MÉMOIRE D'HABILITATION À DIRIGER DES RECHERCHES
(Spécialité informatique)
École polytechnique

EFFECTIVE FORMAL RESOLUTION OF SYSTEMS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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M.	Manuel	Kauers	
M.	Pierre	Rouchon	
M.	Werner M.	Seiler	Rapporteurs
M.	Moulay	Barkatou	
M.	Alin	Bostan	
M.	François	Boulier	
M.	Xavier	Caruso	
M.	Jean-Michel	Coron	
M.	Frédéric	Hélein	Examinateurs

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עת לבקש ועת לאבד עת לשמור ועת להשליך.

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אין כּל־חדש תחת השמש.

כּהלת, א ט [2]

קוקט נישט אויף מײַן גראָען קאָפּ,
צי שטערט דאָס אײַך אין שפּיל?
מײַן נשמה איז נאָך יונג,
ווי צוריק מיט יאָרן פֿיל.

מרדכי געבירטיק, *הוליעט קינדערלעך* [3]

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de Marc Ferro, et des nombreuxses historienes qu'il avait fait venir, et de profiter avec des artistes comme Hervé Loillier d'une formation exceptionnelle.

L'équipe du Forum a été une école d'organisation, avec Jean-Pierre Courel, Christophe David, Éric Denoyer, Hubert Descans, Pascal Faure, Jean-Yves Le Floch, Cécile Martin…

Dans le chemin de la recherche, j'ai apprécié la compagnie et les échanges avec de nombreux camarades : au sein de la promo X83, j'ai pris le chemin des laboratoires avec Thierry Robin, puis Frédéric Hélein lors d'un stage avec Nicole Berline et Jean-Michel Bismuth.

J'ai été intégré à l'équipe des doctorantes déjà présents : Marc Chardin, Willy Moreno-Socías, Pierre-Vincent Koseleff, Jean-François Maltey, Wu Hua, Osmo Pekonen, Thomas Ehrhardt, bientôt rejoints par Éric Boix et Bernard Mourrain, sans oublier les voisines de Jussieu, Joël Marchand, Renaud Rioboo, Annick Valibouze, Jean-Charles Faugère…

Au sein du DEA LAP de Paris VI, suivi avec Marie-Chritine Gontard, Didier Pinchon, Renaud Rioboo, Nitsan Seniak… Daniel Lazard aura eu une forte influence sur moi. J'ai eu la chance de profiter des derniers cours de Jacques Arsac, qui « apprenait à réfléchir » avec une franchise sans compromis : « Polytechnique ? Mon pauvre ! », « L'intelligence artificielle ? Je déduis que c'est de la bêtise naturelle. »

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Mes débuts au LIX m'ont fait cotoyer Michel Weinfeld, Patrick Cousot, … et d'autres jeunes Bruno Salvy, Nitsan Seniak, Jan Stranski, Joris van der Hoeven, …

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Ce fut toujours un plaisir au fil des années de revoir André Galligo, toujours passionné en science et appaisant dans les relations humaines.

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Jacques Calmet s'est beaucoup investi dans ce volume 20 (1), dédié à Jacobi, de AAECC, en particulier pour l'édition délicate des traductions. Teo Mora aura été l'éditeur patient et efficace d'un article d'une longueur atypique qui les prolonge. Merci à Jacques, Marc et Teo pour le vaste travail accompli afin de faire vivre cette revue à travers un travail éditorial de qualité. Et merci à tous les rapporteurs anonymes… Et un très grand merci à Daniel J. Katz, qui fit un travail exceptionel comme raporteur des deux traductions de Jacobi.

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This is a much difficult exercise, when time makes forgetting unavoidable and the number of influences, helps, favors and encouragements to remember so great. Betting all those that I may have omitted to forgive me, let it be an occasion to recall that our work, although very individual and solitary in its core, requires, in order to work on abstractions, to be able to cut oneself from material duties, which would be impossible without the strong support of family, friends, numerous colleagues and the support of a collectivity that cares for public institutions and academic research. Polytechnique and CNRS have been an inestimable support for me during all these years.

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This defense was made possible by the wise advice and dedication of Benjamin Doerr, who did welcome and support my atypical request.

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Marc Giusti guided my first steps in research, and contributed with tenacity to develop and maintain a material and institutional framework to support our work, and meaningful goals that could unite a research community. Moreover he spent much energy to convince me to defend this habilitation.

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Among the friend of Darius Milhaud, the exchanges with Dararith Chau and Benoît Ninin forced me to work, and the multiple projects, theatrical or photographic, with Philippe Arson, have taught me to be serious without taking myself too seriously.

A warm thinking for Manuel Bronstein ז״ל, briefly my classmate at Louis le Grand and met again years later, who would have undoubtedly been part of this jury if the Rebeynu Shel Oylem had not decided otherwise.

From the Army, I will have retained some simple but effective methods of organization, which "sur le tas" proved to be valuable and personalities who knew how to pursue the objective beyond the hazards, some "enfants de troupe" who "paid cash". Captain Husté and his "little blue Schtroupfs", at the artillery school, Captain Orcin at the 3^{e} RA and his wall calendars, the chief Brickler of the BCS and his benevolence and Colonel Galleraud at X.

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The Forum team was a school of organization, with Jean-Pierre Courel, Christophe David, Éric Denoyer, Hubert Descans, Pascal Faure, Jean-Yves Le Floch, Cécile Martin...

In the the way to research, I enjoyed the company and the exchanges with many comrades: within the X83 class, I first went to the laboratories with Thierry Robin, then with Frédéric Hélein during an internship with Nicole Berline and Jean-Michel Bismuth.

I was integrated into the team of PhD students already present: Marc Chardin, Willy Moreno-Socías, Pierre-Vincent Koseleff, Jean-François Maltey, Wu Hua, Osmo Pekonen, Thomas Ehrhardt, soon joined by Éric Boix and Bernard Mourrain, without forgetting the neighbors of Jussieu, Joël Marchand, Renaud Rioboo, my DEA LAP classmate in Paris VI, Annick Valibouze...

Within the DEA LAP of Paris VI, followed with Marie-Chritine Gontard, Didier Pinchon, Renaud Rioboo, Nitsan Seniak... Daniel Lazard had a strong influence on me. I had the chance to attend the last lectures of Jacques Arsac, who "taught to think" with an uncompromising frankness: "Polytechnique? Poor guy!", "Artificial intelligence? I deduce that it is natural stupidity."

Michel Merle and Jean-Pierre Henri kindly put up with our little troupe.

My beginnings at LIX made me meet Michel Weinfeld, Patrick Cousot, ... and other youngs Bruno Salvy, Nitsan Seniak, Jan Stranski, Joris van der Hoeven, ...

I remember with emotions the "Journées du GRECO de Calcul Formel" at CIRM, and all those that I met there: Yvette Amice, James Davenport, Jean Della Dora, Dominique Duval... and I am all the more happy to have joined the team of JNCF organizers: Jérémy Berthomieux, Florent Bréhard, Paola Boito, Eleonora Guerrini, Aude Maignan, Fatemeh Mohammadi et Pierre-Jean Spaenlehauer. Thanks to the CIRM staff for letting us take benefit of this unique environment.

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I received from my father, a fitter who joined Gnôme et Rhône in 1937, "l'art du trait" and a taste for some mathematical problems, like the scales in the corridor, from my mother some style concern, the attraction for the hieroglyphs and some wind from the east.

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En somme, FMRFIJ, sommes-nous des cow-boys de l'Arizona dans un laboratoire,
Ou des cobayes prenant l'horizon pour un labyrinthe?
Robert Desnos, *L'aumonyme* [4]

Introduction

This memoir describes 36 years of work, on many topics. One common thread is a special interest to field or algebra membership, rather than ideal membership. A second is a constant inspiration from problems of control theory that may be expressed in a natural way by inclusions of differential fields. So, "solving" takes here a very wide meaning, that is not limited to the computation of a normal forms in view of numerical integration, but also includes looking for more precise information on the structure of the set of solutions.

Working conditions have changed a lot. The necessity of accessing books and computers is no more the main motivation for our presence in laboratory and remote exchanges with colleagues in foreign countries have become easier—while unfortunately traveling did sometimes become impossible—, as well as the access to documentation. However, the most important documents for my work were not obtained by bibliographic search on internet, and a part of them are still unavailable online.

The reading of a few mathematical works has been decisive. Ritt's *Differential algebra*, discovered thanks to Wu Wentsun during a talk at IHP in 1988, Denef and Lipschitz's *Power Series Solutions of Algebraic Differential Equations*, found when looking for a stapler on the desk of Michel Merle, Jacobi's *De investigando...*, one of Ritt's refences, that I started to read at the CIRM library in 2003, followed by many copies of Jacobi's manuscripts, received by mail from the Berlin archives, or Kondrateva et al. *О границе Якоби для систем дифференциальних многочленов*, mentioned by some ISSAC referee and scanned by the author.

Ritt's *differential algebra* and Vinogradov's *diffiety theory* are two convenient mathematical frameworks for control theory. Differential algebra is often more convenient, as one can rely on easy computer representations of the data—differential polynomial or ideal—and on implemented algorithms to work with, such as Diffalg. But one may easily encounter situations where polynomial or algebraic functions are not enough. In most cases, one may work with functions defined by systems of PDEs, completed with suitable initial conditions. Then, it may be convenient to escape some computational issues in a preliminary study, using diffiety theory.

Some theoretical difficulties play a special role. The loss of Noetherianity for differential ideal or polynomial algebra is related to hard decidability problems: the membership problem for differential ideals, Ritt's problem, i.e. deciding inclusion of two differential algebraic varieties, the dimensional problem or Jacobi's bound in the general case.

I was able to take advantage of the freedom and time offered by a CNRS position to read, translate in French and English before translating them in the mathematical formalism of differential algebra some important contributions of Jacobi and to invest deeply in the still unsuccessful quest of a differential flatness criterion. Indeed, the results presented here do not reflect the amount of work, but the amount of success. So, some time consuming topics such as flatness criteria will play a little role. I still hope to be able to contribute to this puzzling question, being thankful to a rare institution that can support the risk of years of modest uncertain investigations, in difficult areas were main theoretical contributors are Monge, Hilbert or Cartan... Here again, the language issue is not the main difficulty and traveling between mathematical theories and styles requests time for assimilation.

Computers and algorithms made great progress, so that naive attempts may have more chances of success, but one faces with differential algebra tremendous complexity issues, due to the swiftly increasing size of data. Differentiation alone creates an exponential growth of the equations, before any actual elimination could be attempted, so that the TERA philosophy, using smaller data representations, such as Straight Line Programs, remains a challenging goal for implementation. Bypassing such a demanding work, my recent Maple packages mostly rely on numerical evaluations.

Complexity and decidability issues come together with some paradoxes. Regarding identifiability, it is a generic property, so that a non identifiable system will likely become identifiable by using a more accurate and more complicated mathematical model. On the other hand, deciding identifiability will be much more complicated, as well as actual identification of the parameters. Flatness is non generic, so that most known flat systems are simplified systems, for which flat outputs may be guessed using physical considerations. The interest of theoretical criteria and algorithms to compute flat outputs may then be questioned. In the same way, our efforts *to lower asymptotic complexity are sometimes disappointing, as some "naïve" or "lazy" algorithms may be challenging in the range of sizes useful in practice.*

In this short presentation, I will of course underline some of my main results. I will also try to cast some light on a few difficulties that may have remained hidden in the margins of more technical papers, such as the question of initial conditions and that of "generic solutions" for identifiability. Some sections on many topics will be devoted to open problems and perspectives for further work.

A last chapter will give my work plan for the near future.

Due to the multiplicity of interrelated topics, the following linear exposition required arbitrary and artificial choices and may look like a lexicon novel, such as Milorad Pavić Dictionary of Khazars. *A glossary and hypertext links will respectively help reading the paper versions and the PDF one. A blue star indicates papers I have co-signed, available on my web page.*

The next section completes the introduction with the definition of flat systems, using diffiety theory and differential algebra, a way to fix some mathematical framework and notations.

We use the theoretical framework of *diffiety theory* [78, 171], the main difference is that we fix one derivation, instead of considering the distribution, *i.e.* the vector field it generates. More details are available in **[125]***. Fixing one derivation defines *flatness*, fixing just a distribution *orbital flatness*. See sec. 1.3.

