The Euler-Poisson equations; an elementary approach to partial integrability conditions. Goryachev-Chaplygin and beyond

Jean Moulin-Ollagnier, Sasho Ivanov Popov, Jean-Marie Strelcyn

JEAN MOULIN-OLLAGNIER, SASHO IVANOV POPOV and
JEAN-MARIE STRELCYN

The Euler-Poisson equations; an elementary approach to partial integrability conditions. Goryachev-Chaplygin and beyond.

Jean Moulin-Ollagnier
Laboratoire LIX, Ecole Polytechnique,
F 91128 Palaiseau Cedex, France
E-mail: jean.moulin-ollagnier@polytechnique.edu

Sasho Ivanov Popov
Institute of Metal Science, Equipment and Technologies
with Hydro- and Aerodynamics Centre “Acad. A. Balevski”,
Bulgarian Academy of Sciences,
67 Shipchenski Prohod Blvd, 1574 Sofia, Bulgaria
E-mail: sasho_popov2003@yahoo.com

Jean-Marie Strelcyn
CNRS, UMR 7539, Laboratoire Analyse, Géométrie et Applications, LAGA,
Université Sorbonne Paris Nord, F-93430, Villetaneuse, France
E-mail: strelcyn@math.univ-paris13.fr

Contents

Abstract

We consider the Euler-Poisson equations describing the motion of a heavy rigid body about a fixed point with all six parameters in a complex domain. These equations always admit three functionally independent first integrals H_1, H_2, H_3, that is respectively the area([1]), geometrical and conservation of energy first integrals. In four cases (Euler, Lagrange, Kovalevskaya and kinetic symmetry case) a fourth functionally independent first integral appears. In all these four cases this fourth integral can be found among polynomials that do not depend on all variables.

We produce a careful study when, apart from the four cases above, the Euler-Poisson equations, restricted to the level manifolds of H_1, H_2 and H_3 as well as of all their mutual intersections, admit a new first integral which does not depend on all the variables involved. In this way we cover the well known Goryachev-Chaplygin case of partial integrability and discover in the complex domain a new partially integrable case on level manifold $\{H_1 = 0,\ H_2 = 0\}$.

We provide a general method to find all these cases of partial integrability and corresponding partial first integrals. By meticulous and detailed analysis, we show that these two cases are unique when such an additional partial first integral exists. The use of computer algebra is unavoidable to carry out our investigations.

As a further application of the method we used, we also cover the Sretenskii case of partial integrability of the gyrostat equations and describe their new integrable case in the complex domain.

The method we used is of general interest and is probably the most interesting point of this paper. It can also be applied in many other circumstances.

Acknowledgements. We warmly thank Islam Boussaada, Daniel Bennequin, Piotr Biler and Alain Chenciner for interesting discussions and Marie-Claude Werquin for her help in checking our English.

2020 *Mathematics Subject Classification*: Primary 34C45, 34M04, 70E17; Secondary 13P10, 13P15.

Key words and phrases: Euler-Poisson equations, partial first integrals, polynomial systems, Gröbner bases, Lie bracket, Frobenius integrability theorem.

([1]) In [59] "the area first integral" is by inadvertance called "kinetic moment first integral".

Hâtez-vous lentement, et sans perdre courage,
Vingt fois sur le métier remettez votre ouvrage,
Polissez-le sans cesse, et le repolissez,
Ajoutez quelquefois, et souvent effacez.

Nicolas Boileau, Art poétique (1674) ([2])

In memory of our dear friend Andrzej Nowicki
who passed away while we were finishing our work.

1. Introduction

This paper is one more contribution to the study of classical problem of Euler-Poisson equations describing the motion of a heavy rigid body about a fixed point. It can be considered as a natural continuation of paper [59] but it can be read completely independently.

1.1. The problem. Let us briefly describe the content of [59] which is devoted to the search of the so called *fourth integral* of Euler-Poisson equations (see below), but only when this integral does not depend on all the variables. Let us recall some basic facts about the Euler-Poisson equations.

The Euler-Poisson equations are given by the following system

$$\begin{aligned}
I_1\frac{d\omega_1}{dt} &= (I_2 - I_3)\omega_2\omega_3 + Mg(c_3\gamma_2 - c_2\gamma_3),\\
I_2\frac{d\omega_2}{dt} &= (I_3 - I_1)\omega_1\omega_3 + Mg(c_1\gamma_3 - c_3\gamma_1),\\
I_3\frac{d\omega_3}{dt} &= (I_1 - I_2)\omega_1\omega_2 + Mg(c_2\gamma_1 - c_1\gamma_2),\\
\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\gamma_3,\\
\frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \omega_3\gamma_1,\\
\frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - \omega_1\gamma_2.
\end{aligned} \tag{1.1}$$

([2]) In classical English translation of John Dryden (1683):

Gently make haste, of Labour not afraid;
A hundred times consider what you've said:
Polish, repolish, every Colour lay,
And sometimes add; but oft'ner take away.

Studying the Euler-Poisson equations (1.1) from a mechanical point of view, one considers only the real case with $H_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, as well as with the inequalities

$$I_1 > 0, I_2 > 0, I_3 > 0, I_1 + I_2 \geq I_3, I_2 + I_3 \geq I_1 \text{ and } I_3 + I_1 \geq I_2.$$

Let us note that the Euler-Poisson equations with non-zero real parameters I_1, I_2 and I_3 of different signs appear in the theory of equilibria of elastic rods [33, 39].

Equations (1.1) describe the motion of a heavy rigid body of mass M about a fixed point O. We consider a body fixed frame $Oxyz$ with origin in point O and axes coinciding with the principal axes of inertia through O. Here I_1, I_2, I_3 are the principal moments of inertia about point O, c_1, c_2, c_3 the coordinates of the body mass center, g is the acceleration of gravity, $g \neq 0$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the body and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector directed upwards.

It is well known that without any loss of generality one can suppose that $Mg = 1$ and further on we admit that it is so. Indeed, instead of system (1.1) with principal moments of inertia I_1, I_2, I_3, it suffices to consider such a system but with $I_1/(Mg)$, $I_2/(Mg)$, $I_3/(Mg)$ as new principal moments of inertia. As we study the totality of the Euler-Poisson equations (1.1), such a rescaling does not change anything. For shortness we introduce the notation $\mathcal{I}c = (I_1, I_2, I_3, c_1, c_2, c_3)$.

Like in [59], in the present paper we study these equations as a purely mathematical problem considering the general complex case $\mathcal{I}c \in \mathbb{C}^6$, without any restrictions on the parameters except $I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$ which will always be assumed.

Equations (1.1) always have three functionally independent first integrals:

$$\begin{aligned} H_1 &= I_1\omega_1\gamma_1 + I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3, \\ H_2 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \\ H_3 &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2(c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3). \end{aligned} \tag{1.2}$$

In the real case these are the area, geometrical and conservation of energy first integrals of system (1.1).

In real case to be integrable [5, Sec. 28], system (1.1) needs a supplementary fourth first integral H_4, functionally independent of $H_1,\ H_2,\ H_3$, called shortly a *fourth integral.* The only known cases when such fourth integral exists as well in the real case as in complex case are the following four cases: *Euler case*, defined by the condition

$$c_1 = c_2 = c_3 = 0, \tag{1.3}$$

as well as the following two cases, that up to appropriate numeration of principal moments of inertia are: *Lagrange case*, defined by the conditions

$$I_1 = I_2,\ c_1 = c_2 = 0, c_3 \neq 0 \tag{1.4}$$

and *Kovalevskaya case*, defined by the conditions

$$I_1 = I_2 = 2I_3,\ (c_1, c_2) \neq (0,0),\ c_3 = 0. \tag{1.5}$$

Let us note that in the real case, in the Kovalevskaya case we can always take $c_2 = 0$ which is reached by an appropriate rotation of the frame of principal axes of inertia around the axis z. In this case we suppose $c_1 \neq 0$. Let us stress that in the complex case the reduction to $c_2 = 0$ by a linear change of variables is not always possible. The fourth case is the *kinetic symmetry case*, defined by

$$I_1 = I_2 = I_3. \tag{1.6}$$

We denote sets of parameters satisfying cases (1.3) by $\mathcal{E}$ and (1.4) and (1.5) (up to appropriate numeration of principal moments of inertia) by $\mathcal{L}$ and $\mathcal{K}$, respectively.

The fourth integral in cases (1.3)–(1.6) is given as follows:

$$\begin{aligned} &H_4 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \text{ - when } \mathcal{I}c \in \mathcal{E},\\ &H_4 = \omega_3 \text{ - when } \mathcal{I}c \in \mathcal{L},\\ &H_4 = \left(\omega_1^2 - \omega_2^2 - \frac{c_1\gamma_1 - c_2\gamma_2}{I_3}\right)^2 + \left(2\omega_1\omega_2 - \frac{c_2\gamma_1 + c_1\gamma_2}{I_3}\right)^2 \text{ - when } \mathcal{I}c \in \mathcal{K},\\ &H_4 = c_1\omega_1 + c_2\omega_2 + c_3\omega_3 \text{ - in kinetic symmetry case.} \end{aligned} \tag{1.7}$$

For $\mathcal{I}c \in \mathcal{K}$, when $c_2 = 0$, we recover

$$H_4 = \left(\omega_1^2 - \omega_2^2 - \frac{c_1}{I_3}\gamma_1\right)^2 + \left(2\omega_1\omega_2 - \frac{c_1}{I_3}\gamma_2\right)^2$$

that is the standard form of fourth integral in the real Kovalevskaya case where $c_2 = 0$ ([3, 6, 10, 12, 20, 22, 24, 36, 37, 54]).

These four cases are called *classical* cases of integrability of the Euler-Poisson equations.

One sees that for the above four cases the fourth integral does not depend on all variables. So that the question whether there is another case when the fourth integral does not depend on all variables is natural. In [59] for $\mathcal{I}c \in \mathbb{C}^6$, $I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$, we answered this question negatively.

Usually one cites also the so called *Goryachev-Chaplygin case* that up to appropriate enumeration of principal moments of inertia is the following one. Let $I_1 = I_2 = 4I_3$, $(c_1, c_2) \neq (0, 0)$, $c_3 = 0$. In this case the restriction of the Euler-Poisson equations to the five-dimensional level manifold $\{H_1 = 0\}$, admits a supplementary first integral functionally independent of first integrals H_2 and H_3. It is given by the formula:

$$H_4 = I_3\omega_3(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3. \tag{1.8}$$

Like in the above four cases, first integral (1.8) depends on number of variables strictly smaller than the dimension of manifold $\{H_1 = 0\}$.

The first integral like H_4 in Goryachev-Chaplygin case is an example of the so called partial first integral. More precisely, when a smooth dynamical system defined on manifold M, restricted to an invariant submanifold $N \subsetneq M$, admits a first integral φ that is not the restriction to N of some first integral defined on M and φ is functionally independent of restriction to N of all first integrals defined on M, then φ is called a *partial first integral.*

Thus the following problems become natural. Let us consider the complex manifolds of complex dimension five:

$$\{H_1 = U_1\}, \quad \{H_2 = U_2\}, \quad \{H_3 = U_3\},$$

where U_1, U_2, U_3 are arbitrary complex numbers. These level manifolds are always invariant manifolds for the Euler-Poisson equations.

Let $1 \le i, j, k \le 3$.

a) When on the complex five-dimensional level manifold $\{H_i = U_i\}$ there exists a partial first integral of the Euler-Poisson equations restricted to this manifold, that depends on at most four variables and that is functionally independent of H_j and H_k, $j \neq i$, $k \neq i$, $j \neq k$.
b) When on the complex four-dimensional level manifold $\{H_i = U_i,\ H_j = U_j\}$, $i \neq j$, there exists a partial first integral of the Euler-Poisson equations restricted to this manifold, that depends on at most three variables and that is functionally independent of H_k, $k \neq i, j$.
c) When on the complex three-dimensional level manifold $\{H_1 = U_1,\ H_2 = U_2,\ H_3 = U_3\}$, there exists a partial first integral of the Euler-Poisson equations restricted to this manifold, that depends on at most two variables.

In this paper, we give a complete answer to all these questions, apart from four classical cases of integrability.

Indeed, in (a) we recover the Goryachev-Chaplygin case and in (b) we find a supplementary partial first integral on level manifold $\{H_1 = 0, H_2 = 0\}$. By a meticulous and detailed analysis, we show that these two cases are unique for (a), (b) and (c), when an additional partial first integral which does not depend on all variables exists.

Let us underline that in the paper [25] by D. N. Goryachev from 1900 where the case of Goryachev-Chaplygin appears for the first time, as well as in the paper by S. A. Chaplygin [14] from 1901, there is no explanation how this case was found. To the best of our knowledge no such explanation was published until 1983, when S. L. Ziglin in [79] published it for the first time. See also 2005 paper [43] where A. J. Maciejewski and M. Przybylska present such a deduction in a very clever and clear manner. Nevertheless these deductions are trying and in no way can be considered as simple or elementary.

On the contrary, the deduction of the Goryachev-Chaplygin case from the general principles that we present in Sec. 5.2 is short and simple. It only uses facts that were already well known in 1900. Once tedious computations are now easy through the use of elementary computer algebra.

The Euler-Poisson equations have many modifications which describe the different mathematical models related to the movements of rigid bodies with a fixed point [9, 11, 27, 30, 31, 45, 46]. One of the simplest of these is the system of the equations describing the motion of the so-called gyrostat, the equations of which, in the simplest case, are only slightly modified Euler-Poisson equations (1.1). Indeed, the gyrostat equations differ from Euler-Poisson equations only in first three equations, where linear terms $b_3\omega_2 - b_2\omega_3$, $b_1\omega_3 - b_3\omega_1$, $b_2\omega_1 - b_1\omega_2$, $b_1, b_2, b_3 \in \mathbb{C}$, are respectively added to the first three Euler-Poisson equations (1.1). When $b_1 = b_2 = b_3 = 0$ we recover the Euler-Poisson equations (1.1). The gyrostat equations are explicitly written in [21, 24] and in [62, 63] (see also Sec. 2.7 in [12] and for more details [36, 37, 38, 44, 72]). The four classical integrable cases of Euler-Poisson equations admit their natural extensions to gyrostat equations. As proved by L. N. Sretenskii in [62, 63], the same concerns the Goryachev-Chaplygin case of partial integrability. Its gyrostatic analogue is named the Sretenskii case. By applying the method of Sec. 5.2 which leads to the Goryachev-Chaplygin case, in Sec. 6.2 we recover the Sretenskii case. We also find a new case of integrability of gyrostat equations in the complex domain.

As it will be proved in Sec. 6.3, in complex domain the gyrostat equations can have a fourth integral outside four classical cases.

Let us note that this kind of deduction of Goryachev-Chaplygin and Sretenskii cases appeared for the first time in [16], but our approach is more general.

In summary, our problem is to know, having a multiparameter family of ordinary differential equations, how to find the values of the parameters for which the supplementary first integral (i.e. non-obvious or not yet known), that does not depend on all the variables, exists.

Below, when we speak about *smooth functions*, we always mean class C^1 functions in the real case and analytic functions in the complex case. Indeed, in complex case any function having a complex derivative at any point of some open subset of $\mathbb{C}^n$ is analytic on it (see [51]).

Let us stress that we only require the C^1 differentiability of the first integral we are looking for. Although in complex domain C^1 differentiability implies analyticity, we shall never explicitly use this fact. Moreover, all the considerations are *local*. We never use the fact that such first integral is globally defined. We only require that it be defined on an open subset of phase space and not constant on any open subset of it. But the obtained results in all known examples are global because the explicit formulas that we obtain for them, are globally defined. Let us note that in complex case multivalued analytic functions can appear.

The important open question is whether, in the studied examples, there are cases with supplementary partial first integral depending on all the variables while there is no supplementary local partial first integral that does not depend on all variables.

It should be emphasized that there is a substantial difference between [42, 59] and the present paper. In both of the cited papers, the use of the computer algebra could in principle be avoided by tedious hand calculations. This is not the case here, where the huge systems of polynomial equations in several variables that appear, cannot even be written and solved without the use of computer algebra.

1.2. The method. Following [59], let us explain the approach used which is general and can be applied to many frequently encountered systems of ordinary differential equations. We describe it in the real case but it also works in the complex case.

Let

$$\frac{dx}{dt} = G(x) \tag{1.9}$$

be an autonomous system of ordinary differential equations defined on $\mathbb{R}^n$ (or on its open connected subset), $x = (x_1, \ldots, x_n)$, $G = (G_1, \ldots, G_n)$, G is of class C^∞. Let us note that $G = \sum_{i=1}^n G_i \frac{\partial}{\partial x_i}$ is the vector field that defines the system (1.9) and for a smooth function $f = f(x)$, $G(f) = \sum_{i=1}^n G_i \frac{\partial f}{\partial x_i}$.

Function $F \in C^1(U)$, where $U \subset \mathbb{R}^n$ is an open subset, is a *first integral* of system (1.9) if F is constant along the orbits of system (1.9), that is $G(F) = 0$, and F is not constant on any open subset of U.

We are interested here by the first integrals that do not depend on all variables.

1.2.1. Part one. Let us search a first integral F of system (1.9) that does not depend on x_1, that means $F = F(\widehat{x})$, where $\widehat{x} = (x_2, \dots, x_n)$, or equivalently $\frac{\partial F}{\partial x_1} = 0$ identically. Here we have privileged x_1, but similar conditions can be written for every index r, $1 \leq r \leq n$. Then for every x:

$$G(F(x)) = \sum_{i=2}^{n} G_i(x) \frac{\partial F}{\partial x_i}(\widehat{x}) = 0.$$

As F does not depend on x_1, then for every $k \geq 1$ one has

$$\sum_{i=2}^{n} \frac{\partial^k G_i}{\partial x_1^k}(x) \frac{\partial F}{\partial x_i}(\widehat{x}) = 0.$$

In other words, if one notes by Y_k the vector fields

$$Y_k = \sum_{i=2}^{n} \frac{\partial^k G_i}{\partial x_1^k}(x) \frac{\partial}{\partial x_i}, \tag{1.10}$$

then for every $k \geq 0$ one has $Y_k(F) = 0$, where $Y_0 = G$, that is F is a first integral of all these vector fields. All these vector fields are defined on n-dimensional space $\mathbb{R}^n(x)$.

