The reduction to normal form of a non-normal system of differential equations

De æquationum differentialium systemate non normali ad formam normalem revocando

Carl Gustav Jacob Jacobi (1804–1851)
Prof. ord. math. Regiom. (1832–1844) & Berol. (1844–1851)

Summarium

Hanc commentationem in medium protulerunt Sigismundus Cohn, C.W. Borchardt et A. Clebsch e manuscriptis posthumis C.G.J. Jacobi.

Solutio problematis, datum m^2 quantitatum schema quadraticum per numeros minimos ℓ_i singulis horizontalibus addendos ita transformandi, ut m maximorum transversalium systemate præditum evadat, determinat systematis m æquationum differentialium $u_i = 0$ ordinem et brevissimam in formam normalem reductionem: æquationes $u_i = 0$ respective ℓ_i vicibus differentiandæ sunt.

Etiam indicatur quot vicibus iteratis singulæ æquationes differentiales propositæ differentiandæ sint, ut æquationes differentiales nascantur, ad reductionem systematis propositi ad unicam æquationem necessariæ.

Abstract

This paper was edited by Sigismund Cohn, C.W. Borchardt and A. Clebsch from posthumous manuscripts of C.G.J. Jacobi.

The solution of the following problem: "to transform a square table of m^2 numbers by adding minimal numbers ℓ_i to each horizontal row, in such a way that it possess m transversal maxima", determines the order and the shortest normal form reduction of the system: the equations $u_i = 0$ must be respectively differentiated ℓ_i times.

One also determines the number of differentiations of each equation of the given system needed to produce the differential equations necessary to reduce the proposed system to a single equation.

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Translator's comments

This text [Jacobi 2] is based on a single manuscript [II/23 b)], the transcription of which was considered and started by Cohn in 1859 (see his letter to Borchardt [II/13 a)] and Borchardt's abstract [II/25] that refers to his and Cohn's transcription). The first published version appeared a volume edited by Clebsch [VD]. Nothing indicates his participation in the preparation of the text. One may guess that the main part of the work, if not all, was achieved by Borchardt after Cohn's death, but the transcriptions are lost. A strange manuscript [II/24], from some unknown hand, mixes the texts of the two articles related to Jacobi's bound [Jacobi 1 and Jacobi 2], showing that their publication in a single paper has been considered.

This published version follows closely the original texts, and most changes only concern questions of style that are not perceptible in the translation, e.g. permuation of words, the order in Latin sentences being very flexible. The original text is in general easier to translate. Having translated the published paper before I have found the original manuscript, I often had to simplify it to make the translation readable: many times these changes just restored Jacobi's version. Additions made by Borchardt were kept when they can help the reader. [They are indicated by sans sherif letters enclosed between brackets].

Before the translation of the published article [Jacobi 2], we provide the second part of an unpublished manuscript [II/23 a), fos 2217–2220], related to the different normal forms that a given system may admit, and the way to compute them, that may help to understand the mathematical context. This text was possibly intended as a part of an article [Jacobi 3], but the § 17 of the published version only considers changes of normal forms for linear systems.

Thanks are due to Alexandre Sedoglavic for his kind help and encouragements to achieve a preliminary French translation. I must express my most heartfelt thanks to Daniel J. Katz, who also corrected the translation of a companion paper [Jacobi 1]. He provided again a very competent rereading of the latin transcription and of this translation, suggesting many important improvements and correcting numerous mistakes and inaccuracies.

References

Primary material, manuscripts

The following manuscripts from Jacobis Nachlaß, Archiv der Berlin-Brandenburgische Akademie der Wissenschaft, were used to establish this translation.

We thank the BBAW for permission to use this material and its staff for their efficiency and dedication.

[II/13 a)] *Letter from* Sigismund COHN *to* C.W. BORCHARDT. Hirschberg, August, 25th 1859, 3 p.

[II/23 a)] *Reduction simultaner Differentialgleichungen in ihre canonische Form und Multiplicator derselben*, manuscript by Jacobi. Five different fragments: 2214–2216; 2217–2220 (§ 17-18); 2221–2225 (§ 17); 2226–2229; 2230–2232, 2235, 2237, 2236, 2238.

[II/23 b)] *De aequationum differentialium systemate non normali ad formam normalem revocando*, manuscript by Jacobi 2238–2251. The basis of [Jacobi 2].

[II/24] *De aequationum differentialium systemate non normali ad formam normalem revocando*, manuscript from unknown hand, mixing [Jacobi 1] and [Jacobi 2].

[II/25] *De aequationum differentialium systemate non normali ad formam normalem revocando.* Abstract and notes by Borchardt. 8 p.

Publications

[Crelle 27] *Journal für die reine und angewandte Mathematik*, **27**, Berlin, Georg Reimer, 1844.

[Crelle 29] *Journal für die reine und angewandte Mathematik*, **29**, Berlin, Georg Reimer, 1845.

[GW IV] *C.G.J. Jacobi's gesammelte Werke IV*, K. Weierstrass ed., Berlin, Georg Reimer, 1886.

[GW V] *C.G.J. Jacobi's gesammelte Werke V*, K. Weierstrass ed., Berlin, Georg Reimer, 1890.

[VD] *Vorlesungen über Dynamik von C. G. J. Jacobi nebstes fünf hinterlassenen Abhandlungen desselben*, A. Clesch ed., Berlin, Georg Reimer, 1866.

[Jacobi 1] JACOBI (Carl Gustav Jacob), "De investigando ordine systematis aequationum differentialum vulgarium cujuscunque", [GW V] p. 193-216, and *this special issue of AAECC.*

[Jacobi 2] JACOBI (Carl Gustav Jacob), "De aequationum differentialum systemate non normali ad formam normalem revocando", [VD] 550–578 and [GW V] 485-513.

[Jacobi 3] JACOBI (Carl Gustav Jacob), "Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi", [Crelle 27] 199–268, [Crelle 29] 213–279 and 333–376; [GW IV], 317–509.

Translations

[II/23 a) fos 2217–2220]

The transformations of a system of differential equations of an arbitrary order; how the multiplier of the transformed system is obtained from the multiplier of the proposed system.

17.

THE SYSTEM OF DIFFERENTIAL EQUATIONS

$$1) \qquad \frac{d^{n_1}x_1}{dt^{n_1}} = A_1, \frac{d^{n_2}x_2}{dt^{n_2}} = A_2, \dots, \frac{d^{n_m}x_m}{dt^{n_m}} = A_m$$

is proposed, in which only derivatives lower than $\frac{d^{n_1}x_1}{dt^{n_1}}$, $\frac{d^{n_2}x_2}{dt^{n_2}}$ etc. are to be found on the right side. The order of this system is the number

$$n_1 + n_2 + \cdots + n_m$$

that is the sum of the orders n_1, n_2, ..., n_m, up to which go the derivatives of each variables in the proposed equations. One agrees that *the order of an arbitrary system of differential equations is defined as being equal to the number of first order differential equations to which it may be reduced, or also to the number of arbitrary constants that its complete integration admits and requires*[1]. The proposed system of differential equations can always be transformed into some others of the same order, and the order up to which the derivatives of each variable in the transformed equations must go will be, *in general*, left to our discretion, provided that their sum be equal to the system order. Such a transformation is done in the following way.

It is proposed to transform the system of equations 1) into some other, in which the derivatives of variables x_1, x_2, ..., x_m respectively reach the orders i_1, i_2, ..., i_m, where

$$i_1 + i_2 + \cdots + i_m = n_1 + n_2 + \cdots + n_m.$$

Always denoting the derivatives by upper indices in *Lagrange's* way, the given system may be reduced to differential equations of the *first* order between the variables

[1]I do not say *to the order* of a differential equation between two of the proposed variables, to which the proposed system may be reduced by appropriate differentiations and eliminations; indeed, in some particular cases this reduction does not succeed, for example if A_1 is a function of x_1 and t alone, A_2 of x_2 and t alone, ..., A_m of x_m and t alone. J.

$$\text{A)}\qquad \begin{cases} t, \quad x_1, x_1', x_1'', \ldots, x_1^{(n_1-1)}, \\ \qquad x_2, x_2', x_2'', \ldots, x_2^{(n_2-1)}, \\ \qquad \ldots\ldots\ldots\ldots\ldots \\ \qquad x_m, x_m', x_m'', \ldots, x_m^{(n_m-1)}. \end{cases}$$

In the same way, the system that it is proposed to obtain by this transformation is equivalent to a system of first order equations between the variables

$$\text{B)}\qquad \begin{cases} t, \quad x_1, x_1', x_1'', \ldots, x_1^{(i_1-1)}, \\ \qquad x_2, x_2', x_2'', \ldots, x_2^{(i_2-1)}, \\ \qquad \ldots\ldots\ldots\ldots\ldots \\ \qquad x_m, x_m', x_m'', \ldots, x_m^{(i_m-1)}. \end{cases}$$

So, the problem is reduced to transforming a system of first order differential equations between the variables A) into some other one between the variables B).

Let us assume that

the orders	$i_1, i_2, \ldots, i_k$	are resp.	*greater than*	$n_1, n_2, \ldots n_k$
″	$i_{k+1}, i_{k+2}, \ldots, i_{h-1}$	″	*equal to*	$n_{k+1}, n_{k+2}, \ldots n_{h-1}$
″	$i_h, i_{h+1}, \ldots, i_m$	″	*smaller than*	$n_h, n_{h+1}, \ldots n_m;$

We shall need to replace the variables

$$\text{C)}\qquad \begin{cases} x_h^{(i_h)}, x_h^{(i_h+1)}, \ldots, x_h^{(n_h-1)}, \\ x_{h+1}^{(i_{h+1})}, x_{h+1}^{(i_{h+1}+1)}, \ldots, x_{h+1}^{(n_{h+1}-1)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_m^{(i_m)}, x_m^{(i_m+1)}, \ldots, x_m^{(n_m-1)} \end{cases}$$

by these new ones,

$$\text{D)}\qquad \begin{cases} x_1^{(n_1)}, x_1^{(n_1+1)}, \ldots, x_1^{(i_1-1)}, \\ x_2^{(n_2)}, x_2^{(n_2+1)}, \ldots, x_2^{(i_2-1)}, \\ \ldots\ldots\ldots\ldots\ldots \\ x_k^{(n_k)}, x_k^{(n_k+1)}, \ldots, x_k^{(i_k-1)}. \end{cases}$$

In order to obtain the substitutions requested for that goal, I differentiate repeated times the equations

$$x_1^{(n_1)} = A_1, x_2^{(n_2)} = A_2, \ldots, x_k^{(n_k)} = A_k$$

and as soon as the n_μ^{th} derivative of the variable x_μ appears in the right part, I substitute for it its value A_μ from 1). As the given functions A_1, A_2, …, A_n only involve the variables A), the quantities D) are in this way also all equal to functions of A) only. I will call these equations, providing the necessary substitutions, *auxiliary equations. In order that the proposed transformation be possible, it must be that the values of the variables C) to eliminate could be reciprocally obtained from the auxiliary equations by which the values of the new variables D) are determined*[2], so that, using the auxiliary equations, the quantities D) are expressed by A) or also the quantities C) by B). Differentiating again the values of quantities

$$x_1^{(i_1-1)}, x_2^{(i_2-1)}, \dots, x_k^{(i_k-1)},$$

expressed by variables A), and substituting the proposed differential equations 1), the values of the quantities

$$x_1^{(i_1)}, x_2^{(i_2)}, \dots, x_k^{(i_k)}$$

appear, expressed by variables A). These, expressed by variables B) with the help of the auxiliary equations, become

$$B_1, B_2, \dots, B_k.$$

Then, let the quantities

$$\begin{array}{cccc} A_{k+1}, & A_{k+2}, & \dots, & A_{h-1} \\ x_h^{(i_h)}, & x_{h+1}^{(i_{h+1})}, & \dots, & x_m^{(i_m)}, \end{array}$$

be respectively changed, with the help of the auxiliary equations, in functions of the variables B):

$$\begin{array}{cccc} B_{k+1}, & B_{k+2}, & \dots, & B_{h-1} \\ B_h, & B_{h+1}, & \dots, & B_m; \end{array}$$

one will have the system of transformed differential equations,

$$\text{2)} \qquad \frac{d^{i_1}x_1}{dt^{i_1}} = B_1, \frac{d^{i_2}x_2}{dt^{i_2}} = B_2, \dots, \frac{d^{i_m}x_m}{dt^{i_m}} = B_m,$$

where the functions B_1, B_2, …, B_m placed on the right contain only derivatives smaller than those on the left.