In the sequel, we may sometimes denote ∂/∂_x by ∂_x, for short.

Definition 1. — *A diffiety is a $\mathscr{C}^\infty$ manifold V of denumerable dimension equipped with a derivation δ, the* Cartan derivation *of the diffiety.*

The ring of functions $\mathscr{O}(V)$ is the ring of $\mathscr{C}^\infty$ function on V depending on a finite *number of coordinates. The topology on the diffiety is the coarsest topology that makes coordinate functions continuous, i.e. the topology defined by open sets on subspaces of finite dimensions.*

Definition 2. — *The* time diffiety $\mathbf{R}_t$ *is* $\mathbf{R}$ *equipped with the derivation $\delta_t := \partial/\partial t$.*

The trivial diffiety $\mathbf{T}^m$ *is $\left(\mathbf{R}^{\mathbf{N}}\right)^m$ equipped with the derivation $\delta := \sum_{i=1}^m \sum_{k\in\mathbf{N}} z_i^{(k+1)}\partial/\partial_{z_i^{(k+1)}}$.*

So, finite type diffieties are subdiffieties included in the trivial diffiety, or $\mathbf{R}_t\times\mathbf{T}^m$, that is isomorphic to jet space. One is often concerned with the problem of "solving" an ODE system, which means first finding an "explicit normal form" $x_i^{(e_i)} = f_i(x)$. For this, we need to chose an order on derivatives, compatible with the derivation, and assume in the normal form that f_i only depend on derivatives strictly smaller than $x_i^{(e_i)}$.

Definition 3. — *A morphism of diffiety $\phi : V_1 \mapsto V_2$ is a smooth map between manifolds such that $\phi^*\circ\delta_2 = \delta_1\circ\phi^*$, where $\phi^* : \mathscr{O}(V_2)\mapsto\mathscr{O}(V_1)$ is the dual application, defined by $\phi^*(f) = f\circ\phi$ for $f\in\mathscr{O}(V_2)$.*

With this definition, one may define flat systems [43, 46].

Definition 4. — *A diffiety V is flat if there exists a dense open set $W\subset V$ of* flat points. *In that context a point is called flat, if it admits a neighborhood that is diffeomorphic to an open set of $\mathbf{R}_t\times\mathbf{T}^m$. The generators z_i of $\mathbf{T}^m$ are called* linearizing outputs *or* flat outputs.

A point is parameterizable *if it admits a neighborhood that is the image of an open set of $\mathbf{R}_t\times\mathbf{T}^m$ by a diffiety morphism.*

Ritt's *differential algebra* [146] (see also Kolchin [75]) considers rings and fields equipped with one (or more) derivation δ, all morphisms being compatible with the derivation action. If A_δ is a differential ring, the differential A-algebra of differential polynomials in variables X is denoted by $A\{A\}$ and $\mathscr{F}\langle X\rangle$ is its differential fraction field.

The differential ideal generated by Σ is denoted by $[\Sigma]$. If I is a ideal, we denote by $I : S^\infty$ the ideal $\{a|\exists(b,n)\in S\times\mathbf{N}\ ab^n\in I\}$.

A key tool is the notion of *characteristic set*, that are in Jacobi's word "implicit normal form", meaning systems from which an explicit normal form can be computed without further differentiation.

A stronger definition of flat systems can be given in this setting [39].

Definition 5. — *A differential field extension $\mathscr{G}/\mathscr{F}$ is* flat *if the algebraic closure of $\mathscr{G}$ is isomorphic to the algebraic closure of a differentially transcendental field extension $\mathscr{F}\langle z_1,\ldots,z_m\rangle$. The z_i are* flat outputs.

Meaning is differential.
Hal Hartley, *Theory of achievement*

Chapter 1

Decidability issues in modeling and control

Most of my work was motivated by problems from theoretical control, mostly decidability issues. Knowing the equations of a system, one wishes to test if it satisfies some abstract property, that should express some concrete property of the actual physical process modeled by the equations. A first step is to translate this physical property in a suitable mathematical framework. The accuracy of the translation and the algorithmic efficiency depend greatly of this preliminary choice, most precise answers demanding obviously extra work.

1.1 Identifiability

In my PhD **[111]***, I have studied the problem of effective tests of identifiability [167, 168]. One considers a concrete process (physical, chemical, biological, ...) that is described by differential equations

$$x_i' = f_i(x, u, t, \theta),\ 1 \le i \le n, \tag{1.1}$$

where $x \in \mathscr{C}(\mathbf{R}, \mathbf{R}^n)$ is the state vector, $u \in \mathscr{C}(\mathbf{R}, \mathbf{R})$ is the command vector, t is the time and $\theta \in \Theta \subset \mathbf{R}^s$ the vector of parameters, that are assume to belong to some set of *admissible values*. For example, some parameters are often assumed to be positive. The problem is to determine whether it is possible, in theory, to determine the value of parameters, knowing exactly the inputs or commands u and the outputs or measurements

$$y_j = g_j(x, t, \theta),\ 1 \le j \le p. \tag{1.2}$$

Theoretical identifiability does not fully reflect the practical possibility to compute parameters values. Indeed, available data are often poor and noisy. It is nevertheless an essential preliminary step when studying a model, mostly with limited measurements, which is the common situation in biology [16] or epidemiology [163].

1.1.1 Exhaustive summary

At first, I took advantage of existing basic tools such as “exhaustive summaries” [86], that are functions of the parameters fully defining some description of the input-output behavior, such as the transfer function or the Markov matrix, that provide rational exhaustive summaries ρ.

At this stage of my work, I used without questioning it the definition provided by Walter [167, 86, 168], Lecourtier [85] or Rakasanyi [140, 141]. We need to assume implicitly that, the vector of parameters θ being known, the output y only depends of the input u, meaning that initial conditions are already defined in some way. So we have an input-output function $C(\theta) : \mathscr{C}^\infty(\mathbf{R}, \mathbf{R}^m) \mapsto \mathscr{C}^\infty(\mathbf{R}, \mathbf{R}^p)$, $u \mapsto y$.

Definition 6. — *A structure M is defined by (1.1,1.2) and a model $M(\theta)$ of the structure M is defined by the choice of a vector of parameter. A property is structural if it stands for all $\theta \in V \subset \Theta$, where V is a dense open subset of the set of admissible parameters.*

The model $M(\theta)$ (resp. the parameter θ_i) is identifiable if $C(\theta) = C(\vartheta)$ implies $\theta = \vartheta$ (resp. $\theta_i = \vartheta_i$).

The model $M(\theta)$ is locally identifiable if there exists an open set $U \ni \theta$ such that $\vartheta \in U$ and $C(\theta) = C(\vartheta)$ implies $\theta = \vartheta$. With this definition, structural identifiability means that C is injective on some dense open subset $V \subset \Theta$.

Exhaustive summaries are functions $\rho : \Theta \mapsto \mathbf{R}^q$ such that $\rho(\theta) = \rho(\vartheta)$ is equivalent to $C(\theta) = C(\vartheta)$. To fix ideas, exhaustive summaries are well known in the setting of linear systems with initial conditions all equal to 0:

$$\begin{array}{lcl} X' & = & A(\theta)X + B(\theta)U; \\ Y & = & C(\theta)X; \\ X(0) & = & 0. \end{array}$$

Using differential modules as a theoretical framework, we may associate to this system a module on the differential algebra $K(\theta)[s]$, where s stands for the time derivation $\mathrm{d}/\mathrm{d}t$. The transfer function is then defined by the matrix

$$C(\theta)(s\mathrm{Id} - A(\theta)^{-1}B(\theta),$$

which is defined by the data of its coefficients, so that they form an exhaustive summary $\rho(\theta)$. Identifiability is then reduced to the injectivity of ρ. One may also use the Markov coefficients, *i.e.* the coefficients of the matrix

$$(CB \;\; CAB \;\; \cdots \;\; CA^{n-1}B).$$

The question is then reduced to testing that the application $\rho : \mathbf{R}^s \mapsto \mathbf{R}^r$ is injective on some dense open set. Solving such a question was too difficult and would remain challenging, despite the progress made in effective real algebra. However, testing local identifiability is easy as it is enough to test that the Jacobian matrix of ρ has maximal rank s. Looking at the complex problem is much easier. We have then to test that $\mathbf{C}(\rho(\theta)) = \mathbf{C}(\theta)$. This can be solved using some well conducted Gröbner basis computation can provide a satisfactory answer with much shorter computation time. Moreover algebraic relations between θ and other possible extra vector of solutions may be exhibited which is enough in practice.

More generally, one may replace structural identifiability of θ_i by the easier problem of testing membership of θ_i to the field $\mathbf{C}(\rho(\theta))$. Such problem is reduced to checking that a Gröbner basis contains an element of a particular form **[110]***. Details are given in subsection 2.2.

The practical meaning of def. 6 can be questioned. Indeed, in practice one works with a single experiment, which means one single vector of inputs, not the whole input-output behavior. So, one would rather expect the following definition.

Definition 7. — *A model $M(\theta)$ is identifiable if there exists a dense open set $V \subset \Theta \times \mathscr{C}(\mathbf{R}, \mathbf{R}^m)$ such that for $(\theta, u) \in V$ $C(\theta, u) = C(\vartheta, u)$ implies $\theta = \vartheta$.*

It turns out that for linear models such as (1.1,1.2), the two definitions are equivalent. This is easily seen from the formula

$$y^{(r)}(0) = \sum_{k=0}^{r-1} CA^k Bu^{(r-k-1)}(0),$$

which implies that the Markov matrix is known, *e.g.* using $u_j(t) = t^{n(j-1)}$ for $1 \le j \le m$.

1.1.2 Differential algebra

The preceeding approach was much more difficult to follow in the non linear case, for which partial results had been obtained using truncated power series solutions [138]. This provides a summary ρ, meaning that $C(\theta) = C(\vartheta)$ implies $\theta = \vartheta$, and such a summary is exhaustive when the order of the power series development is great enough, but no general criterion was given allowing to stop the computations.

The discovery of differential algebra [144, 146, 75] and of its wide possibilities in the field of control [38], thanks to a talk of Wu Wentsun [170] at IHP in 1988, suggested new algorithmic solutions. The main difficulty was to be able to compute a characteristic set of the differential ideal $\mathscr{I}$, defined by eq. (1.1) and (1.2). As the state equations already form a characteristic set for a suitable ordering, it is enough to achieve a change of ordering, using an algorithm sketched in sec. 2.1.1. Trading again the real case to the complex one, identifiability of a parameter is reduced to its belonging to the differential field generated by the inputs and the outputs, using the method described in 2.2.1.

The cost of such an algorithm is quickly prohibitive, as it requires to eliminate the state variables, the inputs and the outputs. One knows that a good recipe for computational efficiency is to “eliminate elimination”. One will find more details on changes of orderings in 3.4 and in 2.2.3 algorithms to test membership to a differential subfield.

1.1.3 The come-back of exhaustive summaries

A safe way to reduce the computational cost is to eliminate only the state variables. Then, the coefficients depending on the parameters that appear in factor of monomials in the inputs u and outputs y, in some characteristic set of $\mathcal{I} \cap \mathbf{Q}(\theta, t)\{y, u\}$, may provide an exhaustive summary, under some technical hypotheses that make the characteristic set in some way unique. But a deeper hypothesis is requested: the ideal $\mathcal{I}$ must admit a *generic element*, that is a solution that does not satisfy any differential polynomial equation not in $\mathcal{I}$.

Controllability provides an easy sufficient condition. For given initial conditions, strong accessibility, which has been investigated in a paper with Fliess, Lévine, Martin and Rouchon **[44]*** admits very fast computational criterion. Indeed, the rank of the Lie algebra generated by the Cartan field of the diffiety and the partial derivatives with respect to the control must be $n + m + 1$, where n is the state dimension and m the number of controls.

But in biology, many models have no control. In such cases, the existence of generic solution may be deduced from the existence of stable equilibrium points, or limit cycles, that are known to exist, according to the properties of the model and of the biological system itself **[119]***. On may think of the adrenal-post-pituitary system investigated by Bernard-Weil [7]. Some more recent works also used the non-vanishing of the Wrońskian of monomials appearing in the equations that describe the input-output behavior **[87]***.

This method inspired many further works [31, 95, 96, 103, 35], in particular the software DAISY [6] (see also [22] and the references therein).

