Bézout identities and control of the heat equation

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Abstract. Computing analytic Bézout identities remains a difficult task, which has many applications in control theory. Flat PDE systems have cast a new light on this problem. We consider here a simple case of special interest: a rod of length $a+b$, insulated at both ends and heated at point $x=a$. The case $a=0$ is classical, the temperature of the other end $\theta(b,t)$ being then a flat output, with parametrization $\theta(x,t)=\cosh((b-x)(\partial/\partial t)^{1/2}\theta(b,t)$.

When a and b are integers, with a odd and b even, the system is flat and the flat output is obtained from the Bézout identity $f(x)\cosh(ax)+g(x)\cosh(bx)=1$, the computation of which boils down to a Bézout identity of Chebyshev polynomials. But this form is not the most efficient and a smaller expression $f(x)=\sum_{k=1}^{n} c_k \cosh(kx)$ may be computed in linear time.

These results are compared with an approximations by a finite system, using a classical discretization.

We provide experimental computations, approximating a non rational value r by a sequence of fractions b/a, showing that the power series for the Bézout relation seems to converge.
Keywords: Bézout identities, Chebyshev polynomials, flat PDE systems, motion planning, heat equation.

Introduction

Bézout relations and control

It is known that the ring of entire functions $\mathbf{C}\langle z\rangle$ is a unique factorization domain. Moreover, for any subfield $\mathbf{K} \subset \mathbf{C}$, any ideal of $\mathbf{K}\langle z\rangle$ that admits a finite basis is a principal ideal [11, th. 9]. However explicitly finding Bézout identities is difficult. This problem is related with many applicational issues in control theory, such as the design of closed loop controlers. See *e.g.* Berenstein and Yger [2] and the references therein.

This interest became even stronger with the introduction of *flat systems* in the '90s by Fliess, Lévine, Martin and Rouchon [6, 7, 17]. Flat systems are systems the solutions of which can be parameterized by m arbitrary functions, called *linearizing outputs*, this parametrization being locally invertible. It is not known if testing if a non linear system is flat is decidable. This problem is closely related to *Monge problem* [19], considered by Cartan [4, 5] and Hilbert [12]. See also Zervos [28]. This problem is more precisely equivalent to testing *orbital flatness* [7], *i.e.* flatness allowing *time change*, which amount to transformations that also affect the independent variable.

Considering finite dimensional systems, one requires that the parametrization only involves a finite number of derivatives, which also implies the functional unicity of the flat outputs in the single input case. One may notice that a notion of *generalized flatness* has been proposed in the finite dimensional case, allowing parametrization with an potentially infinite number of derivations [21].

The goal of this paper is to provide a fast algorithm for computing GCDs of Chebyshev polynomials, that could be used to approximate the GCD of $\cosh(ax)$ and $\cosh(bx)$ is the general case. This method is inspired by the control of heat equation or wave equation that suggests a simple paper tape folding process. One may refer to [3, 26] for general results on GCD computations. See Chyzak *et al.* [9] for computer algebra algorithms related to parametrizations in differential or Ore algebras.

Flat control systems

In the ordinary linear case, flatness reduces to controllability, which means from a mathematical standpoint that the module associated to the system has no torsion element. In that case, the associated $\mathbf{R}(t)[\mathrm{d}/\mathrm{d}t]$-module is a free module, hence flat [6]. One may notice that, for time varying systems, the flat parametrization may be undefined where some numerators vanish: flat systems are generaly understood as admitting a flat parametrization of a dense open set. See Kaminski *et al.* [14] for the study of flat singularities.

Gluing together two flat single input systems, with linearizing outputs z_1 and z_2, we have $u = L_1 z_1 = L_2 z_2$, assuming that the input u is the same for both systems. This means, in the stationary case where $L_i \in \mathbf{R}[d/dt]$ that we have a parametrization $z_1 = L_2 z\ z_2 = L_1 z$ that gives $u = L_1 L_2 = L_2 L_1 z$, which is injective and surjective iff $GCD(L_1, L_2) = 1$. In such a case, the Bézout identity $M_1 L_1 + M_2 L_2 = 1$ provides the expression of the new flat output $z = M_2 z_1 + M_1 z_2$.

Using Mikusiński's operational calculus [18], flatness has been generalized to linear PDE, considering a robot arm, with small deflexion, described by the Euler-Bernouilli equation [8, 1]. Here the analogy is weaker, as the associated module is not free. On reduces to a free module by enlarging the operator ring with the inverse of some operator π, for which there is some freedom of choice. Considering the heat equation (see Laroche [15, 16]), there is a natural choice for a rod heated at one end and insulated at the other end, for which it is best to chose the temperature of the insulated end a flat output. One may notice on this example that the parametrizations are then given by an *entire analytic operator*. With such a requirement, for a single input system, the flat output is unique, up to a multiplication by an entire analytic operator with entire analytic inverse. A key issue is that fractional derivatives that appear in intermediate computations disappear at the end, with a suitable choice of output, so that the parametrization is also uniquely defined. One may also accept keeping fractionnal derivatives, used with succes by Oustaloup [25], but then their definition is not unique, and one may also accept that the parametrization depends on this choice, as proposed by Rammal *et al.* [23].

Flatness has also been generalized to non-linear PDE systems by reducing to a sequence of finite dimensional flat systems, using suitable discretizations [22, 24].

As we see, many theoretical approches are available, and we are far from a general unified algebraic theory. We will focus here on simple cases where the use of $\mathbf{R}\langle \mathrm{d}/\mathrm{d}t\rangle$-modules is relevant.

Bézout relations for the heat equation

We will be concerned here with the problem of computing Bézout identities for entire analytic operators, and will focus on the case of $\cosh(at)$ and $\cosh(bt)$, investigating first the case when b/a is rational which reduces to the simple case when a and b have no common factor. Then, a rod of length $a+b$ insulated at both ends and heated at point $x=a$ is controllable iff a and b are not both odd.