If among the vector fields $\{Y_k\}_{k \geq 0}$ one can find $(n-1)$ of them that are linearly independent at some point $a \in \mathbb{R}^n(x)$, then by continuity they are also linearly independent on some open neighborhood of a. As $Y_k(F) = 0$ for all $k \geq 0$, one deduces that $\operatorname{grad} F$ vanishes identically on U and consequently $F|_U = const$. Then F is not a first integral of system (1.9) because by definition a first integral is non constant on any open subset of its domain of definition. The same argument works also when arbitrary $n-1$ vector fields $\{Z_i\}_{1 \leq i \leq n-1}$ such that $Z_i(F) = 0$, $1 \leq i \leq n-1$, are given. Such a criterion of non existence of the first integral will be frequently used in future.

Let us suppose now that the vector field G is of the form

$$G(x) = \sum_{i=0}^{p} x_1^i \widetilde{Y}_{p-i}(\widehat{x}) = x_1^p \widetilde{Y}_0(\widehat{x}) + \ldots + x_1 \widetilde{Y}_{p-1}(\widehat{x}) + \widetilde{Y}_p(\widehat{x}) \tag{1.11}$$

for some smooth vector fields $\{\widetilde{Y}_i\}_{0 \leq i \leq p}$ defined on $\mathbb{R}^{n-1}(\widehat{x})$ (or on some open subset of $\mathbb{R}^{n-1}(\widehat{x})$). Then as F does not depend on x_1, one has

$$0 = G(F)(x) = x_1^p \widetilde{Y}_0(F)(\widehat{x}) + \ldots + x_1 \widetilde{Y}_{p-1}(F)(\widehat{x}) + \widetilde{Y}_p(F)(\widehat{x}).$$

As $\widetilde{Y}_p(\widehat{x})$ does not depend on x_1, one deduces that $\widetilde{Y}_p(F) = 0$. Thus $G(x) = x_1 G_1(x)$ where the smooth vector field G_1 is

$$G_1(x) = x_1^{p-1} \widetilde{Y}_0(\widehat{x}) + \ldots + x_1 \widetilde{Y}_{p-2}(\widehat{x}) + \widetilde{Y}_{p-1}(\widehat{x}).$$

As above, one deduces that $\widetilde{Y}_{p-1}(F) = 0$, etc. Finally one deduces that $\widetilde{Y}_i(F) = 0$ for all $0 \leq i \leq p$. Thus all vector fields $\widetilde{Y}_i$ defined on $\mathbb{R}^{n-1}(\widehat{x})$ have a common first integral F that does not depend on x_1.

What follows is completely independent of condition (1.11). Like in [59] the main tools used to decide if two smooth vector fields could have a common first integral are the simplest facts from linear algebra and the following well known fact. If F is a first integral common for two vector fields X and Y defined on some open subset U of $\mathbb{R}^p$, $p \geq 2$, then F is also a first integral of their Lie bracket (also known as Jacobi-Lie bracket) or the commutator of vector fields) $[X, Y]$, defined by $[X, Y](f) = X(Y(f)) - Y(X(f))$ for all twice differentiable functions f. Indeed, if $X(F) = Y(F) = 0$, then evidently $[X, Y](F) = X(Y(F)) - Y(X(F)) = 0$.

For vector fields $X = \sum_{i=1}^{p} X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^{p} Y_i \frac{\partial}{\partial x_i}$, simple computations give

$$[X, Y] = \sum_{i=1}^{p} \left[\sum_{j=1}^{p} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) \right] \frac{\partial}{\partial x_i}.$$

Consequently if X and Y are C^∞, then $[X, Y]$ is also C^∞.

For $x \in U$, let us denote by $\mathcal{D}(x)$ the two-dimensional vector subspace of $\mathbb{R}^p$ spanned by vectors $X(x)$ and $Y(x)$. Let us note $\mathcal{D}_0 = \{\mathcal{D}(x), x \in U\}$. Let us note $\mathcal{D}_1 = \mathcal{D}_0 + [\mathcal{D}_0, \mathcal{D}_0] = \mathcal{D}_0 + \{[A, B]; A, B \in \mathcal{D}_0\}$, where $[A, B] = AB - BA$ is the Lie bracket of vector fields A and B. Let us note $\mathcal{D}_2 = \mathcal{D}_1 + [\mathcal{D}_1, \mathcal{D}_1] = \mathcal{D}_1 + \{[A, B]; A, B \in \mathcal{D}_1\}$, etc, where $A + B = \{a + b;\ a \in A;\ b \in B\}$. Thus $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots$. For some k, necessarily $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{D}_k$. $\mathcal{D}_k$ is nothing but the smallest Lie algebra generated by vector fields X and Y.

Let now $a \in U$, $X(a) \neq 0$ and $Y(a) \neq 0$. The *Frobenius Integrability Theorem* [34, 51] implies that in some neighborhood of point $a \in U$ there exists a function Φ such that $X(\Phi) = Y(\Phi) = 0$, if an only if $\dim \mathcal{D}_k(a) < p$, where $\mathcal{D}(x)$ is the vector space of the vector bundle $\mathcal{D}_k$ over x. The number of functionally independent solutions of equations $X(F) = Y(F) = 0$ defined in some neighborhood of a is equal to $p - \dim \mathcal{D}_k(a)$.

The equation $X(F)(x) = 0$, $x \in U$, can be considered as a linear homogeneous equations with unknowns $\left\{ \frac{\partial F}{\partial x_i}(x) \right\}_{1 \leq i \leq p}$. The same is true for equations $Y(F)(x) = 0$ and $[X, Y](F)(x) = 0$. More generally this is true for all vector fields from $\mathcal{D}_k$. Then if $\dim \mathcal{D}_k(a) = p$, by continuity $\dim \mathcal{D}_k(x) = p$ for x belonging to some neighborhood $V \subset U$ of a. Choosing an arbitrary basis $v_1, \ldots, v_p$ of vector bundle $\mathcal{D}_k(V) = \bigcup_{x \in V} \mathcal{D}_k(x)$ and writing the corresponding linear homogeneous equations with in general variable coefficients and unknowns $\left\{ \frac{\partial F}{\partial x_i}(x) \right\}_{1 \leq i \leq p}$, as $\dim \mathcal{D}_k(x) = p$ for $x \in V$, one deduces that $\frac{\partial F}{\partial x_i}(x) = 0$ for $x \in V$, and finally that $F|_V = \mathit{const}$. This contradicts the assumption that F is a first integral.

Thus the condition $\dim \mathcal{D}_k(a) < p$ is necessary for the existence of first integral. In this case the corresponding system of linear equations has infinitely many non-zero solutions. From Frobenius Integrability Theorem we know that now first integrals exist in some neighborhood of point a. But we do not know if these first integrals are the restrictions of first integrals defined on whole phase space: $\mathbb{R}^n$ for system (1.9).

Let us return to system (1.9) and let us suppose that $\dim \mathcal{D}_k(a) = n - 2$ for some $a \in \mathbb{R}^n$ and thus $\dim \mathcal{D}_k(x) = n - 2$ for x from some neighborhood W of a.

In this case, for $x \in W$, $\left\{ \frac{\partial F}{\partial x_i}(x) \right\}_{2 \leq i \leq n}$ satisfy some system of $n - 2$ linearly independent linear homogeneous equations. Let $\{\varphi_i(x)\}_{2 \leq i \leq n}$ be a fixed solution of this system.

Any other solution is of the form $\{\mu(x)\varphi_i(x)\}_{2\le i\le n}$ for some smooth function μ.

If F is a first integral, then for some smooth function μ, $\frac{\partial F}{\partial x_i}(x) = \mu(x)\varphi_i(x)$, which means that μ is an integrating factor of differential form $\sum_{i=1}^n \varphi_i(x)dx_i$. From Frobenius Integrability Theorem we know that such an integrating factor exists because a first integral exists. Surprisingly, in this work in all cases when this situation arises, that is when $\dim \mathcal{D}_k = n-2$ and $n \le 6$, MAPLE is able to compute *explicitly* the first integral F, globally defined. This is precisely in this way we compute all the unknown first integrals.

1.2.2. Part two. Let us consider now the systems of ordinary differential equations like (1.9) but depending on parameters $\lambda = (\lambda_1, \dots, \lambda_k)$

$$\frac{dx}{dt} = G(x, \lambda), \tag{1.12}$$

where $G \in C^\infty(\mathbb{R}^{n+k}, \mathbb{R}^k)$ and for smooth function $f = f(x)$,

$$G(f, \lambda)(x) = \sum_{i=1}^n G_i(x, \lambda)\frac{\partial f}{\partial x_i}(x).$$

All the content of Sec. 1.2.1 without parameters remains valid also with parameters. So, like (1.10) we have the vector fields $Y_k(x, \lambda)$, $k \ge 0$, etc.

As an example, let us consider the simple case when all functions $G_i = G_i(x, \lambda)$, $1 \le i \le n$, are of the form

$$G_i(x, \lambda) = x_1 g_i(\widehat{x}, \lambda) + h_i(\widehat{x}, \lambda), \quad 1 \le i \le n.$$

Let us search a first integral $F(x, \lambda)$ of system (1.12) that does not depend on x_1. We repeat the whole Sec. 1.2.1 but for now the new data depending on λ appear.

This leads us to the identity

$$0 = G(F(\widehat{x}, \lambda), \lambda) = x_1 \widetilde{Y}_1(F(\widehat{x}, \lambda), \lambda) + \widetilde{Y}_2(F(\widehat{x}, \lambda), \lambda),$$

where

$$\widetilde{Y}_1(\widehat{x}, \lambda) = \sum_{i=2}^n g_i(\widehat{x}, \lambda)\frac{\partial}{\partial x_i}, \quad \widetilde{Y}_2(\widehat{x}, \lambda) = \sum_{i=2}^n h_i(\widehat{x}, \lambda)\frac{\partial}{\partial x_i},$$

etc.

In particular for every $\lambda = (\lambda_1, \dots, \lambda_k)$ and $x \in \mathbb{R}^n$, starting from vector fields $\widetilde{Y}_1(\widehat{x}, \lambda)$ and $\widetilde{Y}_2(\widehat{x}, \lambda)$, computing their commutator and commutators of higher orders we define Lie algebra $\mathcal{D}_{k(\lambda)}(\widehat{x}, \lambda)$ like Lie algebra $\mathcal{D}_k(\widehat{x})$ in Sec. 1.2.1.

We search the smooth function $F(x, \lambda)$ such that for any fixed λ, F is a first integral of system (1.12). This leads to the necessary condition

$$\dim \mathcal{D}_{k(\lambda)}(\widehat{x}, \lambda) \le n-2 \tag{1.13}$$

for the existence of such first integral.

As in all examples treated below, $n \le 6$, without difficulty we compute explicitly a base of vector space $\mathcal{D}_{k(\lambda)}(\widehat{x}, \lambda)$. Let $M(\widehat{x}, \lambda)$ be a matrix whose rows are coordinates of vectors of above base. The condition (1.13) is nothing other but

$$\operatorname{rank} M(\widehat{x}, \lambda) \le n-2 \tag{1.14}$$

for all $\widehat{x} \in \mathbb{R}^{n-1}$. Using MAPLE and the method described in Sec. 3, in all our examples we manage to determine all parameters λ such that (1.14) holds, and thus also that (1.13), is satisfied for all $\widehat{x} \in \mathbb{R}^{n-1}$.

Having a concrete example to examine, we compute the associated vector fields and their commutators. After, using computer algebra, we determine the parameters λ that answer the problem posed; existence or nonexistence of supplementary first integral. For details, see Sec. 5 and 8-9.

All that, with evident changes, remains valid in complex case with $G = (G_1, \ldots, G_n)$ analytic in some open, connected subset of $\mathbb{C}^n$, because Frobenius integrability theorem can be also formulated in complex framework ([34, 51]).

1.3. History. Today, the standard approach for the detection of integrable versus non-integrable cases of ordinary differential equations of quite general nature follows mainly the ideas that begun with S. V. Kovalevskaya (1889) and A. M. Lyapunov (1894) from one side and those of J. Liouville (around 1840), E. Picard (1883-1896) and E. Vessiot (1892) from the other and culminate in the so-called Morales-Ramis theory. The history of this subject, as well as some of its applications, can be found in [47, 48, 49, 23, 2]. See also [66, 67, 70, 71, 69, 40, 7].

The method of compatible vector fields that is used in this paper was initiated independently by three persons: Anatolij Dokshevich [16], Vladimir Bogaevskii [8] and Stefan Rauch-Wojciechowski (in the past Wojciechowski) [75].

Let us make a digression on the problem of priority between A. Dokshevich and V. Bogaevskii. The book where A. Dokshevich paper is published was sent to the print June 30, 1964. The paper of V. Bogaevskii was received by the editor October 20, 1964. In footnote of page 93 of [8] he says that the paper was submitted to the editor before the publication of [16]. Let us stress also that [8] was published in largely known mathematical journal published in Moscow and that [16] was published in proceedings of some conference in Tashkent, at the time the capital of Soviet Uzbekistan. There is no doubt that the two authors independently each other have discovered and applied the method of compatible vector fields.

Let us give a very short review of these papers. Both papers are devoted to the study of real Euler-Poisson equations having supplementary first integral that does not depend on all variables. In both papers the condition $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ is assumed.

In Dokshevich paper [16], using method of compatible vector fields one proves that if the supplementary first integral is of the form $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2)$ then this occurs only in Kovalevskaya case and when $c_2 = 0$, the explicit formula for Kovalevskaya first integral is deduced. The other cases studied in [16] concern the supplementary first integrals of the form $F = F(\omega_3)$, $F = F(\omega_1, \omega_2)$, $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, $F = F(\omega_1, \omega_2, \omega_3, \gamma_3)$ which lead to the cases of integrability of Lagrange and Euler, to the invariant relation of Hess (if $I_1(I_2 - I_3)c_1^2 = I_2(I_1 - I_3)c_2^2$, $c_3 = 0$ then $I_1 c_1 \omega_1 + I_2 c_2 \omega_2 = 0$ is an invariant manifold for the Euler-Poisson equations) and the Goryachev-Chaplygin partial integrability case respectively. The author notes also that Sretenskii case of partial integrability of gyrostat equations can be found among the same lines as Goryachev-Chaplygin case.

Let us quote the last paragraph of [16], where the principle of the method of compatible vector fields is clearly stated.

"From a more general point of view, the idea of the presented technique is as follows. It is required to solve some first order partial differential equation

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} X^i(x_1, \dots, x_n) = 0.$$

We add to it also partial differential equations of the simplest form, for example

$$\frac{\partial F}{\partial x_k} = 0, \quad 1 \le k \le s, \quad s < n.$$

Then it will be required that the built system is compatible. If the compatibility conditions are satisfied, then the solution of the system will be at the same time some solution of the original equation."

In Bogaevskii paper [8] one considers the supplementary first integral of the form $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$ for the Euler-Poisson equations of motion of a rigid body with a fixed point in the potential force field $U = U(\gamma_1, \gamma_2, \gamma_3)$. When $U = c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3$, we recover the standard Euler-Poisson equations (1.1). Applying the method of compatible vector fields one identifies the classical cases of integrability: Euler, Lagrange and Kovalevskaya.

Afterwards, one considers the problem of finding the general form of the potential $U = U(\gamma_1, \gamma_2, \gamma_3)$ when there exists a supplementary first integral $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$. A new generalization of the Kovalevskaya case appears.

Unlike [59] and the present paper, where we sweep up all possible cases of first integrals that do not depend on all variables, in [16] and [8] they only focus on a few concrete cases that enable them to catch integrable cases.

This line of research was pursued by Yu. A. Arkhangelskii [3, 4] directly inspired by [16] and [8] and also by S. I. Popov [55, 56, 57]. For further development see [58, 42, 59] and the present paper.

Around twenty years later, around 1986, Stefan Rauch-Wojciechowski motivated by [73, 74], where implicitly the Euler equations on the dual of Lie algebras appear [52, Ch. 6] and also [10, 12, 19, 36, 65, 53], discovered independently the method of compatible vector fields (the name coined by him) advocating their application to three-dimensional systems [64, 75, 76, 26, 77]. For further development see [50, 28, 35].

The method of compatible vector fields uses exclusively notions and facts already well known by Jacobi even if their formal settings were not perfect. Jacobi and some of his contemporaries already knew and understood vector fields, Jacobi-Lie bracket and the link between compatible vector fields and existence of the common first integrals for them, i.e. Frobenius theorem ([13, Ch. Groupes de Lie et algèbres de Lie, p. 310], [29, Sec. 2.5], [70, 71]). We cannot therefore exclude that the method of compatible vector fields appeared in certain works now forgotten, in the period going from the second half of the nineteen century, or even before, until the publication of [16, 8].

The paper is organized as follows. In Sec. 2 an important technical tool, the so called *permutational symmetries* are shortly described. Sec. 3 is devoted to the use of Gröbner basis to obtain by Maple the solutions of the enormous systems of polynomial equations which appear in this article. The direct approach used in [59] is totally inappropriate here. This is one of the pivots of the paper.