1.1.4 Importance of initial conditions

As already stated, a difficult point is the role of initial conditions in identifiability (*cf. e.g.* [95]).

Dealing with linear systems, the use of the transfer matrix or Markov parameters imposed initial conditions to be 0. The new approach developed with differential algebra requests a generic solution, which may impose some restrictions on initial conditions. A way is to add new parameters ϑ_i and to set $x_i(0) = \vartheta_i$. Then, the identifiability of all parameters θ and ϑ means that the system is identifiable and *observable*, meaning that the values of the state functions can be computed knowing inputs and outputs.

I had proposed a theoretical framework to deal with the problem of initial conditions in full generality **[115]*** et **[116]***. The PhD work of Ariane Germa-Péladan [135], defended in 1997, was inspired by this problem. She designed an complete algorithm to test 0 equality in a ring of series defined by differential equations and initial conditions [134, 136]. This work strongly relies on the tools provided by Denef and Lipshitz [30].

A poster that summarizes some recent works with Hervé Le Meur about the definition of identifiability and the importance of initial conditions has been presented in October 2019 at the Workshop on viral dynamics, in Paris **[87]***.

1.1.5 Local identifiability

The PHD thesis of Alexandre Sedoglavic [158], defended in 2001 and co-directed with Marc Giusti, has produced a polynomial time probabilistic method to test local identifiability **[121]*** [159]. The main idea is to compute a power series solution $x(t)$, together with the power series expressing the derivatives $\partial x_i/\partial \theta_j(x)$, for initial values (considered as new parameters ϑ) and parameters values randomly chosen. We can then compute the Jacobian matrix $(\partial y_\ell^{(k)}/\partial \theta_j)$ for all parameters θ_j, $1 \le j \le s$ and ϑ_i, $1 \le i \le n$, all outputs $y_\ell(t)$ and all derivatives for order $k = 0$ to order $k = s + n - 1$. Then, the system is locally identifiable and observable if this Jacobian matrix has rank $s + n$. Variants allow to test local identifiability alone or local identifiability of one parameter θ_{j_0}. A negative answer is not certain and there is a probability of failure. In practice, random values are chosen in a finite field $\mathbf{F}_q$, for some prime q and this probability decreases with the size of q.

The key point for small complexity is a fast method to compute power series solutions, that relies on a variant of Newton's method that is described in more detail in section 4.2.

1.1.6 Further perspectives

My project with Hervé Le Meur are developed in two directions. The first is to investigate more deeply the *meaning* of identifiability. The situation is paradoxical, as identifiability is a generic property, a more complex model, with more parameters, is more liquely to be identifiable, but will be more difficult to identify in practice. Moreover, some *universal systems*, depending on a finite number of parameters, are known to be able to approximate any continuous function on a compact set with an arbitrary precision. Such systems are identifiable, from a theoretical standpoint, but will provide whimsical values, even with a very small noise, trying to fit the corrupted output as closely as possible.

Does really identifiability mean that the parameters can be computed if one knows the output with an arbitrary precision, or what is its meaning ? One possible answer is that the set of parameters must belong to some *compact set*, to prevent the paradoxical behavior of universal systems. But an identifiability study may also provide more information, such as the existence of a group action, leaving invariant the inputs and the outputs, for a system that is observable, but not identifiable.

A second aspect is more computational and related to the existence of generic solutions, depending on initial conditions. This problem is also closely related to the nature of some biological or pharmacological models. Indeed, some initial conditions are not arbitrary, and should most of the time be 0, when the experimental process starts, for example, the drug concentration at the beginning of the treatment. Computing the generic rank of the Wrońskian is then not enough and non identifiability is related to the existence of a Darboux polynomial. Such investigations are related with some works in progress with Thierry Combot on first integrals of polynomial vector fields. They include the use of fast power series development and may also use certified arithmetic, in the framework of Mathemagix and the NODE ANR project, to be sure that a numerically computed determinant is non zero. See sec. 6.2.

1.2 Alien method and observators

I have participated to the launching of the project ALIEN by Michel Fliess. Our goal was to use powerful methods inspired by the theory of operators of Mikusiński [104] to design fast and reliable methods to build observators and to compute derivatives using integrations, which implies a reduced influence of the noise. Together with Saïd Moutaouakil, a Moroccan student and Brahim Sadik, we have extended **[124]*** the "Alien method" [48] for the parameteric identification of a linear system with delay [4]. The "state reconstructors" of Fliess and Sira-Ramírez [49, 50], may be reinterpreted in some elementary way, using simple cascades of part integrations. The advantage is to offer more freedom in the choice of the functions.

Using a family of functions f_j such that $f_j^{(k)}(T_1) = f_j^{(k)}(T_2) = 0$ for $k \le k$ and setting $I_{x,f} := \int_{T_1}^{T_2} f(\tau)x(\tau)d\tau$, we easily get $I_{x^{(k)},f_j} = (-1)^k I_{x,f_j^{(k)}}$ by part integration. We can then estimate the values of the coefficients a_i and b by solving the system $\sum_{i=0}^{n}\left((-1)^i a_i I_{y,f_j^{(i)}}\right) + bI_{u,f_j} = 0$ for $j = 1,\ldots,m$, by the mean squares method. Then, estimations at time T_2 of the derivative x^k are obtained using functions g_j such that $g_j^{(\kappa)}(T_1) = 0$ for $0 \le \kappa \le k$ and $g_j^{(\kappa)}(T_2) = 0$ for $0 \le \kappa < k$, with $g_j^{(k)}(T_2) \neq 0$.

In practice, we can use $f_j(t) = (T_2 - t)^{n+j} e^{-\lambda(T_2 - t)}$, the integration being done between $-\infty$ and the current time. A good approximation of such integrals is obtained by integrating the system $J'_{x,0} = x - \lambda J_{x,0}$ and $J'_{x,j} = jJ_{x,j-1} - \lambda J_{x,j}$ if $j > 0$, with initial conditions $J_j(0) = 0$: $J_j(t)$ tends quickly to $I_{x,f_{j-n+1}}$ for λ great enough.

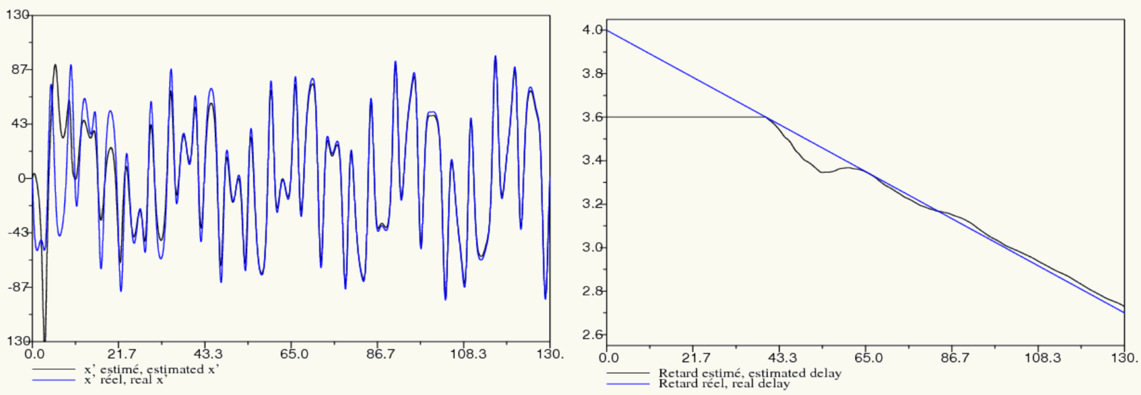


Figure 1.1: Estimations of the delay, of x and of x'.

The method was extended to delay systems of the form $\sum_{i=0}^{n} a_i x^{(i)}(t) + bu(t-h)$, using the approximation $I_{\hat{u},f} = I_{u,f} - hI_{u,f'}$. Simulations show that it allows to follow slowly varying coefficients or delay, as shown by the curves in fig. 1.1, borrowed from **[124]***. The main limitation is that an upper bound on the amplitude of the delay must be *a priori* known [61].

1.3 Flatness and Monge problem

To decide if the general solution of a system of ordinary differential equations can be parameterized by m arbitrary functions is a difficult problem, known as *Monge problem*. Indeed, Monge seems to have been the first to consider such systems [106] in 1784. Hilbert [56] was the first to state explicitly that the parameterization should be locally bijective (*umkehrbar integrallos*), although all the examples previously considered were of this kind. Cartan [21] solved the problem for systems of the form $\mathrm{d}x_i = f_i(x_1, \ldots, x_n)\mathrm{d}u + g_i(x_1, \ldots, x_n)\mathrm{d}v$, $1 \leq i \leq n$. In this setting, the transformation can also change the independent variable.

Such systems where considered by Fliess, Lévine, Martin and Rouchon in the setting of theoretical control, as *systems linearizable by endogenous feedback*, under the name of *flat systems* [39, 41, 43, 46, 89]. Their basic properties makes motion planning and feed-back stabilization easy **[42, 45]***. One must notice that the transformations allowed for flatness *stricto sensu* do not change the independent variable, that is the time. Nevertheless, time changes are also sometimes considered. On speaks then of *orbital flatness* [43].

The criterion of Cartan becomes a flatness criterion for *driftless systems with* 2 *controls*: $x_i' = f_i(x)u + g_i(x)v$ (Martin and Rouchon [98]). This class contains many examples with applications, such as trucks with trailers [150].

1.3.1 The case of EDP

Using the theory of Mikusiński [104], one may generalize flatness to some systems of linear EDPs as done by Fliess *et al.* [47] or Laroche *et al.* [82]. The basic example is the control of the heat equation for a rod of length 1, heated at one end ($x = 0$) and insulated at the other end ($x = 1$). The temperature at the insulated end $\theta(1, t)$ is the *flat output*, allowing the parameterization $\theta(x, t) = \cosh((1 - x)(d/dt)^{1/2})\theta(1, t)$.

Following this example, such a parameterization was extended to the case of a rectangular plate, with heat control on a single edge **[3]***, in collaboration with Nader Belgith, Michel Fliess and Alexandre Sedoglavic, and to the control of a twin roll strip casting process **[37]***, when Christian Fleck was visiting our team and we used to have long discussions on flatness, and other topics...

With Alexandre Sedoglavic, we have considered examples of systems of flat non linear PDE **[120]***, using discretization. We started with a simple linear example, the heat equation for a rod, and a classical discretization, dividing the rod into n segments and approximating $\partial^2\theta(x, t)/(\partial t)^2$ at $x = i/n$ by $(\theta_{n,i+1} - 2\theta_{n,i} + \theta_{n,i-1})n^2$. We found a parameterization

$$\theta_{n,i} = \sum_{j=0}^{i-1} Q(i + j - 1, 2j)\frac{y^{(j)}}{(2j)!}\left(\frac{i + j - 1}{n}\right)^{2j},$$

using Ramanujan Q-distribution:

$$Q(p, q) = \frac{p!}{(p - q)!\ p^q},$$

that converges to the parameterization of Laroche *et al.* when n tends to infinity[82]. This kind of discretization was used by Utz *et al.* [164] in the case of a square plate, with colorific coefficients depending on the temperature.

We proposed then a definition of a flat PDE system, using approximations by a sequence of finite systems and relying on the following requirement: the behavior of the sequence of flat systems must converge to the behavior of the PDE system; their flat output must converge to a limit function of the state; the parameterization must also converge, assuming that the time function assigned to the flat outputs belongs to a reasonable class, such as Gevrey function of some suitable order [82].

This definition was illustrated with a non linear version of the flexible rod considered by Fliess *et al.* [47], where the flexion of the rod is assumed to remain little, allowing to use linear approximation and computations with Mikusiński operators. A straightforward discretization leads to a chained system, which is flat, but we need to assume that the position (x, y) of the attach point of the rod is controlled, together with its angle of rotation. Simulations show a fast convergence, allowing a small number of segments, 20 to produce the following figure.

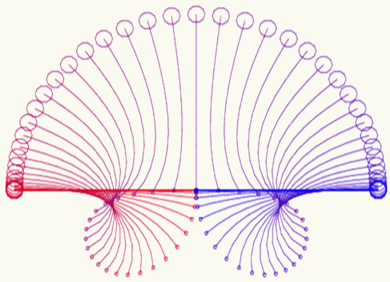


Figure 1.2: Position of the rod through the time.