The problem then boils down to computing the GCD of Chebyshev polynomials, which becomes hard for large degrees. However, this may be done in linear time using a representation of the polynomials in the Bézout identity $M_1 \cosh(at) + M_2 \cosh(bt) = 1$ as

$$M_i(\cosh(t)) = \sum_k a_{i,k} \cosh(kt)$$

that suits better our purpose. Then, one may try to consider the general case by using approximations by discretizations or truncated Fourier series, or by considering rational approximations of an irrational value of a/b, for which fast computations of Bézout identities for Chebyshev polynomials with degrees up to 10^5 are usefull.

Our aim is to provide computational tools allowing mathematical experimentations, resting on Maple implementations.

Plan of the paper

The plan of the paper is the following. In a section 1, one recalls some basic definitions and properties of linear flat systems, focussing on the single input.

In section 2, we investigate the heat equation for a rod, first in the case of a rod heated at one end, then reducing the general case to the case of two rods of different lengths, which leads to the computation of a Bézout relation for cosh operators

In a third section 3, we describe a linear time algorithm for computing the Bézout relation, using a suitable representation.

In section 4, we use a discretization of the rod, showing that its flat output is the same as the flat output coming from the Bézout relation. Of this we deduce a linear time method, based on folding and cutting paper tapes.

Experiments of computations are presented in sec. 4

1 Flatness for linear EDO systems

We recall here a few basic results about flat systems in finite dimension that will be needed in the sequel or help understand the situation.

1.1 General results

As already stated in the introduction, in the linear case, systems are best described by $\mathbf{R}(t)[\mathrm{d}/\mathrm{d}t]$-modules and then flatness is equivalent to controllability, using the following classical result, for which we refer to Jacobson [13, chap. 3 th. 18]. We denote by $[\Sigma]$ the submodule of a module M generated by a family Σ.

Theorem 1. — *If A is a euclidean domain, or more generally a principal ideal domain, possibly non commutative, then any finitely generated A-module M admits a decomposition: $M = L \oplus T$, where L is a free module, and T is torsion.*

Obviously, the existence of a non trivial torsion part means that the system is non controllable. Reducing to a first order system, as we may, a torsion element and all its derivatives do no depend on the inputs, that satisfy no differential equations. So, the system is non flat. Reciprocally, if $T = 0$, the module is free and any basis of L is a flat output, providing a flat parametrization. We denote the derivation $\mathrm{d}/\mathrm{d}t$ by d_t.

Example 2. — Let be the system

$$\begin{aligned} x_1' &= u \\ x_2' &= u. \end{aligned}$$

We associate to it the module M, which is the quotient of

$$(\mathbf{R}(t)[\mathrm{d}_t]x_1 + \mathbf{R}(t)[\mathrm{d}_t]x_2 + \mathbf{R}(t)[\mathrm{d}_t]u)$$

by its submodule $[\mathrm{d}_t x_1 - u, \mathrm{d}_t x_2 - u]$. It is easily seen that

$$M = [x_1] \oplus [x_1 - x_2],$$

where $[x_1]$ is free and $[x_1 - x_2]$ is torsion, as $x_1' - x_2' = 0$.

Example 3. — We now consider the system

$$\begin{aligned} x_1' &= tu \\ x_2' &= u, \end{aligned}$$

and define M accordingly. It is easily seen as M is now free, as $M = [z := tx_1 - x_2]$. Indeed, $z' = x_1$, $z'' = u$ and $x_2 = tz' - z$.

The following theorem provides a simple criterion for flatness in the linear case.

Theorem 4. — *We consider a linear system*

$$x_i' = L_i(t, x, u) := \sum_{k=1}^{n} c_{i,k}(t) x_k + \sum_{j=1}^{m} d_{i,k}(t) u_j, \text{ for } 1 \leq i \leq n, \qquad (1)$$

where the x_i are the state variables and the u_j the controls.

We denote by ∂_{x_i} the partial derivative $\partial/\partial x_i$, ... The derivation d_t on the quotient module is then given by

$$\mathrm{d}_t = \partial_t + \sum_{i=1}^{n} L_i(t, x, u) \partial_{x_i} + \sum_{j=1}^{m} \sum_{k \in \mathbf{N}} u_j^{(k+1)} \partial_{u_j^{(k)}}. \qquad (2)$$

We then define $\Gamma_0 := \langle \partial_{u_1}, \ldots, \partial_{u_m} \rangle$[1]*, and then recursively $\Gamma_{i+1} := \Gamma_i + [\mathrm{d}_t, \Gamma_i]$.*

With these definitions, the torsion elements are first integrals of the derivations in Γ_n, which means that the system is flat iff $\dim \Gamma_n = n + m$, as a $\mathbf{R}(t)$-vector space.

Proof. — First we show that torsion elements cannot depend on the controls and their derivatives. Indeed, the module is finitely generated, so the dimension of the torsion submodule T as a $\mathbf{R}$ vector space is finite. Assume that $u_j^{(k)}$ is the highest derivative of u_k appearing in the elements of T and that is appear in some torsion element y. Then y' must depend on $u_j^{(k+1)}$, using formula 2: a contradiction.

So torsion elements are first integrals of Γ_0. Now, the derivatives of torsion elements are torsion elements and so first integrals of Γ_0 too. This means that for $y \in T$, $\Gamma_0 y = 0$ and $\Gamma_0 \mathrm{d}_t y = 0$, so that $[\mathrm{d}_t, \Gamma_0] y = 0$, so that $\Gamma_1 y = 0$. We can iterate the process, showing that $\Gamma_k y = 0$, for all $k \in \mathbf{N}$. It is easily seen that $\dim \Gamma_k \leq n + m$, so that the sequence $\Gamma_0 \subset \Gamma_1 \subset \cdots$ must be stationary and equal to Γ_{n+m}. The torsion elements y are then such that $\Gamma_{n+m} y = 0$. ∎

One must stress that in the non linear case, the computation of the firt integral is much more complicated and that sometimes no rational or algebraic first integral exists. See Chèze and Combot [10].

[1] The notation $\langle \Sigma \rangle$ denote the $\mathbf{R}(t)$-vector space generated by Σ.

1.2 The single input case

We give two simple results in the single input case, that we will need in the sequel.