Sec. 5 is devoted to the study of five-dimensional invariant manifolds $\{H_i = U_i\}$, $1 \le i \le 3$, that is the problem (a) formulated before. This leads us to recover in a natural way the Goryachev-Chaplygin case. This is the content of Sec. 5.2. In Sec. 6 we sketch the study of gyrostat equations and of derivation of Sretenskii case. In Secs. 5.4 and 5.3 without giving the tedious and long proofs, we shortly report what happens on manifolds $\{H_3 = U_3\}$ and $\{H_2 = 0\}$, respectively. The case of manifolds $\{H_2 = U_2 \ne 0\}$ was completely elucidated in Sec. 5 of [59]. In Sec. 7 we determine the so called domain of the Goryachev-Chaplygin partial first integral. In Secs. 8 and 9 we study what happens on four and three-dimensional invariant manifolds $\{H_i = U_i,\ H_j = U_j\}$, $1 \le i, j \le 3$, $i \ne j$, and $\{H_1 = U_1,\ H_2 = U_2,\ H_3 = U_3\}$, respectively.

We refer to [59] for some supplementary details.

The method we used is of general interest and is probably the most interesting point of this paper. It can also be applied in many other circumstances (see for example [10] - [12]).

2. Permutational symmetries

The Euler-Poisson equations (1.1) possess invariant property which we called *permutational symmetry.* The permutational symmetries can be described in a general framework as follows. Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, and let $V(x, \lambda) = (V_1(x, \lambda), \ldots, V_n(x, \lambda))$ depend smoothly on x. Let us consider the following system of differential equations

$$\frac{dx}{dt} = V(x, \lambda). \tag{2.1}$$

Let σ be an element of the symmetric group S_n, i.e., the group of all permutations of $\{1, \ldots, n\}$. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ we will note $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

The permutation $\sigma \in \mathrm{S}_n$ will be called a *permutational symmetry* of system (2.1) if for all $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}^n$, one has

$$V_k(\sigma(x), \sigma(\lambda)) = \varepsilon V_{\sigma(k)}(x, \lambda), \qquad 1 \le k \le n, \tag{2.2}$$

where $\varepsilon = \pm 1$ is a constant depending on k but independent of x and λ. It is obvious that all permutational symmetries of given equation form a group.

Let us recall that the subset $M \subset \mathbb{C}^n$ is an *invariant subset* of system (2.1) if M is formed by the entire orbits of it. That means that if for some $t_0 \in \mathbb{C}$, $x(t_0) \in M$, then $x(t) \in M$ for all $t \in \mathbb{C}$ such that $x(t)$ is well defined.

Let us formulate the following theorem, already proved in [42] and [59] respectively. We formulated it in the complex setting, but it remains valid also in the real framework as well. For the sake of completeness, we also report its proof.

THEOREM 2.1. *The permutational symmetries of the Euler-Poisson equations (1.1) are the following:*

$$\begin{aligned} \sigma_1 &= \{(1,2,3),(1,2,3)\}, \quad \varepsilon = 1,\\ \sigma_2 &= \{(1,3,2),(1,3,2)\}, \quad \varepsilon = -1,\\ \sigma_3 &= \{(2,3,1),(2,3,1)\}, \quad \varepsilon = 1,\\ \sigma_4 &= \{(2,1,3),(2,1,3)\}, \quad \varepsilon = -1,\\ \sigma_5 &= \{(3,1,2),(3,1,2)\}, \quad \varepsilon = 1,\\ \sigma_6 &= \{(3,2,1),(3,2,1)\}, \quad \varepsilon = -1, \end{aligned} \tag{2.3}$$

where $\sigma\{(i_1,i_2,i_3),(j_1,j_2,j_3)\}$, $1 \le i_r, j_r \le 3$, $1 \le r \le 3$, *is the permutation*

$$\sigma(s_1,s_2,s_3,t_1,t_2,t_3) = (s_{i_1},s_{i_2},s_{i_3},t_{j_1},t_{j_2},t_{j_3}).$$

Proof. The permutation σ_1 with $\varepsilon = 1$ is evidently a permutational symmetry for the Euler-Poisson equations. One can see from these equations that σ_2 with $\varepsilon = -1$ is a permutational symmetry too. The same is true for σ_3 with $\varepsilon = 1$.

Taking into account the equalities

$$\sigma_4 = \sigma_2 \circ \sigma_3, \quad \sigma_5 = \sigma_3^2, \quad \sigma_6 = \sigma_3 \circ \sigma_2$$

we deduce that σ_4 with $\varepsilon = -1$, σ_5 with $\varepsilon = 1$ and σ_6 with $\varepsilon = -1$ are permutational symmetries for the Euler-Poisson equations.

To complete the proof of the theorem it remains only to note that if the permutation

$$\sigma(1,2,3,4,5,6) = (l_1,l_2,l_3,l_4,l_5,l_6)$$

is a permutational symmetry for the Euler-Poisson equations then $l_1,l_2,l_3 \in \{1,2,3\}$ and $l_4,l_5,l_6 \in \{4,5,6\}$. Thus $\sigma = \{(l_{i_1},l_{i_2},l_{i_3}),(l_{j_1},l_{j_2},l_{j_3})\}$. Now, from the Euler-Poisson equations one deduces easily that $i_k = j_k$, $1 \le k \le 3$. ∎

It is interesting to observe that the three sets $\mathcal{E}$, $\mathcal{L}$ and $\mathcal{K}$ are invariant with respect to the permutational symmetries. The same concerns the kinetic symmetry case.

In other words, all permutational symmetries of Euler-Poisson equations (1.1) coincide with symmetric group S_3, where the same permutation is simultaneously applied to variables $\{\omega_1,\omega_2,\omega_3\}$ and $\{\gamma_1,\gamma_2,\gamma_3\}$ and to parameters $\{I_1,I_2,I_3\}$ and $\{c_1,c_2,c_3\}$.

It is also important to note that the first integrals H_1, H_2 and H_3 are invariant with respect to all permutational symmetries of Euler-Poisson equations. This means that for any such permutational symmetry σ one has:

$$H_i(I,c,\omega,\gamma) = H_i(\sigma(I),\sigma(c),\sigma(\omega),\sigma(\gamma)), \quad 1 \le i \le 3,$$

where for permutation $\sigma \in S_3$, $\sigma(a_1,a_2,a_3) = (a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)})$.

This leads to the following general statement that will be frequently used in the future.

Let us define the function Φ_0, $\Phi_0(x,\lambda) = 1$ for all $x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}^n$. Let $U_0 = 1$ and let us note $M(U_0,\lambda) = \{x \in \mathbb{C}^n;\ \Phi_0(x,\lambda) = 1\} = \mathbb{C}^n$.

Let $\lambda \in \mathbb{C}^n$ be fixed. Let $\Phi_i = \Phi_i(x,\lambda)$, $1 \le i \le k < n$, be a finite number of first integrals of system (2.1), that are all invariant with respect to all permutational symmetries σ of the system (2.1), that is

$$\Phi_i(x,\lambda) = \Phi_i(\sigma(x),\sigma(\lambda)), \quad 1 \le i \le k. \tag{2.4}$$

For $U_1,\dots,U_k \in \mathbb{C}^n$ and $k \ge 0$ let us note:

$$M(U_0,\dots,U_k,\lambda) = \{x \in \mathbb{C}^n;\ \Phi_i(x,\lambda) = U_i,\ 0 \le i \le k\}. \tag{2.5}$$

In the future, without repeating it each time, we will only consider the cases where $M(U_0,\dots,U_k,\lambda)$ is either $\mathbb{C}^n$ (when $k=0$) or a non-empty submanifold (perhaps with singularities) of $\mathbb{C}^n$ (when $k \ge 1$).

All these submanifolds of $\mathbb{C}^n$ are invariant manifolds of the system (2.1) and from (2.4) it follows that they are all invariant with respect to all permutational symmetries of system (2.1).

THEOREM 2.2. *Let $k \ge 0$. Let σ be some permutational symmetry of system (2.1). Let us consider the system (2.1) restricted to the invariant manifold $M(U_0,\dots,U_k,\lambda)$ and its local first integral $F = F(x)$ defined on some open subset $W_F \subset M(U_0,\dots,U_k,\lambda)$. Then the function $G = F \circ \sigma^{-1}$, i.e. $G(x) = F(\sigma^{-1}(x))$, defined on the open subset $\sigma(W_F) = \{x \in M(U_0,\dots,U_k,\lambda);\ \sigma^{-1}(x) \in W_F\}$ of $M(U_0,\dots,U_k,\lambda)$ is a local first integral of the system*

$$\frac{dx}{dt} = V(x,\sigma(\lambda)). \tag{2.6}$$

restricted to $M(U_0,\dots,U_k,\lambda)$.

Proof. As F is a first integral of system (2.1), restricted to W_F then for every $x \in W_F$

$$\sum_{k=1}^{n} V_k(x,\lambda)\left(\frac{\partial F}{\partial x_k}\right)(x) = 0.$$

As σ is a permutation of $\{1,\dots,n\}$, the last equality is equivalent to

$$\sum_{k=1}^{n} V_{\sigma(k)}(x,\lambda)\left(\frac{\partial F}{\partial x_{\sigma(k)}}\right)(x) = 0.$$

Taking into account (2.2), we can write this as:

$$\sum_{k=1}^{n} V_k(\sigma(x),\sigma(\lambda))\left(\frac{\partial F}{\partial x_{\sigma(k)}}\right)(x) = 0.$$

The last equality is satisfied for every $x \in W_F$. Then putting instead of x, $\sigma^{-1}(x)$ we obtain that for every $x \in \sigma(W_F)$

$$\sum_{k=1}^{n} V_k(x,\sigma(\lambda))\left(\frac{\partial F}{\partial x_{\sigma(k)}}\right)(\sigma^{-1}(x)) = 0.$$

On the other hand a function $G = G(x)$ is a first integral of system (2.6) if

$$\sum_{k=1}^{n} V_k(x,\sigma(\lambda))\left(\frac{\partial G}{\partial x_k}\right)(x) = 0.$$

Thus to finish the proof it remains to prove that for $G = F \circ \sigma^{-1}$ and $1 \leq k \leq n$ one has

$$\left(\frac{\partial F}{\partial x_{\sigma(k)}}\right)(\sigma^{-1}(x)) = \left(\frac{\partial G}{\partial x_k}\right)(x),$$

but this is obvious. ∎

For $k = 0$, Theorem 2.2 coincides with Theorem 2.1 from [42] and also from [59].

Theorem 2.2 shows that from the point of view of integrability/non-integrability the systems (2.1) and (2.6), both restricted to $M(U_0, \ldots, U_k, \lambda)$ are exactly of the same nature.

In the future, when considering the local first integrals, the word "local" will frequently be omitted.

Thereafter, we will always have $\Phi_i = H_i$, $1 \leq i \leq 3$.

3. Solving some systems of polynomial equations

The method applied to solve all systems of polynomial equations encountered in this paper uses the theory of Gröbner bases of polynomial rings ([15, 18, 60]).

Let us recall some basic facts concerning them and their MAPLE implementations. For all computations we use exclusively the monomial order

$$\mathrm{tdeg}(U_1, U_2, U_3, I_1, I_2, I_3, c_1, c_2, c_3)$$

with ordering $U_1 > U_2 > U_3 > I_1 > I_2 > I_3 > c_1 > c_2 > c_3$.
https://www.maplesoft.com/support/help/Maple/view.aspx?path=Groebner/MonomialOrders

For a fixed monomial order a Gröbner basis of an ideal of polynomial ring $\mathbb{Q}[U, I, c]$ is characterized by the property that the leading monomial of every polynomial in the ideal is divisible by the leading monomial of some polynomial in the Gröbner basis.

A *Maple reduced* Gröbner basis is such a Gröbner basis that if we remove a polynomial from it, the remaining polynomials no longer form a Gröbner basis and it has the additional property that no monomial of any polynomial in the basis is divisible by any of the leading monomials (other than itself). If all polynomials in a Maple reduced Gröbner basis have leading coefficient 1, then this basis is unique up to permutation of its elements and is called *reduced* Gröbner basis. Let us stress that the reduced Gröbner basis always exists.

As proved by the following simple example, in general, Maple reduced Gröbner basis is not the reduced Gröbner basis.

The MAPLE command `Groebner[Basis]`
https://www.maplesoft.com/support/help/Maple/view.aspx?path=Groebner/Basis
computes Maple reduced Gröbner bases for ideals of polynomial rings.

Let us consider the polynomial ring $\mathbb{C}[x, y, z]$ where $x > y > z$ and its ideal L generated by polynomials

$$\{3y^2 - 8z^3, xy^2 + yz^3, x^2 - 2xz + 5\}.$$

With monomial order tdeg(x, y, z), the command `Groebner[Basis]` gives the following Maple reduced Gröbner basis of L:

$$[x^2 - 2xz + 5, 8z^3 - 3y^2, 8xy^2 + 3y^3, 9y^4 + 48y^3z + 320y^2]$$

with leading coefficients $[1, 8, 8, 9]$. With monomial order plex(x, y, z) we obtain the Maple reduced basis

$$\begin{gathered}[1600z^3 - 96z^8 + 240z^6 + 9z^9, -40z^5 + 32z^7 - 3z^8 + 80yz^3,\\ 3y^2 - 8z^3, 120z^5 - 96z^7 + 9z^8 + 640xz^3, x^2 - 2xz + 5]\end{gathered}$$

with leading coefficients $[9, 80, 3, 640, 1]$.

We observe that the obtained Maple reduced Gröbner bases consist of polynomials, each with integer coprime coefficients and positive leading coefficient.

The reason that MAPLE in its definition of the reduced Gröbner bases does not require that the leading coefficients are 1 is due to avoidance of use of rational non-integer numbers.

All factorizations are over $\mathbb{Q}$, that is in the polynomial ring

$$\mathbb{Q}[U, I, c] = \mathbb{Q}[U_1, U_2, U_3, I_1, I_2, I_3, c_1, c_2, c_3].$$

Let us consider polynomials $P_i = P_i(U, I, c) = P_i(U_1, U_2, U_3, I_1, I_2, I_3, c_1, c_2, c_3) \in \mathbb{Q}[U, I, c]$, $1 \le i \le n$. We want to find all complex solutions of the system $P_i(U, I, c) = 0$, $1 \le i \le n$, such that $I_j \ne 0$, $1 \le j \le 3$. Such solutions will be called *good solutions.* To find them we proceed as follows (steps A.1–A.3) and in all cases encountered we achieve a success.

Let us note $\mathbb{C}^9_g = \{(U, I, c) \in \mathbb{C}^9;\ I_i \ne 0,\ 1 \le i \le 3\}$. The good solutions are in $\mathbb{C}^9_g$.

A.1. With MAPLE command `factor`, we factorize over $\mathbb{Q}$ all polynomials $P_i(U, I, c)$, $1 \le i \le n$,

$$P_i = I_1^{\alpha_{i1}} I_2^{\alpha_{i2}} I_3^{\alpha_{i3}} \prod_{k=1}^{r_i} D_{ik}^{\beta_{ik}},$$

where $\beta_{ik} \in \mathbb{N} = \{1, 2, \ldots\}$, $\alpha_{i1},\ \alpha_{i2},\ \alpha_{i3} \in \mathbb{N} \cup \{0\}$, $D_{ik} \in \mathbb{Q}[U, I, c]$. Moreover, for $k \ne l$, polynomials D_{ik} and D_{il} are relatively prime and irreducible in $\mathbb{Q}[U, I, c]$, $1 \le k, l \le r_i$, $1 \le i \le n$.

Then

$$\begin{gathered}\{(U, I, c) \in \mathbb{C}^9_g;\ P_i(U, I, c) = 0,\ 1 \le i \le n\} =\\ \{(U, I, c) \in \mathbb{C}^9_g;\ \widehat{P}_i(U, I, c) = 0,\ 1 \le i \le n\},\end{gathered} \tag{3.1}$$

where $\widehat{P}_i = \prod_{k=1}^{r_i} D_{ik}$ is a square-free factorization of $\prod_{k=1}^{r_i} D_{ik}^{\beta_{ik}}$. Let us stress that in (3.1) we have identity of zeros but perhaps not of their multiplicities.

It is clear that the following inclusion of ideals in the ring $\mathbb{Q}[U, I, c]$ takes place:

$$\{P_1, \ldots, P_n\} \subset \{\widehat{P}_1, \ldots, \widehat{P}_n\}, \tag{3.2}$$

where $\{R_1, \ldots, R_q\}$ denotes the ideal in $\mathbb{Q}[U, I, c]$ generated by the polynomials

$$R_1, \ldots, R_q \in \mathbb{Q}[U, I, c].$$

A.2. Using MAPLE command `Groebner[Basis]` we compute in the ring $\mathbb{Q}[U,I,c]$ a Maple reduced Gröbner basis $\{Q_1,\dots,Q_m\}$ of ideal $\{\widehat{P}_1,\dots,\widehat{P}_n\}\subset\mathbb{Q}[U,I,c]$. The polynomials $Q_1,\dots,Q_m$ can have multiple factors in $\mathbb{Q}[U,I,c]$.

Formulas (3.1) and (3.2) imply respectively that

$$\begin{aligned}&\{(U,I,c)\in\mathbb{C}_g^9;\ P_i(U,I,c)=0,\ 1\le i\le n\}=\\&\{(U,I,c)\in\mathbb{C}_g^9;\ Q_j(U,I,c)=0,\ 1\le j\le m\}\end{aligned}$$

and

$$\{P_1,\dots,P_n\}\subset\{Q_1,\dots,Q_m\}.$$

As $\{R_1,\dots,R_u\}\subset\{\widehat{R}_1,\dots,\widehat{R}_u\}$ we have

$$\{P_1,\dots,P_n\}\subset\{\widehat{Q}_1,\dots,\widehat{Q}_m\}. \tag{3.3}$$

and

$$\begin{aligned}&\{(U,I,c)\in\mathbb{C}_g^9;\ P_i(U,I,c)=0,\ 1\le i\le n\}=\\&\{(U,I,c)\in\mathbb{C}_g^9;\ \widehat{Q}_j(U,I,c)=0,\ 1\le j\le m\}\end{aligned} \tag{3.4}$$

The passage from the system $P_1=0,\dots,P_n=0$ to the system $\widehat{Q}_1=0,\dots,\widehat{Q}_m=0$ will be called a *simplification.*

According to (3.4) the system obtained by simplification has the same good solutions as the source system and in all encountered cases the obtained system of equations is simpler than the source one.