1.3.2 Singularities of flat systems

Definition

Flat parameterizations can be defined on a dense open set, but there may exist *flat singular points*, for which no flat parameterization exists. A systematic study of such singularities with Jean Lévine and Yirmeyahu J. Kaminski started with a first introductive paper **[69]***, focusing on the classical car example, described by the following equations, according to fig. 1.3.

$$\begin{aligned}\dot{x} &= u\cos\theta\\ \dot{y} &= u\sin\theta\\ \dot{\theta} &= \frac{u}{l}\tan\varphi\end{aligned}$$

We prove that the only intrinsic flat singularities are the points where $x' = y' = 0$.

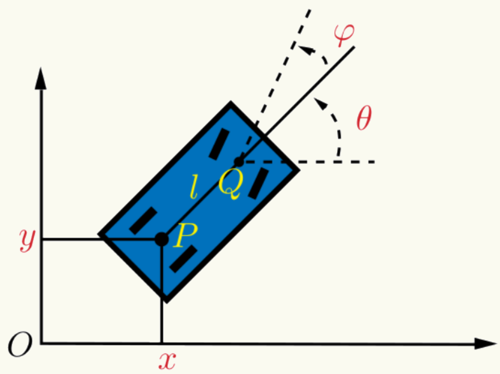


Figure 1.3: Car Model: the state vector is made of the coordinates (x, y) of the rear axle's center and of the angle θ between the car's axis and the x-axis. The controls are the speed u and the angle φ between the wheels' axis and the car's axis. The length l is the distance between the two axles.

Affine systems

In a second paper, we have investigated affine systems with n state variables and $n-1$ controls **[70]***, of the form

$$\dot{x} = f_0(x) + \sum_{j=1}^{n-1} u_j f_j(x) \triangleq g(x, u).$$

We have proved that the sufficient condition for strong-accessibility

$$\dim\langle f_1, \dots, f_{n-1}, [g, f_k]\rangle = n,$$

for some $1 \le k < n$, where g is the Cartan field, is also a sufficient condition for flatness[1]. Moreover, the flat outputs can be obtained as *independent first integrals* of the above field f_k in an open neighborhood of each point where the condition holds.

We then show that the system can remain flat at some points of the state space where the dimension of the vector space generated by the control vector fields $f_1, \dots, f_{n-1}$ drops down, providing an explicit sufficient condition of flatness in such a case. We define $\Gamma_0 := \langle f_i | 1 \le i \le n-p \rangle$. We then recursively define $\Gamma_{i+1} := \Gamma_i + [g, \Gamma_i]$. The system is flat in the neighborhood of any point η where $\dim \Gamma_p = n$ and $\dim \Gamma_k$, for $0 \le k \le p$ is locally stable in the neighborhood of η and Γ_k is involutive.

The construction is as follows. Let $r_k := \dim \Gamma_{k+1} - \dim \Gamma_k$, the sequence r_k is decreasing. Let then k_1, ..., k_s be the integers such that $r_{k_\ell} > r_{k_\ell+1}$[2]. We define Z_s as being a set of $r_{k_s} - r_{k_s+1}$ independent first integrals, common to the fields in Γ_{k_s-1}. We then recursively define Z_ℓ, for $1 \le \ell \le s$ so that $\bigcup_{q=s}^{\ell} \bigcup_{\kappa=0}^{k_s-k_q} Z_k^{(\kappa)}$ form a maximal set of first integrals, common to the fields in $\Gamma_{k_\ell-1}$, meaning that $\sharp Z_{k_\ell} = r_{k_\ell} - r_{k_\ell+1}$. We just have to complete $\bigcup_{k=1}^{s} Z_k$ with u_{n-p+1}, ..., u_{n-1}, to get a complete set of $m-1$ linearizing outputs.

This result is close the condition of linearization by static feedback of Jakubczyk and Respondek [65].

Chained systems. The aircraft case

In a recently submitted paper with Y.J. Kaminski [71], we address the problem of plane control, setting it in the general framework of chained or triangular systems [90, 91, 54], that we consider as special cases of systems for which a suitable choice of variables leads to a Jacobi number (see sec. 3.2) equal to 0, with a non vanishing truncated determinant, so that testing this kind of flatness reduces to a combinatorial problem.

To get a flat chained model, one needs to neglect some terms in the model, namely the thrusts created by the actuators. We study how a suitable feed-back allows the control of the full model, keeping flat output very close to the values used for the flat motion planning provided by the simplified flat model.

Our theoretical setting provides moreover alternative flat outputs, using the bank angle which allows gravity free flight, rather than the sideslip angle used by Martin [97], which is then singular. The engine thrust is also a flat output, singular when the sideslip angle is 0, but usable to control an aircraft with an engine failure during a forward slip landing maneuver.

1.3.3 Two questions of Ritt

Joseph Ritt [146] has proposed to investigate open questions of special interest for differential algebra, among which the existence of an analog of Lüroth theorem (see subsec. 2.2.2 in the algebraic case) in differential dimension 2, a problem considered by Buium [18] as out of reach of available methods, and the existence of an analog of a theorem of Max Noether [108, 109] showing that the Cremona group in two variables is generated by the permutation of variables and "Jonquières" automorphisms, that are applications of the form $(x, y) \mapsto (x, Q(x, y))$, where Q is a rational fraction of degree 1 in y. Inspired by a flat system due to Pierre Rouchon: $z' = x'y'$, and using a necessary flatness condition due to Sluis and Rouchon [161, 149], counter examples could be given in both cases **[117, 118]***.

1.3.4 Generalized flatness

In a communication **[132]*** to ISSAC'22, I have shown that the flat parameterization of a simplified aircraft (see above subsec 1.3.2), as proposed by Martin [97], could be extended to the real model, probably non flat, at least non flat for those flat outputs. Moreover, if controls do not change too fast, a suitable feedback keeps the trajectory close to the trajectory obtained with this generalized flat motion planning.

This new motion planning seems to be a fixed point of some differential operator that increases at each step the order of derivation of the flat outputs involved in the computations. This generalized flatness is thus based on using functions that potentially depend on an infinite number of derivatives, which is excluded by the classical theory, relying on differential algebra [143] or diffieties [78]. On may notice that the criterion of Sluis and Rouchon strongly depends on the assumption that the order is finite.

The precision is very good in practice with 4 iterations, so that this new kind of parameterization opens new perspectives for the theory and for applications.

[1] These result looks much like some conditions given by Li and Respondek in the case of two-inputs driftless systems [90].

[2] We may associate a Young tableau to the r_k, giving to k_ℓ the multiplicity $r_{k_\ell} - r_{k_\ell+1}$, the k_ℓ correspond to the dual tableau.

1.4 Appedge and automatic differentiation

In the framework of the CNES–ONERA project CARINS, that produced a software dedicated to the simulation of liquid propellant engines, such as the Vulcain cryogenic engines used on Ariane rockets, I have established strong links with John Masse, whose start-up Appedge was initially integrated into the X incubator.

Our activities have resulted in the writing of a survey, which summarizes a common experience on the difficulties encountered with automatic differentiation in specific circumstances: derivation of solutions of algebraic equations or ODEs, presence of discontinuities, iterated derivatives of large order. A version is available online, along with the Maple and Matlab programs used **[99]***. A small vulgarization paper has also been produced on this topic **[100]***.

I don't want to belong to any club that would have me as a member.
Groucho Marx, *Groucho and me*

Chapter 2

Membership problems

The membership problems for an algebraic structure appear in many decision problems and the membership to an algebraic or differential ideal is decisive for the formal or numerical solution of systems of equations. The standard[1] or Gröbner bases [17], using monomial rewriting close to Euclidean division, and the characteristic sets [144, 146], relying on Euclidean pseudo-divisions, are the most common methods.

2.1 Characteristic set

The reduction to 0 of a polynomial P by a characteristic set A of an ideal $\mathscr{I}$ shows that $H_A^s P \in I$, where H_A is the product of the initial coefficients of elements of A (and their derivatives in the differential case). This test allows to test the membership to a prime ideal, since $H_A \notin I$, which corresponds to most practical situations.

2.1.1 Changes of ordering

I have shown in my thesis how to compute a characteristic set for a new order, without using factorization. The basic idea is to check that the initials, *i.e.* the initial coefficients of the polynomials with respect to their main derivatives, and the separants, *i.e.* the initial coefficients of derivatives, are invertible. If this is not the case, a factorization is obtained for the cost of a simple GCD computation. In this way, we avoid the necessity to factor systematically, as in Ritt [146], which makes the computation much more difficult, even with the most recent factorization algorithms, as one needs to factor in towers of algebraic extensions.

The most general result is that we can obtain an algorithm for computing the characteristic set of a prime differential ideal for an arbitrary order, provided that we have an oracle testing membership to the ideal. It is indeed the case if the starting system is already a characteristic set for another order, which is the case of state equations considered in control theory.

2.1.2 Boulier's algorithm

This idea was successfully developed under the impulse of François Boulier, starting with his impressive PHD thesis [10], leading to a complete new algorithm for representing the solutions of a differential system by characteristic sets, and requiring no factorization **[11, 13]***, using the fact that some non-prime radical ideals, of which prime factors have characteristic sets with the same leading derivatives, can be represented by a a single characteristic set. This algorithm, first implemented in Maple (DiffAlg), has been extended and enriched, notably by Évelyne Hubert [59, 60] and remains the main reference for actual computations in differential algebra. In addition to the algebraic Gröbner bases, it relies on two essential preliminary results: Rosenfeld [148]'s lemma, reducing to the algebraic equations and Lazard's lemma [84], stating that the non vanishing of the Jacobian determinant $|J|$ implies that an algebraic system P generates a radical ideal, namely $(P) : |J|^\infty$ is radical. An alternative proof of this result was given in **[130]***, that makes a link with Newton algorithm. See Morrison [107] for another algebraic proof.

The publication of **[13]***, very close to the preprint produced in 1997, was delayed by some complicated reviewing process, until the special issue *Jacobi* of AAECC offered us a nice opportunity, as the ideas devel-

[1]The denomination standard bases comes from Hironaka and is reserved by some authors to the case of power series. This term has long been some lexical specificity of our team, in an brave attempt to fight Stigler's law of eponymy, or perhaps better Merton's.

oped by the Master, strongly relying on a sequence of differentiations and eliminations are so close to our computational tools **[126, 127]***. See chap. 3.

2.1.3 Complexity issues

In the case of PDE, the complexity is intrinsically high in the worst case. During his thesis work, in 1995, Brahim Sadik [151, 152] has shown, by adapting the monoïd of Mayr and Meyer [102], that testing the existence of solutions for a systems of linear PDEs with second members, has a complexity that is at least exponential. This is due to the fact that Mayr and Meyer construction is strongly related to the module of relations between the equations, that become relations between second members when algebraic equations in variables x_1, ..., x_n are translated to PDE using derivations $\partial/\partial x_i$.

Of course, most systems of practical importance behave much better, but avoiding exponential growth of non linear equations with the differentiation order requires to forget dense representation (see sec. 4.1).

Any way of avoiding useless computation must be used. In collaboration with Amir Hashemi, a differential analogue of a criterion of Buchberger [17] for detecting unnecessary critical pair calculations, that was first stated in the linear case **[13]***, has been extended to products of linear factors **[55]***. We discussed this point with Amir during my trip in Iran, where I presented Boulier's algorithm, the CNRS "fonctionnaire de défense" being less fearful than his colleague of Lille university. We managed to work out a proof by mail, Amir being unable to get a visa to France.

2.2 Subfield membership

Testing subfield membership can be reduced in many ways to ideal membership. The most immediate one is to use "tag variables" T_i. Thus, in order to test if $f_0 \in k(f_1, \ldots, f_m)$, where $f_i = P_i(x)/Q_i(x)$, it is enough to consider the ideal[2]

$$\Gamma := (Q_i T_i - P_i | 0 \le m) : \left(\prod_{i=1}^{m} Q_i\right)^{\infty}$$

and to compute a characteristic set or a standard basis of Γ for an order such that $x_1 > \cdots > x_n > T_1 > \cdots > T_m > T_0$. The answer is positive if the result contains a polynomial of the form $S(T)T_0 - R(T)$ and then $f_0 = R(f)/S(f)$. This is the method of Shannon and Sweedler [155].