THEOREM 5. — *Let be a linear flat single input system (1), its flat outputs are non trivial first integrals of the* $\mathbf{R}(t)$*-vector space* Γ_{n-1}*, which is of dimension* n*, so flat outputs linear in the* x_i *are unique up to multiplication by a factor.*

PROOF. — As the system is flat, Γ_n must have full rank $n+1$, according to th. 4. As $\dim \Gamma_{i+1} - \dim \gamma_i$ is at most the number of controls, so 1, we need have $\dim \Gamma_i = i+1$ for all $0 \leq i \leq n$ and so $\dim \Gamma_{n-1} = n$. Linear first integrals of Γ_{n-1} are defined by a linear system of n independent equation in $n+1$ variables, so that a linear non trivial solution in the x_i must exist, which is unique, up to multiplication by a constant in $\mathbf{R}$.

Let z be such a non trivial first integral. The module is free, so that z, z', ..., z^{n-1} are independent. One easily checks that $z^{(r)}$ is a first integral of Γ_{n-r-1}, so that they are linear combinations of the x_i and first integrals of Γ_0, so that we may recompute the x_i as linear combinations of z, ..., $z^{(n-1)}$ and then u, using $z^{(n)}$. This precisely means that z is a flat output. ∎

Our second results considers gluing two flat systems with the same single input u and provides an obvious criterion for the resulting system to be flat. For simplicity, we retreat here to stationary systems, that is systems with constant coefficients, and the commutative case, as all systems in the sequel will be of this kind.

THEOREM 6. — *Let be two flat systems with the same control* u*, and flat outputs* z_1 *and* z_2*. We use the derivatives* z_i*, ...,* $z_i^{(n-1)}$ *as state variables for system* $i = 1, 2$ *and define a module by the two expressions of* u*:*

$$u = L_i(z_i), \text{ for } i = 1 \text{ or } i = 2, \qquad (3)$$

where L_i *is a linear operator in* $\mathbf{R}(t)[\mathrm{d}_t]$ *of order* n_i*.*

The system (3) is flat iff $\mathrm{GCD}(L_1, L_2) = 1$*.*

PROOF. — If $\mathrm{GCD}(L_1, L_2) = M$, with M non trivial, then let $T_i := L_i/M$, for $i = 1, 2$, and $T := T_1 z_1 - T_2 z_2$. We obviously have $MT = 0$, so that T is torsion. Reciprocally, if $\mathrm{GCD}(L_1, L_2) = 1$, we have a Bézout relation $M_1 L_1 + M_2 L_2 = 1$, so that $z := M_1 z_2 + M_2 z_1$ is a flat output for the full system, with a parametrization given by $z_1 = L_2 z$ and $z_2 = L_1 z$. ∎

2 The heat equation for a rod

Considering here a partial differential equation, we need to consider modules over the ring of entire functions $\mathbf{R}\langle\partial_t\rangle$, or sometimes $\mathbf{R}\langle\partial_t^{1/2}\partial\rangle$ during computations.

2.1 The simple case

We consider the heat equation on a rod of length a that is heated at the end $x = a$, and insulated at $x = 0$. We follow the presentation of Laroche *et al.* This is decribed by the system

$$\begin{array}{lcl} \partial_t\theta(x,t) & = & \partial_x^2\theta(x,t) \\ \theta(a,t) & = & u(t) \\ \partial_x(0,t) & = & 0, \end{array} \tag{4}$$

denoting $\partial/\partial x$ by $\partial_x, \ldots$ In the Mikusiński domain, one may define $\partial_t^{1/2}\theta$, which must then be equal to $\pm\partial_x\theta$. Then, the general solution is of the form

$$[c_+\exp((x-x_0)\partial_t^{1/2}) + c_-\exp((x-x_0)\partial_t^{1/2})]\theta(x_0,t), \tag{5}$$

with $c_+ + c_- = 1$. Choosing $x_0 = a$, we get

$$\theta(x,t) = \cosh(x\partial_t^{1/2})\theta(0,t). \tag{6}$$

Indeed, as $\partial_x(0,t) = 0$, $\partial_t^p\partial_x\theta(0,t) = \partial_x^{2p+1}\theta(0,t) = 0$, so that all odd derivatives vanish at this point in the general solution (5). We need choose for the value of the flat output $\theta(0,t)$ functions $f(t)$ that provide converging series. It is shown in [16, th. 1] that this is granted for Gevrey α functions, with $\alpha < 2$[2].

2.2 The general case

We consider here the case of the heat equation for a rod of length $a+b$ insulated at both ends $x = 0$ and $x = a+b$, and heated at point $x = a$, so the control is $u(t) = \theta(a,t)$. We have two copies of the problem

[2]We recall that function f Gevrey or order α if there exist M and R such that for all $m \in \mathbf{N}$ $f^m(t) \leq M\frac{(m!)^\alpha}{R}$.

investigated at subsec 2.1 and we can rely on the parametrization already found, using

$$\begin{array}{rcl}\theta(x,t) & = & \cosh(x\partial_t^{1/2})\theta(0,t) \text{ for } 0 \leq x \leq a \\ \theta(x,t) & = & \cosh((a+b-x)\partial_t^{1/2})\theta(a+b,t) \text{ for } a \leq x \leq a+b,\end{array} \tag{7}$$

but we need have the compatibility relation

$$\begin{array}{rcl}u(t) = \theta(a,t) & = & \cosh(a\partial_t^{1/2})\theta(0,t) \\ & = & \cosh(b\partial_t^{1/2})\theta(a+b,t).\end{array} \tag{8}$$

We may proceed as done in [8, 1] and allow ourselves to invert some operator. Let

$$z(t) = \operatorname{acosh}(a\partial_t^{1/2})^{-1}\theta(a+b,t), \tag{9}$$

according to eq. (8), this implies

$$z(t) = \cosh(a\partial_t^{1/2})^{-1}\theta(a+b,t), \tag{10}$$

so that the compatibility condition stands, using the parametrization

$$\begin{array}{rcl}\theta(x,t) & = & \cosh(x\partial_t^{1/2})\cosh(b\partial_t^{1/2})\theta(0,t) \text{ for } 0 \leq x \leq a \\ \theta(x,t) & = & \cosh((a+b-x)\partial_t^{1/2})\cosh(a\partial_t^{1/2})\theta(a+b,t) \\ & & \text{for } a \leq x \leq a+b.\end{array} \tag{11}$$

