Differential Ideal Theory

Algebraic Systems and Differential Ideals.

The pioneering work of Janet ([Janet 1929]) gave a clear link between the theory of analytic partial differential equations and the one of algebraic ideals. But Ritt was the first to introduce differential algebra (c.f. [Ritt 1932, 1950]) as a new field of mathematics. He also introduced difference algebra (c.f. [Cohn 1966] and the references therein).

Given a differential or partial differential field, e.g. $\mathcal{F} := R(X_1, \ldots, x_n)$ equipped with the partial derivations $\partial/\partial X_i$, a system of differential algebraic equations in m differential indeterminates $f_1, \ldots, f_m$ is a subset Σ of the differential polynomial algebra $\mathcal{F}\{f_j\}$. One will search for zeros in some universal extension of $\mathcal{F}$. Such a set is the differential algebraic variety $V(\Sigma)$ defined by the system. We have $V(\Sigma) = V([\Sigma])$, where $[\Sigma]$ is the differential ideal generated by Σ, i.e. the set of all linear combinations of derivatives of elements of Σ. If $\mathcal{F}$ is of characteristic 0, then $V(\Sigma) = V(\sqrt{\Sigma})$.

A differential ideal I such that $I = \sqrt{i}$ is said to be perfect. A perfect differential ideal is a finite intersection of prime differential ideals, meaning that a differential algebraic variety is a finite union of irreducible components. This is a corollary of the Ritt–Raudenbush theorem which asserts that a perfect differential ideal is the radical of a finitely generated differential ideal. One may refer to [Kaplansky 1957] for an introduction, to [Kolchin 1973] for much more details and to [Buium 1994] for more recent theoretical developments (chapter 2 is also a quite readable introduction).

Gröbner Bases.

It is known that the theory of standard or Gröbner bases allows to find a canonical set of generators for every algebraic ideal and to answer many basic problems such as ideal membership or the computation of the dimension. Unfortunately, if Gröbner bases do exist in the differential case, they are no longer finite (c.f. [Carrà Ferro 1987; Ollivier 1990a,b]). This is related to the fact that the Ritt–Raudenbush theorem only provides a weak analog of noetherianity. It may be proved that differential ideal membership is undecidable (c.f. [Gallo et al. 1991]). It is not known whether it is decidable for finitely generated differential ideals.

Characteristic Sets.

The key theoretical and algorithmic tool for differential ideals is the notion of characteristic sets. One may notice that the work of Wu Wentsun on automatic theorem proving was inspired by Ritt's work (c.f. [Chou 1988]). Ritt gave an algorithm to decompose a perfect ideal into a finite intersection of prime ideals, defined by the datum of some characteristic set. This requires some hypothesis on the ground field in order to perform factorization. Seidenberg (c.f. [Seidenberg 1956]) gave an algorithm for elimination in differential algebra without factorization, but it does not produce characteristic sets.

The recent work of Boulier shows that we can decompose any perfect differential ideal into an intersection of perfect differential ideals defined by characteristic sets, without factorization. This leads to greater efficiency. The method has been implemented in a MAPLE package (c.f. [Boulier et al. 1995, 1997]). This package has been improved by Hubert, who also had theoretical contributions, allowing a full decoupling between the differential and the algebraic completion [Hubert 1999] .

A great number of papers on the computation of characteristic sets for regular ideals, without factorization, followed Boulier's pionneering work: see [Bouziane et al. 2001; Maarouf 1996; Maarouf et al. 1998; Sadik 2000b]. On may notice the bound obtained by Sadik (see [Sadik 2000a]) on the order of derivations necessary to compute a characteristic set of a generic ordinary differential ideal.

Ritt's Problem.

A great unsolved problem is that of testing inclusion. Given two prime differential ideals I_1, I_2 defined by characteristic sets, can we decide whether $I_1 \subset I_2$? This is equivalent to finding an effective version of the Ritt–Raudenbush theorem, i.e. knowing a characteristic set of a prime differential ideal I to find a finite set Σ such that $I = \sqrt{[E]}$. See [Peladan–Germa 1995; Hubert 1996, 1999] for more details on the subject. This problem is related to that of testing equalities in differential rings defined by differential algebraic systems and initial conditions.

Complexity.

Improving Seidenberg's method, Grigor'ev obtained a triple exponential upper bound for quantifier elimination of nonlinear ordinary differential equations (c.f. [Grigor'ev 1987]). Using the classical construction of Mayr and Meyer, Sadik (c.f. [Sadik 1995]) proved a double exponential lower bound for the complexity of the differential Nullstellensatz for system of linear partial differential equations.

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