2.2.1 Algebraic case

This method has the advantage of providing the explicit expression of the fraction f_0, depending on the generators f_i. This can also be a disadvantage, if only a binary answer is desired, because this expression can be of exponential degree in the number of variables. I have therefore introduced an alternative method, using variables y_1, ..., y_n playing a role that is (anti)symmetric with respect to the variables x_1, ..., x_n. We then consider the ideal

$$\Delta := (P_i(x)Q_i(y) - P_i(y)Q_i(x)) : \left(\prod_{i=1}^{m} Q_i(x)\right)^{\infty},$$

in the algebra $k(y_1, \ldots, y_n)[x_1, \ldots, x_n]$, or better $k(f_1, \ldots, f_m)[x_1, \ldots, x_n]$, since the ideal becomes prime, when restricted to its field of definition.

At the time when I wanted to implement this method, only the symbolic computation system Scratchpad II, developed by IBM, and now freely available under the name AXIOM, allowed this kind of computations, thanks to its strong typing. See sec. 5.3.1.

In the special case of testing identifiability, if f is an exhaustive summary, identifiability is equivalent to $\Delta = (x_i - y_i | 1 \le i \le n)$. The result is small, and the size of the objects tends to decrease during the computation, although the theoretical bounds are much higher. More generally, we have $f_0 \in k(f)$ iff $P_0(x)Q_0(y) - P_0(y)Q_0(x) \in \Delta$.

In the case where f defines a birational application, we can bound the degree of the inverse by d^{n-1}, according to a result proved by Ofer Gabber [2], where d is the degree of the direct application and n the number of variables, which allows to bound by d^n the degree of required intermediate computations **[110]***, and so their cost, meaning that S-polynomials of higher degrees can be neglected during a Gröbner basis construction.

[2] With the notation: $\mathcal{I} : S^\infty := \{P | \exists q \in \mathbf{N}\ PS^q \in \mathcal{I}\}$.

2.2.2 Lüroth theorem

Lüroth theorem [93] (see subsec. 1.3.3 in the differential case) affirms that any subfield $k \subsetneq K \subset k(x)$ is generated by a single element $K = k(y)$. This theorem has been extended by Igusa [62] and Samuel [153] to the multivariate case $k \subsetneq K \subset k(x_1, \ldots, x_n)$, if the degree of transcendence of K/k is 1. With Brahim Sadik, we have given a simple and constructive proof based on the computation of the ideal Δ associated to K **[131]***.

2.2.3 Differential case

The differential case with a subfield $\mathcal{G} := \mathcal{F}\langle f_1, \ldots, f_m\rangle$, with $f_i = P_i(x)/Q_i(x)$, is quite similar to the algebraic case. An analogue of Shannon and Sweedler's method is then to consider the differential ideal

$$\Gamma := [Q_i T_i - P_i | 0 \le i \le m] : \left(\prod_{i=1}^{m} Q_i\right)^{\infty}.$$

It still provides an explicit expression $f_0 = R(f)/S(f)$ when $f_0 \in \mathcal{G}$, which is the case iff some element of the shape $S(T)T_0R(T)$ belongs to the characteristic set of Γ for an ordering such that $T_0 > T_1, \ldots, T_m > x_1, \ldots, x_n$.

We can again speed up the computations, when we do not need such explicit expressions, by considering the differential ideal

$$\Delta := [P_i(x)Q_i(y) - P_i(y)Q_i(x)] : \left(\prod_{i=1}^{m} Q_i(x)\right)^{\infty}, \tag{2.1}$$

in the algebra $\mathcal{F}\langle f(y)\langle x\rangle$. We have the property $f_0 \in \mathcal{G}$ if $Q(y)P(x) - Q(x)P(y) \in \Delta$, which provides a membership criterion.

Moreover, in the algebra $\mathcal{F}\langle f(y)\rangle\{x\}$, we have the property $P(y,x) \in \Delta$ iff $P(y,y) = 0$. This simple membership criterion allowed to compute a characteristic set for Δ simply by checking that the initials and separants were invertible, before the full theory of Boulier's algorithm was developed. See subsec. 2.1.1.

It may be shown that if a differential rational application ϕ of order e is invertible, its inverse is at most of order ne. So, in order to test invertibility of a differential rational mapping, we may bound the computation of a differential Gröbner basis (see subsec. 2.3.1) by ne. **[113]***. The proof of this result follows that of Ofer Gabber in the algebraic case (see subsubsec. 2.2.1). The key ideal is to consider n generic hyperplanes H_i, where n is the space dimension. The degree (resp. order) of $\phi^{-1}(H_1)$ is the degree (resp. order) of $\phi^{-1}(H_1) \cap \bigcap_{i=2}^{n} H_i$, which is the same as the degree (resp. order) of $H_1 \cap \bigcap_{i=2}^{n} \phi(H_i)$. In the algebraic case, the Bézout bound shows that this degree is at most d^{n-1}, if d is the degree of ϕ. In the differential case, using Ritt's analog of Bézout's theorem [146], we get a bound en. As we are in the quasi-regular case, Jacobi's bound may be used (see chap. 3), that gives an improved version, closer to the algebraic case: $(n-1)e$.

2.3 Standard or Gröbner bases and membership problems

I have participated in the definition of variants of the notion of Gröbner bases or standard bases for various structures, differential ideals **[113]***, first introduced by Carrà Ferro also [20]) and subalgebras **[112]***, first introduced by Kapur ad Madlener [72] (see also Robbiano and Sweedler [147]). The main obstacle is that these standard bases are in general infinite.

2.3.1 Gröbner bases of differential ideals

In the differential case, the membership to a differential ideal is in general undecidable, as shown by Gallo and Mishra **[51]***. See also subsec. 2.3.3.

Using the reverse lexicographic degree order, which I had excluded by a too restrictive definition, Alexey Zobnin [172, 173] was able to show that x^p is a standard basis of the differential ideal it generates. This property allows us to reinterpret a classical result of differential algebra, the Levi theorem [88, 146], that is a direct consequence of Gröbner basis reduction.

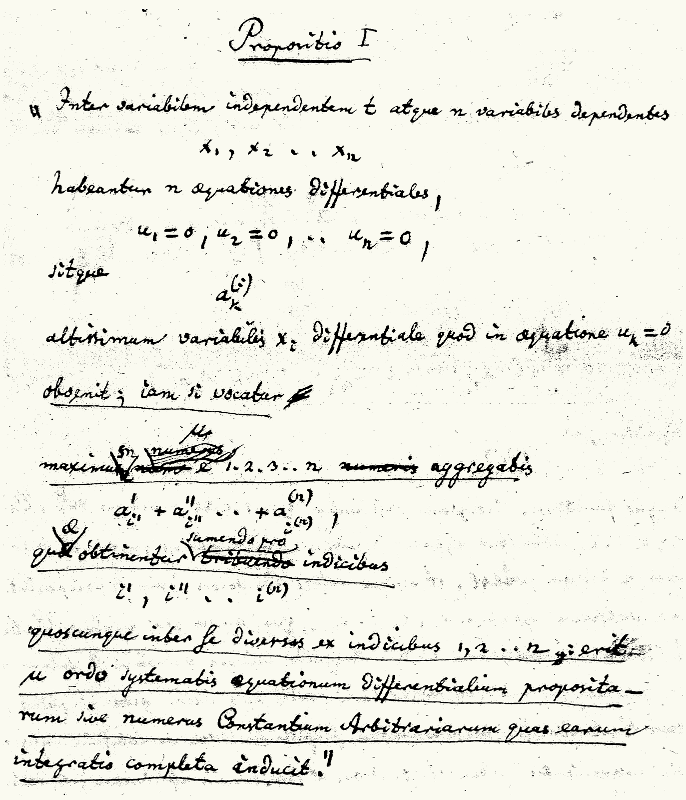
2.3.2 SAGBI

Some algebras of invariants have finite bases. One can then compute standard bases for symmetric systems, directly in the algebra of invariants **[51]***, a construction which was later developed by Miller [105] and which can also be used as an argument in some theoretical works [142].

2.3.3 More perspectives

Ualbai Umirbaev [162] has shown that the membership problem for finitely generated ideals was undecidable, reducing the question to a stopping criterion for a 2-tape Minski machine, which solves one of Ritt problem in the case of PDE. The ordinary case remains open.

> *W ciele człowieka słowo pęka na dwoje, na substancję i istotę. Gdy ta pierwsza zanika, druga, pozostając bez kształtu, daje się wchłonąć tkankom ciała, jako że istota nieustannie poszukuje materialnego nośnika; nawet jeśli ma się to stac przyczyną wielu nieszczęść.*
>
> Olga Tokarczuk, *Księgi Jakubowe* [5]



Propositio I

"Inter variabilem independentem t atque n variabiles dependentes $x_1, x_2 \dots x_n$ habeantur n aequationes differentiales, $u_1=0, u_2=0, \dots u_n=0$, sitque $a_k^{(i)}$ altissimum variabilis x_i differentiale quod in aequatione $u_k=0$ obvenit; iam si vocatur maximum μ summarum e $1.2.3 \dots n$ aggregatis $a'_{i'} + a''_{i''} \dots + a^{(n)}_{i^{(n)}}$, quae obtinentur sumendo pro indicibus $i', i'' \dots i^{(n)}$ quoscunque inter se diversos ex indicibus $1, 2 \dots n$; erit μ ordo systematis aequationum differentialium propositarum sive numerus Constantium Arbitrariarum quas earum integratio completa inducit."

Proposition I. *"Let $u_1 = 0$, $u_2 = 0$, …, $u_n = 0$, be n differential equations between the independent variable t and the dependent variables x_1, x_2, …, x_n and let $a_k^{(i)}$ be [the order of] the maximal derivative of the variable x_k that appears in the equation $u_i = 0$. Then, calling μ the maximum of sums $a'_{i'} + a''_{i''} + \cdots + a^{(n)}_{i^{(n)}}$, obtained when summing for indices $i', i'', \dots, i^{(n)}$, all different the one from the other, among the indices $1, 2, \dots, n$; μ will be the order of the proposed system of differential equations, or also the number of arbitrary constants appearing in its complete integration."*

Chapter 3

Works of Jacobi

Jacobi's bound [145] is a sharp bound on the order of the components defined by some ODE system. Still conjectural in the general case, it can be proved for *quasi-regular* components $\mathscr{P}$ of a system (see Kondrateva *et al.* [76]), that are those such that the module of the differentials $d\mathscr{P}$ is equal to the module generated by dP (see Johnson [66, 67]).

If $a_{i,j}$ is the order of the equation P_i in the variable x_j, the order of the system is bounded by the *tropical determinant* [94], in contemporary terms, of the order matrix $(a_{i,j})$: $\max_{\sigma\in S_n} a_{i,\sigma(i)}$: multiplications are replaced by sums and sums by max. Jacobi gave an efficient method to compute this expression, long forgotten, and similar to the Hungarian method of Kuhn [79], inspired by the work of Egerváry [34], which is an important step in the field of combinatorial optimization [156].

I have translated the original texts from Latin to French and then to English, including unpublished documents **[126, 127]***, from the archives of the Academy of Sciences of Berlin: [II/13 b)], [II/23 a)] and [II/23 b)]. The precursor role of Jacobi is now recognized [80, 19, 33, 74].

In a recent paper **[130]***, I have provided proofs, in the formalism of differential algebra, of the main results stated by Jacobi in the texts I have translated.

3.1 Tropical determinants and shortest paths

I have produced a complexity analysis of the algorithm computing the tropical determinant, comparing it to contemporary results. His method is based on the computation of a *minimal canon*. It consists in adding constants to the rows of the matrix, so that each column has a maximal element, these elements being placed in all mutually different rows. Jacobi calls them *transversal maxima*.

We can associate to each canon a minimal cover in the sense of Egerváry, *i.e.* integers α_i and β_i such that $a_{i,j} \le \alpha_i + \beta_j$, with $\sum_{i=1}^{n}(\alpha_i + \beta_i)$ minimal. For a canon ℓ_i, we can define $\alpha_i := (\max_{k=1}^{n} \ell_k) - \ell_i$ and $\beta_j := \max_{i=1}^{n} a_{i,j} - \alpha_i$. Conversely, to any minimal cover, we can associate a canon.

The Jacobi method is thus similar in its spirit to the Hungarian method of Kuhn, but, moreover, both follow the same basic requirement: the ℓ_i are increased of the minimal quantity requested to build a canon in Jacobi's approach, while in Kuhn's one decreases a minimal number of μ_i. This strong analogy was noticed and more deeply analyzed bu Kuhn himself [80].