It is easily seen that the operators $\cosh(a\partial_t^{1/2})$ and $\cosh(b\partial_t^{1/2})$ have a non trivial GCD iff $\cosh ax$ and $\cosh bx$ have. This can only happen when a/b is rational. Without loss of generality, we can reduce with a change of time scale to the case when a and b are integers without common factors. Then $\cosh bx = T_b(\cosh x)$, so that we are reduced to computing the GCD of T_a and T_b which is non trivial iff a and b are odd. In this case, the function $\hat{\theta}(x,t) = e^{-\pi^2 t/4}\cos(\pi x/2)$ is a solution of the full PDE system and limit conditions, with $u(t) = \hat{\theta}(a,t) = 0$, so that $\hat{\theta}$ is torsion: $\partial_t\theta = -\pi^2\hat{\theta}/4$. The PDE system is not controllable for a and b both odd.

Hence we can focus on the case a even and b odd, for which we have controllability and can compute a Bézout identity $L_1 T_a + L_b T_b = 1$ allowing to express the flat output z in the following way:

$$\begin{array}{l}z(x,t) = L_1(\cosh(x\partial_t^{1/2}))\theta(0,t) \text{ for } 0 \leq x \leq a \\ z(x,t) = L_1(\cosh((a+b-x)\partial_t^{1/2}))\theta(a+b,t) \text{ for } a \leq x \leq a+b.\end{array} \tag{12}$$

In fact, we will need to consider accurate rational approximations of real number $r = b/a$ and so great values of integers a and b for which a naive computation becomes soon impossible.

Remark 7. — To be perfectly rigorous, we work here in the ring A of entire differential operators $\mathbf{R}\langle\partial_t\rangle$. As already stated, any ideal of $\mathbf{K}\langle z\rangle$ that admits a finite basis is a principal ideal [11, th. 9], so that any finite type A-module M admits a decomposition $M = F \oplus T$, where F is free and T is torsion, according to th. 1. In our case, we consider the quotient

$$(Ae_1 + Ae_2)/[\cosh(b\partial_t^{1/2})e_1 + \cosh(a\partial_t^{1/2})e_2],$$

where the generators e_1 and e_2 are meant to represent the time functions $\theta(0,t)$ and $\theta(a+b,t)$, if one wishes to recover some mathematically non rigorous but easily understood physical interpretation.

3 A linear time algorithm

3.1 Description of the algorithm

In the case where a and b are integers such that $\mathrm{GCD}(T_a, T_b) = 1$, we are loooking for a Bézout relation $L_1 T_a(\cosh x) + L_2 T_b(\cosh x) = 1$, with $\deg L_1 \leq b-1$ and $\deg L_2 \leq a-1$. We want to use a representation of L_1 and L_2 as

$$L_i := \sum_{k=1}^{b-1} c_{i,k} \cosh(kx). \tag{13}$$

The basis is to use the classical formula $2\cosh(ix)\cosh(jx) = \cosh((i+j)x) + \cosh(|i-j|x)$.

Remark 8. — a) One knows that if a or b is even (resp. odd), then T_a or T_b involves only terms of even (resp. odd) degree so that $L1$ or L_2 are of even (resp. odd) degree.

b) As $\deg L_1 \leq b-1$ and $\deg L_2 \leq a-1$, the terms involved in the Bézout relation are of even degree k with $0 \leq k \leq a+b-1$.

Theorem 9. — *Assume that a and b have no common factor and that one is odd and the other is even, then there exists integer sequences α_i, c_i, k_i and f_i, for $1 \leq i \leq (a+b+1)/2$, such that for all $1 \leq i_0 \leq (a+b+1)/2$*

$$\sum_{i=1}^{i_0} c_i \cosh(f_i x)\cosh(\alpha_i x) = 1 + d_{i_0}\cosh(k_{i_0} x), \tag{14}$$

where α_i is equal to a or b, the k_i are even and $0 \leq k_i \leq a+b-1$, $0 \leq f_i < b$ (resp. $0 \leq f_i < a$) when $\alpha_i = a$ (resp. b), $c_i = \pm 2$ and $d_{i_0} = c_{i_0}/2$ if $1 \leq i_0 < (a+b+1)/2$, $c_{(a+b+1)/2} = \pm 1$ and $d_{(a+b+1)/2} = 0$, so that the sum is equal to 1 when $i_0 = (a+b+1)/2$. By convention, we set $k_0 = 0$.

Proof. — The proof is done by induction on i_0. When $i = 1$, the constant term 1 must come from

$$2\cosh(ax)\cosh(ax) \text{ or } 2\cosh(bx)\cos(bx).$$

Assuming that $a < b$, as we may up to a permutation, then

$$2\cosh(bx)\cosh(bx) = 1 + \cosh(2bx)$$

that includes a term of degree $2b > a+b-1$, which is excluded. So we need use $2\cosh(ax)\cosh(ax)$ when $a < b$, which makes appear a term $\cosh(2ax)$. We set then $f_1 := a$, $c_1 = 2$, $\alpha_1 = a$ and $k_1 = 2a$, so that k_1 is even. (This is step 1. of algorithm 10.)