The fact that one can get back to the proposed equations from the transformed differential equations by differentiations alone, appears by representing the systems 1) and 2) as systems of differential equations of the first order respectively between the variables A) and bewteen the variables B). In the case of *first* order equations, the way for going back from the transformed differential equations to the proposed ones is in fact straightforward. Indeed, n differential equations between t, x_1, x_2, …, x_n, of the following form

$$\text{3)} \qquad u_i = \frac{dx_i}{dt} - X_i = 0,$$

[2]Hence, e. g., the values of quantities D) must involve *all* the variables C). J.

are transformed into some others, between the variables t, y_1, y_2, $\ldots$, y_n, denoting by y_1, y_2, $\ldots$, y_n given functions of the variables t, x_1, x_2, $\ldots$, x_n. Taking

$$4) \qquad Y_i = \frac{\partial y_i}{\partial t} + \frac{\partial y_i}{\partial x_1}X_1 + \frac{\partial y_i}{\partial x_2}X_2 + \cdots + \frac{\partial y_i}{\partial x_n}X_n,$$

gives the transformed differential equations

$$5) \qquad 5) \quad v_i = \frac{dy_i}{dt} - Y_i = 0,$$

where quantities Y_1, Y_2, $\ldots$, Y_n are expressed by the variables t, y_1, y_2, $\ldots$, y_n. To go back from these equations to the proposed ones, one needs to restore in the place of each y_i, the functions to which they are equal, from which the quantities Y_i are changed to the values 4); this being done, as

$$\frac{dy_i}{dt} = \frac{\partial y_i}{\partial t} + \frac{\partial y_i}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial y_i}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial y_i}{\partial x_n}\frac{dx_n}{dt},$$

the equation 5), if we again set $u_i = \frac{dx_i}{dt} - X_i$, is changed in the following,

$$6) \qquad 6) \quad 0 = \frac{\partial y_i}{\partial x_1}u_1 + \frac{\partial y_i}{\partial x_2}u_2 + \cdots + \frac{\partial y_i}{\partial x_n}u_n.$$

Assigning to the index i the values 1, 2, $\ldots$, n one obtains from the preceding equation n equations, linear in each of the u_1, u_2, $\ldots$, u_n and deprived of constant terms; whenever the determinant of these equations

$$\sum \pm \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\cdots\frac{\partial y_n}{\partial x_n}$$

does not vanish, one necessarily has

$$u_1 = 0, u_2 = 0, \ldots, u_n = 0,$$

which is the proposed system of differential equations. And it is known that this determinant does not vanish whenever y_1, y_2, $\ldots$, y_n be independent functions of x_1, x_2, $\ldots$, x_n, that is whenever that one can, from the values of the quantities y_1, y_2, $\ldots$, y_n, expressed by t, x_1, x_2, $\ldots$, x_n get reciprocally the values of the quantities x_1, x_2, $\ldots$, x_n, expressed by t, y_1, y_2, $\ldots$, y_n, or also whenever that no equation can appear between t, y_1, y_2, $\ldots$, y_n alone. Such a supposition is implicitely understood, when considering to transform n differential equations between t, x_1, x_2, $\ldots$, y_x into some others between t, y_1, y_2, $\ldots$, y_n.

In the same way, one may go back to the proposed ones from the differential equations 2) every time that the auxiliary equations, by which the variables D) are expressed by A), are such that one can obtain from them the values of the variables C), which was requested above, or equivalently such that one cannot get from them an equation between the quantities B) alone. This is a necessary and sufficient condition for having the proposed transformation. And, this transformation being done, going back from the transformed equations 2) to the proposed ones, may be conceived in this way: the equations 2) are first reduced to a system of differential equations of the first order, that will be between the variables B), and then, without considering their signification as derivatives, the new quantities C) are introduced in it, at the place of the quantities D), using the same auxiliary equations by which the transformation was done.

Hence, one will get a system of first order equations between the quantities A), which shows that these quantities are equal to the derivatives expressed by their upper indices and that substituting for the quantities A) the derivatives to which they are equal provides the proposed differential equations 1) themselves.

Let Δ be the determinant of the quantities D) formed with respect to the quantities C), which according to what precedes cannot vanish, so that the proposed tranformation may be achieved, let then M and N be the multipliers of the proposed 1) and transformed 2) differential equations. We have according to § 9[3]

$$M = \Delta . N;$$

then according to formula 5)[4] § 14[3] $\frac{d\mathrm{Lg}M}{dt}$ and $\frac{d\mathrm{Lg}N}{dt}$ are defined by the formulæ,

$$\begin{aligned}
\frac{d\mathrm{Lg}M}{dt} &= -\left\{ \frac{\partial A_1}{\partial x_1^{(n_1-1)}} + \frac{\partial A_2}{\partial x_2^{(n_2-1)}} + \cdots + \frac{\partial A_m}{\partial x_m^{(n_m-1)}} \right\} \\
\frac{d\mathrm{Lg}N}{dt} &= -\left\{ \frac{\partial B_1}{\partial x_1^{(i_1-1)}} + \frac{\partial B_2}{\partial x_2^{(i_2-1)}} + \cdots + \frac{\partial B_m}{\partial x_m^{(i_m-1)}} \right\}.
\end{aligned}$$

This provides the memorable formula,

$$\begin{aligned}
&\int \left\{ \frac{\partial A_1}{\partial x_1^{(n_1-1)}} + \frac{\partial A_2}{\partial x_2^{(n_2-1)}} + \cdots + \frac{\partial A_m}{\partial x_m^{(n_m-1)}} \right\} dt \\
7)\qquad = &\int \left\{ \frac{\partial A_1}{\partial x_1^{(i_1-1)}} + \frac{\partial A_2}{\partial x_2^{(i_2-1)}} + \cdots + \frac{\partial A_m}{\partial x_m^{(i_m-1)}} \right\} dt + \mathrm{Lg}\Delta,
\end{aligned}$$

by which one integral is reduced to the other.

[3] This numeration is compatible with that of [Jacobi 3]. T.N.

[4] The formula number is missing in [GW IV] but appears in [Crelle 29]. T.N.

18.

The equations, from which by suitable eliminations the proposed transformation was obtained, were formed by differentiiating repeatedly some particular differential equations in 1) and when, just after each differentiation, some derivative $x_\mu^{n_\mu}$ appeared in their right side, substituting its value A_μ. But it is sometimes interesting to separate the tasks of differentiations and substitutions, so that only by differentiations of the proposed differential equations, without doing any substitutions, one forms a system of equations, from which the transformed differential equations may then be obtained by substitutions or eliminations alone, without doing any more differentiations. If one wishes to perform the transformation in that way, this problem is the foremost to be solved

> “if one differentiates one of the equations 1), e.g. $x_1^{(n_1)} = A_1$, f_1 times, to differentiate all the others in order to obtain *all* the equations, in which appears no derivative of the variables x_2, x_3, …, x_m greater that those that appear in the equation
>
> $$\frac{d^{f_1} x_1^{(n_1)}}{dt^{f_1}} = \frac{d^{f_1} A_1}{dt^{f_1}}.”$$

One may conceive the solution of this problem in the following way.

In the proposed differential equations 1) and in the others that come from their differentiation, we decide to denote the derivatives of x_1 up to the ${n_1}^{\text{th}}$, of x_2 up to the ${n_2}^{\text{th}}$, …, of x_m up to the ${n_m}^{\text{th}}$ in *Lagrange's* way *using indices*; but to denote the greater derivatives that appear differentiating the equations 1) by the usual symbol d. Using this notation in forming the expression $\frac{d^{f_1} A_1}{dt^{f_1}}$, I look for the highest derivative of the quantities

$$x_2^{(n_2)}, x_3^{(n_3)}, \dots, x_m^{(n_m)}.$$

that appears in it. When it is the f_2^{th} of $x_2^{(n_2)}$, I differentiate

$$x_2^{(n_2)} = A_2,$$

f_2 times and look for the highest derivative of

$$x_3^{(n_3)}, x_4^{(n_4)}, \dots, x_m^{(n_m)}.$$

that appears in the expressions

$$\frac{d^{f_1} A_1}{dt^{f_1}}, \frac{d^{f_2} A_2}{dt^{f_2}}.$$

When it is the $f_3{}^{\text{th}}$ of $x_3^{(n_3)}$, I look again for the highest derivative of $x_4^{(n_4)}$, $x_5^{(n_5)}$ etc. appearing in the expressions

$$\frac{d^{f_1}A_1^{n_1}}{dt^{f_1}}, \frac{d^{f_2}A_2}{dt^{f_2}}, \frac{d^{f_3}A_3}{dt^{f_3}}.$$

This being posed, defining new differentiations, we achieve in this way the whole system of differential equations that was requested in the proposed problem

$$8)\quad \begin{array}{l} x_1^{(n_1)} = A_1, \dfrac{dx_1^{(n_1)}}{dt} = \dfrac{dA_1}{dt}, \dfrac{d^2x_1^{(n_1)}}{dt^2} = \dfrac{d^2A_1}{dt^2}, \ldots, \dfrac{d^{f_1}x_1^{(n_1)}}{dt^{f_1}} = \dfrac{d^{f_1}A_1}{dt^{f_1}}, \\ x_2^{(n_2)} = A_2, \dfrac{dx_2^{(n_2)}}{dt} = \dfrac{dA_2}{dt}, \dfrac{d^2x_2^{(n_2)}}{dt^2} = \dfrac{d^2A_2}{dt^2}, \ldots, \dfrac{d^{f_2}x_2^{(n_2)}}{dt^{f_2}} = \dfrac{d^{f_2}A_2}{dt^{f_2}}, \\ \qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.} \\ x_m^{(n_m)} = A_m, \dfrac{dx_m^{(n_m)}}{dt} = \dfrac{dA_m}{dt}, \dfrac{d^2x_m^{(n_m)}}{dt^2} = \dfrac{d^2A_m}{dt^2}, \ldots, \dfrac{d^{f_m}x_m^{(n_m)}}{dt^{f_m}} = \dfrac{d^{f_m}A_m}{dt^{f_m}}. \end{array}$$