Moreover, Jacobi provides two algorithms allowing a faster computation of the minimal canon, 1) in the case where a canon is already known 2) in the case when the permutation σ providing a maximal sum $\sum_{i=1}^{n} a_{i\sigma(i)}$ is known. These algorithms can be be reinterpreted as methods for computing a shortest path in a graph [157] 1) when all distances are positive, which is close to Dijkstra algorithm [32]; 2) when negative distances can exists, which is close to Bellman algorithm [5]. Jacobi is therefore also a precursor of graph theory.

3.2 The proof of the bound

The proof scheme sketched by Jacobi for the bound can be made perfectly rigorous in the case of a quasi-regular component $\mathscr{P}$ of the differential ideal $\{P\}$, *i.e.* a component such that the differentials $d_{\mathscr{P}} P_i$ are differentially independent. In this case, we even have a necessary and sufficient condition for the bound to be reached, given by the non nullity of the truncated determinant of the system $\nabla := \left|\partial P_i/\partial x_j^{(\alpha_i+\beta_j)}\right|$, a result that could not be obtained in the approach of Kondrateva *et al.* [76, 77].

Jacobi provides some arguments for the proof, the first of which is quite similar to the quasi-regularity hypothesis, proceeding by reduction to the linearized system, that Jacobi interprets as the system satisfied by the derivatives of the solutions with respect to a parameter[1]. We reused the idea in the framework of of automatic differentiation **[99]***. See sec. 1.4.

The second argument is sketched and crossed out in the manuscript. The basic claim is that one may reduce to a system with constant coefficients, which did not seem to be a problem for the XIX century editors, but was possibly the reason why Ritt considered the proof as "whimsical". An easy explanation in contemporary terms is that, if G is a standard basis of a differential module it is still a standard basis if we consider its coefficients as constant, and the order is unchanged.

Then, Jacobi obtains the order as the degree of the characteristic polynomial, whose dominant coefficient is precisely the truncated determinant ∇.

With Brahim Sadik **[125]***, we have proved and generalized this result, under the assumption of quasi-regularity, in the framework of diffiety theory [78, 171].

Generalizations to an underdetermined system are easy. One may obtain a general bound by adding as many rows of zeros as needed in the order matrix. Another way is to consider all bounds obtained by choosing a subset of variables. Chained and triangular systems of control theory are systems for which the minimum values of these Jacobi bounds in 0. See sec. 1.3.2.

3.3 Shortest reduction

If the truncated determinant is indeed non-zero, we can compute a normal form, or a characteristic set in the setting of differential algebra, for a suitable order, *i.e.* an order such that $\mathrm{ord}^{\mathrm{J}} x_j^{(k)} = k - \beta_j$, by deriving each equation P_i at most λ_i times, where λ is the Jacobi minimal canon and α, β the associated minimal cover. This order is generically minimal and we have made explicit the genericity conditions. This method has been redicovered independently by Shaleninov [154] and Pryce [139].

It is in fact easily seen that the method for computing a characteristic set works for any canon. Only minimality is lost. Jacobi was possibly aware of this possibility, as some parts of his manuscripts suggests. I used it to prove the bound for computing a differential resolvent, which seems to be the meaning of some sketch of arguments in the manuscripts. See sec. 3.5.

One is sometimes able to achieve some changes of ordering, just by choosing suitable canons, but generic systems have a single canon, up to equivalence.

3.4 Change of orderings

Jacobi also gave conditions for the existence of normal forms with prefixed head derivatives, expressed by the non-nullity of some determinants, and sharp bounds on the order of derivatives of the initial equations, necessary in order to compute some different normal form.

The local identifiability criterion of Sedoglavic **[121]*** can be reinterpreted as a some avatar of such general conditions, as well as methods for fast computation of differential Hilbert functions due to Matera and Sedoglavic [101].

These results evoke some contemporary methods for computing efficiently Gröbner bases, as done by Faugère [36] or characteristic sets, as done by Boulier [14], by changes of orderings, starting with an order for which the computation is the easiest, and which may corresponds, if the truncated determinant of the system is non-zero, to a normal form for a Jacobi order, computed by the shortest reduction.

3.5 Resolvants

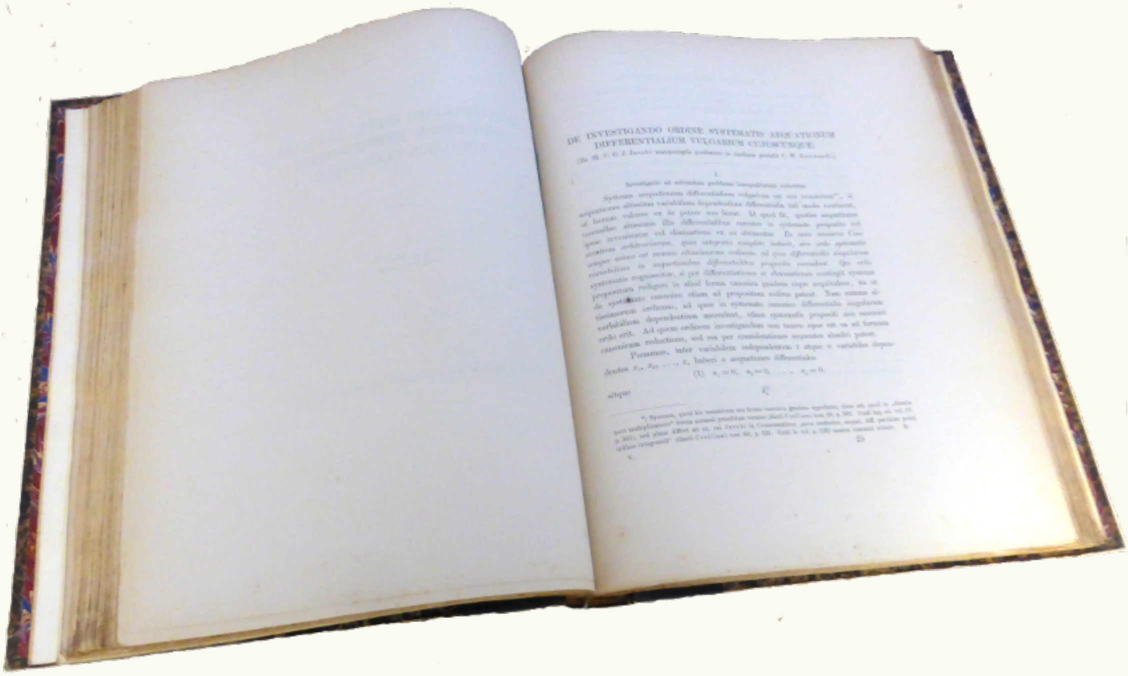
If the determinant of the system is non-zero, Jacobi showed how one can compute, if it exists, a differential resolvent depending on some variable x_j. See Cluzeau and Hubert [26] or D'Alfonso *et al.* [28] for recent works on this topic.

[1] An early article of Ritt [143], inspired by his time computing artillery tables during WWI, relies on a special case of this idea.

In order to compute a resolvent in x_{j_0}, one needs to differentiate the equation P_i, $\mathcal{O}_{i,j_0}$ times, where $\mathcal{O}_{i,j_0}$ is the tropical determinant of the matrix $(a_{i,j})$ from which the row i and the column j_0 have been removed.

Jacobi's evaluation only stands under some more technical hypothesis, or when using the *weak bound*, that is when setting $\operatorname{ord}_{x_j} P_i = 0$ instead of the *strong bound* $\operatorname{ord}_{x_j} P_i = -\infty$ when x_j and its derivatives do not appear in P_i.

This result can be compared to the computation of the differential determinant by Li *et al.* [92].



Car l'un de nous avait inventé pour les mots
Le piège à loup de la vitesse
Louis Aragon, *Le roman inachevé* [6]

Chapter 4

Fast methods and SLP

4.1 Differential Kronecker method

The resolution of systems of differential algebraic equations, requiring to derive the initial equations to reach a normal form, remains a recurrent problem in many practical fields, including chemical engineering (Alloula *et al.* [1]).

In the nonlinear case, it is difficult to escape an exponential complexity, intrinsically due to the size of the result. Even worse, differentiating a non linear equation already produces a result of an exponential size, according to the order of differentiation, before any actual elimination process could start.

The only escape way is to look for a different representation of data, *e.g.* using a representation of polynomials by straight line programs (SLP) that compute them. This point of view has led for the algebraic systems to the Kronecker algorithm of Grégoire Lecerf [52], following the TERA (Turbo Evaluation and Rapid Algorithms) philosophy promoted by Giusti and Heintz. A generalization to the differential framework has been realized in collaboration with A. Sedoglavic and Argentinian colleagues L. D'Alfonso, G. Jeronimo and P. Solernó **[29]***, which has strong links with previous works on differential resolvents [26, 28]. See sec. 3.5.

This first attempt should be extended to the PDE case and some spurious technical hypotheses removed. See sec. 6.3.4.

4.1.1 The algebraic case

The basic idea of Kronecker in the algebraic case, in order to solve a set of $n-m$ equations P_i in n variables x_j, is to build a resolvent, *i.e.* to parameterize the algebraic variety of degree d defined by P, using a single equation $Q(T, x_1, \ldots, x_m)$ of degree d, considered as an equation in a single variable T, which is a generic linear combination of the initial variables. Working with a representation of polynomials as SLP, one needs to use a probabilistic test for 0 equality, just by evaluation using random data.

We need regularity hypotheses, allowing to assume that Q is square-free, as we have

$$Dx_i = F_i(T, x_{n-m+1}, \ldots, x_n),$$

where D is the discriminant of Q. After some generic recombination of equations and variables, we can assume that the sequence P_i is regular enough, the precise requirement being known as *Noether position.* We can assume for simplicity that $T = x_1$, and that all variables except x_1 and x_2 have been evaluated to generic values a_j, so that we start by solving a system of two equations $P_1(x_1, x_2, a) = 0$ and $P_2(x_1, x_2, a) = 0$, which is done by a resultant computation, that gives $Q_1(x_1)0$ and $D_1 x_2 = F_2(x_1)$. A *lifting process* allows to extend these relations, taking in account the variable x_3 as a parameter.

So we get two equations $\tilde{Q}_1(x_1, x_3)$ and $\tilde{D}_1 x_2 = \tilde{F}_2(x_1, x_3)$, allowing a fast elimination of x_2 in $P_3(x_1, x_2, x_3, a)$, which provides a new equation in x_1 and x_3 alone: $\tilde{P}_3(x_1, x_3)$.

We go on like this, using a sequence of resultants of two equations in two variables and liftings, that solves the zero dimensional case, keeping a good control of complexity. A final lifting provides the dependency with respect to extra variables in the case of positive dimension.

This allows to get a parameterization with a complexity which is polynomial in the degree of the variety, which is at most d^{n-m}, if the initial polynomial P_i have degree at most d, according to the Bézout theorem.

4.1.2 Differential case

The basic idea to extend the Kronecker algorithm to the differential case is easy. One starts with a set of differential equations, that we consider like algebraic equations and for which we compute a Kronecker parameterization. Then, the trivial derivation δ (see def. 2) of jet space in the initial coordinates has a natural action on the new coordinates, and the derivative of all equations that describe the Kronecker parameterization should be 0, modulo these equations.

If not, new equations are created and one may continue the algebraic Kronecker process with them, provided that suitable regularity hypotheses stand for them. This is where technical hypotheses appear, that we hope to be able to dispense with. See sec. 6.3.4.

4.2 Power series solutions

The analog of Newton's method due to and Kung [15] works only for the order 1. Alexandre Sedoglavic has used and implemented in his thesis [158] a method working in quasi-linear time, that is up to a logarithm of the order, without publishing details and just claiming he used Brent and Kung. Some years later, colleagues demanded details, as "one knows that Brent and Kung does not generalize to higher order." Indeed, the first version of the program failed, more or so due to the fact that $Y(t) = \exp\left(\int A(t)\right)$ is not a solution of $Y'(t) = AY(t)$, as matrix multiplication is not commutative. A patch was soon found, so that it seemed unimportant.