Assume that we have (14) with k_i, f_i, c_i and α_i according to our requirements for all i up to i_0. There are 2 possible values for $0 \leq k_{i_0+1} \leq a+b-1$, so that

$$c_{i_0}/2\cosh(k_{i_0} - c_{i_0}\cosh(f_{i_0+1}x)\cosh(\alpha_{i_0}x) = c_{i_0+1}\cosh(k_{i_0+1}x). \qquad (15)$$

i) We can always use $k_{i_0+1} = |2a - k_{i_0}|$, which is such that

$$c_{i_0}/2\cosh(k_{i_0} - c_{i_0}\cosh(f_{i_0+1}x)\cosh(ax) = c_{i_0+1}\cosh(k_{i_0+1}x),$$

providing a contribution of $-c_{i_0}\cos(|a-k_i|x)$ to L_1. We would set then $\alpha_{i_0} = a$, $f_{i_0+1} := |a - k_{i_0}|$, $c_{i_0+1} := -c_{i_0}$ and $d_{i_0+1} := c_{i_0+1}/2$.

ii) a) If $k_{i_0} \geq b - a + 1$, then we can also choose $k_{i_0+1} = 2b - k_i \leq b + a - 1$, so that

$$c_{i_0}/2\cosh(k_{i_0}x) - c_{i_0}\cos(|b-k_i|x)\cosh(bx) = -c_{i_0}/2\cosh(k_{i+1}x),$$

providing a contribution of $-c_{i_0}\cosh(|b-k_i|x)$ to L_2. We could set then $\alpha_{i_0+1} = b$, $f_{i_0+1} := |b-k_i|$, $c_{i_0+1} := -c_{i_0}$ and $d_{i_0+1} := c_{i_0+1}/2$.

ii) b) If $k_{i_0} \leq b - a - 1$, we may set $k_{i_0+1} = 2a + k_{i_0}$, so that

$$c_{i_0}/2\cosh(k_i x) - c_{i_0}\cos((a+k_{i_0})x)\cosh(ax) = -c_{i_0}/2\cosh(k_{i+1}x),$$

providing a contribution of $-c_{i_0}\cos((a+k_i)x)$ to L_1. We could set then $\alpha_{i_0+1}=a$, $f_{i_0+1}:=a+k_{i_0}$, $c_{i_0+1}:=-c_{i_0}$ and $d_{i_0+1}:=c_{i_0+1}/2$.

Among these two possible values for k_{i_0+1}, that are seen to be even when k_{i_0} is even, one is the value of k_{i_0-1}, so the other value must be chosen for k_{i_0+1}. We always have $k_{i_0+1}\neq k_{i_0}$ and $k_{i_0-1}\neq k_{i_0}$, except in two cases. The first is $k_0=0$, set above by convention, which is in fact the value for k_2 coming from rule i) whith $k_1=2a$, that the convention $k_0=0$ excludes. The second case is $k_{i_0}=a$ if a is even, in which case i) sets $k_{i_0+1}=a$ or b if is b is even, in which case ii) a) sets $k_{i_0+1}=b$. As the sequence k_i starts with $k_0=0$ that is a stationary value, and the only a finite number of values are possible for the k_i, it must end at the second stationary value for some $k_p=a$ (resp. $k_p=b$) when a (resp. b is even).

Then, we only have to set $c_p=-c_i/2$, so that the sum (14) is equal to 1 and we may set $d_p=0$, the choice of k_p being then unimportant.

We only have left to show that the maximal index p is indeed equal to $(a+b+1)/2$. Consider the equivalence relation $\equiv$ in $\mathbf{Z}$ such that $x\equiv y$ if $x=-y$ or $x-a=-y+a$ or $x-b=-y+b$. According to rules i) and ii), for any value k_i in the sequence, any value $0\leq k\leq a+b-1$ such that $k\equiv k_i$ also belong to the sequence. Now, as $\mathrm{GCD}(a,b)=1$ there are only two equivalence classes: the class of 0 and the class of 1, so that all even values $0\leq k\leq a+b-1$ must belong to the sequence. The paper folding process of sec. 4.2 is an illustration of this property. ∎

From the previous proposition, we deduce the following algorithm, which we have implemented in a Maple package.

Algorithm 10. — **Input** Two integers a and b with $a<b$, one even, the other odd.

Output The factors L_1 and L_2 in a Bézout relation for the Chebyshev polynomials T_a and T_b, represented by two arrays A_1 and A_2 with $A_i[i]=c_i$ if $c_i\cosh(ix)$ appears in L_i.

Step 1. $k_0:=0$, $k_1:=2a$, $A_1[a]:=2$;

if a is even then $k_{final}:=a$ **else** $k_{final}:=b$ **fi**;

Step 2. while $k_i\neq k_{final}$ **do**

Determine α_{i+1}, k_{i+1}, c_{i+1} and f_{i+1} as in th. 9 using rules i) or ii)in the proof.

Set $A_{1,i}[f_i]:=c_i$ if $\alpha_i=a$;

Set $A_{2,i}[f_i] := c_i$ if $\alpha_i = b$;

$i := i + 1$; **od**;

Step 3. ($k = k_{final}$) **if** a is even **then** $A_1[a] := -c_{(a+b-1)/2}/2$ **else** $A_2[b] := -c_{(a+b-1)/2}/2$ **fi**;

return A_1 and A_2.

3.2 Complexity issues and implementation

It is easily seen that the total number of operations in algorithm 10 is proportional to the number $(a + b + 1)/2$ of steps, so $O(a + b)$. As the number of steps is also the number of terms in the output, the complexity is linear in the size of the result and no great improvement can be expected.

One must notice that a naive use of Maple addition in the previous algorithm leads to a quadratic complexity, as the cost of addition is linear, but we just need a power series expansion up to a chosen order. Computing just the arrays is very fast.

We give here a few curves showing CPU time, starting with the computation of the Bézout relation using Maple `gcdex` function and our implementation fig 1. We see that our implementation is much faster for getting the same result. The irregularities in the right curve is possibly due to the particularity of Maple's quite unpredictable internal term ordering, implying term permutations. Of course, just because of the size of intermediate computations, noticing that the first coefficient of T_a is 2^{a-1}, general GCD algorithms cannot compete, as they do not use a suitable data representation. One may notice however that they can provide already interesting results for pratical purpose in an acceptable time.

The following curve on the right exhibits the quadratic behaviour obtained by computing explicitly the factors L_i of the Bézout relation. On the left, we compute the power series development of the factors, up to order 20, and the complexity keeps linear.

The algorithm 10 has been implemented in a function `BezoutBis` of a Maple package `Chaleur`, with a few related functions. Global variables `use_pol` and `use_ser`, set to *true* or *false* allow to compute or not the result as a sum of $\cosh(ix)$ or as the corresponding series. With this im-

Figure 1: Left, Maple `gcdex` function. Right our implementation. We compute $\mathrm{GCD}(T_{2i}, T_{2i+1})$. Time are given in sec, depending on i.