I observe that, if, when searching for the highest derivative, many highest derivatives of the same order are to be found, the way of proceeding does not change. Let, e.g., the highest derivatives of the two quantities $x_2^{(n_2)}$ and $x_3^{(n_3)}$ in the expression $\frac{d^{f_1}A_1}{dt^{f_1}}$ be of the same order f_2, the highest derivatives of the remaining variables x_4, x_5 etc. being of a smaller order; in the expression $\frac{d^{f_2}A_2}{dt^{f_2}}$, the hightest derivatives of the quantities $x_3^{n_3}$, $x_4^{n_4}$, …, $x_m^{(n_m)}$ will be of a smaller order than the f_2^{th}, so that the f_2^{th} derivative of the quantity $x_3^{(n_3)}$ will the highest derivative among those of the quantities $x_3^{(n_3)}$, $x_4^{(n_4)}$ etc. appearing in the two expressions $\frac{d^{f_1}A_1}{dt^{f_1}}$ and $\frac{d^{f_2}A_2}{dt^{f_2}}$. For that reason, if among all the derivatives of the quantities $x_2^{(n_2)}$, $x_3^{(n_3)}$, …, $x_m^{(n_m)}$ appearing in the expression $\frac{d^{f_1}A_1}{dt^{f_1}}$, two highest f_2^{th} derivatives of the quantities $x_2^{(n_2)}$ and $x_3^{(n_3)}$ are found, we may continue like this, looking in the expressions

$$\frac{d^{f_1}A_1}{dt^{f_1}}, \frac{d^{f_2}A_2}{dt^{f_2}}, \frac{d^{f_2}A_3}{dt^{f_2}}$$

for all the derivatives of the quantities $x_4^{(n_4)}$, $x_5^{(n_5)}$ etc. that are of the highest order. One proceeds in the same way if in observing the highest order of derivatives, many are found to be of the same highest order. The numbers f_1, f_2, …, f_m decrease continuously, but so that exceptionally some may be mutually equal.

These preliminaries done, let us now investigate *how many times each of the proposed equations must be differentiated to obtain a system of differential equations from which the transformed differential equations 2) may appear.* Let us assume again that the orders i_1, i_2, …, i_k are respectively greater than

n_1, n_2, …, n_k, the remaining i_{k+1}, i_{k+2} etc. respectively equal or smaller than n_{k+1}, n_{k+2} etc. I take the maximum of the numbers

$$i_1 - n_1, i_2 - n_2, \ldots, i_k - n_k;$$

when it is

$$i_1 - n_1 = f_1$$

I form the system of equations 8), according to the given method. When in them

$$f_2 \geq i_2 - n_2, f_3 \geq i_3 - n_3, \ldots, f_k \geq i_k - n_k,$$

what was proposed will be satisfied. In the contrary case, among the numbers $i_2 - n_2$, $i_3 - n_3$, …, $i_k - n_k$ that are respectively greater than the numbers f_2, f_3, …, f_k, I look for the maximum. Let it be e.g. $i_4 - n_4$, differentiating again the proposed differential equations 1), I form by the exposed method the system of all the equations in which one finds no derivative of the variables x_2, x_3, x_5, …, x_m, greater than those that appear in the equation

$$\frac{d^{i_4} x_4}{dt^{i_4}} = \frac{d^{i_4 - n_4} A_4}{dt^{i_4 - n_4}}.$$

I repeat the same operation until in the formed systems of equations, the orders up to which the derivatives of x_1, x_2, …, x_k go are respectively equal or greater than i_1, i_2, …, i_k. By this method, as I produce the appropriate number of equations from which the transformed differential equations 2) may arise by simple eliminations, then all the equations that are formed by the different differentiations will be necessary to perform these eliminations. We may at the same time dispose the equations in such a order that the quantities appearing on the left of each equation do not appear in all the preceding equations. These quantities are the derivatives of the variables x_1, x_2, …, x_m respectively greater than the $(n_1 - 1)^{\text{th}}$, $(n_2 - 1)^{\text{th}}$, …, $(n_m - 1)^{\text{th}}$. When the equations are disposed in this order, they are easilly determined by lower derivatives, that is by quantities A), as each equation, by substitutions of the equations preceding it alone, provides at once the determination of the new quantities. In this way, when the quantities D) by […][5]

[5]The sentence is interrupted at the bottom of the page. T.N.

[Jacobi 2]

[The reduction in normal form of a non normal system of differential equations]

The multiplier of a system of differential equations not being in normal form

LOOKING for the multiplier of isoperimetrical differential equations creates much greater difficulties if the highest derivatives of the functions x, y, z, etc. appearing in the expression U are not of the same order.[6] In this case, the isoperimetrical system of differential equations will not be, as I have always assumed up to now, in such a form that the highest derivative of each dependent variables could be taken as unkowns, the values of which will be determined by the equations themselves.

We reduce then the isoperimetrical differential equations to the form that I have indicated only after some differentiations and eliminations; this makes complicated the search for the value of the multiplier. As a reward for this work, I have obtained all the necessary material to expose with care the reduction in normal form of a non normal system of differential equations. In this search, I came to general propositions that one will see to fill some gap in the theory of ordinary differential equations, a summary of which I will indicate here briefly.

[6] See [GW IV, p. 395]. T.N.

[§. 1.

The order of a system of m differential equations and its fastest normal form reduction are determined by the resolution of the following problem: to transform a given square table of m^2 quantities by adding to each line minimal numbers $\ell_1, \ell_2, \dots, \ell_m$ in order to equip it with a system of transversal maxima.

The resolution is illustrated with an example.]

We call again the independent variable t, its functions, or variables considered as dependent, $x_1, x_2, \dots, x_m$. Let there be m differential equations between these variables:

$$u_1 = 0, \dots, u_m = 0.$$

Let $a_{i,\kappa}$ be the highest order of a derivative of variable x_κ in equation $u_i = 0$. I say that:

1) the order of the proposed system of differential equations, or equivalently the number of arbitrary constants that their complete integration requires, is equal to the *maximum* of all the sums:

$$a_{i_1,1} + a_{i_2,2} + \cdots + a_{i_m,m},$$

if we choose the indices $i_1, i_2, \dots, i_m$ all different the one from the other, in all possible ways, among the indices $1, 2, \dots, m$.

I will denote by $\mathcal{O}$ this *maximum*, that is the system order; $\mathcal{O}$ will be equal to the sum of orders of the highest derivatives of each variable appearing in a normal system to which the proposed system may be reduced, a sum that in the proposed system of differential equations is greater than $\mathcal{O}$.

There exist different normal forms, and always at least two, to which the system may be reduced, reductions that require the help of various differentiations and eliminations. In this field, this proposition is fundamental,

2) among the various ways to differentiate the proposed differential equations to obtain the auxiliary equations with the help of which the proposed system may be reduced to normal form by eliminations alone, there exists a *unique way* that requires *the least number* of differentiations, meaning that, by any other way, some of the proposed differential equations, or all of them, must be differentiated a greater number of times, and none can be differentiated a smaller number of times.

We denote this fastest way under the name of *shortest reduction*. In this reduction, there will always be one or more of the proposed differential equations that will not be differentiated at all, meaning that none of its derivatives contributes to the system of auxiliary equations. So, if we assume that in order to form the auxiliary equations, the equation $u_i = 0$ must be differentiated ℓ_i successive times, among the positive integers :

$$\ell_1, \ell_2, \ldots, \ell_m$$

one or more must be equal to zero. In order to find these numbers, on which fully depends the shortest reduction, one must solve this problem.

Problem.

"Given m^2 arbitrary quantities $a_{i,\kappa}$, where i and κ must take the values $1, 2, \ldots, m$, let $a_{i,\kappa} + \ell_i = p_{i,\kappa}$; to search for m minimal positive or zero quantities $\ell_1, \ell_2, \ldots, \ell_m$ such that we may choose, among the m^2 quantities $p_{i,\kappa}$, m quantities :

$$p_{i_1,1}, p_{i_2,2}, \ldots, p_{i_m,m}$$

placed in different horizontal and vertical series, each of which taking the maximal value among the quantities of the same vertical or being less than no other quantity of the same vertical."

Solution.

I will briefly indicate the main steps of the solution of the given problem. Let us arrange the quantities $a_{i,\kappa}$ in a square table:

$$\begin{array}{lllll} A. & a_{1,1}, & a_{1,2}, & \ldots, & a_{1,m}, \\ & a_{2,1}, & a_{2,2}, & \ldots, & a_{2,m}, \\ & \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ & a_{m,1}, & a_{m,2}, & \ldots, & a_{m,m}. \end{array}$$

If we find a horizontal series of which no term is maximal (by this word, I will always understand that it is not less than any of the others terms) among all those of the same *vertical*, I add to all the terms of this horizontal series the same positive quantity of minimal value such that one of its terms become equal to the maximum of its vertical.

After the indicated preparation, if we have changed the given square to the following,

$$\begin{array}{lllll} B. & b_{1,1}, & b_{1,2}, & \ldots, & b_{1,m}, \\ & b_{2,1}, & b_{2,2}, & \ldots, & b_{2,m}, \\ & \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ & b_{m,1}, & b_{m,2}, & \ldots, & b_{m,m}, \end{array}$$

there will be no horizontal series of the square B, in which there is no maximal term among all these of its vertical. I associate to such a square the following definitions that are to be well remembered.

I call a system of *transversal maxima* a system of quantities $b_{i,\kappa}$ that are all placed in different horizontal and vertical series and *maximal* among all the terms placed in the same vertical.

I take in the square B the maximal number of transversal maxima and, when the same number of transversal maxima may be produced in many ways, I chose arbitrarily one of these systems, the terms of which I mark with an asterisk. The maximal number of these transversal maxima may be or 1, or 2 [7] etc. or m ; if their number is m, the given problem is solved. If this number is smaller than m, I make it so that some horizontal series are increased by minimal numbers, such that a greater number of transversal maxima is found in the obtained new square. Repeating this process, it is necessary that one gets a square in which the number of transversal maxima is m, and at this stage the solution of the problem is found. I say here that *a horizontal series is increased*, if the same positive quantity is added to all its terms.

I call respectively H and V the horizontal and vertical series to which the chosen system of transversal maxima belongs and the other horizontal and vertical series H' and V'. I also mark with an *asterisk* the maximal terms in each vertical of V'[8]. I call *starred maxima* the terms marked with an asterisk.

Let us assume that in the horizontal series h_1 there is a starred maximum to which is equal a term of some horizontal series h_2 placed in the same vertical; that in the series h_2 there is a starred maximum to which is equal a term of the same vertical placed in some horizontal series h_3, [...] etc. If in this way we reach the horizontal series h_α, where h_α denotes one of the series $h_2, h_3, \ldots, h_m$, I will say that *there is a path from h_1 to h_α*. If one says that there is a path from h_1 to h_α, the series h_2, h_3, $\ldots$, $h_{\alpha-1}$ belong to the series of H; the series h_α may belong to the series of H or to those of H'. If there is no path to a series of H' from a horizontal series in which there are two or more starred maxima and if there is no maximal term of some

[7][The applied preparation, by which the square table A has been changed in the table B, makes 2 the minimal value of this number, this value appearing if all the maxima are placed in the same horizontal series and if, moreover, in some vertical all the terms are equal the one with the other. See [Jacobi 1].]