The idea is just to work in the basis defined by the solutions $Y(t)$, so that we have to consider linear systems of the form $Y' = t^{2^r-1}A(t)Y$, for which $\mathrm{Id} + t^{2^r}A$ is a basis of solutions. A complete study of its complexity has been published in a choral paper **[9]*** with many authors of complementary skills, the writing of which is a good memory. It remains the asymptotically fastest method, even if for orders actually needed in practice the "relaxed" method of Joris van der Hoeven Hoeven [165, 166] is often better.

L'inventable est invantable.
Alors moi je le déclarationne,
Quand on n'a rien à dire
Faut savoir se publicitaire
Une fois pour toute !

Marc Favreau, *Rien d'étonnant avec Sol !* [7]

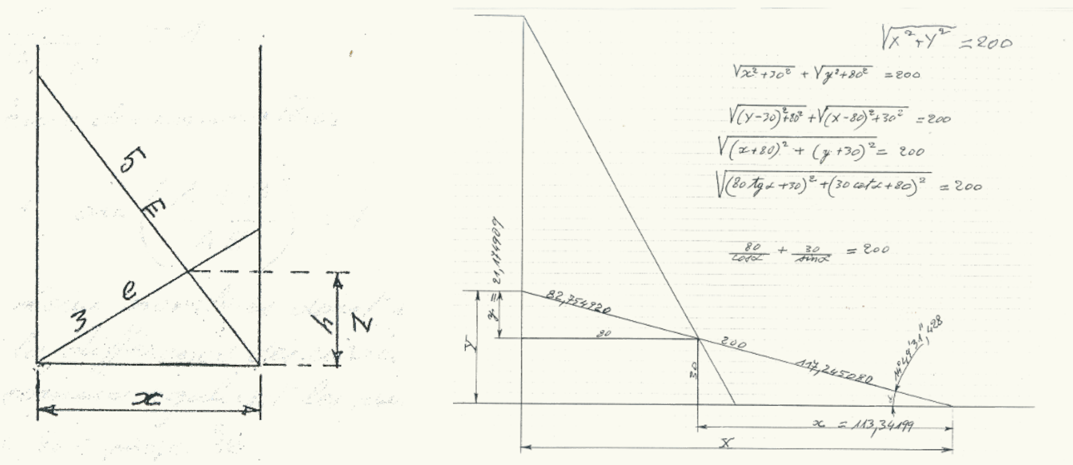


Figure 4.1: Problems with ladders

Chapter 5

Broadcasting

I address in this chapter miscellaneous topics with the common concern of getting our result outside the academic world, or at least outside our research community.

5.1 Problems for vulgarization and teaching

I liked much some problems of Jacques Arsac, aimed at making students find solutions by themselves, such a computing the median element of a list in linear time, *i.e.* without sorting it. I needed to design such problems myself, starting by a few geometrical questions, inherited from my father, that look much in their spirit to the Chinese classic of algebra, the *Jade mirror of the four unknowns*, so close to characteristic sets computing.

5.1.1 Ladders in a corridor

Knowing the length of the ladder and the height of their crossing point, find the width of the corridor (fig. 4.1).

This is a problem that I have used a lot for Maple initiations. First, one needs to find a way to write the equations, then to solve them. It is of degree 4 in x^2, so that a radical solution exists, which is quite big and does not help much. A good way to make the student think first of what ze expects from computing. A good question for real world problems too.

5.1.2 Ladder on a box

A ladder touches the wall, the ground and the corner of a box. One knows the size of the box and the length of the ladder (fig. 4.1). Find the positions of the contact point, x with the wall and y with the ground.

Another problem of degree 4, for which one finds, for a long enough ladder, two solutions with y and x positive. It was used as a example of Gröbner basis computation in a small vulgarization text, written with PhD students of my team **[169]***.

5.1.3 Kapitsa pendulum

One knows that if the attach point of a pendulum has fast small amplitude vertical oscillations, this creates some artificial force, keeping the pendulum stable in vertical position. A good exercise is to compute the average force created by the high frequency oscillations and to compare the behavior of the full system and of the averaged one.

This was inspired by the high frequency control methods exposed in Fliess *et al.* [40] and used as an example in a short text aimed at illustrating the possibility of computer algebra in engineering **[123]***, using free software. The equations were computed with Maxima and the numerical integrations with Scilab. See 5.3.2. The computation of the differential equation is the easiest using the Lagrangian formalism, that is a special case of *isoperimetric equations* that inspired Jacobi's bound. See chap. 3.

5.1.4 A double pendulum after an elastic collision

This was a problem of mechanics I was unable to solve at the entrance exam "rue d'Ulm". The double pendulum is vertical, a body of infinite mass moving with horizontal speed 1 hits the extremity of the second pendulum and the angular speeds of both pendulums, just after impact, are to be found.

I asked many physicists without getting an answer, until I implemented the solution in Maple, that is in fact easy using the Lagrangian formalism and classical part integrations. This was an illustration of the difficulty of naive automatic differentiation, when the program to differentiate is an integrator, moreover with discontinuities **[100]***.

5.2 Contracts

My greatest disappointment was the impossibility to access suitable experimental data to test identification tools. In the worst case, the actual set of equations to work with never came during the contract duration. Sometimes, data was two poor. For example, just one blood pressure data is not enough if the cardiac output is unknown. I hope to have more chance in the future.

I am pleased to see that rules may change for the better and that we can now get support for team projects, on our very own themes and goals, like Node (see chap. 6).

I will mostly focus here on the CARINS software project [133]. CARINS, which stands for "CARamel INStationnaire" was inspired by a previous software for dimensioning and performance analysis of liquid propellant engines in stationary situations, *i.e.* when all rates of flow are constant. The goal was to develop a tool relying on the same physical models, but allowing simulations of the engine behavior during the whole flight, and mostly at ignition, in order to understand better some engine failures due to excessive pressure.

The idea was to link engine components using a graphic interface to build a graph describing the whole engine, from which computer algebra could build a complete set of differential equation, that could be sent to a classical FORTRAN solver. This simple scheme was made much more complicated by two main difficulties: first, some engine components were not described by differential equations, but by "black boxes" FORTRAN subprograms; second, the differential equations structure did not ensure that an explicit differential system could be built. The specifications of the project had overestimated the possibilities of computer algebra, planning fast computations for general problems with a tremendous complexity.

To make things worse, it was required to chose a free software, that imposed to retreat to Maxima, developed in the United States for the DoE in the 70ies, with limited possibilities for differential equations. We had to fight to impose reducing non explicit equations to a few components, for which simplified systems compatible with numerical implicit solvers requirement could be precomputed. But we did not manage to include some control which could have been useful *e.g.* for stability issues: this was the task of another department…

Testing the first software prototypes was a challenge, facing many interface troubles, and lacking engine components that were to be implemented later. Elementary linear flat systems of increasing complexity obtained by connecting larger and larger numbers of heat tanks proved to be efficient, as the differential equations are more and more stiff, while the theoretical final values are known. A 30 years old Maxima bug was found in this way: a digit was lost during the conversion of integers to floats.

5.3 Implementations

5.3.1 Scratchpad II

After practicing a lot the earlier version of Macaulay, in order to experiment at the beginning of my PhD, I turned to Scratchpad II, that, despite its many drawbacks, allowed to work in the proper algebraic setting, with reduced computation time, thanks to its strong typing. (See sec. 2.2.1.) The main inconvenient was the difficulty of building objects of a suitable type, due to the limitations of the interpreter, but this could be in most cases solved by implementing a conversion function, from type `Expression` to the type of the wanted domain. The reverse function always existed, for it was used for screen display, but this one was most of the time missing. I wrote my own versions of my favorite packages, including such functions, which were a little tedious but not difficult to design, but saved me a lot of time.

At this time, this software was only available on IBM computers under rustic VM/CMS operating system, that could address no more than 16 Mo of memory, meaning at most 12 Mo for computations: the size of a photograph.

The biggest computation I did, that helped Michel Lassalle [83] to correct a conjectural formula, requested to store results on disk, which was slow. By chance the recurrence allowed to restart computation from the last stored value, so that it could be achieved in few weeks, despite the frequent maintenance stopping and power breakdowns.

5.3.2 Scilab

The main part of my further implementations was numerical or semi-numerical, due to the necessity of testing real time methods for solving control problems. Sometimes, I used both Maxima and Scilab (see sec. 5.1.3). A larger Scilab implementation was done for the numerical simulations used in a paper on delay identification **[124]***. See sec. 1.2. One must notice that we could not use the integrators available in Scilab, due to the high level of noise requested in the simulation to illustrate the robustness of the method, so that I needed to implement a plain Euler method with frequency equal to noise frequency.

5.3.3 Maple

I learned Maple for the necessity of teaching, did not like it much, missing Scratchpad's types, but managed to do with it. One interesting feature is a good numerical integrator, so that one can compute algebraic formulas and integrate them without changing software and syntax, with the burden of writing "sed" filters.

Besides the package for automatic differentiation of solutions of ODE systems, used to illustrate a paper written with John Masse and his daughter Clara **[99]***, quite big packages were written to illustrate papers on aircraft control, using flat control **[71]*** and extended flatness **[132]***. The debugging phase was accelerated by the collaboration with Yirmeyahu Kaminski, who designed his own implementation in Python, allowing to cross-check our results.

This implementation would have been useless without actual numerical data for aircraft aerodynamics. A single publication with non linear models, for old planes like the Twin Otter or the F-4 and F-16, could be found on the NASA site [53].

Как говорил гроссмейстер Тартаковер,
Уж лучше план плохой чем никакого…

Псой Короленко [8]

Chapter 6

Perspectives for future works and ongoing collaborations

A large part of this research program corresponds to work that is already advanced, or even in the process of being written. Some other topics will demand implementations, translations or archives consultations that obviously require longer delays, and some could even demand new theoretical tools…

Regarding identifiability, observators or extended flatness, my project will find a natural place in the ANR project NODE (Numeric-symbolic resolution of differential equations), the guideline being to obtain, beyond the mathematical technique, algorithms that can be implemented and that allow to go from symbolic methods to numerical simulations or real time control.

I will certainly not achieve all this by myself, nor with my possible students and collaborators before I retire and turn to other activities, so that, publishing these ideas, everyone can feel free to read, understand and develop them as ze likes, or ask for details and collaborations.

6.1 Flat systems

For more than twenty years, most of my work on "endogenous equals exogenous" and flatness criteria has been done in common with Brahim Sadik.

6.1.1 "Endogenous" equals "exogenous"

As already said in sec. 1.3, a flat system is such that its solutions can, on a dense open dense, be parameterized by m functions, called *linearizing outputs*, m being the number of controls, this parameterization being locally bijective. "Endogenous equals exogenous" means in the language of control that the existence of a parameterization implies the existence of a locally bijective parameterization. Namely, systems linearizable by endogenous feedback are flat systems, while parameterizable systems are linearizable by exogenous (or dynamic) feedback. See def. 4.

Differential Lüroth–Ritt theorem

Using differential algebra, there is a stronger definition of differential flatness [39], requiring that the parameterization of an algebraic system should be algebraic. A simple system, such as

$$x_2' = -x_2 x_1' + x_2$$

is flat with flat output $x_2 e^{x_1}$, so that it admits no algebraic flat output, flat outputs being unique up to functional equivalence in differential dimension 1.

In this framework, "endogenous equals exogenous" is close to Lüroth–Ritt theorem [146]. As in the algebraic case (see sec. 2.2.2), it can be generalized to a differential field extension $\mathscr{F} \subsetneq K \subset \mathscr{F}\langle x_1, \ldots, x_n\rangle$ of differential degree of transcendence equal to 1. This result is currently being written with Brahim Sadik, as well as a paper devoted to the computing the intersection of two algebraic fields $k \subsetneq K_1 \subset k(x)$ and $k \subsetneq K_2 \subset k(x)$, using resultant computations and the ideal Δ method (2.1).

Reduction to one control

A fairly straightforward method for proving "endogenous equals exogenous" is to reduce to the single control case, *i.e.* to the differential dimension 1. To do this, assuming that a parameterization $x_i = X_i(z_1, \dots, z_m)$ for some system of order 1 does exist, which is of order q_j in z_j, it is sufficient to impose new differential equations $z_j^{(q_j+1)} = z_{j+1}$, for $1 < m$. With these relations, the image of the new system is of differential dimension 1 and parameterizable, and therefore flat. The linearizing output of this system of order 1 can be taken as the first linearizing output of the original system and the additional equations provide quite simply the missing linearizing outputs.