As the ring $\mathbb{Q}[U,I,c]$ is Noetherian, then after a finite number of consecutive simplifications, we will arrive (see (3.3)) to the system $S_1=0,\dots,S_t=0$, that will not be modified by another simplification, that is, every polynomial $S_i,\ 1\le i\le t$, is square-free, without factors of the form $I_1^{\alpha_{i1}}I_2^{\alpha_{i2}}I_3^{\alpha_{i3}}$ and the polynomials $\{S_i\}_{1\le i\le t}$ form a Maple reduced Gröbner basis of the ideal $\{S_1,\dots,S_t\}$.

We call the system of equations $S_1=0,\dots,S_t=0$ *reduced system* or *reduction* of the source system $P_1=0,\dots,P_n=0$. The reduced system $\{S_j=0\}$ has the same set of good solutions as the source system $\{P_i=0\}$. The simplest MAPLE computational criterion that the system $S_1=0,\dots,S_t=0$ is a reduction of the source system is that its simplification coincides with it. This criterion will be constantly used by us.

A.3. The final step is then to describe the set of all complex solutions of the reduced system $\{S_j=0\},\ 1\le j\le t$.

It is clear that when $t=1$ and $S_1=1$, then the source system does not admit any good solution.

Fortunately, in an unexplained and unexpected way, in all other cases encountered below, the reduced systems are simple, of low degrees and all $\{S_j\}_{1\le j\le t}$ are factorized in product of factors that depend on only one kind of unknowns $\{U_1,U_2,U_3\}$, $\{c_1,c_2,c_3\}$ or $\{I_1,I_2,I_3\}$. Moreover every factor belong to the following short list of possibilities:

$$U_1,\ \ U_2,\ \ U_3,\ \ a_1I_1+a_2I_2+a_3I_3,\ \ c_1,\ \ c_2,\ \ c_3\ \ \text{and}\ \ b_1c_1^2+b_2c_2^2+b_3c_3^2, \tag{3.5}$$

where a_i and b_i, $1\le i\le 3$, are some integers. There is only one exception in Sec. 8.2.1 where in one of the equations of the reduced system a factor appears that depends simultaneously on I_i and c_i, $1\le i\le 3$, and it is $(I_2-I_3)c_1^2+(I_1-I_3)c_2^2+(I_2-I_1)c_3^2$.

In many cases the situation is even simpler because some of polynomials S_j, $1 \leq j \leq t$, merely coincide with some of the possibilities from list (3.5). For example, in Sec. 7 (see formula (7.8)) polynomial $S_1 = c_3$.

Thus, without any difficulty, all the good solutions can be found either by hand or by applications of elementary computer algebra, MAPLE for example.

4. Some algebra

The following two simple Propositions will be used repeatedly until the end of the article. The first one is well known and follows from the well known elementary properties of resultant ([15, Chap. 3, §6] and [18]).

Let $\mathbb{K}$ be a field of characteristic 0 and $\mathbb{K}[x]$ be as usual the ring of polynomials of one variable x with coefficients in $\mathbb{K}$.

PROPOSITION 4.1. *Let $g \in \mathbb{K}[x]$ be a polynomial and $h(x) = \frac{dg}{dx}$. Let ρ be the resultant of g and h and $\rho \neq 0$. Let $\overline{x}$ be some root of g, $g(\overline{x}) = 0$. Then*

(i) $h(\overline{x}) \neq 0$,
(ii) g has no multiple roots.

Proof. (i) It follows immediately from the well known fact. If $f, g \in \mathbb{K}[x]$ then f and g have a common factor in $\mathbb{K}[x]$ if and only if their resultant is 0 or equivalently if f and g have a common root (perhaps in algebraic closure of the field $\mathbb{K}$ if this field is not algebraically closed).

(ii) It follows from the evident fact that if g has a multiple root, then $h(x) = \frac{dg}{dx}$ has the same root and thus g and h has a common factor. ∎

The second Proposition is completely evident but for convenience it is called Proposition.

Let $\mathbb{K}$ be a field of characteristic 0. Let $f, g \in \mathbb{K}[x]$ are polynomials of one variable x and $g \neq 0$. By Euclidean division we know that for some polynomials $q, r \in \mathbb{K}[x]$ one has

$$f(x) = q(x)g(x) + r(x), \quad \deg r < \deg g \quad \text{or } r = 0.$$

PROPOSITION 4.2. *Let*

(i) all roots of g are simple and are in $\mathbb{K}$,
(ii) all roots of g are also roots of f.

Then g divides f in $\mathbb{K}[x]$, that means that the remainder r (which is in $\mathbb{K}[x]$) vanishes identically.

In the following, for fixed $n \geq 1$, let $\mathbb{K}_n = \mathbb{A}\mathrm{lg}(s_1, \ldots, s_n)$ be the field of algebraic functions of complex variables $(s_1, \ldots, s_n) \in \mathbb{C}^n$ ([1], [61], [68]). The field $\mathbb{K}_n$ is of characteristic 0.

Let us explain this more in details. Following [1], let $P_0, \ldots, P_k \in \mathbb{C}[x_1, \ldots, x_n]$ be complex polynomials of variables $x_1, \ldots, x_n$, and with $P_k(x_1, \ldots, x_n) \not\equiv 0$. A function

$y = y(x_1, \ldots, x_n)$ of the variables $x_1, \ldots, x_n$ is called *complex algebraic function* if

$$P_k(x_1, \ldots, x_n)y^k + P_{k-1}(x_1, \ldots, x_n)y^{k-1} + \cdots + P_0(x_1, \ldots, x_n) = 0 \tag{4.1}$$

for all $(x_1, \ldots, x_n) \in \mathbb{C}^n$ and if the above polynomial of y is irreducible in $\mathbb{C}[x_1, \ldots, x_n]$. The number k is called *degree* of algebraic function y. If $k = 1$, an algebraic function is a rational function $y = -P_0(x_1, \ldots, x_n)/P_1(x_1, \ldots, x_n)$. For $k = 2, 3, 4$, an algebraic function can be expressed by square and cube roots of rational functions in the variables $x_1, \ldots, x_n$. If $k \geq 5$, this is impossible in general ([18]).

If $k \geq 2$ an algebraic function is multivalued (like for example $y = \sqrt{x}$) and in an open dense subset of $\mathbb{C}^n$, it locally admits holomorphic (analytic) determinations called also branches. This follows from complex implicit function theorem ([34], [51]).

Let us also note that any non-zero complex polynomial (4.1) can be factorized in irreducible factors ([18]). Thus, the equation (4.1) defines algebraic functions even if the polynomial (4.1) is not irreducible.

Let us note for short $x = (x_1, \ldots, x_n)$. Let us compute the partial derivatives $\frac{\partial y}{\partial x_i}(x)$, $1 \leq i \leq n$, for an algebraic function of degree k. By deriving the formula (4.1) with respect to x_i, $1 \leq i \leq n$, one easily deduces that

$$\frac{\partial y}{\partial x_i}(x) = -\frac{\frac{\partial P_k}{\partial x_i}(x)y^k + \frac{\partial P_{k-1}}{\partial x_i}(x)y^{k-1} + \cdots + \frac{\partial P_0}{\partial x_i}(x)}{kP_k(x)y^{k-1} + (k-1)P_{k-1}(x)y^{k-2} + \cdots + P_1(x)}. \tag{4.2}$$

The partial derivatives of higher order of the algebraic function $y = y(x)$ can be computed by consecutive derivations of the formula (4.2).

As the degree of algebraic function $y = y(x)$ is k, the denominator of (4.2) which is a non-vanishing algebraic function is non-zero on open dense subset of $\mathbb{C}^n$, where the formula (4.2) gives the searched derivative, that is also an algebraic function of $x = (x_1, \ldots, x_n)$.

Now, let us consider on some open subset U of $\mathbb{C}^n$, some holomorphic determination of multivalued algebraic function $y = y(x)$, that we shall note $f = f(x)$. Then, if in formula (4.2) instead of y one takes the function f, the formula remains valid.

Consequently, instead of analyzing separately all holomorphic determinations of an algebraic function $y = y(x)$, it suffices to consider the multivalued algebraic function $y = y(x)$ as a whole, the derivatives of which are given by the formula (4.2).

We shall also apply the following well known and easy to prove Proposition.

Proposition 4.3. *Let $n \geq 2$ and let $V \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial that is not a square of another polynomial. Then $\sqrt{V} \notin \mathbb{C}(x_1, \ldots, x_n)$ that means that $\sqrt{V}$ is not a rational function of $x_1, \ldots, x_n$.*

5. Five-dimensional invariant manifolds $\{H_i{=}U_i\}$, $1 \leq i \leq 3$. Goryachev-Chaplygin and Sretenskii cases

5.1. Extraction procedure. In this section we study the existence of a local partial first integral of the Euler-Poisson equations (1.1) restricted to the invariant complex five-dimensional manifolds $\{H_1 = U_1\}$ and $\{H_3 = U_3\}$. We study when on each of them there exists a local partial first integral that depends on at most four variables and such that on $\{H_1 = U_1\}$ it is functionally independent of H_2 and H_3 and on $\{H_3 = U_3\}$ of H_1 and H_2 respectively. The same problem can be stated also for manifold $\{H_2 = U_2\}$ where the functional independence of H_1 and H_3 is required. For $U_2 = 1$ this case has been considered in Sec. 5 of [59] and the general case of $U_2 \neq 0$ can easily be reduced to the case $U_2 = 1$. Thus it remains to study only the case $\{H_2 = 0\}$.

Let us fix i, $1 \leq i \leq 3$. According to (2.5)

$$M(U_0, U_i, \mathcal{I}c) = \{x \in \mathbb{C}^6;\ H_i((\omega, \gamma), \mathcal{I}c) = U_i\},$$

where $(\omega, \gamma) = (\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$ and $\dim M(U_0, U_i, \mathcal{I}c) = 5$.

We search all functions F of four variables $F = F(s_1, s_2, s_3, s_4)$ where $(s_1, s_2, s_3, s_4) \in (\omega, \gamma)$, of class $\mathbb{C}^1$, such that $\operatorname{grad} F$ does not vanish identically on each open subset of $M(U_0, U_i, \mathcal{I}c)$, which are local partial first integrals of the Euler-Poisson equations (1.1) restricted to $M(U_0, U_i, \mathcal{I}c)$.

Let $i = 1$. The unique intrinsic property of $\mathbb{C}^1$ function F that is a local first integral is that $\operatorname{grad} F$ does not vanish identically on any open subset of its domain of definition, that in this case is equal to $M(U_0, U_1, \mathcal{I}c)$. This implies that some of the partial derivatives of F may be identically zero. Thus the results of Sec. 5.2 also remain valid for the functions of at most four variables.

As $\frac{\partial F}{\partial s_1}\frac{ds_1}{dt} + \frac{\partial F}{\partial s_2}\frac{ds_2}{dt} + \frac{\partial F}{\partial s_3}\frac{ds_3}{dt} + \frac{\partial F}{\partial s_4}\frac{ds_4}{dt} = 0$, where $\frac{ds_i}{dt}$, $1 \leq i \leq 4$, are given by the right hand sides of the equations of Euler-Poisson (1.1), then the order of variables s_i, $1 \leq i \leq 4$, in $F(s_1, s_2, s_3, s_4)$ is irrelevant for F to be a first integral.

We have exactly 15 different four elements subsets of (ω, γ) and thus 15 cases of functions of four elements to examine. We will describe now an extraction procedure based on permutational symmetries which reduces the above 15 cases to only four.

These 15 functions of four variables (up to the order of variables) are shown in the table below.

Table 5.1

Functions	Case
$F(\omega_1, \omega_2, \omega_3, \gamma_i)$, $1 \leq i \leq 3$	(i)
$F(\omega_1, \omega_3, \gamma_1, \gamma_3)$, $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, $F(\omega_2, \omega_3, \gamma_2, \gamma_3)$	(ii)
$F(\omega_1, \omega_2, \gamma_1, \gamma_3)$, $F(\omega_1, \omega_3, \gamma_1, \gamma_2)$, $F(\omega_2, \omega_3, \gamma_1, \gamma_2)$, $F(\omega_1, \omega_2, \gamma_2, \gamma_3)$, $F(\omega_1, \omega_3, \gamma_2, \gamma_3)$, $F(\omega_2, \omega_3, \gamma_1, \gamma_3)$	(iii)
$F(\omega_i, \gamma_1, \gamma_2, \gamma_3)$, $1 \leq i \leq 3$	(iv)

It is easy to see that under the group of permutational symmetries (2.3) of the Euler-Poisson equations for every case (i)–(iv) from Table 5.1 each function from the case is consequently transformed into all remaining functions from the same case.

Thus in virtue of Theorem 2.2 we can restrict ourselves to the study of only four functions where every one belongs to a different case from Table 5.1 and is chosen arbitrary from the functions of this case.

We will call such four functions F_i, $1 \leq i \leq 4$, (up to the order of variables) a *basis*.

As Table 5.1 shows, the functions

$$F(\omega_1, \omega_2, \omega_3, \gamma_3),\ F(\omega_1, \omega_3, \gamma_1, \gamma_3),\ F(\omega_1, \omega_2, \gamma_1, \gamma_3),\ F(\omega_1, \gamma_1, \gamma_2, \gamma_3) \tag{5.1}$$

form a basis.

To be a local first integral of some vector field, first integral defined in some open subset of some manifold, is an intrinsic property, that is independent of the system of coordinates used. Thus in $M(U_0, U_i, \mathcal{I}c)$ instead of coordinates $(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$ inherited from the Euler-Poisson equations, we can consider for example the system of coordinates $(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_3)$, where the coordinate (variable) γ_2 can be eliminated thanks to identity $H_1 = U_1$. The same concerns all remaining coordinates.

Using coordinates $(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_3)$ on $M(U_0, U_i, \mathcal{I}c)$ we can verify if the first three functions of the basis (5.1) are partial first integrals or not. For the last function of basis (5.1) we will use the coordinates $(\omega_1, \omega_3, \gamma_1, \gamma_2, \gamma_3)$.

The following general remark concerns also Sec. 8 and Sec. 9. If we are interested in partial first integrals that depend on at most three variables, for instance $F(\omega_2, \omega_3, \gamma_3)$, we can consider it as a particular case of $F(\omega_1, \omega_2, \omega_3, \gamma_3)$ (case (i)), of $F(\omega_2, \omega_3, \gamma_2, \gamma_3)$ (case (ii)) and of $F(\omega_2, \omega_3, \gamma_1, \gamma_3)$ (case (iii)). From the study of each of these functions, we can conclude about the existence of the sought partial first integral $F(\omega_2, \omega_3, \gamma_3)$.

5.2. Invariant manifold $\{H_1{=}U_1\}$. Determination of the Goryachev-Chaplygin case. Here we show the method we use on the example $\{H_1 = U_1\}$. This invariant manifold gives not only results for non-existing of the sought partial first integrals at $U_1 \neq 0$ but when $U_1 = 0$ it also gives a nice derivation of the Goryachev-Chaplygin case.

5.2.1. Elimination of γ_2. Let us express γ_2 from the equation $H_1 = U_1$. We have

$$\gamma_2 = \frac{U_1 - I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3}{I_2\omega_2}. \tag{5.2}$$

We put the expression for γ_2 from (5.2) in the Euler-Poisson equations (1.1) and remove the fifth equation. In this way we obtain

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{c_3U_1 + I_2\left(I_2 - I_3\right)\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - I_2c_2\omega_2\gamma_3 - I_3c_3\omega_3\gamma_3}{I_1I_2\omega_2},\\
\frac{d\omega_2}{dt} &= \frac{\left(I_3 - I_1\right)\omega_1\omega_3 + c_1\gamma_3 - c_3\gamma_1}{I_2},\\
\frac{d\omega_3}{dt} &= \frac{-c_1U_1 + I_2\left(I_1 - I_2\right)\omega_1\omega_2^2 + I_1c_1\omega_1\gamma_1 + I_2c_2\omega_2\gamma_1 + I_3c_1\omega_3\gamma_3}{I_2I_3\omega_2},\\
\frac{d\gamma_1}{dt} &= \frac{-I_1\omega_1\omega_3\gamma_1 - I_2\omega_2^2\gamma_3 - I_3\omega_3^2\gamma_3 + \omega_3U_1}{I_2\omega_2},\\
\frac{d\gamma_3}{dt} &= \frac{I_1\omega_1^2\gamma_1 + I_2\omega_2^2\gamma_1 + I_3\omega_1\omega_3\gamma_3 - \omega_1U_1}{I_2\omega_2}.
\end{aligned} \tag{5.3}$$

Looking for a partial first integral of system (5.3) which depends on four variables indicated in brackets, we come to the following five possible cases:

1. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$, (case (i))
2. $F(\omega_1, \omega_2, \omega_3, \gamma_3)$, (case (i))
3. $F(\omega_1, \omega_2, \gamma_1, \gamma_3)$, (case (iii))
4. $F(\omega_1, \omega_3, \gamma_1, \gamma_3)$, (case (ii))
5. $F(\omega_2, \omega_3, \gamma_1, \gamma_3)$, (case (iii))

where "case(*)" indicates in which case of Table 5.1 the corresponding partial first integral appears.