[8]A litteral reading of the original text *terminos in una verticalium V'* is *maximal terms in one of the verticals of V'*. But a correct description of the algorithm requires to mark *all* such terms. T.N.

series of H' in some series of V', this is a criterion certifying that a *maximal* number of transversal maxima has been chosen.

This being posed, I distribute all the horizontal series in three classes.

> I associate to the *first class* of horizontal series the series in which one finds *two or more* starred maxima, and also all the horizontal series to which there is a path from these series; no series of the first class belong to H'.
>
> I associate to the *second class* of horizontal series the series of H not belonging to the first class, from which there is no path to a series of H'.
>
> I associate to the *third class* of horizontal series all the series of H' and the series of H from which there is a path to some series of H'.

The partition of horizontal series being made, I increase all the series belonging to the third class of a same quantity, the smallest such that one term of these series become equal to some starred maxima of the first or second class placed in the same vertical. If this starred maximum belongs to some horizontal series of the second class, this one, in the new square obtained, goes to the third class and nothing else changes in the series repartition[9]. In this case, the operation must be iterated, the new series being transfered from the second to the third class, until one term of the series of the third class become equal to some starred maxima of a series in the *first class.* If it does not happen sooner, this will necessarily happen when all the second class series are gone to the third. And we obtain at the same time a square in which there is a greater number of transversal maxima than in the square B. Then, with a new disposition of the starred maxima and a new distribution of the horizontal series in three classes, a new square must be formed by the same method, in which the number of transversal maxima will be increased again; one must go on until one reaches a square in which there are m transversal maxima. The square found in this way will be obtained from the proposed square A by adding to the horizontal series minimal positive quantities that will be the requested ℓ_1, ℓ_2, $\ldots$, ℓ_m.

Due to the complexity of the rule, it is convenient to present a single example, contained in the following figure. The proposed square itself is A. I have underlined the terms in it which are maximal in their vertical, and have done likewise for the square derived from it. I have denoted the horizontal series by the letters $a, b, \ldots, k$. Among them, b, c, e, f, i, k do not contain any

[9] Up to the fact that all the series from which there is a path to this second class series will go to the third class with it. T.N.

The eight tables of the example appear on fo 2250v; the recto contains the list of the doubles of prime numbers congruent to 1 *modulo* 8*, from* 2018 *to* 20018*. A marginal note of Jacobi in German in the margin of fo 2239v indicates that they must appear in this part of the text. T.N.*

A.

	α	β	γ	δ	ε	ζ	η	ϑ	ι	κ
a	14	23	1	5	73	$\underline{91}$	10	$\underline{34}$	5	$\underline{99}$
b	25	32	2	4	62	81	9	23	4	88
c	14	1	7	16	21	7	13	12	3	77
d	11	$\underline{53}$	$\underline{61}$	4	3	1	12	1	4	91
e	9	21	23	18	27	3	6	9	12	15
f	4	16	18	13	5	12	23	21	14	81
g	25	43	13	16	$\underline{83}$	10	$\underline{91}$	3	7	13
h	$\underline{27}$	7	17	$\underline{37}$	73	8	11	24	$\underline{23}$	22
i	25	12	18	27	32	18	24	23	14	88
k	16	28	30	25	34	10	13	16	19	42

B.

		V	V	V'	V	V'	V'	V	V'	V	V
I	a	14	23	1	5	73	$\underline{91}^*$	10	$\underline{34}^*$	5	$\underline{99}^*$
III	b	$\underline{27}^*$	34	4	6	64	83	11	25	6	90
III	c	$\underline{27}$	14	20	29	34	20	26	25	16	90
I	d	11	$\underline{53}^*$	$\underline{61}^*$	4	3	1	12	1	4	91
III	e	20	32	34	29	38	14	17	20	$\underline{23}^*$	26
III	f	13	25	27	22	14	21	32	30	$\underline{23}$	90
I	g	25	43	13	16	$\underline{83}^*$	10	$\underline{91}^*$	3	7	13
II	h	$\underline{27}$	7	17	$\underline{37}^*$	73	8	11	24	$\underline{23}$	22
III	i	$\underline{27}$	14	20	29	34	20	26	25	16	90
III	k	20	32	34	29	38	14	17	20	$\underline{23}$	46
			19	27	8	19	8	59	4		9

C.

		V	V	V'	V	V'	V'	V	V	V	V
I	a	14	23	1	5	73	$\underline{91}^*$	10	$\underline{34}$	5	$\underline{99}^*$
III	b	$\underline{31}^*$	38	8	10	68	87	15	29	10	94
III	c	$\underline{31}$	18	24	33	38	24	30	29	20	94
I	d	11	$\underline{53}^*$	$\underline{61}^*$	4	3	1	12	1	4	91
III	e	24	36	38	33	42	18	21	24	$\underline{27}^*$	30
III	f	17	29	31	26	18	25	36	$\underline{34}^*$	$\underline{27}$	94
I	g	25	43	13	16	$\underline{83}^*$	10	$\underline{91}^*$	3	7	13
II	h	27	7	17	$\underline{37}^*$	73	8	11	24	23	22
III	i	$\underline{31}$	18	24	33	38	24	30	29	20	94
III	k	24	36	38	33	42	18	21	24	$\underline{27}$	50
			15	23	4	15	4	61	5		5

D.

		V	V	V'	V	V'	V	V	V	V	V
II	a	14	23	1	5	73	$\underline{91}$	10	$\underline{34}$	5	$\underline{99}^*$
II	b	$\underline{35}$	42	12	14	72	$\underline{91}^*$	19	33	14	98
III	c	$\underline{35}^*$	22	28	$\underline{37}$	42	28	34	33	24	98
I	d	11	$\underline{53}^*$	$\underline{61}^*$	4	3	1	12	1	4	91
III	e	28	40	42	$\underline{37}$	46	22	25	28	$\underline{31}^*$	34
II	f	17	29	31	26	18	25	36	$\underline{34}^*$	27	94
I	g	25	43	13	16	$\underline{83}^*$	10	$\underline{91}^*$	3	7	13
III	h	27	7	17	$\underline{37}^*$	73	8	11	24	23	22
III	i	$\underline{35}$	22	28	$\underline{37}$	42	28	34	33	24	98
III	k	28	40	42	$\underline{37}$	46	22	25	28	$\underline{31}$	54
			13	19		10	63	57	1		1

E.

		V	V	V'	V	V'	V	V	V	V	V
III	a	14	23	1	5	73	$\underline{91}$	10	$\underline{34}$	5	$\underline{99}^*$
III	b	35	42	12	14	72	$\underline{91}^*$	19	33	14	98
III	c	$\underline{36}^*$	23	29	$\underline{38}$	43	29	35	$\underline{34}$	25	$\underline{99}$
I	d	11	$\underline{53}^*$	$\underline{61}^*$	4	3	1	12	1	4	91
III	e	29	41	43	$\underline{38}$	47	23	26	29	$\underline{32}^*$	35
III	f	17	29	31	26	18	25	36	$\underline{34}^*$	27	94
I	g	25	43	13	16	$\underline{83}^*$	10	$\underline{91}^*$	3	7	13
II	h	28	8	18	$\underline{38}^*$	74	9	12	25	$\underline{24}$	23
III	i	$\underline{36}$	23	29	$\underline{38}$	43	29	35	$\underline{34}$	25	$\underline{99}$
III	k	29	41	43	$\underline{38}$	47	23	26	29	$\underline{32}$	55
			11	18		9		55			

F.

		V	V	V'	V	V	V	V	V	V	V
III	a	23	32	10	14	82	$\underline{100}$	19	$\underline{43}$	14	$\underline{108}^*$
III	b	44	51	21	23	81	$\underline{100}^*$	28	42	23	107
III	c	$\underline{45}^*$	32	38	$\underline{47}$	52	38	44	43	34	$\underline{108}$
I	d	11	$\underline{53}^*$	$\underline{61}^*$	4	3	1	12	1	4	91
III	e	38	50	52	$\underline{47}$	56	32	35	38	$\underline{41}^*$	44
III	f	26	38	40	35	27	34	45	$\underline{43}^*$	36	103
II	g	25	43	13	16	$\underline{83}$	10	$\underline{91}^*$	3	7	13
II	h	37	17	27	$\underline{47}$	$\underline{83}^*$	18	21	34	33	32
III	i	$\underline{45}$	32	38	$\underline{47}^*$	52	38	44	$\underline{43}$	34	$\underline{108}$
III	k	38	50	34	$\underline{47}$	56	32	35	38	$\underline{41}$	64
			2	9		1		46			

G.

		V	V	V'	V	V	V	V	V	V	V
III	a	24	33	11	15	$\underline{83}$	$\underline{101}$	20	$\underline{44}$	15	$\underline{109}^*$
III	b	45	52	22	24	82	$\underline{101}^*$	29	43	24	108
III	c	$\underline{46}^*$	33	39	$\underline{48}$	53	39	45	$\underline{44}$	35	$\underline{109}$
I	d	11	$\underline{53}^*$	$\underline{61}^*$	4	3	1	12	1	4	91
III	e	39	51	53	$\underline{48}$	57	33	36	39	$\underline{42}^*$	45
III	f	27	39	41	36	28	35	46	$\underline{44}^*$	37	104
II	g	25	43	13	16	$\underline{83}$	10	$\underline{91}^*$	3	7	13
II	h	37	17	27	47	$\underline{83}^*$	18	21	34	33	32
III	i	$\underline{46}$	33	39	$\underline{48}^*$	53	39	45	$\underline{44}$	35	$\underline{109}$
III	k	39	51	35	$\underline{48}$	57	33	36	39	$\underline{42}$	65
			1	8				45			

H.