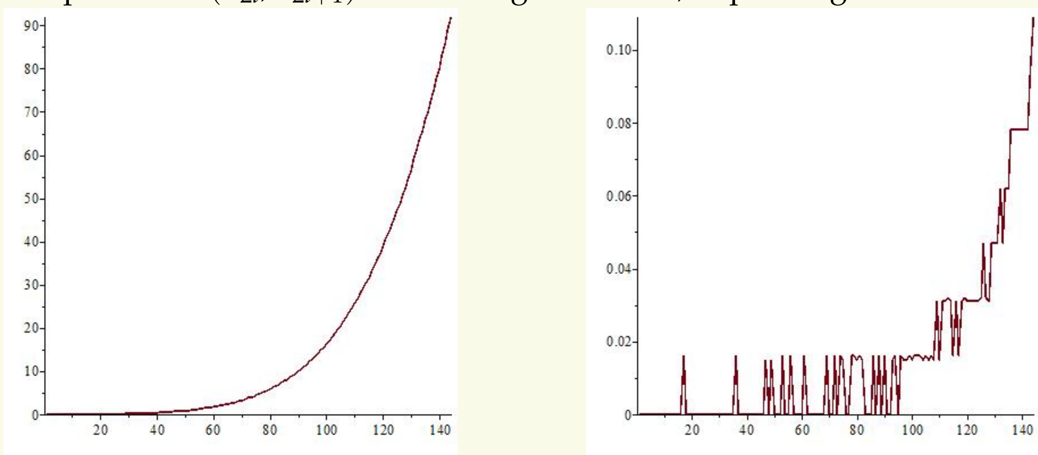
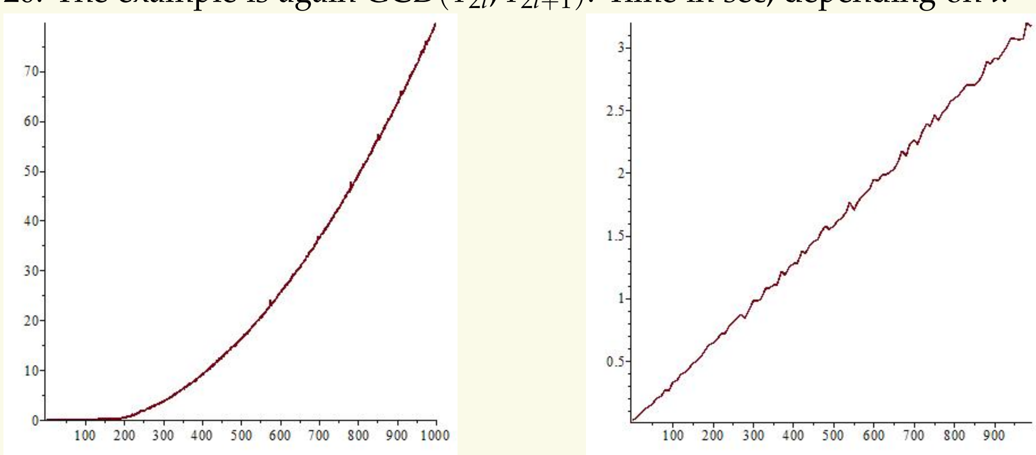


Figure 2: Left, we compute the factors L_1, leading to a quadratic complexity. Right, we only compute their power series development, up to order 20. The example is again $\mathrm{GCD}(T_{2i}, T_{2i+1})$. Time in sec, depending on i.



plementation, we can reach degrees up to 10^6, just computing the arrays of coefficients or up to 10^5 computing power series of order 20 in a few minutes.

The Maple package is available at adress:
`http://www.lix.polytechnique.fr/~ollivier/Chaleur/`

4 Discretization

Assuming that a and b have no common factor, we use a classical parametrization, that is adapted from [22]:

$$\theta_i' = 2\theta_i - \theta_{i-1} - \theta_{i+1}, \text{ for } 0 \leq i\, qa \text{ and } qa < i \leq q(a+b), \tag{16}$$

with $\theta_{qa} = u$ and, by convention, $\theta_{-1} = \theta_1$ and $\theta_{q(a+b)+1} = \theta_{q(a+b)-1}$.

We will show that this discretized model is flat when a and b are not both odd and that its flat output corresponds to the one obtained in the preceeding section for the PDE system 4.

For this, one may consider the following system:

$$\begin{array}{rcl} \theta_i' & = & \theta_{i+1}, \text{ for } i = 0, \\ \theta_i' & = & \theta_{i-1}, \text{ for } i = a+b, \\ \theta_i' & = & (\theta_{i-1} + \theta_{i+1})/2, \text{ for } 0 \leq i\, qa \text{ and } qa < i \leq q(a+b). \end{array} \tag{17}$$

Remark 11. — It is build so that $\theta_1 = \mathrm{d}_t\theta_0$ and $\theta_{n+1} = 2\mathrm{d}_t\theta_n - \theta_{n-1}$, where we recognize the classical recurrence defining Chebyshev polynomials. So we have $\theta_i = T_i(\mathrm{d}_t)\theta_0$, for $0 \leq i \leq a$ and in the same way $\theta_{a+b-i} = T_i(\mathrm{d}_t)\theta_{a+b}$.

Then, easy computations show that for both systems the sets Γ_i are the same. More precisely, we have the following proposition.