The functions of types 2, 3 and 4 belong to the basis (5.1). We should study all of them. We start with a partial first integral of type 2.

Type 2. Let us look for a partial first integral of system (5.3) that does not depend on γ_1, i.e. of type 2. Moreover we want this integral to be functionally independent of H_2 and H_3. Let us suppose that the function

$$F(\omega_1, \omega_2, \omega_3, \gamma_3) \tag{5.4}$$

is such a partial first integral of (5.3). It satisfies the following identity

$$\begin{aligned}\frac{dF}{dt} = & \frac{c_3 U_1 + I_2\left(I_2 - I_3\right)\omega_2^2\omega_3 - I_1 c_3 \omega_1\gamma_1 - I_2 c_2 \omega_2 \gamma_3 - I_3 c_3 \omega_3 \gamma_3}{I_1 I_2 \omega_2} \frac{\partial F}{\partial \omega_1} \\ & + \frac{\left(I_3 - I_1\right)\omega_1\omega_3 + c_1\gamma_3 - c_3\gamma_1}{I_2} \frac{\partial F}{\partial \omega_2} \\ & + \frac{-c_1 U_1 + I_2\left(I_1 - I_2\right)\omega_1\omega_2^2 + I_1 c_1 \omega_1 \gamma_1 + I_2 c_2 \omega_2 \gamma_1 + I_3 c_1 \omega_3 \gamma_3}{I_2 I_3 \omega_2} \frac{\partial F}{\partial \omega_3} \\ & + \frac{I_1 \omega_1^2 \gamma_1 + I_2 \omega_2^2 \gamma_1 + I_3 \omega_1 \omega_3 \gamma_3 - \omega_1 U_1}{I_2 \omega_2} \frac{\partial F}{\partial \gamma_3} = 0,\end{aligned}$$

or equivalently

$$I_1 I_2 I_3 \omega_2 \frac{dF}{dt} = I_1 \gamma_1 Y_1(F) + Y_2(F) = 0, \tag{5.5}$$

where Y_1 and Y_2 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_3)$

$$\begin{aligned} Y_1 = & -I_3 c_3 \omega_1 \frac{\partial}{\partial \omega_1} - I_3 c_3 \omega_2 \frac{\partial}{\partial \omega_2} + (I_1 c_1 \omega_1 + I_2 c_2 \omega_2) \frac{\partial}{\partial \omega_3} + I_3 (I_1 \omega_1^2 + I_2 \omega_2^2) \frac{\partial}{\partial \gamma_3}, \\ Y_2 = & I_3 \left[I_2 (I_2 - I_3) \omega_2^2 \omega_3 - I_2 c_2 \omega_2 \gamma_3 - I_3 c_3 \omega_3 \gamma_3 + c_3 U_1 \right] \frac{\partial}{\partial \omega_1} \\ & - I_1 I_3 \omega_2 \left[(I_1 - I_3) \omega_1 \omega_3 - c_1 \gamma_3 \right] \frac{\partial}{\partial \omega_2} \\ & - I_1 \left[I_2 (I_2 - I_1) \omega_1 \omega_2^2 - I_3 c_1 \omega_3 \gamma_3 + c_1 U_1 \right] \frac{\partial}{\partial \omega_3} + I_1 I_3 \omega_1 \left(I_3 \omega_3 \gamma_3 - U_1 \right) \frac{\partial}{\partial \gamma_3}. \end{aligned}$$

As (5.5) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on γ_1 we have

$$Y_1(F) = Y_2(F) = 0. \tag{5.6}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]/I_3$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$Y_3 = m_{31}\frac{\partial}{\partial\omega_1} + m_{32}\frac{\partial}{\partial\omega_2} + m_{33}\frac{\partial}{\partial\omega_3} + m_{34}\frac{\partial}{\partial\gamma_3},$$
$$Y_4 = m_{41}\frac{\partial}{\partial\omega_1} + m_{42}\frac{\partial}{\partial\omega_2} + m_{43}\frac{\partial}{\partial\omega_3} + m_{44}\frac{\partial}{\partial\gamma_3},$$

where

$$\begin{aligned}
m_{31} &= -I_1I_2I_3c_2\omega_1^2\omega_2 - I_1I_3^2c_3\omega_1^2\omega_3 - I_2^2I_3c_3\omega_2^2\omega_3 + I_1I_2(I_2-I_3)c_1\omega_1\omega_2^2\\
&\quad - I_1I_3c_1c_3\omega_1\gamma_3 + I_2^2(I_2-2I_3)c_2\omega_2^3 - I_2I_3c_2c_3\omega_2\gamma_3 - I_3^2c_3^2\omega_3\gamma_3 + I_3c_3^2U_1,\\
m_{32} &= I_1\omega_2\big[I_2I_3c_1\omega_2^2 - I_1(I_1-2I_3)c_1\omega_1^2 - I_2(I_1-I_3)c_2\omega_1\omega_2 + I_3(I_1-I_3)c_3\omega_1\omega_3\big]\\
m_{33} &= I_1\big[I_1I_3c_1\omega_1^2\omega_3 + I_1c_1^2\omega_1\gamma_3 + I_2c_1c_2\omega_2\gamma_3 - I_2(I_2-2I_3)c_1\omega_2^2\omega_3\\
&\quad + I_2(I_1-I_3)c_2\omega_1\omega_2\omega_3 - 3I_2(I_1-I_2)c_3\omega_1\omega_2^2 + I_3c_1c_3\omega_3\gamma_3 - c_1c_3U_1\big],\\
m_{34} &= I_1I_3\big[I_1I_3\omega_1^3\omega_3 + I_1c_1\omega_1^2\gamma_3 - 2I_2c_1\omega_2^2\gamma_3 + 3I_2c_2\omega_1\omega_2\gamma_3\\
&\quad + I_2(2I_1-2I_2+I_3)\omega_1\omega_2^2\omega_3 + I_3c_3\omega_1\omega_3\gamma_3 - c_3U_1\omega_1\big],\\
m_{41} &= I_3c_2\big[-2I_1^2I_3c_1\omega_1^3 - I_1I_2(3I_2-I_3)c_1\omega_1\omega_2^2 - I_1I_3c_1c_2\omega_1\gamma_3 - 3I_2^2(I_2-I_3)c_2\omega_2^3\\
&\quad - I_2I_3c_2c_2\omega_2\gamma_3 + I_2I_3(I_2-I_3)c_2\omega_2^2\omega_3 - I_3^2c_2^2\omega_3\gamma_3 + U_1I_3c_2^2\big],\\
m_{42} &= I_1\omega_2I_3c_3\big[-2I_2I_3c_1\omega_2^2 + (3I_1-5I_3)I_1c_1\omega_1^2\\
&\quad + 3(I_1-I_3)I_2c_2\omega_1\omega_2 - (I_1-I_3)I_3c_3\omega_1\omega_3\big],\\
m_{43} &= I_1\big[2I_1^2I_2c_1c_2\omega_1^2\omega_2 + 2I_1^2I_3c_1^2\omega_1^3 + I_1I_3c_1^2c_3\omega_1\gamma_3 - 2I_2^2(I_2-2I_3)c_1c_2\omega_2^3\\
&\quad + I_2I_3c_1c_2c_3\omega_2\gamma_3 + 3I_2I_3(I_2-I_3)c_1c_3\omega_2^2\omega_3 - 3I_2I_3(I_1-I_3)c_2c_3\omega_1\omega_2\omega_3\\
&\quad + I_3^2c_1c_3^2\omega_3\gamma_3 - I_2\omega_1\omega_2^2(2I_1I_2c_1^2 - 2I_1I_2c_2^2 - 4I_1I_3c_1^2 - 9I_1I_3c_3^2 + 2I_2I_3c_2^2\\
&\quad + 9I_2I_3c_3^2) - I_3c_1c_3^2U_1\big],\\
m_{44} &= I_1I_3\big[2I_1^2I_3c_1\omega_1^4 + 6I_1I_2I_3c_2\omega_1^3\omega_2 - 4I_2^2I_3c_1\omega_2^4 + 2I_1I_2(2I_1-2I_2-I_3)c_1\omega_1^2\omega_2^2\\
&\quad + I_1I_3c_1c_3\omega_1^2\gamma_3 + 2I_2^2(2I_1-2I_2+3I_3)c_2\omega_1\omega_2^3 + 4I_2I_3c_1c_3\omega_2^2\gamma_3\\
&\quad - 3I_2I_3c_2c_3\omega_1\omega_2\gamma_3 - 8I_2I_3(I_1-I_2)c_3\omega_1\omega_2^2\omega_3 + I_3^2c_3^2\omega_1\omega_3\gamma_3 - I_3c_3^2U_1\omega_1\big].
\end{aligned}$$

Equations (5.6) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{5.7}$$

Equations (5.6) and (5.7) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\omega_3}, \frac{\partial F}{\partial\gamma_3}\right)$, which do not vanish identically on any open subset of domain of definition of F, because F is non-constant on any such open subset.

If a new integral F exists, system (5.6)–(5.7) has at least one non-zero solution. Let us consider the 4×4 matrix A whose rows are the coefficients of vector fields Y_1, Y_2, Y_3 and Y_4. The condition under which system (5.6)–(5.7) has at least one non-zero solution is

$$\operatorname{rank} A \le 3.$$

We equate to zero the determinant D of the 4×4 matrix A of the coefficients of system (5.6)–(5.7) and study when identity

$$D = \det(A) \equiv 0 \tag{5.8}$$

is fulfilled. We compute D and obtain

$$D = I_1^2 I_2^2 I_3^2 \omega_2^3 \widehat{D},$$

where the expression for $\widehat{D}$ is a polynomial in variables ω_1, ω_2, ω_3 and γ_3 having 72 monomials and thus with 72 coefficients depending on $\mathcal{I}c$ and U_1. It is clear that to solve (5.8) is equivalent of finding all values of the parameters $\mathcal{I}c$ and U_1 for which the 72 coefficients of $\widehat{D}$ are zero. The expression for $\widehat{D}$ is too long and that is why we do not write it here. To solve this system of 72 equations we proceed as described in Sec. 3.

After four consecutive simplifications of the source system of 72 equations we obtain the reduced system having only nine equations:

$$\begin{aligned}
&(I_2 - I_3)c_2c_3 = 0, \quad (I_1 - I_3)c_2c_3 = 0,\\
&(I_2 - I_3)c_1c_3 = 0, \quad (I_1 - I_3)c_1c_3 = 0, \quad (I_1 - I_2)c_1c_2 = 0,\\
&(I_2 - 4I_3)(I_1 - I_3)c_2 = 0, \quad (I_1 - I_3)(I_1 - 4I_3)c_2 = 0,\\
&(I_2 - I_3)(I_2 - 4I_3)c_1 = 0, \quad (I_2 - I_3)(I_1 - 4I_3)c_1 = 0.
\end{aligned}$$

We solve these nine equations by the MAPLE command `solve` and obtain five solutions. Two of them lead to the Lagrange case and one - to the kinetic symmetry case. In this way we come to the following two cases that should be studied separately:

1. $I_1 = I_2 = 4I_3$, $c_3 = 0$, $(c_1, c_2) \neq (0, 0)$ and U_1 are arbitrary,
2. $c_1 = c_2 = 0$, $c_3 \neq 0$, $I_1 \neq 0$, $I_2 \neq 0$, $I_3 \neq 0$ and U_1 are arbitrary.

Let us study these cases.

Case 1. $I_1 = I_2 = 4I_3$, $c_3 = 0$, $(c_1, c_2) \neq (0, 0)$ and U_1 are arbitrary. At this condition we have $\widehat{D} = 0$ and therefore the vector fields Y_i, $1 \leq i \leq 4$, are linearly dependent.

Let us note by D_{ab} the determinant of 3×3 matrix obtained from 4×4 matrix A by canceling row a and column b. Elementary MAPLE computations show that the determinant D_{43}:

$$D_{43} = 768 I_3^7 \omega_2^2(\omega_1^2 + \omega_2^2)(-c_2\omega_1 + c_1\omega_2)(I_3\omega_1^2\omega_3 + I_3\omega_2^2\omega_3 - c_1\omega_1\gamma_3 - c_2\omega_2\gamma_3)$$

never vanishes identically unless $c_1 = c_2 = 0$, i.e. the Euler case. Thus the vector fields Y_i, $1 \leq i \leq 3$, are linearly independent on open dense subset of the space $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_3)$ for every $U_1 \in \mathbb{C}$, in particular for $U_1 = 0$.

We compute the Lie bracket $Y_5 = [Y_2, Y_3]$ and obtain

$$Y_5 = m_{51}\frac{\partial}{\partial \omega_1} + m_{52}\frac{\partial}{\partial \omega_2} + m_{53}\frac{\partial}{\partial \omega_3} + m_{54}\frac{\partial}{\partial \gamma_3},$$

where

$$\begin{aligned}
m_{51} = I_3\omega_2\big[&9I_3c_1\omega_1^2\omega_2\omega_3 - 9I_3c_1\omega_2^3\omega_3 - 4I_3c_2\omega_1^3\omega_3 + 14I_3c_2\omega_1\omega_2^2\omega_3\\
&+ 2c_1c_2\omega_1^2\gamma_3 + c_1c_2\omega_2^2\gamma_3 - (3c_1^2 + 2c_2^2)\omega_1\omega_2\gamma_3\big],\\
m_{52} = I_3\omega_2\big[&-2I_3c_1\omega_1^3\omega_3 + 16I_3c_1\omega_1\omega_2^2\omega_3 - 15I_3c_2\omega_1^2\omega_2\omega_3 + 3I_3c_2\omega_2^3\omega_3\\
&- 2c_1^2\omega_1^2\gamma_3 - c_1c_2\omega_1\omega_2\gamma_3 + (-4c_1^2 - 3c_2^2)\omega_2^2\gamma_3\big],
\end{aligned}$$

$$\begin{aligned} m_{53} = {} & I_3^2 c_1\omega_1^3\omega_3^2 - 17I_3^2 c_1\omega_1\omega_2^2\omega_3^2 + 9I_3^2 c_2\omega_1^2\omega_2\omega_3^2 - 9I_3^2 c_2\omega_2^3\omega_3^2 \\ & + 4I_3 c_1 c_2\omega_1\omega_2\omega_3\gamma_3 + c_1^3\omega_1\gamma_3^2 + c_1^2 c_2\omega_2\gamma_3^2 - (c_1^2 - 3c_2^2)I_3\omega_2^2\omega_3\gamma_3 \\ & + 2c_1^2 U_1\omega_1^2 - 2c_1^2 U_1\omega_2^2 + 4c_1 c_2 U_1\omega_1\omega_2, \\ m_{54} = {} & I_3\big[I_3^2\omega_1^4\omega_3^2 - 2I_3^2\omega_1^2\omega_2^2\omega_3^2 - 3I_3^2\omega_2^4\omega_3^2 - 20I_3 c_1\omega_1\omega_2^2\omega_3\gamma_3 \\ & + 14I_3 c_2\omega_1^2\omega_2\omega_3\gamma_3 - 6I_3 c_2\omega_2^3\omega_3\gamma_3 + c_1^2\omega_1^2\gamma_3^2 + 2c_1 U_1\omega_1^3 \\ & - 4c_1 U_1\omega_1\omega_2^2 + 4c_2 U_1\omega_1^2\omega_2 - 2c_2 U_1\omega_2^3 + (4c_1^2 + 3c_2^2)\omega_2^2\gamma_3^2\big]. \end{aligned}$$

Equations (5.6)–(5.7) imply that $Y_5(F) = 0$. In this way we obtain the following four equations

$$Y_1(F) = Y_2(F) = Y_3(F) = Y_5(F) = 0. \tag{5.9}$$

If a supplementary partial first integral F exists, system (5.9) has at least one non-zero solution. We consider the 4×4 matrix B of the coefficients of this system and look for such values of the parameters for which

$$\operatorname{rank} B \leq 3. \tag{5.10}$$

We have

$$\det(B) = -3840 I_3^7\omega_2^4 U_1 (c_2\omega_1 - c_1\omega_2)^3 (I_3\omega_1^2\omega_3 - c_1\omega_1\gamma_3 + I_3\omega_2^2\omega_3 - c_2\omega_2\gamma_3).$$

Thus (5.10) will be fulfilled if and only if $U_1 = 0$, because $(c_1, c_2) \neq (0, 0)$.

Let $U_1 = 0$. Thus (5.10) is fulfilled. As $Y_1,\ Y_2,\ Y_3$ are linearly independent, then Y_5 is linearly dependent on them. Moreover, as we have already mentioned, Y_4 is also linearly dependent on Y_i, $1 \leq i \leq 3$ (see (5.8)). Thus equations

$$Y_i(F) = 0, \quad 1 \leq i \leq 3, \tag{5.11}$$

are in involution. They give a system of three first order linear homogeneous partial differential equations for determining the function F. We note here that the local solvability of system (5.11) around any point $(\omega_1, \omega_2, \omega_3, \gamma_3)$ where vector fields Y_1, Y_2 and Y_3 are linearly independent, follows from the Frobenius Integrability Theorem (see [51, 52]). Hence equations (5.11) have, at least locally, a non-trivial solution. We shall now present two ways, (a) and (b), to identify F.

(a) We solve system (5.11) by the MAPLE command `pdsolve`. In this way we obtain the solution

$$F = G\left[I_3\omega_3(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3\right], \tag{5.12}$$

where G is an arbitrary smooth function. By direct computations one can verify that function $I_3\omega_3(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3$ that corresponds to $G(x) = x$ is really a first integral of system (5.3) at the conditions $I_1 = I_2 = 4I_3$, $c_3 = 0$, $U_1 = 0$, which is functionally independent of first integrals H_2 and H_3 both restricted to $\{H_1 = 0\}$. In this way, by our approach we recover the Goryachev–Chaplygin partially integrable case (1.8).