		α	β	γ	δ	ε	ζ	η	ϑ	ι	κ
S_3	a	25	34	12	16	$\underline{84}$	102^*	21	$\underline{45}$	16	$\underline{110}$
S_2	b	46	53^*	23	25	83	$\underline{102}$	30	44	25	109
S_5	c	47^*	34	40	$\underline{49}$	54	40	46	$\underline{45}$	36	$\underline{110}$
S_1	d	11	$\underline{53}$	61^*	4	3	1	12	1	4	91
S_6	e	40	52	54	$\underline{49}$	58	34	37	40	43^*	46
S_4	f	28	40	42	37	29	36	47	45^*	38	105
S_1	g	25	43	13	16	83	10	91^*	3	7	13
S_4	h	38	18	28	48	84^*	19	22	35	34	33
S_4	i	$\underline{47}$	37	40	$\underline{49}$	54	40	46	$\underline{45}$	36	$\underline{110}$
S_5	k	40	52	36	49^*	58	34	37	40	$\underline{43}$	66

underlined term. Subtracting the terms of b of the underlined terms of their verticals, one gets the differences

$$2, 21, 59, 33, 21, 10, 82, 11, 19, 11,$$

of which 2 is the smallest, so I increase by 2 the series b. The terms of the series c respectively differ from the underlined terms of their verticals by the quantities

$$13, 52, 54, 21, 62, 84, 78, 22, 20, 22;$$

as 13 is the smallest of them, I increase the series c of the quantity 13. In the same way, I deduce the square B by increasing e, f, i, k of the respective quantities $11, 9, 2, 4$ and I denote its construction by the symbol:

$$B. \qquad (a,\ b+2,\ c+13,\ d,\ e+11,\ f+9,\ g,\ h,\ i+2,\ k+4).$$

In the square B, one can assign 6 transversal maxima and not more; the vertical series in which they are placed are surmounted with a V, the remaining with a V'. I denote these maxima with an asterisk. If one finds an underlined term in some series of V', I denote it also with an asterisk. I attach the horizontal series a, d, g, in which two or more starred maxima are found, to the class I. In the 7 verticals to which belong these maxima, on finds no other underlined term, so there is no path from one of these series to some other and a, d and g constitute alone the first class. The series c, f, i, k, as one finds in them no starred term, belong to class III. Then, there is a path from e to the series f and k and from b to c and i; so the series b and e also belong to the third class. For, according to the definition above, there is a path to a horizontal series s from some other s_1, if there is in s an underlined non-starred term and in the same veritical a starred term belonging to the series s_1. As the series a, d, g belong to the first and the series b, c, e, f, i, k to the third class, the series h remains that constitutes the second class. Now, in each vertical series in which there is a starred maximum of a first or second class series, one takes a *nearest lower* term of a series of the third class and one notes the difference of the two terms under the vertical series. From these differences:

$$\begin{array}{llll} 53-34=19, & 61-34=27, & 37-29=8, & 83-64=19, \\ 91-83=\ \ 8, & 91-32=59, & 34-30=4, & 99-90=\ \ 9, \end{array}$$

one takes the smallest 4 ; one obtains the next square by increasing all the series of the third class of the same quantity 4. This square may be denoted by the symbol

$$C. \qquad (a,\ b+6,\ c+17,\ d,\ e+15,\ f+13,\ g,\ h,\ i+6,\ k+8).$$

We see that in the square C, one finds 7 transversal maxima and that a new starred term appeared in series f; this series goes from the third to the second class. I write below the quantities by which the starred terms belonging to series of the first and second class dominate in the square C the *nearest lower* [third class] terms of the same vertical. As the smallest of these quantities is 4, augmenting by 4 all the series of class III, I form the square

$$D. \qquad (a,\ b+10,\ c+21,\ d,\ e+19,\ f+13,\ g,\ h,\ i+10,\ k+12),$$

in which there are already 8 transversal maxima. According to the given rules, the disposition of asterisks must be modified a little in the square D; this being done, the series a, b and h are found to move from classes I, III and II to classes II, II and III. The starred terms of classes I and II exceed the immediately lower terms [of class III and] of the same verticals by the numbers 13, 19, 10, 63, 57, 1, 1; augmenting by their minimum 1 all the series of class III, I deduce the square

$$E. \qquad (a,\ b+10,\ c+22,\ d,\ e+20,\ f+13,\ g,\ h+1,\ i+11,\ k+13),$$

in which the number of transversal maxima is the *same*. The structure of square E does not differ from that of the square D except in the fact that three class II series a, b and f moved to class III. *Viz.*, f and a moved to class III because their starred terms 34 and 99 are equal to the terms of the series i and c placed in the same verticals; regarding b, it moved to class III because its starred term 91 is equal to the term of the series a in the same vertical, that already went to class III. From square E, one deduces by the enunciated rules the square

$$F. \qquad (a+9,\ b+19,\ c+31,\ d,\ e+29,\ f+22,\ g,\ h+10,\ i+20,\ k+22),$$

in which there are 9 transversal maxima; from the square F, one deduces the square

$$G. \qquad (a+10,\ b+20,\ c+32,\ d,\ e+30,\ f+23,\ g,\ h+10,\ i+21,\ k+23),$$

in which there are also 9 transversal maxima; at last, from G comes the requested square

$$H. \qquad (a+11,\ b+21,\ c+33,\ d,\ e+31,\ f+24,\ g,\ h+11,\ i+22,\ k+24),$$

in which there are 10 transversal maxima, which is just the number of horizontal and vertical series. The symbolic representation of square H[10] shows that

$$11, 21, 33, 0, 31, 24, 0, 11, 22, 24$$

are the minimal numbers to be added to the series of the proposed square A, such that a new square is obtained in which maximal terms of the different

[10][the explanation of the signs S_1, S_2 etc. used in the table of square H will the given in the next paragraph.]

vertical series all belong to different horizontal series and that no other such square may be deduced from A by adding to one of the horizontal series a smaller number that the assigned one.

If the number of quantities of which the squares are made is very great, it will not be difficult to invent devices by which the trouble of writing numbers be avoided, for among their great mass, only a few are necessary to form a new square.

[§. 2.

We expose the rule for finding minimal numbers ℓ_1, ℓ_2, ..., ℓ_m, being given an arbitrary system of such numbers or being only given the terms of the square table that provide the transversal maxima after the addition of $\ell_1, \ell_2, \ldots, \ell_m$.

An example of the rule is added.]

AGAIN, let ℓ_i be positive or null quantities and, having set

$$a_{i,\kappa} + \ell_i = p_{i,\kappa},$$

let the square

$$\begin{matrix} p_{1,1} & \cdots & p_{1,m} \\ p_{2,1} & \cdots & p_{2,m} \\ \cdots & \cdots & \cdots \\ p_{m,1} & \cdots & p_{m,m} \end{matrix}$$

be such that the maximal terms of the different vertical series also belong to different horizontal series, so that one may find one or more [complete][11] systems of transversal maxima. Distinguishing one of them by asteriks and underlining the remaining maxima, that are equal to them in each vertical, we shall have this sure criterion by which one may know if such a square is derived from A, which is formed of quantities $a_{i,\kappa}$, by adding minimal positive or null quantities ℓ_i to the horizontal series. One takes indeed the horinzontal series for which $\ell_i = 0$, or equivalently that are the same than in the proposed square A. I shall denote by S_1 such series, of which at least one must exist. One takes the underlined terms in the series of S_1[12] and the starred terms

[11] [That is, composed of m terms.]

[12] I call vertical or horizontal series of a term, the horizontal or vertical series in which it lies. I call *transversal terms*, terms all placed in different horizontal and vertical series; I simply call maxima of different verticals all placed in different horizontals *transversal maxima*. J.

in the vertical of these ones; I denote by S_2 the horizontal series of these starred terms that do not already belong to S_1 itself. We take again the starred terms in the vertical series to which belong the underlined terms of S_2 and I denote by S_3 their horizontal series that are different from S_1 and S_2. *If, in this way, we exhaust all the horizontal series, the square formed by the quantities $p_{i,\kappa}$ is deduced from the proposed one, formed by the quantities $a_{i,\kappa}$, by adding minimal positive or null quantities to its horizontal series.* So, in our example, *all* the horizontal series are referred to the systems S_1, S_2 etc. successively found in the following way:

S_1	S_2	S_3	S_4	S_5	S_6
d	b	a	f	c	e
g			h	k	
			i		

Hence, one may certainly conclude that, in our example, one uses minimal positive quantities to be added to the horizontal series, in order to build this solution of the proposed problem.

The method by which the simplest solution may be deduced from an arbitrary one relies on these same principles, by which we obtained a criterion for whever the problem has been solved in the simplest way, that is by minimal positive quantities ℓ_i. Setting

$$a_{i,\kappa} + h_i = q_{i,\kappa},$$

where the quantities h_i are positive or null, and forming a square of the quantities $q_{i,\kappa}$ in the same way as the square A is formed of quantities $[a_{i,\kappa}]$, we assume that one may take maxima in its different series that are also all placed in different horizontal ones. I denote with asterisks an arbitrary [complete] system of such transversal maxima. By subtracting from all the $q_{i,\kappa}$ the smallest of the quantities h_i, which I call h, one produces a square of which one or more horizontal series is unchanged, [i.e.] are the same as in the square A; I denote again these series by S_1. Underlining then the maximal terms in their vertical different from the starred ones, one deduces successively from the series S_1, according to the rule given above, the systems of horizontal series $S_1, S_2, \ldots, S_\alpha$. If by them we get all the horizontal series, the simplest solution is found, but if there remains a horizontal series in which there is no starred term being placed in the same vertical as an underlined term of the series $S_1, S_2, \ldots, S_\alpha$, I subtract from all these series the same quantity h', the smalest such that one of their starred terms become equal to one of the terms of the same vertical belonging to one of the series $S_1, S_2, \ldots, S_\alpha$ or such that one of them becomes equal to the corresponding series of the square A. So, the number of horizontal series belonging to the sets $S_1, S_2, \ldots, S_\alpha$ will be made greater than in the square formed of the quantities $q_{i,\kappa} - h$. Continuing, if needed, this process, the horizontal series excluded from the sets $S_1, S_2, \ldots, S_\alpha$ will remain fewer and fewer and one comes soon to a square in which the series $S_1, S_2, \ldots, S_\alpha$ contain *all* horizontal series.

If, adding some quantities $h_1, h_2, \ldots, h_m$ to the horizontal series of the square A, one gets a square possessing m transversal maxima, the sum of the terms that occupy in the square A the same place as these transversal maxima in the derived square, will have a maximal value among all the sums of m transversal terms of the square A. Hence the inequality problem,

> a square A formed of m^2 terms being given, to find m transversal terms of A possessing a *maximal* sum,

will have as many solutions as one may find systems of transversal maxima in the derived square. One finds all these systems if we keep in the derived square only the terms being maximal in their vertical, give all the others a null value and form the determinant of all these terms. Indeed, the different terms of this determinant give the different solutions of this problem. One may reciprocally show that any solution of the foregoing inequality problem gives a system of transversal maxima of the derived square. In our example, one needs to form the determinant of the underlined terms of the square H, the remaining terms of this square being given the value zero. This determinant may be successively reduced to the simpler determinants formed of the squares

I.

	α	β	δ	ζ	ι	κ
a				102		110
b		53		10		
c	47		49			110
e			49		43	
i	47		49			110
k			49		43	

,

II.

	α	δ	ζ	ι	κ
a			102		110
c	47	49			110
e		49		43	
i	47	49			110
k		49		43	

,

III.

	α	δ	ι	κ
c	47	49		110
e		49	43	
i	47	49		110
k		49	43	

.