Proposition 12. — *For systems (17) and (16), assuming $a < b$, the sets Γ_i are such that $\Gamma_0 := \langle \partial_u \rangle$ and $\Gamma_i = \langle c_{1,i}\partial_{x_{k_{1,i}}} + \partial_{x_{k_{2,i}}}$, for $1 \leq i \leq a+b$, where $k_{2,i} = a+i$ for $1 \leq i \leq b$ and $k_{2,i} = a+2b-i$, for $b \leq i \leq a+b$. For $c_{1,i}$ and $k_{1,i}$ the rule is the following: $c_{1,i} = 0$ for $a = 2pa$. Assume that $i = 2pa + k$ for $p \in \mathbf{N}$ and $<\leq k < 2a$. If p is odd, then $c_{1,i} = 1$, if not $c_{1,i} = -1$. If $1 < k \leq a$, then $k_{1,i} = a - k$ and if $a \geq k < 2a$, then $k_{1,i} = k - a$.*

Theorem 13. — *Let $L_1T_{qa} + L_2T_{qb} = 1$ be a Bézout relation for the Chebyshev polynomials T_a and T_b, a and b without common factor and not both odd, with $L_i = \sum_{k=0}^{b-1} c_{i,k}\cosh(kqx)$, then a flat output for the discrete system (16) is*

$$\sum_{k=0}^{a-1} c_{2,k}x_{q(a+b-k)} + \sum_{k=0}^{b-1} c_{1,k}x_{qk}. \tag{18}$$

Proof. — By prop. 12, the systems (17) and (16) have the same sets Γ_i, and so, according to th. 5, the same flat outputs. By rem. 11 and prop. 6, a flat output of (17) is (18). ∎

4.1 Analogy with the wave equation

One may view the propagation of the indices as a wave, starting at the heated point, that reflects on the insulated end. When it goes back to the heated point, then it reflects too, but with an opposite sign.

This may be easier to understand using an analogy with the wave equation, which has the same flat output.

$$\begin{array}{lcl} \partial_t \theta(x,t) & = & \partial_x^2 \theta(x,t) \\ \theta(a,t) & = & u(t) \\ \partial_x(0,t) & = & 0, \end{array} \tag{19}$$

Such a system is a delay system, with a flat parametrization, meaning that the associated module is free: $\theta(x,t) = \cosh(x\mathrm{d}_t)\theta(0,t) = \theta(0,t-x) + \theta(0,t+x)$. Indeed, in the theory of Mikusiński [18], the operator $\exp(\mathrm{d}_t)$ is a delay operator and $\exp(x\mathrm{d}_t)f(t) = f(t+x)$. See Mounier *et al.* [20, 27] for more detail on wave control.

As explained in rem. 7, we work on the ring A of integer differential operators $\mathbf{R}\langle\partial_t\rangle$ and consider the quotient module

$$M := Ae_1 + Ae_2/[\cosh(b\partial_t)e_1 + \cosh(a\partial_t)e_2],$$

where the generators e_1 and e_2 are meant again to represent the time functions $\theta(0,t)$ and $\theta(a+b,t)$. In this more rigorous setting, the module M is indeed free when a/b is not the quotient of two odd integers.

Remark 14. — A naive discretization, such as:

$$\theta_i'' = 2\theta_i - \theta_{i-1} - \theta_{i+1}, \text{ for } 0 \leq i\, qa \text{ and } qa < i \leq q(a+b), \tag{20}$$

would be flat, allowing to reach any point in state space in any non zero time. So it fails to model the incompressible delay for wave propagation. See Zuazua [29] for such issues.

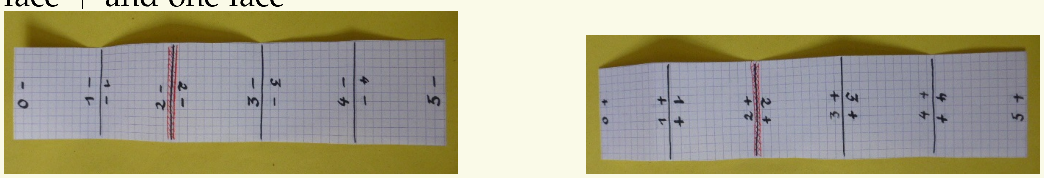
4.2 Computations with a paper tape

This section may sound anachronical, but as designing new physical devices for computations is not devoid of interest, a short presentation of a this simple computational tool may help to understand the basic idea of the algorithm and as a contribution to the study of computing as a physical process, even if we do not actually want to use it!

The process, based on the propagation of the differential operators in prop. 12, is indeed close to the wave equation that is equivalent to algorithm 10 and may help to visualize how the computation of a flat output using a paper tape divided in $a+b$ boxes. On each side of the border between boxes, we write the index i and a sign, which is always $+$ on one side and $-$ on the other side, as shown in fig. 3. We show here the computation of the GCD of T_2 and T_3 that gives:

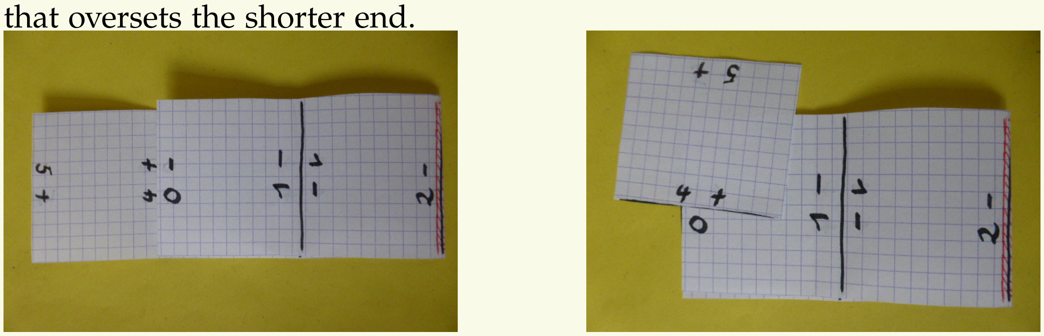
$$(2\cosh(2x)+1)T_2(x) - 2\cosh(x)T_3(x). \tag{21}$$

Figure 3: A tape of paper with boxes, indices on both side of borders, one face $+$ and one face $-$



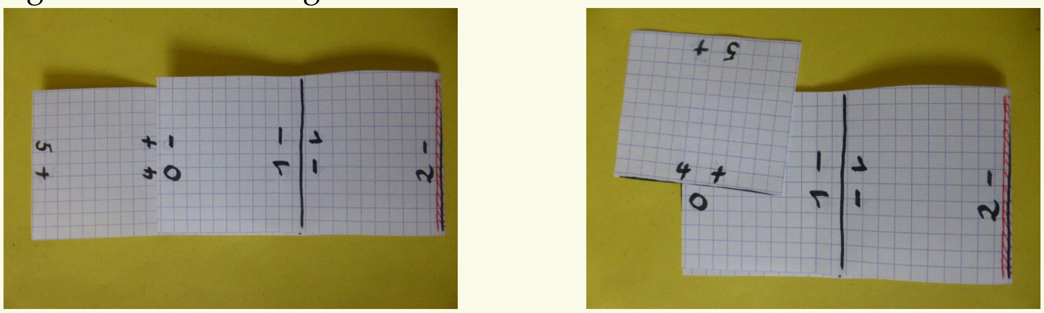
We mark the heated point with a red line, and fold the paper tape at this place. During the process, we have a long end and a short end. If the long end oversets the heated point, we fold it. If it oversets an end, we cut it and rotate it of π rad in the same plane, as shown in fig. 4.