(b) Although the use of the MAPLE command `pdsolve` immediately gives a solution of system (5.11), it is not difficult to solve it by hand starting from the following simple remark.

Let us consider the following linear partial differential equation with constant coefficients

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y}=0, \tag{5.13}$$

where $p\neq 0$, q are constants and $f=f(x,y)$ is a smooth function defined on some open subset of $\mathbb{C}^2$.

A linear change of variables $u=qx-py$, $v=x$ transforms equation (5.13) into $\frac{\partial\varphi(u,v)}{\partial v}=0$, where $f(x,y)=f(v,\frac{qv-u}{p})=\varphi(u,v)$. Equation (5.13) is then transformed into $\frac{\partial\varphi(u,v)}{\partial v}=0$. The general solution of this equation is $\varphi(u,v)=\Phi(u)$, where Φ is an arbitrary smooth function. Consequently, the general solution of (5.13) is

$$f(x,y)=\Phi(qx-py). \tag{5.14}$$

Let us return to system (5.11). When $I_1=I_2=4I_3$, $c_3=0$ and $U_1=0$, one has $I_3\omega_2Y_1=Z$, where

$$Z=(c_1\omega_1+c_2\omega_2)\frac{\partial}{\partial\omega_3}+I_3(\omega_1^2+\omega_2^2)\frac{\partial}{\partial\gamma_3}.$$

$Y_1(F)=0$ if and only if $Z(F)=0$. The equation $Z(F)=0$ is of type (5.13), with $x=\omega_3$, $y=\gamma_3$, $p=c_1\omega_1+c_2\omega_2$ and $q=I_3(\omega_1^2+\omega_2^2)$. Thus by (5.14) the general solution of equation $Z(F)=0$ is given by formula (5.12). Now, all the rest is exactly the same as in (a).

Let us stress that in fact we never used the Frobenius theorem. Indeed, the desired partial first integral was obtained by direct computation.

Case 2. $c_1=c_2=0$, $c_3\neq 0$, $I_1\neq 0$, $I_2\neq 0$, $I_3\neq 0$ and U_1 are arbitrary. Now the first integral H_3 is of type (5.4). If a new integral F of this type exists, system (5.6)–(5.7) has at least two non-zero solutions. The condition under which system (5.6)–(5.7) has at least two linearly independent solutions is

$$\operatorname{rank} A\leq 2. \tag{5.15}$$

We compute the determinant D_{44} of the matrix obtained from A by crossing out its last row and last column and obtain

$$D_{44}=-I_1I_2I_3^2(I_1-I_2)c_3^2\omega_1\omega_2^3\omega_3\left[I_1(2I_1-3I_3)\omega_1^2+I_2(2I_2-3I_3)\omega_2^2-4I_3c_3\gamma_3)\right].$$

Condition (5.15) implies that D_{44} is identically equal to zero. One easily sees that as $c_3\neq 0$, the last is possible only when $I_1=I_2$, i.e. $\mathcal{I}c\in\mathcal{L}$. Thus a new partial first integral of the studied type does not exist for system (5.3).

Type 3. Here we look for a first integral of system (5.3) of type 3 $F(\omega_1,\omega_2,\gamma_1,\gamma_3)$, i.e. a first integral that does not depend on ω_3 requiring that it is functionally independent of H_2 and H_3. It satisfies the following identity

$$\frac{dF}{dt}=\frac{c_3U_1+I_2\left(I_2-I_3\right)\omega_2^2\omega_3-I_1c_3\omega_1\gamma_1-I_2c_2\omega_2\gamma_3-I_3c_3\omega_3\gamma_3}{I_1I_2\omega_2}\frac{\partial F}{\partial\omega_1}$$
$$+\frac{\left(I_3-I_1\right)\omega_1\omega_3+c_1\gamma_3-c_3\gamma_1}{I_2}\frac{\partial F}{\partial\omega_2}$$

$$+\frac{-I_1\omega_1\omega_3\gamma_1 - I_2\omega_2^2\gamma_3 - I_3\omega_3^2\gamma_3 + U_1\omega_3}{I_2\omega_2}\frac{\partial F}{\partial \gamma_1}$$
$$+\frac{I_1\omega_1^2\gamma_1 + I_2\omega_2^2\gamma_1 + I_3\omega_1\omega_3\gamma_3 - \omega_1 U_1}{I_2\omega_2}\frac{\partial F}{\partial \gamma_3} = 0,$$

or equivalently

$$I_1 I_2 \omega_2 \frac{dF}{dt} = \omega_3^2 Y_1(F) + \omega_3 Y_2(F) + Y_3(F) = 0, \tag{5.16}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_3)$

$$\begin{aligned}
Y_1 &= -I_1 I_3 \gamma_3 \frac{\partial}{\partial \gamma_1},\\
Y_2 &= (\omega_2^2 I_2^2 - I_2\omega_2^2 I_3 - c_3 I_3 \gamma_3)\frac{\partial}{\partial \omega_1} - I_1\omega_1\omega_2(-I_3 + I_1)\frac{\partial}{\partial \omega_2}\\
&\quad + (U_1 - I_1\omega_1\gamma_1)I_1\frac{\partial}{\partial \gamma_1} + \omega_1 I_3\gamma_3 I_1 \frac{\partial}{\partial \gamma_3}\\
Y_3 &= (c_3 U_1 - c_3 I_1\omega_1\gamma_1 - c_2 I_2\omega_2\gamma_3)\frac{\partial}{\partial \omega_1} + I_1\omega_2(c_1\gamma_3 - c_3\gamma_1)\frac{\partial}{\partial \omega_2}\\
&\quad - I_1\omega_2^2\gamma_3 I_2 \frac{\partial}{\partial \gamma_1} + (I_2\omega_2^2\gamma_1 - U_1\omega_1 + I_1\omega_1^2\gamma_1)I_1\frac{\partial}{\partial \gamma_3}.
\end{aligned}$$

As (5.16) is an identity with respect to all the variables and as $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ do not depend on ω_3 we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.17}$$

We compute the Lie brackets $Y_4 = [Y_2, Y_3]/I_1$ and obtain

$$\begin{aligned}
Y_4 = &\big[I_2^2 c_3\omega_2^2\gamma_1 + I_1(I_1 + I_3)c_3\omega_1^2\gamma_1 - 2I_2(I_2 - I_3)c_1\omega_2^2\gamma_3\\
&+ I_2(I_1 - 2I_3)c_2\omega_1\omega_2\gamma_3 + I_3 c_3^2\gamma_1\gamma_3 - (I_1 + I_3)U_1 c_3\omega_1\big]\frac{\partial}{\partial \omega_1}\\
&+ \big[I_1 I_3 c_1\omega_1\omega_2\gamma_3 + I_1 I_3 c_3\omega_1\omega_2\gamma_1 + I_2(I_3 - I_1)c_2\omega_2^2\gamma_3 - I_3 c_3 U_1\omega_2\big]\frac{\partial}{\partial \omega_2}\\
&+ \big[-I_1^2 c_3\omega_1\gamma_1^2 - I_1 I_2 c_2\omega_2\gamma_1\gamma_3 + I_1 I_2(I_1 - 3I_3)\omega_1\omega_2^2\gamma_3 + I_1 c_3 U_1\gamma_1\big]\frac{\partial}{\partial \gamma_1}\\
&- \big[(I_1 + I_3)I_1^2\omega_1^3\gamma_1 + (3I_1 - 2I_2 + I_3)I_1 I_2\omega_1\omega_2^2\gamma_1 + I_1 I_3 c_3\omega_1\gamma_1\gamma_3\\
&- I_2 I_3 c_2\omega_2\gamma_3^2 - I_1(I_1 + I_3)U_1\omega_1^2 - I_2(I_1 - I_2 + I_3)U_1\omega_2^2\big]\frac{\partial}{\partial \gamma_3}.
\end{aligned}$$

Equations (5.17) imply that

$$Y_4(F) = 0. \tag{5.18}$$

Equations (5.17) and (5.18) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically on any open subset of domain of definition of F, because F is non-constant on any such open subset.

If a new integral F exists then system (5.17)–(5.18) has at least one non-zero solution. Let us consider the 4×4 matrix A whose rows are the coefficients of vector fields Y_1, Y_2,

Y_3 and Y_4. We know that the condition under which system (5.17)–(5.18) has at least one non-zero solution is

$$D = \det(A) \equiv 0. \tag{5.19}$$

We compute D and obtain

$$D = I_1^2 I_2^2 I_3 \omega_2^2 \gamma_3 \widehat{D}.$$

The expression for $\widehat{D}$ is long and we do not show it here. This expression is a polynomial in variables ω_1, ω_2, γ_1 and γ_3 having 26 monomials and thus with 26 coefficients depending on $\mathcal{I}c$ and U_1. It is clear that solving (5.19) is equivalent to finding all values of the parameters $\mathcal{I}c$ and U_1 for which the 26 coefficients of $\widehat{D}$ are zero. To solve this system of 26 equations we proceed as described in Sec. 3.

After three consecutive simplifications of the source system we obtain the reduced system consisting of the following five equations:

$$c_2c_3 = 0, \quad c_1c_3 = 0, \quad (I_1 - I_2)c_3 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_2 - I_3)c_1 = 0.$$

We solve these five equations by the MAPLE command `solve` and obtain five solutions. Three of them give the Lagrange case, one - the Euler case and one - the kinetic symmetry case.

Thus a new partial first integral of type 3 does not exist.

Type 4. Now let us study the existence of a first integral of system (5.3) of type 4, i.e. $F(\omega_1, \omega_3, \gamma_1, \gamma_3)$ requiring that it is functionally independent of H_2 and H_3. We have the following identity

$$\begin{aligned}\frac{dF}{dt} &= \frac{c_3U_1 + I_2\,(I_2 - I_3)\,\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - I_2c_2\omega_2\gamma_3 - I_3c_3\omega_3\gamma_3}{I_1I_2\omega_2}\frac{\partial F}{\partial \omega_1}\\ &\quad + \frac{I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1c_1\omega_1\gamma_1 + I_2c_2\omega_2\gamma_1 + I_3c_1\omega_3\gamma_3 - c_1U_1}{I_2I_3\omega_2}\frac{\partial F}{\partial \omega_3}\\ &\quad + \frac{-I_1\omega_1\omega_3\gamma_1 - I_2\omega_2^2\gamma_3 - I_3\omega_3^2\gamma_3 + U_1\omega_3}{I_2\omega_2}\frac{\partial F}{\partial \gamma_1}\\ &\quad + \frac{I_1\omega_1^2\gamma_1 + I_2\omega_2^2\gamma_1 + I_3\omega_1\omega_3\gamma_3 - \omega_1U_1}{I_2\omega_2}\frac{\partial F}{\partial \gamma_3} = 0,\end{aligned}$$

or equivalently

$$I_1I_2I_3\omega_2\frac{dF}{dt} = I_2\omega_2^2Y_1(F) + I_2\omega_2Y_2(F) + (U_1 - I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3)Y_3(F) = 0, \tag{5.20}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_3, \gamma_1, \gamma_3)$:

$$\begin{aligned}Y_1 &= \omega_3I_3(I_2 - I_3)\frac{\partial}{\partial \omega_1} + I_1\omega_1(I_1 - I_2)\frac{\partial}{\partial \omega_3} - \gamma_3I_1I_3\frac{\partial}{\partial \gamma_1} + \gamma_1I_1I_3\frac{\partial}{\partial \gamma_3},\\ Y_2 &= c_2\left(-I_3\gamma_3\frac{\partial}{\partial \omega_1} + I_1\gamma_1\frac{\partial}{\partial \omega_3}\right)\\ Y_3 &= I_3c_3\frac{\partial}{\partial \omega_1} - I_1c_1\frac{\partial}{\partial \omega_3} + I_1I_3\omega_3\frac{\partial}{\partial \gamma_1} - I_1I_3\omega_1\frac{\partial}{\partial \gamma_3}.\end{aligned}$$

As (5.20) is an identity with respect to all the variables and as $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ do not depend on ω_2 we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.21}$$

We compute the Lie brackets $Y_4 = [Y_1, Y_2]/I_1 I_2 I_3$ and obtain

$$Y_4 = -c_2 \left(\gamma_1 \frac{\partial}{\partial \omega_1} + \gamma_3 \frac{\partial}{\partial \omega_3} \right).$$

Equations (5.21) imply that

$$Y_4(F) = 0. \tag{5.22}$$

Equations (5.21) and (5.22) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_3} \right)$, which do not vanish identically on any open subset of domain of definition of F, because F is non-constant on any such open subset.

If a new integral F exists then system (5.21)–(5.22) has at least one non-zero solution. Let us consider the 4×4 matrix A whose rows are the coefficients of vector fields Y_1, Y_2, Y_3 and Y_4. We know that the condition under which system (5.21)–(5.22) has at least one non-zero solution is

$$D = \det(A) \equiv 0.$$

We compute D and obtain

$$D = -I_1^2 I_3^2 c_2^2 \gamma_3 \omega_1 (I_1 \gamma_1^2 + I_3 \gamma_3^2).$$

This determinant is not zero if $c_2 \neq 0$. Thus in this case a new partial first integral cannot exist. We should consider the case $c_2 = 0$.

Therefore let $c_2 = 0$. We compute the Lie brackets $Y_5 = [Y_1, Y_3]/(I_1 I_3)$ and $Y_6 = [Y_1, Y_5]$. We have

$$\begin{aligned} Y_5 &= (I_2 - I_3) c_1 \frac{\partial}{\partial \omega_1} - (I_1 - I_2) c_3 \frac{\partial}{\partial \omega_3} \\ &\quad + I_1 (I_1 - I_2 - I_3) \omega_1 \frac{\partial}{\partial \gamma_1} - I_3 (I_1 + I_2 - I_3) \omega_3 \frac{\partial}{\partial \gamma_3}, \\ Y_6 &= I_3 (I_2 - I_3)(I_1 - I_2) c_3 \frac{\partial}{\partial \omega_1} - I_1 (I_1 - I_2)(I_2 - I_3) c_1 \frac{\partial}{\partial \omega_3} \\ &\quad - I_1 I_3 (I_2^2 - I_1 I_2 + 2 I_1 I_3 + I_2 I_3 - 2 I_3^2) \omega_3 \frac{\partial}{\partial \gamma_1} \\ &\quad - I_1 I_3 (2 I_1^2 - I_1 I_2 - 2 I_1 I_3 - I_2^2 + I_2 I_3) \omega_1 \frac{\partial}{\partial \gamma_3}. \end{aligned}$$

We consider the system

$$Y_1(F) = 0, \quad Y_3(F) = 0, \quad Y_5(F) = 0, \quad Y_6(F) = 0.$$

As we know its determinant δ should be zero. We compute δ and obtain

$$\delta = I_1^2 I_3^2 \widehat{\delta},$$

where

$$\widehat{\delta} = I_1^2 (I_1 - I_2)(I_1 - I_2 - I_3)(2 I_1 - 2 I_2 - I_3) c_3 \omega_1^3$$

$$
\begin{aligned}
&- I_1I_3(I_2 - I_3)(3I_1^2 - I_1I_2 - 3I_1I_3 - 2I_2^2 + 2I_2I_3)c_1\omega_1^2\omega_3 \\
&- I_1I_3(I_1 - I_2)(2I_1I_2 - 3I_1I_3 - 2I_2^2 - I_2I_3 + 3I_3^2)c_3\omega_1\omega_3^2 \\
&+ I_1(2I_1 - 2I_2 - I_3)(I_1I_2c_1^2 - I_1I_3c_1^2 - I_1I_3c_3^2 + I_2I_3c_3^2)\omega_1\gamma_3 \\
&+ I_3^2(I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)c_1\omega_3^3 \\
&+ I_3(I_1 + 2I_2 - 2I_3)(I_1I_2c_1^2 - I_1I_3c_1^2 - I_1I_3c_3^2 + I_2I_3c_3^2)\omega_3\gamma_1.
\end{aligned}
$$

It is clear that the equation $\delta = 0$ is equivalent to $\widehat{\delta} = 0$. As it is seen from the expression for $\widehat{\delta}$ it is a polynomial in variables $\omega_1, \omega_3, \gamma_1$ and γ_3 having six monomials and thus with six coefficients depending on $\mathcal{I}c$. Thus we should solve a system of six equations with respect to the parameters $\mathcal{I}c$. To solve this system we apply a simplification. After four consecutive simplifications we obtain the reduced system consisting of the following five equations:

$$
\begin{aligned}
&(I_1 - I_3)c_1c_3 = 0, \quad (I_1 - I_2)(2I_2 - I_3)c_3 = 0, \quad (2I_1 + 2I_2 - 3I_3)(I_1 - I_2)c_3 = 0, \\
&(I_2 - I_3)(2I_2 - I_3)c_1 = 0, \quad (I_1 - I_3)(I_2 - I_3)c_1 = 0.
\end{aligned}
$$

We solve these five equations by the MAPLE command `solve` and obtain the following six solutions:

$$
\begin{aligned}
&\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = 0\} \\
&\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = c_3\} \\
&\{I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = 0,\ c_3 = c_3\} \\
&\{I_1 = I_1, I_2 = I_3, I_3 = I_3, c_1 = c_1, c_3 = 0\} \\
&\{I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = c_1,\ c_3 = c_3\} \\
&\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = c_3\}.
\end{aligned}
$$

Taking into account that we consider now the case $c_2 = 0$ we see that the first solution leads to the Euler case, the second and fourth ones - to the Lagrange case. The third and fifth solutions give the Kovalevskaya case and the last one - the kinetic symmetry case.

Thus a new partial first integral of type 4 does not exist.