We indicate here the terms of the square by the indication of the vertical and horizontal series to which they belong, the former denoted by the letters a, b, c, etc. and the latter by the letters α, β, γ, etc. In the square H, the terms (d, γ), (g, η) are the only underlined ones in their verticals, the terms (f, ϑ), (g, η), (h, ε), the only underlined ones in their horizontal series. So, all the terms forming the determinant must have the common factor

$$(d, \gamma)(h, \varepsilon)(g, \eta)(f, \vartheta).$$

This factor eliminated, the determinant formed of the quantities of square I remains, that arises from the elimination of the horizontal series d, f, g, h and of the verticals $\gamma, \varepsilon, \eta, \vartheta$. In this square, the term (b, β) is the only non zero term in its vertical, so that, removing this common factor, there remains to search the determinant of the quantities of square II. In this square again, the term (a, ζ), alone in its vertical, is removed; there remains to form the determinant of quantities III

$$\begin{array}{rl} & -(c,\alpha)(e,\delta)(k,\iota)(i,\kappa) - (i,\alpha)(k,\delta)(e,\iota)(c,\kappa) \\ & +(c,\alpha)(k,\delta)(e,\iota)(i,\kappa) + (i,\alpha)(e,\delta)(k,\iota)(c,\kappa) \\ = & -\left\{(c,\alpha)(i,\kappa) - (i,\alpha)(c,\kappa)\right\}\left\{(e,\delta)(k,\iota) - (k,\delta)(e,\iota)\right\}^{13}. \end{array}$$

As it contains four terms, there will be in the square A four systems of transversal maxima possessing a maximal sum, *viz.*

$$\begin{array}{ccl} (b,\beta) & + & (d,\gamma) + (h,\varepsilon) + (a,\zeta) + (g,\eta) + (f,\theta) \\ + & 1) & (c,\alpha) + (e,\delta) + (k,\iota) + (i,\kappa) \\ \text{or} & 2) & (c,\alpha) + (k,\delta) + (e,\iota) + (i,\kappa) \\ \text{or} & 3) & (i,\alpha) + (k,\delta) + (e,\iota) + (c,\kappa) \\ \text{or} & 4) & (i,\alpha) + (e,\delta) + (k,\iota) + (c,\kappa) \end{array}$$

that are numericaly expressed in our example by

$$\begin{array}{cclcl} 32 & + & 61 + 73 + 91 + 91 + 21 & = & 369 \\ + & 1) & 14 + 18 + 19 + 88 & = & 139 \\ \text{or} & 2) & 14 + 25 + 12 + 88 & = & 139 \\ \text{or} & 3) & 25 + 25 + 12 + 77 & = & 139 \\ \text{or} & 4) & 25 + 18 + 19 + 77 & = & 139, \end{array}$$

so that the maximal sum of transversal terms is 508. Reciprocally, if we know in any way some transversal terms of the proposed square A possessing a maximal sum, one obtains by the addition of minimal quantities ℓ_i to the horizontal series of the proposed square A a square in which all the maxima of the different vertical series are also placed in different horizontal series.

I denote of course with asterisks these given transversal terms possessing a maximal sum and I add to the horizontal series quantities such that their starred terms become equal to the maxima of their respective vertical series. I write each augmented series under the remaining ones and compare it to the remaining ones, the preceding ones and the following ones. For this, I indicate the horizontal series denoted by the letters a, b, etc. by the same letters, once the augmentation is made, and I also keep the asterisks of the

[13] The manuscript gives the result up to the sign.

starred terms. The following table will illustrate this way of proceeding in our example. We assume to be given the transversal terms possessing a maximal sum

$$\begin{array}{cccccccccc} (a,\zeta), & (b,\beta), & (c,\alpha), & (d,\gamma), & (e,\delta), & (f,\theta), & (g,\eta), & (h,\varepsilon), & (i,\kappa), & (k,\iota), \\ 91 & 32 & 14 & 61 & 18 & 21 & 91 & 73 & 88 & 19. \end{array}$$

		α	β	γ	δ	ε	ζ	η	ϑ	ι	κ
(1)	a	14	23	1	5	73	91*	10	34	5	99
(2)	b	25	32*	2	4	62	81	9	23	4	88
(3)	c	14*	1	7	16	21	7	13	12	3	77
(4)	d	11	53	61*	4	3	1	12	1	4	91
(5)	e	9	21	23	18	27	3	6	9	12	15
(6)	f	4	16	18	13	5	12	23	21*	14	81
(7)	g	25	43	13	16	83	10	91*	3	7	13
(8)	h	27	7	17	37	73*	8	11	24	23	22
(9)	i	25	12	18	27	32	18	24	23	14	88*
(10)	k	16	28	30	25	34	10	13	16	19*	42
(11)	b	46	53*	23	25	83	102	30	44	25	109
(12)	a	25	34	12	16	84	102*	21	45	16	110
(13)	c	46*	33	39	48	53	39	45	44	35	109
(14)	e	39	51	53	48*	57	33	36	39	42	45
(15)	f	28	40	42	37	29	36	47	45*	38	105
(16)	h	38	18	28	48	84*	19	22	35	34	33
(17)	i	47*	34	40	49	54	40	46	45	36	110*
(18)	c	47	34	40	49*	54	40	46	45	36	110
(19)	e	40	52	54	49	58	34	37	40	43	46
(20)	k	40	52	54	49	58	34	37	40	43*	66

In the vertical ζ, the starred term is itself maximal, so at first the [horizontal series a][14] does not change; in the vertical β, the maximum is 53, so the horizontal b must be written below, augmented by the number 21, which forms line (11). Going back to the first term, we find in the series ζ the maximum 102, so a must be increased by 11, which provides line (12). Progressing up to term (c, α), we find in α the maximum 46 placed on line (11), so c must be augmented by [32][15], which provides line (13). In the same way, the series d and g remain unchanged, I respectively increase the series e, f, h, i by numbers $30, 24, 11, 22$, which provides lines $(14), (15), (16), (17)$. Then, one finds in line (17) the term 47 in the vertical α, greater than the starred term

[14]Jacobi wrote: *series horizontalibus non mutatur*; I follow Borchardt's correction: *series horizontalis a non mutatur*. T.N.

[15]The manuscript has *13*. T.N.

of the same vertical 46 placed in line (13), I add 1 to line (13), which provides line (18). In (17) and (18), the term 49 of the series δ is greater than the starred term of this same vertical, placed in (14), so I increase line (14) itself by one, which produces line (19). At last, I proceed to the term $(k, \iota) = 19$; and, as the maximum of the vertical ι is 43, placed in (19), I form line (20) by adding 24 to the series k. By this, the work will be achived. We have indeed found series

	a,	b,	c,	d,	e,	f,	g,	h,	i,	k,
forming lines	(12),	(11),	(18),	(4),	(19),	(15),	(7),	(16),	(17),	(20),

the starred terms of which are maximal in their verticals, as was requested. We see that these series constitute the square H found above by another method.

By using what precedes, one gets a new solution of the problem proposed above: if the question is, some quantities being added to the horizontal series of the square A, to get a square, of which the maximal terms in their verticals all belong to different horizontal series and some such quantities are known, how to find minimal ones. For, as according to the assumption made, one knows a square derived of A possessing m transversal maxima, one also knows in A m transversal terms possessing a maximal sum. These being known, following the rule given above, one easily derives from A, by the addition of minimal positive quantities, a square possessing m transversal maxima. At the same time, we see how, a system of transversal terms of A possessing a maximal sum being known, we easily find all the remaining ones. For, knowing such a system, we see that one deduces from A a square possessing m transversal maxima; in which, if we only keep the maximal terms in each vertical, the remaining being made equal to zero, every non zero term of the determinant of the square formed by these quantities provides every system of transversal maxima, and so every system of transversal terms of the square A possessing a maximal sum; the terms of both systems occupy indeed the same places in the two squares.

[§. 3.

The solution of the problem related to a square table of m^2 quantities is applied to a system of m differential equations. The normal form or forms to which the proposed system may be brought back by a shortest reduction.
Other reductions to normal form.]

THE PROPOSED DIFFERENTIAL EQUATIONS

$$u_1 = 0, u_2 = 0, \ldots, u_m = 0,$$

had to be respectively differentiated ℓ_1, ℓ_2, ..., ℓ_m times to be brought back to another system in normal form, by a shortest reduction. The numbers ℓ_1, ℓ_2, ..., ℓ_m are the same as those whose computation I described above. These being fully determined, the system of auxiliary differential equations requested for the shortest reduction, formed of these derivatives, will be also fixed. But, most of the time, there are many different normal forms to which the proposed differential equations may be reduced with this system of auxiliary equations. Again let $a_{i,\kappa}$ be the order of the highest derivative of the variable x_κ that appears in equation $u_i = 0$ and let us again place the quantities $a_{i,\kappa}$ into a square A, of which the terms $a_{i,1}, a_{i,2}, \ldots, a_{i,m}$ constitute the i^{th} horizontal series and the terms $a_{1,\kappa}, a_{2,\kappa}, \ldots, a_{m,\kappa}$ the κ^{th} vertical. We take in the square A an arbitrary system of transversal terms possessing a maximal sum

$$a_{\alpha_1,1}, a_{\alpha_2,2}, \ldots, a_{\alpha_m,m},$$

the proposed differential equations may be brought back by a shortest reduction to these, in normal form:

$$x_1^{(a_{\alpha_1,1})} = X_1, x_2^{(a_{\alpha_2,2})} = X_2, \ldots, x_m^{(a_{\alpha_m,m})} = X_m,$$

where the derivatives of the different variables appearing on the left are the highest that appear in the reduced system, of which the functions X_1, X_2, ..., X_m placed on the right may be assumed to be absolutely free. And one will have as many different such systems, to which the proposed differential equations may be brought back by a shortest reduction, as one has systems of transversal terms possessing a maximal sum in the square A.[16] These are found as explained above. Having formed the auxiliary equations used for a shortest reduction, we assume that we find the highest derivative of the variable x_κ in the proposed equations $u_i = 0, u_{i_1} = 0$, etc. or in the auxiliary

[16]This proposition does not stand in all cases, as shown by the example $x_1'' + x_2'' + x_3'' = 0$, $x_2' = 0$, $x_2 + x_3 = 0$. T.N.

equations derived from them by iterated differentiations; in the places of the square that belong to the κ^{th} vertical series and to the i^{th}, i_1^{th}, etc. horizontal series, I put a unit or any other [non zero] quantity, and I put zero in the other places of the κ^{th} vertical. This being done for each of the variables x_κ, I form the determinant of the terms of this square. One of its non-zero term, if it is formed of quantities of the first, second, ..., m^{th} vertical respectively belonging to the ${\alpha_1}^{\text{th}}$, ${\alpha_2}^{\text{th}}$, ..., ${\alpha_m}^{\text{th}}$ horizontal series, will give a normal form in which the highest derivatives of the variables $x_1, x_2, \dots, x_m$ are respectively the same as in the equations

$$u_{\alpha_1} = 0, u_{\alpha_2} = 0, \dots, u_{\alpha_m} = 0.$$

As for another term of the determinant one has another succession of the indices α_1, α_2, ..., α_m, each normal form to which the proposed equations may be reduced by a shortest reduction, is given for that reason by each non-vanishing term of the determinant.

The method by which, adding minimal positive quantities to horizontal series, one deduces a square in which all the maxima of verticals are placed in different horizontal series will be made easier if one knows in some way a system of m transversal terms of the square possessing a maximal sum. By this easier method, one may find how many times each proposed equation must be differentiated in a shortest reduction in order to form the auxiliary equations, whenever one will have in any way some normal form to which the proposed equations are reduced by such a reduction. Such a normal form is known if the proposed differential equations are such that in each of them a derivative of a different variable reaches the highest order. For, indeed, these derivatives of the different variables, the highest in the different proposed equations, will also be the highest in a normal form, to which the proposed differential equations can be brought back by a shortest reduction. One easilly sees that the orders of these derivatives constitute in the square A a system of m transversal terms.