This approach does not solve the problem of fiding time invariant flat outputs if the parametrization is time-varying, a problem raised by Pereira da Silva and Rouchon [137].

6.1.2 Algorithmic flatness criteria

The criterion for static feedback linearization of Jakubczyk and Respondek [65], which is obviously a sufficient flatness criterion, or the necessary and sufficient flatness criterion for two input driftless systems, inspired by Cartan [21, 98] provide in addition involutive vectors fields whose linearizing outputs are the common first integrals. Our approach is to obtain more general criteria, while keeping such a constructive property.

Generalization of the lemma of Sluis–Rouchon

This statement characterizes in fact the parameterizable systems Σ and expresses the commutation of $\partial/\partial z_j^{(r+1)}$ and $\partial/\partial z_j^{(r)}$, where r is the order of the parameterization. It can be generalized by expressing the commutation of the derivations $\partial/\partial z_j^{(r+1+k)}$ and $\partial/\partial z_j^{(r)}$, when considering the system $\Sigma^{(k)}$. We obtain then a non trivial condition only when k is strictly less than the degree of the system Σ in its main derivatives.

Orbital flatness

Orbital flatness corresponds to the transformations used by Monge which do not fix the independent variable, thus allowing changes of variables, which transforms the time. The Cartan criterion thus appears as the orbital equivalent of the criterion of Jakubczyk and Respondek with one control, using the fact that static feedback is in this case equivalent to dynamic feedback, using a theorem of Charlet, Lévine and Marino [23]. We can indeed associate to any system $x_i' = f_i(x, u, t)$, a new "tachyconical"[1] system $x_i' = u_0 f_i$, $t' = u_0$ and the original system is orbitally flat iff the associated tachyconical system is flat in the classical sense.

A first objective would be to look for the tachyconical equivalent for the criterion of Jakubczyk and Respondek in any differential dimension.

Differential dimension 2 and beyond

The generalized Sluis–Rouchon lemma provides a valuable hint in the case when the system is of degree at least 3 (see above sec. 6.1.2). The case of degree 2 remains undetermined. This is the main difficulty that we have encountered with Brahim Sadik. But once we have identified this problem, a proper solution was not difficult to design, since we have in fact more freedom to choose the fields defining the linearizing defining the linearizing outputs.

With Brahim Sadik, we are working on the final version of an algorithm, focusing first on the easier case of systems with 2 controls.

6.1.3 Flat PDE systems

Flat nonlinear partial derivative systems have been little considered since our preliminary investigations with Alexandre Sedoglavic **[120]***. I would like to come back to it with a general definition of

[1] To express the idea that first derivatives belong to a cone, defined by the system.

flatness in this framework, based on the existence of flat discretizations whose linearizing outputs and flat parameterizations converge.

As a first step in this direction, I am considering the case of the heat equation for a rod of length $a + b$ insulated at both ends and heated at a point of coordinate a. If a and b are integers without common divisors, the system is controllable and flat when a and b are not both odd, and the flat output is obtained by computing the GCD of Chebyshev polynomials.

6.1.4 An avatar of differential Galois theory for diffieties

The pseudogroup of local isomorphisms of a flat diffiety is particularly rich. On the other hand, for a generic diffiety, it is reduced to the identity. This suggests to develop an original approach for flatness in the framework of diffieties, using some avatar of Galois theory, where the constraint is linked to use functions depending only on a *finite* number of coordinates. Moreover, a link exists with the classical differential Galois theory, Chelouah and Chitour [24] having studied a non flat system, that admits solutions in a Picard-Vessiot extension.

6.1.5 Generalized flatness

The notion of generalized flatness **[132]***, where the parameterizations depend on an infinite number of derivatives, requires a deeper investigations, starting with the study of more examples, such as a truck with two trailers, which is not flat in the general case.

On the theoretical side, we can start by studying generalized linearizing outputs in the case of linear systems and investigate the link with path planning methods by homotopy [25]. For practical computations, we can use and enlarge, in the framework of the project Node, the formal/numerical methods and certified numerical computations developed in Mathemagics.

6.1.6 Difference flatness

A notion of flatness for difference systems, that I have briefly described **[129]***, may be relevant for some kinds of discrete time systems. It differs from the definition proposed by Kolar *et al* [73], which requires a comparative study.

One may notice that the difference flatness notion of Kolar *et al.* is generic, as mine is not, due to some analogue of the Sluis–Rouchon lemma (see sec. 1.3.3 and sec. 6.1.2), which suggests that what they define is closer to generalized flatness (see sec. 1.3.4 and sec. 6.1.2).

6.2 First integrals

Flatness criteria such as the Cartan criterion provide families of vector fields whose linearizing outputs are the common first integrals. The flat outputs used in practice are often polynomials or fractions of small degree in the state variables. This motivates the search for algorithms allowing to test the existence of rational first integrals of small degree. The most general methods rely on the vanishing of a Wrońskian (the so-called "extatic curve") and it seems difficult to escape an exponential complexity with respect to the number of variables. See Bostan *et al.* [8] for an efficient algorithm in two variables.

An aricle on the case of an algebraic surface, the simplest case after the affine plane, is being written with Thierry Combot.

6.3 Jacobi's bound

6.3.1 Dimensional conjecture

The dimensional conjecture states that the differential codimension of a prime component defined by a system of r equations is at at most r. This is a decisive question, which allows among other things to avoid unnecessary computations. The scheme of a simultaneous proof of this conjecture and of

Jacobi's bound in the general case, sketched in a poster at Beijing **[128]*** in 2010, still seems to be likely to succeed. Cohn [27] has shown that the Jacobi bound implies the dimensional conjecture.

6.3.2 Singularities

The main difficulties of differential algebra are related to singularities, starting with the Ritt problem, that is to decide inclusion of two prime differential ideals defined by characteristic sets, that can be reduced to deciding if some zero of the separants belongs to the main component. See Hubert [58].

A first step in a systematic study of singularities would be to continue the approach initiated by Johnson [68], with a deeper investigation of the module associated to a singular point. A case of special interest would be to test if a module over a ring of power series is a free module, which provides a necessary condition for local differential flatness. See sec. 1.3.2.

6.3.3 Jacobi's bound and characteristic set computations

Jacobi's bound can advantageously replace Ritt's "Bézout" bound in all cases when it is proved. In this way, one can obtain many new sharp bounds on orderings of intermediate computations during a characteristic set computation. They can obviously also be used to improve the differential Kronecker method. See sec. 4.1 and sec. 6.3.4.

Systems with a non vanishing truncated determinant are a case of practical interest, common in practice, for which the shortest reduction (sec. 3.3) provide an easy starting point for algorithms like Pardi! See sec. 3.4 and Boulier *et al.* [12].

Difference Jacobi bound

Jacobi's bound can be extended to difference equations by considering the order matrix. We can go further, using the fact that there are two order matrices, one corresponding to the maximal orders, the other to the minimal orders of the transforms involved in the equations. A better candidate for a bound is therefore the difference of the tropical determinants of these two matrices. A direct proof of this result, avoiding technical tools from model theory used by Hrushovski [57], would be welcome.

Edition of letters and manuscripts

I have translated a great amount of Jacobi's manuscripts, obtained from the archives in Berlin. The two publications in AAECC **[126, 127]*** contains the English translations of the principal results. I have noticed that, despite my efforts, the French translations were much better and clearer, so that Kuhn and Pankratev prefered to work with them. I plan to publish a bilingual edition of the whole corpus, completed with notes and historical comments on the editing process of Jacobi's posthumous works. To fullfill this goal, a visit to Berlin to try fixing some reading difficulties, looking at the originals, will be necessary and could be completed with extra investigations in Wrocław and Olsztyn.

I also need to make available, with suitable comments, the content of the Jacobi fund in Mittag-Leffler Institute, scanned with the kind help of Guillermo Moreno-Socías.

6.3.4 Differential Kronecker

With Pablo Solernó, Gabriela Jeronimo and Amir Hashemi, we plan to continue the work initiated on the differential Kronecker method **[29]***, first avoiding unnecessary technical assumptions that limit its applicability, then extending it to PDE systems.

In this framework, our criterion **[55]*** might be useful, as well as Jacobi's bound to improve complexity in the ordinary case, noticing that the regularity assumptions required for the Kronecker method imply quasi-regularity of the differential system.

Sharp order majorations coming from Jacobi's bound (sec. 3.2) may help in the ordinary case. See sec. 6.3.3. Indeed, once we can know the order of derivation required for all initial equations, we can reduce quite easily to the algebraic Kronecker method. The easiest case is provided by the shortest reduction, when the system determinant does not vanish. See sec. 3.3. Basically, one would like to work with natural hypotheses such as: $[P] : S^\infty$ is radical, were S defines an *a priori* known degenerate locus.

6.4 Identifiability and identification

6.4.1 Structural identifiability

A paper with Hervé Le Meur, reviewing methods to test identifiability and trying to clarify the meaning of this notion is being written. To some extend, identifiability really means that parameters can be computed knowing input-output behavior with an arbitrary precision, *provided that the parameters belong to a compact*. The existence of an input-output behavior imply to consider initial conditions and to distinguish those who are *a priori* known, according to the experimental protocol, and those who need to be computed. We consider applications in biology, in particular in virodynamics.

6.4.2 State reconstructors and real time identification.

The generalization of the observators of Fliess and Sira-Ramírez [48, 49], for systems with delays [4], sketched in a short note **[124]*** deserves to be developed. In particular, one can use a large class of functions in the integrations by parts, which allows to choose them to minimize the influence of noise, leaving the fructuous but sometimes two narrow framework of Mikusiński's theory, to return to some earlier methods for computing derivatives by integration, as in Lanczos [81].

One field of experimentation is the analysis of musical sounds and their frequencies, in particular when frequencies and amplitudes are variable. Let us note that it is then possible to concentrate the study on a frequency band by treating not only the signals of higher frequencies, but also of lower frequencies, as noises.

זאָלט איר פֿון די אותיות כּוח שעפּן,
קוקט אין זיי אַרײַן!

מאַרק װאַרשאַװסקי, *אויפֿן פּריפּעטשיק* 9

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בראשית ברא֯ אולקים
את: א ביז ת—
אות
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ירמיה העשעלעס, לידער 10

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Translations of quotations

1. *A time to get, and a time to lose; a time to keep, and a time to cast away.* Ecclesiastes 3:6.
2. *Nothing completely new under the sun.* Ecclesiastes 1:9.
3. *Don't look to my grey head,/ Does it bother you in the game ?/ My soul is still young/ Like many years before.* Mordkhe Gebirtig, *Rejoyce little children.*
4. *In short, FMRFIJ*, are we Arizona cowboys in a laboratory/ Or guinea pigs taking the horizon for a labyrinth?* Robert Desnos, *L'aumonyme***
* A play on words: the letters read in French «éphémère effigie», *i.e.* ephemeral effigy.
** A play on words: «aumone» (alms, charity) and «homonyme» (homonym).
5. *In the human body, the word splits in two, into substance and essence. As the former disappears, the latter, remaining without shape, gives itself to be absorbed into the tissues of the body, as the essence constantly seeks a material carrier; even if it should become the cause of much misfortunes.* Olga Tokarczuk, *The Books of Jacob.*
6. *For one of us had invented for the words/ The wolf trap of speed* Louis Aragon, *The unfinished novel.*
7. *The inventable is unpraisable.* / So I declare it,/ When you have nothing to say/ You have to know how to advertise** / Once and for all!* Marc Favreau, *Nothing surprizing with Sol!*
* A play on words: «inventer» (invent) and «vanter» (praise).
** A play on words: «Publicitaire» (advertizer) and «se taire» (to shut up).
8. *Like said the grandmaster Tartakover,/ a bad plan is better than no plan* Psoy Korolenko.
9. *From the letters, you will take strength/ Look in them.* Mark Varshavski, *Oyfn pripetchik=On the hearth.*
10. *A the beginning Helohim created/ A-Z* : from A to Z/ Letter/ Sign and omen.* Jeremiah Hesheles, *Songs.* * Play on words: את that introduces in Hebrew the direct object in the well known begining of the Genesis: «God created the heaven and the earth» is writen with the first letter and the last letter of the Hebrew alphabet.

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יידישע אותיות: טײמס ניו ראָמאַן

That's a fact. But it's not the last word.
What is the last word
I don't know. Faith, maybe.
Which one: faith or maybe?
Hal Hartley, *The unbelievable truth*