5.2.2. Elimination of $\boldsymbol{\omega_2}$. Let us express ω_2 from the equation $H_1 = U_1$. We have

$$
\omega_2 = \frac{U_1 - I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3}{I_2\gamma_2}. \tag{5.23}
$$

We put the expression for ω_2 from (5.23) in the Euler-Poisson equations (1.1) and remove the secon equation. In this way we obtain

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{(I_2-I_3)\omega_3\big[-I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3 + U_1\big] + c_3I_2\gamma_2^2 - c_2I_2\gamma_2\gamma_3}{I_1I_2\gamma_2},\\
\frac{d\omega_3}{dt} &= \frac{(I_1-I_2)\omega_1\big[-I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3 + U_1\big] + c_2I_2\gamma_1\gamma_2 - c_1I_2\gamma_2^2}{I_2I_3\gamma_2},\\
\frac{d\gamma_1}{dt} &= \frac{I_1\omega_1\gamma_1\gamma_3 + I_2\omega_3\gamma_2^2 + I_3\omega_3\gamma_3^2 - U_1\gamma_3}{I_2\gamma_2},\\
\frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \omega_3\gamma_1,\\
\frac{d\gamma_3}{dt} &= \frac{-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 - I_3\omega_3\gamma_1\gamma_3 + U_1\gamma_1}{I_2\gamma_2}.
\end{aligned} \tag{5.24}$$

Looking for a first integral of system (5.24) which depends on four variables indicated in brackets, we come to the following five possible cases:

1. $F(\omega_1,\omega_3,\gamma_1,\gamma_2])$, (case (iii))
2. $F(\omega_1,\omega_3,\gamma_1,\gamma_3)$, (case (ii))
3. $F(\omega_1,\omega_3,\gamma_2,\gamma_3)$, (case (iii))
4. $F(\omega_1,\gamma_1,\gamma_2,\gamma_3)$, (case (iv))
5. $F(\omega_3,\gamma_1,\gamma_2,\gamma_3)$. (case (iv))

In Sec. 5.2.1 we have already studied cases (i), (ii) and (iii) from Table 5.1. It remains only case (iv). The functions of types 4 and 5 belong to this not yet studied case. We should examine one of these two partial first integrals, it does not matter which. We choose type 4, because their study is exactly of the same nature.

Type 4. Let us study the existence of a first integral of system (5.24) of type 4, i.e. $F(\omega_1,\gamma_1,\gamma_2,\gamma_3)$ requiring that it is functionally independent of H_2 and H_3. We have

$$\begin{aligned}
\frac{dF}{dt} = {} & \frac{(I_2-I_3)\omega_3\big[-I_1\omega_1\gamma_1 - I_3\omega_3\gamma_3 + U_1\big] + c_3I_2\gamma_2^2 - c_2I_2\gamma_2\gamma_3}{I_1I_2\gamma_2}\frac{\partial F}{\partial\omega_1}\\
&+ \frac{I_1\omega_1\gamma_1\gamma_3 + I_2\omega_3\gamma_2^2 + I_3\omega_3\gamma_3^2 - U_1\gamma_3}{I_2\gamma_2}\frac{\partial F}{\partial\gamma_1} + (\omega_1\gamma_3 - \omega_3\gamma_1)\frac{\partial F}{\partial\gamma_2}\\
&+ \frac{-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 - I_3\omega_3\gamma_1\gamma_3 + U_1\gamma_1}{I_2\gamma_2}\frac{\partial F}{\partial\gamma_3} = 0,
\end{aligned}$$

or equivalently

$$I_1I_2\gamma_2\frac{dF}{dt} = \omega_3^2Y_1(F) + \omega_3Y_2(F) + Y_3(F) = 0, \tag{5.25}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1,\gamma_1,\gamma_2,\gamma_3)$:

$$\begin{aligned}
Y_1 &= -I_3(I_2-I_3)\gamma_3\frac{\partial}{\partial\omega_1},\\
Y_2 &= (I_2U_1 - I_2I_1\omega_1\gamma_1 + I_3I_1\omega_1\gamma_1 - I_3U_1)\frac{\partial}{\partial\omega_1} + I_1(I_2\gamma_2^2 + I_3\gamma_3^2)\frac{\partial}{\partial\gamma_1}\\
&\quad - \gamma_1I_2\gamma_2I_1\frac{\partial}{\partial\gamma_2} - I_3\gamma_1\gamma_3I_1\frac{\partial}{\partial\gamma_3},
\end{aligned}$$

$$\begin{aligned} Y_3 = & -I_2\gamma_2(-c_3\gamma_2 + c_2\gamma_3)\frac{\partial}{\partial \omega_1} - \gamma_3 I_1(-I_1\omega_1\gamma_1 + U_1)\frac{\partial}{\partial \gamma_1} \\ & + I_2\gamma_2 I_1\omega_1\gamma_3\frac{\partial}{\partial \gamma_2} + (-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 + U_1\gamma_1)I_1\frac{\partial}{\partial \gamma_3} \end{aligned}$$

As (5.25) is an identity with respect to all the variables and as $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ do not depend on ω_3 we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.26}$$

Equations (5.26) can be considered as a system of three homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically on any open subset of domain of definition of F, because F is non-constant on any such open subset.

It is clear that the first integral H_2 is of type 4 and therefore $\operatorname{grad} H_2$ is a solution of system (5.26). If a new integral F exists then system (5.26) has at least two non-zero solutions. This is possible if and only if

$$\operatorname{rank} A \le 2, \tag{5.27}$$

where A is the 3×4 matrix whose rows are the coefficients of vector fields Y_1, Y_2 and Y_3.

Let us consider the 3×3 matrix A_{123} obtained from A by crossing out it last column. A necessary condition for the fulfillment of (5.27) is

$$D_{123} = \det(A_{123}) = 0.$$

We compute D_{123} and obtain

$$D_{123} = I_1^2 I_2 I_3(I_2 - I_3)\gamma_2\gamma_3^2(-I_1\omega_1\gamma_1^2 - I_2\omega_1\gamma_2^2 - I_3\omega_1\gamma_3^2 + U_1\gamma_1).$$

It is easily seen that $D_{123} = 0$ is possible if and only if $I_2 = I_3$. At this last condition we compute the Lie bracket $Y_4 = [Y_2, Y_3]/(I_1 I_3)$ and obtain

$$\begin{aligned} Y_4 = & \, 2I_3(-c_3\gamma_2 + c_2\gamma_3)\gamma_1\gamma_2\frac{\partial}{\partial \omega_1} + I_1\gamma_3\Big[I_1\omega_1(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) - U_1\gamma_1\Big]\frac{\partial}{\partial \gamma_1} \\ & + I_1\gamma_2\gamma_3\Big[\omega_1\gamma_1(I_1 - I_3) - U_1\Big]\frac{\partial}{\partial \gamma_2} \\ & + I_1\Big[(I_3 - 2I_1)\omega_1\gamma_1\gamma_2^2 - I_1\omega_1\gamma_1(\gamma_1^2 + \gamma_3^2) + U_1(\gamma_1^2 + \gamma_2^2)\Big]\frac{\partial}{\partial \gamma_3}. \end{aligned}$$

Now $Y_1 = 0$ and we consider the following system:

$$Y_2(F) = Y_3(F) = Y_4(F) = 0.$$

By the same reason as above we should require that

$$\operatorname{rank} B \le 2, \tag{5.28}$$

where B is the 3×4 matrix whose rows are the coefficients of vector fields Y_2, Y_3 and Y_4.

We consider the 3×3 matrix B_{123} obtained from B by crossing out it last column. Condition (5.28) implies

$$\widehat{D}_{123} = \det(B_{123}) = 0. \tag{5.29}$$

Computing $\widehat{D}_{123}$ we obtain

$$\widehat{D}_{123} = I_1^2 I_3^2 (c_3\gamma_2 - c_2\gamma_3)\gamma_2^2\gamma_3\big[-3I_1\omega_1\gamma_1^3 - (2I_1 + I_3)\omega_1\gamma_1(\gamma_2^2 + \gamma_3^2) + U_1(3\gamma_1^2 + \gamma_2^2 + \gamma_3^2)\big].$$

One immediately sees that the condition (5.29) leads to $c_2 = c_3 = 0$ which together with $I_2 = I_3$ leads to the Lagrange case. Thus a new partial first integral of type 4 does not exist.

The results from Secs. 5.2.1 and 5.2.2 show that we have completely studied all the four cases of the basis (5.1). Now from Theorem 2.2 we conclude that outside of the four integrable cases of the Euler-Poisson equations (1.1), outside of the Goryachev-Chaplygin case ($I_1 = I_2 = 4I_3,\ (c_1, c_2) \neq (0,0),\ c_3 = 0$ or $I_1 = I_3 = 4I_2,\ (c_1, c_3) \neq (0,0),\ c_2 = 0$ or $I_2 = I_3 = 4I_1,\ (c_2, c_3) \neq (0,0),\ c_1 = 0$), the Euler-Poisson equations restricted to the invariant manifold $\{H_1 = U_1\}$ never have a local partial first integral depending on at most four variables and functionally independent of H_2 and H_3.

5.3. Invariant manifold $\{H_2{=}0\}$. We will now study what happens on submanifold $\{H_2 = 0\}$. Here we proceed as in Sec. 5.2. We should stress the following easily seen but important fact that now a first integral belonging to case (iv) from Table 5.1 does not exist because all possible eliminations from the equation $H_2 = 0$ are eliminations of some γ_i, $1 \le i \le 3$. We consider here the elimination of γ_3. The completely analogous results concerning the elimination of γ_1 or γ_2 follows from Theorem 2.2. But they can also be obtained by exactly the same way as the elimination of γ_3 that we describe below.

Let us express γ_3 from the equation $H_2 = 0$. We obtain

$$\gamma_3 = \sqrt{-\gamma_1^2 - \gamma_2^2}. \tag{5.30}$$

γ_3 is now considered as an algebraic function (see Sec. 4) of variables (γ_1, γ_2).

Putting the expression for γ_3 from (5.30) in the Euler-Poisson equations (1.1) and removing the sixth equation we have

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{(I_2 - I_3)\,\omega_2\omega_3 + c_3\gamma_2 - c_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1},\\
\frac{d\omega_2}{dt} &= \frac{(I_3 - I_1)\,\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2},\\
\frac{d\omega_3}{dt} &= \frac{(I_1 - I_2)\,\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2}{I_3},\\
\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2},\\
\frac{d\gamma_2}{dt} &= \omega_1\sqrt{-\gamma_1^2 - \gamma_2^2} - \omega_3\gamma_1.
\end{aligned} \tag{5.31}$$

Looking for a first integral of system (5.31) which depends on at most four variables we come to the following five possible cases:

1. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$, (case (i))
2. $F(\omega_1, \omega_2, \omega_3, \gamma_2)$, (case (i))
3. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, (case (ii))
4. $F(\omega_1, \omega_3, \gamma_1, \gamma_2)$, (case (iii))

5. $F(\omega_2, \omega_3, \gamma_1, \gamma_2)$. (case (iii))

Then it suffices to examine here the functions of types 1, 3 and 5.

Type 1. Here we use the idea from [59] applied there for the proof of Theorem 1.1.B. Let us look for a first integral of system (5.31) that is of type 1, $F(\omega_1, \omega_2, \omega_3, \gamma_1)$, i.e. which does not depend on γ_2 and which is functionally independent of H_1 and H_3. Thus F satisfies the following identity

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{(I_2 - I_3)\omega_2\omega_3 + c_3\gamma_2 - c_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1}\frac{\partial F}{\partial \omega_1} \\ & + \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\frac{\partial F}{\partial \omega_2} \\ & + \frac{(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2}{I_3}\frac{\partial F}{\partial \omega_3} + \left(\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\right)\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

or equivalently

$$\frac{dF}{dt} = \gamma_2 Y_1(F) + \sqrt{-\gamma_1^2 - \gamma_2^2}\; Y_2(F) + Y_3(F) = 0, \tag{5.32}$$

where Y_1, Y_2 and Y_3 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$

$$\begin{aligned} Y_1 &= \frac{c_3}{I_1}\frac{\partial}{\partial \omega_1} - \frac{c_1}{I_3}\frac{\partial}{\partial \omega_3} + \omega_3\frac{\partial}{\partial \gamma_1}, \\ Y_2 &= -\frac{c_2}{I_1}\frac{\partial}{\partial \omega_1} + \frac{c_1}{I_2}\frac{\partial}{\partial \omega_2} - \omega_2\frac{\partial}{\partial \gamma_1}, \\ Y_3 &= \frac{(I_2 - I_3)\omega_2\omega_3}{I_1}\frac{\partial}{\partial \omega_1} + \frac{(I_3 - I_1)\omega_1\omega_3 - c_3\gamma_1}{I_2}\frac{\partial}{\partial \omega_2} + \frac{(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1}{I_3}\frac{\partial}{\partial \omega_3}. \end{aligned} \tag{5.33}$$

Let us write (5.32) in the following way

$$\gamma_2 Y_1(F) + Y_3(F) = -\sqrt{-\gamma_1^2 - \gamma_2^2}\; Y_2(F).$$

Raising the last equation to the second degree we obtain

$$\gamma_2^2\left[Y_1(F)^2 + Y_2(F)^2\right] + 2\gamma_2 Y_1(F)Y_3(F) + \gamma_1^2 Y_2(F)^2 + Y_3(F)^2 = 0, \tag{5.34}$$

where $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$ depend only on $(\omega_1, \omega_2, \omega_3, \gamma_1)$.

As (5.32) is an identity with respect to all the variables ω_1, ω_2, ω_3, γ_1 and γ_2 the same concerns (5.34). Moreover (5.34) is a polynomial with respect to γ_2 because the coefficients of the powers of γ_2 do not depend on γ_2.

Let us fix ω_1, ω_2, ω_3 and $\gamma_1 \neq 0$. We prove that

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{5.35}$$

For this purpose, we examine the polynomial (5.34) studying separately two cases.

A) The first two coefficients of (5.34) vanish. That means that

$$Y_1(F)^2 + Y_2(F)^2 = 0, \quad Y_1(F)Y_3(F) = 0. \tag{5.36}$$

Thus either $Y_1(F) = 0$ or $Y_3(F) = 0$. If $Y_1(F) = 0$, then from first equation of (5.36) one obtains $Y_2(F) = 0$ and thus from (5.34), $Y_3(F) = 0$. If $Y_3(F) = 0$, then from (5.34) one has $\gamma_1^2 Y_2(F)^2 = 0$. As $\gamma_1^2 \neq 0$, $Y_2(F) = 0$ and thus also $Y_1(F) = 0$. Thus in case(A) (5.35) holds.

B) At least one of the first two coefficients of (5.34) is non-vanishing. In this case (5.34) is a first or second order non-zero polynomial in γ_2. For fixed (ω, γ_1) such a polynomial admits at most two roots. But this contradicts the fact that for these (ω, γ_1), (5.34) is identically satisfied for all γ_2.

This proves that case (B) cannot occur and consequently that (5.34) implies (5.35).

Let us compute the Lie bracket $Y_4 = [Y_2, Y_3]$. We obtain

$$Y_4 = \frac{(I_2 - I_3)c_1\omega_3}{I_1 I_2}\frac{\partial}{\partial \omega_1} + \frac{(I_1 - I_3)c_2\omega_3 + I_1 c_3 \omega_2}{I_1 I_2}\frac{\partial}{\partial \omega_2} + \frac{I_1(I_1 - I_2)c_1\omega_1 + I_2(I_2 - 2I_1)c_2\omega_2}{I_1 I_2 I_3}\frac{\partial}{\partial \omega_3} + \frac{(I_3 - I_1)\omega_1\omega_3 - c_3\gamma_1}{I_2}\frac{\partial}{\partial \gamma_1}.$$

Equations (5.35) imply that $Y_4(F) = 0$ so that we have the following system

$$Y_1(F) = Y_2(F) = Y_3(F) = Y_4(F) = 0. \tag{5.37}$$

Equations (5.37) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}\right)$, which do not vanish identically.

As in Sec. 5.2, we should equate to zero the determinant D of the 4×4 matrix A of the coefficients of system (5.37). We compute D and obtain

$$\begin{aligned} D = \frac{1}{I_1^2 I_2^2 I_3} \big[& I_1 I_2 (I_1 - I_2) c_3^2 \omega_1 \omega_2^3 - I_1 I_2 (2I_1 - I_2 - I_3) c_2 c_3 \omega_1 \omega_2^2 \omega_3 \\ & + I_1 I_2 (I_1 - I_3) c_2^2 \omega_1 \omega_2 \omega_3^2 + I_1 I_2 (I_2 - I_3) c_1 c_3 \omega_2^3 \omega_3 \\ & + I_1 (I_1 - I_3)(I_2 - I_3) c_1 c_2 \omega_1^2 \omega_3^2 + I_1 (I_2 - I_3) c_1 c_2 c_3 \omega_1 \omega_3 \gamma_1 \\ & - I_2 (I_2 - I_3)(I_1 - I_2 + I_3) c_1 c_2 \omega_2^2 \omega_3^2 - I_2 (I_1 - I_2) c_2 c_3^2 \omega_2^2 \gamma_1 \\ & + I_2 (2I_1 - I_2 - I_3) c_2^2 c_3 \omega_2 \omega_3 \gamma_1 - \left(I_1 I_2 c_1^2 + I_1 I_2 c_2^2 - I_1 I_3 c_1^2 - I_2 I_3 c_2^2\right) c_2 \omega_3^2 \gamma_1 \, \big]. \end{aligned}$$

It is identically equal to zero and therefore all of its coefficients should be zeros. D is a polynomial in variables ω_1, ω_2, ω_3 and γ_1 having ten monomials and thus with ten coefficients depending on $\mathcal{I}c$. It is clear that to solve equation $D = 0$ is equivalent to finding all values of the parameters $\mathcal{I}c$ for which the ten coefficients of D are zero. To solve this system of ten equations we proceed as in Sec. 5.2.