Let us assume, e.g. that 10 differential equations $u_1 = 0$, $u_2 = 0$, ..., $u_{10} = 0$ between the independent variable t and the 10 dependent variables $x_1, x_2, \dots, x_{10}$ are given and that the numbers of the square A in our example indicate the highest orders up to which the derivatives of each dependent variable in the different equations go, so that e.g. the highest derivatives of variables $x_1, x_2, \dots, x_{10}$ that appear in [equation] $u_1 = 0$ are

$$x_1^{(14)}, x_2^{(23)}, x_3^{(1)}, x_4^{(5)}, x_5^{(73)}, x_6^{(91)}, x_7^{(10)}, x_8^{(34)}, x_9^{(5)}, x_{10}^{(99)}.$$

As the last square H is deduced from the proposed one A by adding to the horizontal series the numbers

$$11, 21, 33, 0, 31, 24, 0, 11, 22, 24,$$

a shortest reduction is obtained with the auxiliary equations formed by differentiating the proposed equations

$$u_1 = 0, u_2 = 0, u_3 = 0, u_5 = 0, u_6 = 0, u_8 = 0, u_9 = 0, u_{10} = 0$$

respectively $11, 21, 33, 31, 24, 11, 22, 24$ times, the two equations $u_4 = 0$ and $u_7 = 0$ not being used to form auxiliary equations. With these auxiliary equations, the proposed equations may be reduced by simple eliminations to *four* different normal forms. In all these ones, among the highest derivatives that are to be expressed by lower order derivatives and the variables themselves, one finds, as explained above

$$x_2^{(32)}, x_3^{(61)}, x_5^{(73)}, x_6^{(91)}, x_7^{(91)}, x_8^{(21)};$$

then in the *first* normal form: $x_1^{(14)}, x_4^{(18)}, x_9^{(19)}, x_{10}^{(88)}$;
second $x_1^{(14)}, x_4^{(25)}, x_9^{(12)}, x_{10}^{(88)}$;
third $x_1^{(25)}, x_4^{(25)}, x_9^{(12)}, x_{10}^{(77)}$;
fourth $x_1^{(25)}, x_4^{(18)}, x_9^{(19)}, x_{10}^{(77)}$.

So, the complete integration of the 10 proposed differential equations requires 508 arbitrary constants, this number being the sum of the orders of the highest derivatives of the different variables in the normal forms. All the highest derivatives appear in the proposed differential equations, but except $x_3^{(61)}, x_6^{(91)}, x_7^{91}$, they are not the highest.

We consider an arbitrary reduction and, among all the [auxiliary and proposed] differential equations, we choose the m highest derivatives of the proposed equations; some will be among the proposed equations, if they are not used to form auxiliary equations by differentiation. In each of these m equations, we gather the orders of the highest derivatives of each variable and we dispose them in square in the usual way: it may be proved that in such a square the maxima of the different vertical series are necessarily also placed in different horizontal series. And by the rules given above, we can go back from such a square to some other, deduced from A, by using minimal ℓ_i numbers. Hence, *from an arbitrary normal form reduction of the proposed differential equation, one may deduce a shortest one.*

[§. 4.

Reduction of the proposed system to a single differential equation. A rule for finding the reduction is given and illustrated with an example. An elegant form under which the rule may be expressed.]

A system of differential equations may in general be reduced to a single differential equation in two variables. Let these two variables be the independent variable t and the dependent one x_1; this unique differential equation must be completed by other equations, by which the remaining variables are expressed as functions of t, x_1 and derivatives of x_1, not reaching the order of the differential equation between t and x_1. As it is usual that this type of normal form be considered before others by mathematicians, I will indicate how many times each proposed differential equations $u_1 = 0$, $u_2 = 0$, $\ldots$, $u_m = 0$ must be differentiated to produce the differential equations necessary for this reduction.

We assume that the proposed differential equations $u_1 = 0$, $u_2 = 0$, $\ldots$, $u_m = 0$ must be respectively differentiated ℓ_1, ℓ_2, $\ldots$, ℓ_m times in order to obtain the auxiliary equations required for a shortest reduction. I have taught above how these numbers $\ell_1, \ell_2, \ldots, \ell_m$ are found. Adding ℓ_1, ℓ_2, $\ldots$, ℓ_m to the horizontal series of the square A, I form another square A', in which I distinguish with an asterisk some [complete] system of transversal maxima and I underline the remaining maxima of the various verticals. If all variables are to be eliminated, except the independent variable t and the dependent variable x_κ, I look for the starred term of the κ^{th} vertical, which is in the i^{th} horizontal series; in the i^{th} horizontal series, I look for the underlined terms, in the vertical of each of them, for the starred terms, in the horizontal series of which again for the underlined terms, and so on. In this framework, it is useless to go back again to the starred terms already considered.

Continuing this operation, as far as possible, I will say that all the horizontal series to which one comes by this process are *attached* to the i^{th} from which we have started. I increase these series as well as the i^{th} of a same quantity, the smallest such that one of their terms being neither starred nor underlined becomes equal to a starred term in its vertical. The horizontal series of this term being added to the series attached to the i^{th} series, I increase again the i^{th} series and all these being attached to it, the number of which have just been increased, by the smallest quantity such that one of their terms being neither starred nor underlined becomes equal to a starred term of its vertical; this being done, the number of series attached to the i^{th}

increases again; and so I increase again and again the number of these series, until one comes to a square A'' in which all horizontal series are attached to the i^{th}.

I deduce then from A'' a square A''' by increasing [all] the horizontal series by a same quantity, such that the term of the i^{th} horizontal series, belonging to the κ^{th} vertical be made equal *to the greatest sum that a system of m transversal terms of the square A may have.* The numbers by which the horizontal series of the square A must be increased so that the square A''' appears indicate how many times each of the proposed differential equations must be differentiated to produce the auxiliary equations needed to get, by eliminations alone, a differential equation between the variables t and x_κ alone and the other equations by which the remaining variables are expressed as functions of t, x_κ and derivatives of x_κ.

In our example, A' is the square denoted by H. We assume that the κ^{th} vertical is the series ζ, whose starred term 102 belongs to the horizontal series a, in which are placed the underlined terms 84, 45, 110 belonging to the verticals ε, ϑ, κ, whose starred terms belong to the series h, f, i in which one has the underlined terms 47 and 49, belonging to verticals α and δ (I do not use 45, because its vertical was already considered); the starred terms of verticals α and δ belong to series c and k, in this last, we have the underlined term 43, belonging to the vertical ι, whose starred term is placed in e, a series that contains the unique underlined term 49, whose vertical was already considered. Hence, we find the series attached to a: h, f, i, c, k, e. Increasing all the series a, h, f, i, c, k, e by one , b is added to the series attached to a, for with this increment, the term 52 of series e or k, belonging to the vertical β becomes 53, which number is equal to the starred term of the vertical β that belongs to the horizontal b. I increase again the series a, h, f, i, c, k, e, b by the number 6, this being done, d is added to the series attached to a; at last, I increase by the number 37 all the series except g, so that g itself joins the series attached to a. Hence the square A'' is obtained from the series of A' or H:

a, h, f, i, c, k, e by adding 44
from the series b by adding 43
from the series d by adding 37,

the series g staying unchanged. As $102 + 44 = 146, 508 - 146 = 362$, the horizontal series of the square A'' must be increased by the number 362 to obtain A'''. Denoting, as above, the square A' by

$$A' \quad (a+11,\ b+21,\ c+33,\ d,\ e+31,\ f+24,\ g,\ h+11,\ i+22,\ k+24),$$

we obtain for the squares A'' and A'''

$$A'' \quad (a+55, b+64, c+77, d+37, e+75, f+68, g, h+55, i+66, k+68)$$

$A'''\,(a+417, b+426, c+439, d+399, e+437, f+430, g+362, h+417, i+428, k+430).$

So, in our example, to eliminate all variables except t and x_6 from the 10 proposed differential equations, they must be differentiated respectively 417, 426, 439, 399, 437, 430, 362, 417, 428, 430 times to produce the requested auxiliary differential equations.

By the same method, we obtain the squares A'', in which all the horizontal series are respectively attached to each of the series $a, b, c, \ldots, k$, by adding to the series of A'

$$\begin{array}{l}
a, h, f, i, c, k, e,\ \ +44;\quad b\ \ +43;\quad d\ \ +37;\quad g\ \ 0,\\
b, a, h, f, i, c, k, e,\ \ +44;\quad d\ \ +37;\quad g\ \ 0,\\
c, k, f, i, e,\ \ +44;\quad b, a, h\ \ +43;\quad d\ \ +37;\quad g\ \ 0,\\
d, b, a, c, e, f, k, h, i, k,\ \ +44;\quad g\ \ 0,\\
e, k\ \ +45;\quad b, a, h, f, i, c\ \ +44;\quad d\ \ +38;\quad g\ \ 0,\\
f\ \ +44;\quad e, i, c, k\ \ +39;\quad b, a, h,\ \ +38;\quad d\ \ +32; g\ \ 0\\
g\ \ +9;\quad h\ \ +8;\quad k, e\ \ +7;\quad b, a, f, i, c\ \ +6;\quad d\ \ 0,\\
h\ \ +46;\quad k, e\ \ +45;\quad b, a, f, i, c\ \ +44;\quad d\ \ +38; g\ \ 0,\\
i, c, k, f, e\ \ +44;\quad b, a, h\ \ +43;\quad d\ \ +37;\quad g\ \ 0,\\
k, e\ \ +45;\quad b, a, h, f, i, c\ \ +44;\quad d\ \ +38;\quad g\ \ 0.
\end{array}$$

We see that the third and the ninth, the fifth an the tenth squares are obtained from A' in the same way. We indicate how the squares A'' are deduced from the proposed square A by the following tables:

	$S.$					A''					
x_6	146	$(a+55,$	$b+64,$	$c+77,$	$d+37,$	$e+75,$	$f+68,$	$g,$	$h+55,$	$i+66,$	$k+68),$
x_2	97	$(a+55,$	$b+65,$	$c+77,$	$d+37,$	$e+75,$	$f+68,$	$g,$	$h+55,$	$i+66,$	$k+68),$
x_1	91	$(a+54,$	$b+64,$	$c+77,$	$d+37,$	$e+75,$	$f+68,$	$g,$	$h+55,$	$i+66,$	$k+68),$
x_3	105	$(a+55,$	$b+65,$	$c+77,$	$d+37,$	$e+75,$	$f+68,$	$g,$	$h+55,$	$i+66,$	$k+68),$
x_9	88	$(a+55,$	$b+64,$	$c+77,$	$d+38,$	$e+76,$	$f+68,$	$g,$	$h+55,$	$i+66,$	$k+68),$
x_8	89	$(a+49,$	$b+59,$	$c+72,$	$d+32,$	$e+70,$	$f+68,$	$g,$	$h+49,$	$i+61,$	$k+63),$
x_7	100	$(a+17,$	$b+27,$	$c+39,$	$d,$	$e+38,$	$f+30,$	$g+9,$	$h+19,$	$i+28,$	$k+31),$
x_5	130	$(a+55,$	$b+65,$	$c+77,$	$d+38,$	$e+76,$	$f+68,$	$g,$	$h+57,$	$i+66,$	$k+69),$
x_{10}	154	$(a+54,$	$b+64,$	$c+77,$	$d+37,$	$e+75,$	$f+68,$	$g,$	$h+54,$	$i+66,$	$k+68),$
x_4	94	$(a+55,$	$b+65,$	$c+77,$	$d+38,$	$e+76,$	$f+68,$	$g,$	$h+55,$	$i+66,$	$k+69).$

In the first, second, …, tenth horizontal series of the square A' or H, we have the starred terms

102, 53, 47, 61, 43, 54, 91, 84, 110, 49,

belonging to the

sixth, second, first, third, ninth, eighth, seventh, fifth, tenth, fourth

verticals. Adding respectively to these terms

44, 44, 44, 44, 45, 44, 9, 46, 44, 45,

the numbers

146, 97, 91, 105, 88, 89, 100, 130, 154, 94,

appear, which I have placed, denoted by S, in a marginal column, together with the variables corresponding to the various verticals.