Figure 4: One folds the tape at the heated point in red and cuts the part that oversets the shorter end.



We iterate the process until both ends have the same length, which is the GCD of a and b. Then, we look the end opposite to the heated point, so the odd end, if a and b have no common factor. The number of $+$

Figure 5: We rotate the cut part of π and repeat the process. The sum of signs at the odd end give the result.



or $-$ for the written indices provide the requested coefficients of the flat output, up to the sign. The values on the picture show the opposite of 21.

Folding corresponds to the change of sign in reflection passing by index a in prop. 12.

5 Computational investigations

Numerical simulations are used to provide empirical estimations of the Bézout relations for operators $\cosh(x)$ and $\cosh(rx)$, with r irrational, using rational approximations of r. Our running example is $r = \sqrt{2}$.

The rational approximations used are provided by the continued fraction expansion:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

using Maple implementation in the package `NumberTheory`. Of them, we extract fractions b/a where b and a are not both odd, so that $\mathrm{GCD}(T_a, T_b) = 1$. We considered values in this list:

$$\frac{3}{2}, \quad \frac{17}{12}, \frac{99}{70}, \quad \frac{577}{408}, \frac{3363}{2378}, \frac{19601}{13860}, \frac{114243}{80782}.$$

and compute the power series corresponding to the factors in the Bézout relation, using 20 digits floats, after substituting x/a to x. We get:

$$\begin{array}{lllll} \frac{3}{2} & L_1 = 3. & +1 * x^2 & +.83x^4 + \cdots \\ & L_2 = -2. & -.25 * x^2 & -.52 * 10^{-2} * x^4 + \cdots \\ \frac{17}{12} & L_1 = 1. & -1.33 * x^2 & -.28 * x^4 + \cdots \\ & L_2 = & .83 * x^2 & +.72 * 10^{-1} * x^4 + \cdots \end{array} \tag{22}$$

The two results already look very different, as we are expecting a convergent sequence. This is due to the fact that $a = 2$ is even, but not a multiple of 4, so that the constant term in T_2 is -1, while in T_{12}, which is a multiple of 4, the constant term is 1. We have obtained two seemingly converging sequences, according to the case. Considering polynomials, the GCDs are normalized by imposing bounds on the degree, which does not work considering GCDs of integer analytic functions. The obtained results can only be interpreted as "convergent" modulo the trivial relation

$$\cosh(\sqrt{2}x)\cosh(x) - \cosh(x)\cosh(\sqrt{2}x) = 0.$$

We have chosen to normalize the relations, so that L_2 has a constant term equal to 0, which is the case for a mutiple of 4.

In this way, we have results that seem to converge and give the following estimations for the series, using the approximation $\frac{114243}{80782}$:

$$\begin{aligned} L_1 \quad = \quad & 1. - 1.3333333332822534857 * x^2 \\ & -.27941176475495906369 * x^4 \\ & -.17992011626456865856e - 1 * x^6 \\ & -.56399934990584172717 * 10^{-3} * x^8 \\ & -.10660593920209884517 * 10^{-4} * x^{10} \\ & -.13660155439036212056 * 10^{-6} * x^{12} \\ & -.12773250009420268932 * 10^{-8} * x^{14} \\ & -.9172810105402920608 * 10^{-11} * x^{16} \\ & -.5253788359821529842 * 10^{-13} * x^{18} \\ & -.24709351400071579999 * 10^{-15} * x^{20}, \end{aligned}$$

$$
\begin{aligned}
L_2 \quad = \quad & .83333333328225349697 * x^2 \\
& +.71078431383315849009 * 10^{-1} * x^4 \\
& +.18972403780540266659 * 10^{-2} * x^6 \\
& +.26307421016151944601 * 10^{-4} * x^8 \\
& +.23019494752812483754 * 10^{-6} * x^{10} \\
& +.14160103021824419386 * 10^{-8} * x^{12} \\
& +.65710552317259491 43 * 10^{-11} * x^{14} \\
& +.24170000965421789288 * 10^{-13} * x^{16} \\
& +.72956699588008530 34 * 10^{-16} * x^{18} \\
& +.18497349717353377031 * 10^{-18} * x^{20}.
\end{aligned}
$$

The total real computation time for this fraction is 428.17 sec.

Further computations would be needed to investigate the convergence of these series. The terms are decreasing, but not so fast.

The following table 6 provides the last term of L_1, corresponding to degree 20 for all our approximations

Figure 6: Terms of degree 20 in L_1.

3/2	.27327502044120918666E − 15
17/12	.27632394776909554494E − 15
99/70	.24164130261737767018E − 15
577/408	.24708886190685028010E − 15
3363/2378	.24708886190685028010E − 15
19601/13860	.24709351973896486850E − 15
114243/80782	.24709351400071579999E − 15

Conclusion

At this stage, we have been able to design an empirical process to look for power series expansions of a very specific class of entire analytic functions, relying on remarkable identities. Such computational tools exceed the needs of practical control theory but may cast some light on some theoretical control problem.

On the other hand, control theory provides intuitions to address the problem of Bézout relations on a wider setting, trying to work in a direct way with discretizations or Fourier series expansions, topics on which we already started some investigations, not conclusive at this stage. Without the comfort of theoretical methods to check of the validity of computations, being able to get compatible results by independent ways is essential for experimentations that may help to improve and make more robust the definitions of controlability and flatness available for PDE systems.

One thing is clear: data representation is essential for computational complexity.

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