\tau F(y)=0$ defines a new set of $n-1$ coordinates $\tilde{y}_1,\ldots,\tilde{y}_{n-1}$. We may iterate the process, until $n=m$. Then, the $\tilde{y}_i$ will form then a linearizing output.

Conclusion. A few open problems

We have seen how the notion of flatness may be extended in a natural way to difference systems, while keeping some analogs of results known in the differential case. We may still expect a possible analog of Cartan's criterion [1].

The two main open questions related to differential flat system may also be generalized to the difference situation:
— If $\mathscr{G}/\mathscr{F}$ is flat and $\mathscr{F}\subset\mathscr{G}_2\subset\mathscr{G}$ is an intermediate field extension, is $\mathscr{G}/\mathscr{F}$ flat? (Endogenous=exogenous.)
— Is there an algorithm to decide if a given system is flat?

We have also seen that some analogs of differential flatness criteria such as Rouchon's criterion may also deserve some more study in order to become efficient in non trivial situations.

References

[1] CARTAN (Élie), "Sur l'intégration de certains systèmes indéterminés d'équations différentielles", *Journal für die reine und angewandte Mathematik*, 145, 86–91, 1915.

[2] CHYZAK (Frédéric), QUADRAT (Alban), ROBERTZ (Daniel), "Oremodules: A symbolic package for the study of multidimensional linear systems", In J. Chiasson and J. J. Loiseau, editors, *Applications of Time Delay Systems, Lecture Notes in Control and Information Sciences*, vol. 352, pages 233–264, Springer, Berlin/Heidelberg, 2007.

[3] CHARLET (B.), LÉVINE (Jean), textscMarino (R.), "Sufficient Conditions for Dynamic State Feedback Linearization", *Siam J. Control & Optimiz.* **29**, 38–57, 1991 .

[4] COHN (Richard M.), *Difference algebra*, Interscience Publishers, New-York, 1965.

[5] FLIESS (Michel), LÉVINE (Jean), MARTIN (Philippe) et ROUCHON (Pierre), "Sur les systèmes non linéaires différentiellement plats" *C. R. Acad. Sci. Paris*, I–315, 619–624, 1992.

[6] FLIESS (Michel), LÉVINE (Jean), MARTIN (Philippe) et ROUCHON (Pierre), "Flatness and defect of nonlinear systems: introductory theory and applications", *Internat. J. Control*, 61, 1995, 1327–1887.

[7] FLIESS (Michel), LÉVINE (Jean), MARTIN (Philippe), OLLIVIER (François) and ROUCHON (Pierre), "A remark on nonlinear accessibility conditions and infinite prolongations", *Systems and Control Letters*, **31**, 1997, 77–83.

[8] GAO (Xiao-Shan), LUO (Yong), YUAN (Chunming), "A characteristic set method for ordinary difference polynomial systems" *Journal of Symbolic Computation*, Volume 44, Issue 3, Elsevier, March 2009, 242–260.

[9] HILBERT (David), "Über den Begriff der Klasse von Differentialgleichungen", *Mathematische Annalen*, 73, 95–108, 1912.

[10] KONDRATIEVA (M. V.), LEVIN (A. B.), MIKHALEV (A. V.) and PANKRATIEV (E. V.), *Differential and difference dimension polynomials*, Mathematics and its Applications, **461**, Kluwer Academic Publishers, 1999.

[11] LEVIN, (Alexander), *Difference algebra*, Algebra and Applications, **8**, Springer, 2008.

[12] MONGE, (Gaspard), "Supplément où l'on fait savoir que les équations aux différences ordinaires, pour lesquelles les conditions d'intégrabilité ne sont pas satisfaites sont susceptibles d'une véritable intégration…", *Histoire de l'Académie royale des sciences*, Paris, 1787, 502–576.

[13] OLLIVIER (François), "Une réponse négative au problème de Lüroth différentiel en dimension 2", *C. R. Acad. Sci. Paris*, t. 327, Série I, 881-886, 1998.

[14] QUADRAT (Alban) and ROBERTZ (Daniel), "Computation of bases of free modules over the Weyl algebras", *Journal of Symbolic Computation*, **42**, (11-12), 1113–1141, 2007.

[15] RITT (Joseph Fels), DOOB (Joseph Leo), "Systems of algebraic difference equations", *Amer. J. Math.*, 55, 505–514, 1933.

[16] RITT (Joseph Fels), "Algebraic difference equations", *Bulletin of the A.M.S.*, **40**, 1934, 303–308.

[17] RITT (Joseph Fels), RAUDENBUSH Jr. (Henry W.), "Ideal theory and algebraic differennce equations", *Transactions of the A.M.S.*, **46**, 1939, 445–452.

[18] RITT (Joseph Fels), *Differential Algebra*, Amer. Math. Soc. Colloq. Publ., vol. 33, A.M.S., New-York, 1950.

[19] ROUCHON (Pierre), "Necessary condition and genericity of dynamic feedback linearization", *J. of Math. Systems, Estimations and Control*, vol. 4, no. 2, 257-260, Birkäuser, Boston, 1994.

[20] ZERVOS, (Panajiotis), *Le problème de Monge*, Mémorial des Sciences Mathématiques, fascicule LIII, Gauthier-Villars, Paris, 1932.