In some square A'', let S be the starred term of the horizontal series to which the remaining ones are attached: one will be able to go from S to any other starred term by a continuous path from a starred term to an underlined one of the same horizontal and from an underlined term to a starred one of the same vertical. We present below, e.g., the first square obtained

A'' $(a+55, b+64, c+77, d+37, e+75, f+68, g, h+55, i+66, k+68)$

or

	α	β	γ	δ	ε	ζ	η	ϑ	ι	κ
a					128	146*		89		154
b		96*								
c	91*			93				89		154
d			98*							
e		96	98	93					87*	
f							91	89*		
g							91*			
h					128*					
i	91			93				89		154*
k		96	98	93*					87	

in which I only put the starred terms and the underlined ones, that is those equal to the starred terms of the same vertical (omitting to underline them). In this square, all the horizontal series follow from a, the starred term of which is 146. From it, one goes to the other starred terms:

$$146, 154, 93, 96; 146_\zeta, 154, 91_\alpha; 146, 154, 93, 98; 146, 154, 93, 87;$$
$$146, 154, 89; 146, 154, 89, 91_\eta; 146, 128; 146, 154; 146, 154, 93.$$

Two terms T and U placed one after the other are starred terms such that a term from the horizontal series of T, placed in the vertical of U be equal to U, that is underlined, which is the indicated passing rule.

If we suppress from the square A'' the vertical series of the term S, from which we have started and an arbitrary horizontal, we will easily determine in the remaining square, a system of transversal maxima. We denote by $\widehat{TU}$

the term equal to U in the horizontal series of T and placed in the vertical of U and we assume that the starred term of the suppressed horizontal series is $S^{(f)}$; so according to the given rule, we go from S to $S^{(f)}$ by the intermediate starred terms $S', S'', \ldots, S^{(f-1)}$. This being set, the remaining starred terms of the proposed square A'', will also be the transversal maxima of the remaining square; but instead of S, S', $\ldots$, $S^{(i)}$, $\ldots$, one needs to take the terms

$$\overset{\frown}{SS'}, \overset{\frown}{S'S''}, \overset{\frown}{S''S'''}, \ldots, \overset{\frown}{S^{(f-1)}S^{(f)}},$$

which are equal to S', S'', $\ldots$, $S^{(f)}$. From this proposition, in the squares that remain, having suppressed the vertical series of the term S together with an arbitrary horizontal, the sums of the transversal maxima will be the same, *viz.* the sum of the proposed square A'' decreased by S.

We consider an arbitrary square A''_κ, in which the starred term of the horizontal series to which all the remaining ones are attached, belongs to the κ^{th} vertical, which term I denote by S_κ. This square A''_κ is the one that must be formed when it is proposed to eliminate all variables except t and x_κ. We assume then that the square A''_κ comes from the addition of the quantities

$$h_1^{(\kappa)}, h_2^{(\kappa)}, \ldots, h_m^{(\kappa)}$$

to the horizontal series of the square A. We call $\mathcal{O}$ the order of the considered system of differential equations, that is the maximal sum of transversal terms in the square A, and let $\mathcal{O} - S_\kappa = P_\kappa$; according to the above results on the formation of auxiliary equations necessary for the proposed elimination, the i^{th} differential equation is to be differentiated $P_\kappa + h_i^{(\kappa)}$ times. To which number $P_\kappa + h_i^{(\kappa)}$ one may attribute a remarkable meaning. In the square A''_κ the sum of the transversal maxima, that is the maximal sum of the transversal terms, is

$$\mathcal{O} + h_1^{(\kappa)} + h_2^{(\kappa)} + \cdots + h_m^{(\kappa)},$$

so, if we suppress the κ^{th} vertical series and the i^{th} horizontal, the maximal sum of the transversal terms in the remaining square will be, according to the proposition found,

$$\mathcal{O} - S_\kappa + h_1^{(\kappa)} + h_2^{(\kappa)} + \cdots + h_m^{(\kappa)} = P_\kappa + h_1^{(\kappa)} + h_2^{(\kappa)} + \cdots + h_m^{(\kappa)}$$

and for this reason, if we suppress from the square A the κ^{th} vertical series and the i^{th} horizontal, the maximal sum of the transversal terms in the remaining square will be $P_\kappa + h_i^{(\kappa)}$. Hence we have found this solution to the problem enunciated here:

Problem.

Between the independent variable t and the m dependent variables x_1, x_2, ..., x_m, let there be the differential equations

$$u_1 = 0, u_2 = 0, \dots, u_m = 0;$$

if it is asked to reduce them to a single differential equation between t and x_κ, new auxiliary differential equations are to be formed, by differentiating the proposed differential equations, with the help of which a differential equation between t and x_κ is obtained by simple eliminations, without any further differentiation; we search how many times the equation $u_i = 0$ must be differentiated to form this system of auxiliary equations.

Solution.

A square containing m vertical series and as many horizontal series is formed; in the α^{th} vertical and the a^{th} horizontal, we place the order of the highest derivative of the variable x_α that appears in equation $u_a = 0$. Having suppressed from this square the i^{th} horizontal series and the κ^{th} vertical series, we look for the maximal sum $\sigma_{i,\kappa}$ that $m-1$ of its terms placed in different horizontal series and different vertical series may reach: in order to form the system of auxiliary equations with the use of which the differential equation between t and x_κ will be obtained, the equation $u_i = 0$ must be differentiated $\sigma_{i,\kappa}$ times. The sought number $\sigma_{i,\kappa}$ will also be equal to the order of the differential equations that appear if we withdraw $u_i = 0$ from the proposed equations and replace x_κ by a constant.

The numbers $\sigma_{i,\kappa} = P_\kappa + h_i^{(\kappa)} = \mathcal{O} - S_\kappa + h_i^{(\kappa)}$ are provided by the square A''_κ, the construction of which from A' I explained above. Above I gave the values of the numbers S_κ and $h_i^{(\kappa)}$ corresponding to the proposed example; with these numbers, one hundred inequality problems are solved, *viz.* suppressing at the same time from the proposed table a vertical and a horizontal series, to find in the remaining hundred squares the maximal sum of transversal terms. Transversal terms possessing the maximal sum are easily found in each of these squares, if one goes back to what I explained above about the way of going from a term S of the square A'' to another arbitrary starred term $S^{(f)}$ by intermediate starred terms.

[§. 5.

One determines the condition that lowers the order of the proposed system of differential equations.]

It may happen, in some special cases, that the order of the systems of differential equations do not reach the value of the maximal sum of the transversal terms of the square A. This particular property is indicated by a precise mathematical condition. Again let $x_\kappa^{(a_{i,\kappa})}$ be the highest derivative of variable x_κ that one finds in the equation $u_i = 0$; I form the determinant of partial derivatives

$$\begin{array}{llll}
\frac{\partial u_1}{\partial x_1^{(a_{1,1})}}, & \frac{\partial u_1}{\partial x_2^{(a_{1,2})}}, & \ldots, & \frac{\partial u_1}{\partial x_m^{(a_{1,m})}}, \\
\frac{\partial u_2}{\partial x_1^{(a_{2,1})}}, & \frac{\partial u_2}{\partial x_2^{(a_{2,2})}}, & \ldots, & \frac{\partial u_2}{\partial x_m^{(a_{2,m})}}, \\
\ldots\ldots & & & \\
\frac{\partial u_m}{\partial x_1^{(a_{m,1})}}, & \frac{\partial u_m}{\partial x_2^{(a_{m,2})}}, & \ldots, & \frac{\partial u_m}{\partial x_m^{(a_{m,m})}}
\end{array}$$

and I only keep its terms

$$\pm \frac{\partial u_1}{\partial x_{i'}^{(a_{1,i'})}} \frac{\partial u_2}{\partial x_{i''}^{(a_{2,i''})}} \cdots \frac{\partial u_1}{\partial x_{i^{(m)}}^{(a_{m,i^{(m)}})}}$$

in which the sum of orders

$$a_{1,i'} + a_{2,i''} + \cdots + a_{m,i^{(m)}}$$

reaches the *maximal* value $\mathcal{O}$; I suppress all the other terms from the determinant. I denote by ∇ the sum of the remaining terms that is in some way a truncated determinant;

$$\nabla = 0$$

will be the condition for the proposed system of differential equations to have a special structure lowering its order. If ∇ does not vanish, the order of the system always reaches the value $\mathcal{O}$ assigned by the general theory that I have exposed. I call the quantity ∇ *the determinant of the proposed system of differential equations.* In our example,

$$\nabla = \frac{\partial u_1}{\partial x_6^{(91)}} \cdot \frac{\partial u_2}{\partial x_2^{(32)}} \cdot \frac{\partial u_4}{\partial x_3^{(61)}} \cdot \frac{\partial u_6}{\partial x_8^{(21)}} \cdot \frac{\partial u_7}{\partial x_7^{(91)}} \cdot \frac{\partial u_8}{\partial x_5^{(73)}}$$
$$\times \left\{ \frac{\partial u_3}{\partial x_1^{(14)}} \cdot \frac{\partial u_9}{\partial x_{10}^{(88)}} - \frac{\partial u_9}{\partial x_1^{(25)}} \cdot \frac{\partial u_3}{\partial x_{10}^{(77)}} \right\} \left\{ \frac{\partial u_5}{\partial x_4^{(18)}} \cdot \frac{\partial u_{10}}{\partial x_9^{(19)}} - \frac{\partial u_{10}}{\partial x_4^{(25)}} \cdot \frac{\partial u_5}{\partial x_9^{(12)}} \right\}.$$

The four terms of this expression, that appear once the braces expanded, correspond to the four systems of transversal terms of the square A possessing a maximal sum that I have investigated above. So, every time that in our example none of the equalities

$$\frac{\partial u_3}{\partial x_1^{(14)}} \cdot \frac{\partial u_9}{\partial x_{10}^{(88)}} - \frac{\partial u_9}{\partial x_1^{(25)}} \cdot \frac{\partial u_3}{\partial x_{10}^{(77)}} = 0,$$
$$\frac{\partial u_5}{\partial x_4^{(18)}} \cdot \frac{\partial u_{10}}{\partial x_9^{(19)}} - \frac{\partial u_{10}}{\partial x_4^{(25)}} \cdot \frac{\partial u_5}{\partial x_9^{(12)}} = 0,$$

is satisfied, the system of equations is of order 508, or also their complete intergration requires 508 arbitrary constants. But if one of the two preceding equations is satisfied, the order of the system is always lower than the value 508. In this case, the proposed differential equations require a preparation that must be made before manipulating them. The non-vanishing of the determinant of the proposed differential equations is a condition without which one cannot deduce the order of the system. Every time the problem of determining transversal terms of the square A possessing a maximal sum has a *unique* solution, the order of the proposed system of differential equations is equal to this maximal sum and it cannot happen that it become smaller. Indeed, the determinant contains a single term and cannot vanish.