After three consecutive simplifications of the source system we obtain the reduced system having five equations:

$$\begin{aligned} &(I_1 - I_2)c_3 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_2 - I_3)c_2 c_3 = 0, \\ &(I_2 - I_3)c_1 c_3 = 0, \quad (I_2 - I_3)c_1 c_2 = 0. \end{aligned}$$

We solve these five equations by the MAPLE command `solve` and obtain the following four solutions:

1. $I_1 = I_2, \; c_1 = c_2 = 0,$
2. $I_1 = I_3, \; c_1 = c_3 = 0,$
3. $I_1 = I_2 = I_3,$
4. $c_2 = c_3 = 0.$

The first three of them lead to the Lagrange and kinetic symmetry cases. We should study only the fourth solution.

Let $c_2 = c_3 = 0$. In this case Y_4 is dependent on $Y_1,\ Y_2$ and Y_3 and system (5.35) has a solution $\operatorname{grad} H_3$. However, H_3 is not a fourth integral. Thus, if a fourth integral F exists, system (5.35) has at least two linearly independent solutions. We consider the 3×4 matrix A of the coefficients of this system. It is clear that our problem has a solution if and only if

$$\operatorname{rank} A \leq 2. \tag{5.38}$$

Now we are going to study when (5.38) is fulfilled. For this purpose we calculate all possible determinants of order three which can be obtained from the matrix A. For $1 \leq i \leq 4$, by D_i, we denote the determinant obtained from matrix A by crossing out its i-th column. We have

$$D_1 = -\frac{(I_2 - I_3)\, c_1}{I_2 I_3}\omega_1\omega_2\omega_3, \quad D_2 = \frac{(I_2 - I_3)\, c_1}{I_1 I_3}\omega_2^2\omega_3,$$
$$D_3 = -\frac{(I_2 - I_3)\, c_1}{I_1 I_2}\omega_2\omega_3^2, \quad D_4 = \frac{(I_2 - I_3)\, c_1^2}{I_1 I_2 I_3}\omega_2\omega_3.$$

It is easy to see that the equations $D_i = 0$, $1 \leq i \leq 4$, are satisfied only if either $c_1 = 0$ which with the condition $c_2 = c_3 = 0$ leads to the Euler case or $I_2 = I_3$ which leads to the Lagrange case.

Thus a new partial first integral of type 1, i.e. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$ does not exist.

Type 3. Let us look for a first integral of the system (5.31) that is of type 3, $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, i.e. which does not depend on ω_3 and which is functionally independent of H_1 and H_3. Thus F satisfies the following identity

$$\begin{aligned}\frac{dF}{dt} &= \frac{(I_2 - I_3)\,\omega_2\omega_3 - c_2\sqrt{-\gamma_1^2 - \gamma_2^2} + c_3\gamma_2}{I_1}\frac{\partial F}{\partial \omega_1} \\ &\quad + \frac{(I_3 - I_1)\,\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\frac{\partial F}{\partial \omega_3} \\ &\quad + \left(\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\right)\frac{\partial F}{\partial \gamma_1} + \left(\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2} - \omega_3\gamma_1\right)\frac{\partial F}{\partial \gamma_2} = 0,\end{aligned}$$

which can be presented in the following way

$$\frac{dF}{dt} = \omega_3 Y_1(F) + Y_2(F) = 0, \tag{5.39}$$

where Y_1 and Y_2 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_2)$

$$Y_1 = \frac{(I_2 - I_3)\,\omega_2}{I_1}\frac{\partial}{\partial \omega_1} + \frac{(I_3 - I_1)\,\omega_1}{I_2}\frac{\partial}{\partial \omega_2} + \gamma_2\frac{\partial}{\partial \gamma_1} - \gamma_1\frac{\partial}{\partial \gamma_2},$$
$$\begin{aligned}Y_2 &= -\frac{c_2\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_2}{I_1}\frac{\partial}{\partial \omega_1} + \frac{c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\frac{\partial}{\partial \omega_2} \\ &\quad - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\frac{\partial}{\partial \gamma_1} + \omega_1\sqrt{-\gamma_1^2 - \gamma_2^2}\frac{\partial}{\partial \gamma_2}.\end{aligned}$$

As (5.39) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on ω_3 we have

$$Y_1(F) = Y_2(F) = 0. \tag{5.40}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$\begin{aligned}
Y_3 &= \frac{(I_3 - I_2)c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - I_3c_3\gamma_1}{I_1I_2}\frac{\partial}{\partial\omega_1} - \frac{(I_1 - I_3)c_2\sqrt{-\gamma_1^2 - \gamma_2^2} + I_3c_3\gamma_2}{I_1I_2}\frac{\partial}{\partial\omega_2}\\
&+ \frac{(I_1 - I_2 - I_3)\,\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2}\frac{\partial}{\partial\gamma_1} - \frac{(I_1 - I_2 + I_3)\,\omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1}\frac{\partial}{\partial\gamma_2},\\
Y_4 &= \frac{I_3c_3\,(I_2 - I_1 - I_3)\,\gamma_2 + (I_1 - I_3)\,(I_2 - I_3)\,c_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1^2I_2}\frac{\partial}{\partial\omega_1}\\
&+ \frac{I_3c_3\,(I_2 - I_1 + I_3)\,\gamma_1 - (I_1 - I_3)\,(I_2 - I_3)\,c_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1I_2^2}\frac{\partial}{\partial\omega_2}\\
&+ \frac{\left(2I_1I_2 + I_2I_3 - 2I_2^2 + I_3^2 - I_1I_3\right)\omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1I_2}\frac{\partial}{\partial\gamma_1}\\
&+ \frac{\left(2I_1^2 - 2I_1I_2 - I_3^2 - I_1I_3 + I_2I_3\right)\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_1I_2}\frac{\partial}{\partial\gamma_2}.
\end{aligned}$$

Equations (5.40) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{5.41}$$

Equations (5.40) and (5.41) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\gamma_1}, \frac{\partial F}{\partial\gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

If a new integral F exists, system (5.40)–(5.41) has at least one non-zero solution. As in Sec. 5.2 we consider the 4×4 matrix A of the coefficients of this system. The condition under which system (5.40)–(5.41) has at least one non-zero solution is $\operatorname{rank} A \le 3$.

Therefore we equate to zero the determinant $D = \det(A)$ and study when identity

$$D \equiv 0 \tag{5.42}$$

is fulfilled. We compute D and obtain

$$D = \frac{\gamma_1^2 + \gamma_2^2}{I_2^3I_3^3}\widehat{D},$$

where

$$\widehat{D} = D_1\sqrt{-\gamma_1^2 - \gamma_2^2} + D_2.$$

The expressions for D_1 and D_2 are polynomials in variables ω_1, ω_2, γ_1 and γ_2.

It is clear that (5.42) is equivalent to $\widehat{D} = 0$, that is $D_1\sqrt{-\gamma_1^2 - \gamma_2^2} + D_2 = 0$. If $D_1 = 0$ identically, $D_2 = 0$ identically too. Let us suppose that $D_1 \neq 0$. Then we have

$$\sqrt{-\gamma_1^2 - \gamma_2^2} = -\frac{D_2}{D_1}. \tag{5.43}$$

Applying Proposition 4.3 to $V = -\gamma_1^2 - \gamma_2^2$ one sees that (5.43) can never occur because $\sqrt{V} \notin \mathbb{C}(\gamma_1, \gamma_2)$. Consequently $D_1 = D_2 = 0$. Thus we require that all the coefficients of D_1 and D_2 be zero. First we consider polynomial D_1. It has six monomials and thus six coefficients depending on $\mathcal{I}c$. We want to find all values of the parameters $\mathcal{I}c$ for which the six coefficients of D_1 are zero, i.e.

$$\begin{aligned}
&I_1^2(I_1-I_3)(2I_1-I_2-2I_3)(I_1-I_2-I_3)c_2=0,\\
&I_1I_2(I_2-I_3)(3I_1^2-3I_1I_2-I_1I_3+2I_2I_3-2I_3^2)c_1=0\\
&I_1I_2(I_1-I_3)(3I_1I_2-2I_1I_3-3I_2^2+I_2I_3+2I_3^2)c_2=0\\
&-I_1(2I_1^2I_2c_1^2+2I_1^2I_2c_2^2-2I_1^2I_3c_1^2-I_1I_2^2c_1^2-I_1I_2^2c_2^2-I_1I_2I_3c_1^2\\
&-4I_1I_2I_3c_2^2+2I_1I_2I_3c_3^2+2I_1I_3^2c_1^2+I_2^2I_3c_2^2-2I_2^2I_3c_3^2+2I_2I_3^2c_2^2)=0,\\
&-I_2^2(I_2-I_3)(I_1-I_2+I_3)(I_1-2I_2+2I_3)c_1=0,\\
&-I_2(I_1^2I_2c_1^2+I_1^2I_2c_2^2-I_1^2I_3c_1^2+2I_1^2I_3c_3^2-2I_1I_2^2c_1^2-2I_1I_2^2c_2^2\\
&+I_1I_2I_3c_2^2-2I_1I_2I_3c_3^2+4I_1I_2I_3c_1^2-2I_1I_3^2c_1^2+2I_2^2I_3c_2^2-2I_2I_3^2c_2^2)=0.
\end{aligned}$$

After five consecutive simplifications we obtain the reduced system that consists of seven equations, that is

$$\begin{aligned}
&(I_1-I_2)c_3=0,\quad (I_1-I_2)c_1c_2=0,\quad (I_1-I_3)(I_2-2I_3)c_2=0,\\
&(I_1-I_3)(I_1-2I_3)c_2=0,\quad (I_2-I_3)(I_2-2I_3)c_1=0,\quad (I_2-I_3)(I_1-2I_3)c_1=0,\\
&(I_2-I_3)(I_2-2I_3)c_2c_3=0.
\end{aligned}$$

Solving this system by the MAPLE command `solve` we obtain six solutions. Removing the solutions that lead to the Euler, Lagrange, Kovalevskaya and kinetic symmetry cases it remains only one solution

$$I_1=2I_3,\ I_2=2I_3,\ I_3,\ c_1,\ c_2,\ c_3 \text{ are arbitrary.}$$

Thus we should consider only the case

$$I_1=I_2=2I_3. \tag{5.44}$$

At the condition (5.44) we have also $D_2=0$ and therefore vector fields Y_i, $1\le i\le 4$, are linearly dependent (they satisfy equation $4Y_4+Y_2=0$). That is why we compute the Lie bracket $Y_5=[Y_2,Y_3]$ and obtain

$$\begin{aligned}
Y_5=&\frac{\omega_1(c_2\gamma_1+c_1\gamma_2)+2c_3\omega_2\sqrt{-\gamma_1^2-\gamma_2^2}+\omega_2(c_2\gamma_2-c_1\gamma_1)}{4I_3}\frac{\partial}{\partial\omega_1}\\
&-\frac{\omega_1(c_1\gamma_1-c_2\gamma_2)+2c_3\omega_1\sqrt{-\gamma_1^2-\gamma_2^2}+\omega_2(c_2\gamma_1+c_1\gamma_2)}{4I_3}\frac{\partial}{\partial\omega_2}\\
&+\frac{I_3\omega_1^2+I_3\omega_2^2-c_3\sqrt{-\gamma_1^2-\gamma_2^2}}{2I_3}\left(\gamma_2\frac{\partial}{\partial\gamma_1}-\gamma_1\frac{\partial}{\partial\gamma_2}\right)
\end{aligned}$$

and consider the following four equations:

$$Y_i(F)=0,\ 1\le i\le 3,\quad Y_5(F)=0. \tag{5.45}$$

As above we equate to zero the determinant $\Delta=\det(B)$, where B is the matrix of the coefficients of system (5.45) and study when the identity

$$\Delta\equiv 0$$

is fulfilled. We compute Δ and obtain

$$\Delta = -\frac{\sqrt{-\gamma_1^2 - \gamma_2^2}}{8I_3^2}\widehat{\Delta},$$

where

$$\begin{aligned}\widehat{\Delta} = c_3 \big(&c_2\omega_1^2\gamma_1^3 + c_1\omega_1^2\gamma_1^2\gamma_2 + c_2\omega_1^2\gamma_1\gamma_2^2 + c_1\omega_1^2\gamma_2^3 - 2c_1\omega_1\omega_2\gamma_1^3 + 2c_2\omega_1\omega_2\gamma_1^2\gamma_2 \\ &- 2c_1\omega_1\omega_2\gamma_1\gamma_2^2 + 2c_2\omega_1\omega_2\gamma_2^3 - c_2\omega_2^2\gamma_1^3 - c_1\omega_2^2\gamma_1^2\gamma_2 - c_2\omega_2^2\gamma_1\gamma_2^2 - c_1\omega_2^2\gamma_2^3\big)\,.\end{aligned}$$

It is clear that the equation $\Delta = 0$ is equivalent to $\widehat{\Delta} = 0$. It is easily seen from the expression for $\widehat{\Delta}$ that $\widehat{\Delta}$ vanishes identically only if $c_3 = 0$ or if $c_1 = c_2 = 0$. Taking into account the condition (5.44) we see that if $c_3 = 0$ we come to the Kovalevskaya case and if $c_1 = c_2 = 0$ - to the Lagrange case.

Thus a new partial first integral of type 3, i.e. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$ can only exist in the two cases known above.

Type 5. Let us look for a first integral of the system (5.31) that is of type 5, $F(\omega_2, \omega_3, \gamma_1, \gamma_2)$, i.e. which does not depend on ω_1 and which is functionally independent of H_1 and H_3. Thus F satisfies the following identity

$$\begin{aligned}\frac{dF}{dt} = &\frac{(I_3 - I_1)\,\omega_1\omega_3 + c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\frac{\partial F}{\partial \omega_2} \\ &+ \frac{(I_1 - I_2)\,\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2}{I_3}\frac{\partial F}{\partial \omega_3} \\ &+ \left(\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\right)\frac{\partial F}{\partial \gamma_1} + \left(\omega_1\sqrt{-\gamma_1^2 - \gamma_2^2} - \omega_3\gamma_1\right)\frac{\partial F}{\partial \gamma_2} = 0,\end{aligned}$$

which can be presented in the following way

$$\frac{dF}{dt} = \omega_1 Y_1(F) + Y_2(F) = 0, \tag{5.46}$$

where Y_1 and Y_2 are the following vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_2)$

$$\begin{aligned}Y_1 &= \frac{(I_3 - I_1)\,\omega_3}{I_2}\frac{\partial}{\partial \omega_2} + \frac{(I_1 - I_2)\,\omega_2}{I_3}\frac{\partial}{\partial \omega_3} + \sqrt{-\gamma_1^2 - \gamma_2^2}\frac{\partial}{\partial \gamma_2}, \\ Y_2 &= \frac{c_1\sqrt{-\gamma_1^2 - \gamma_2^2} - c_3\gamma_1}{I_2}\frac{\partial}{\partial \omega_2} + \frac{c_2\gamma_1 - c_1\gamma_2}{I_3}\frac{\partial}{\partial \omega_3} \\ &\quad + \left(\omega_3\gamma_2 - \omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\right)\frac{\partial}{\partial \gamma_1} - \omega_3\gamma_1\frac{\partial}{\partial \gamma_2}.\end{aligned}$$

As (5.46) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on ω_1 we have

$$Y_1(F) = Y_2(F) = 0. \tag{5.47}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$\begin{aligned}Y_3 = &\frac{(I_1 - I_3)\,c_2\gamma_1 - I_1c_1\gamma_2}{I_2I_3}\frac{\partial}{\partial \omega_2} + \frac{(I_1 - I_2)\,c_3\gamma_1 - I_1c_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2I_3}\frac{\partial}{\partial \omega_3} \\ &+ \frac{I_2\,(I_1 - I_2 + I_3)\,\omega_2\gamma_2 + I_3\,(I_1 + I_2 - I_3)\,\omega_3\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2I_3}\frac{\partial}{\partial \gamma_1}\end{aligned}$$

$$
\begin{aligned}
&- \frac{(I_1 - I_2 + I_3)\,\omega_2\gamma_1}{I_3}\frac{\partial}{\partial\gamma_2},\\
Y_4 = {} & \frac{(I_1 - I_2)(I_1 - I_3)\,c_3\gamma_1 - I_1(I_1 + I_2 - I_3)\,c_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2^2 I_3}\frac{\partial}{\partial\omega_2}\\
&- \frac{(I_1 - I_2)(I_1 - I_3)\,c_2\gamma_1 - I_1(I_1 - I_2 + I_3)\,c_1\gamma_2}{I_2 I_3^2}\frac{\partial}{\partial\omega_3}\\
&- \left[\frac{I_1(I_1 - I_2 + I_3) + 2I_3(I_2 - I_3)}{I_2 I_3}\omega_3\gamma_2\right.\\
&- \left.\frac{I_1(I_1 + I_2 - I_3) - 2I_2(I_2 - I_3)}{I_2 I_3}\omega_2\sqrt{-\gamma_1^2 - \gamma_2^2}\right]\frac{\partial}{\partial\gamma_1}\\
&+ \frac{I_1(I_1 - I_2 + I_3) + 2I_3(I_2 - I_3)}{I_2 I_3}\omega_3\gamma_1\frac{\partial}{\partial\gamma_2}.
\end{aligned}
$$

Equations (5.47) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{5.48}$$

Equations (5.47) and (5.48) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\omega_3}, \frac{\partial F}{\partial\gamma_1}, \frac{\partial F}{\partial\gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

If a new integral F exists, system (5.47)–(5.48) has at least one non-zero solution. As in Sec. 5.2 we consider the 4×4 matrix A of the coefficients of this system. The condition under which system (5.47)–(5.48) has at least one non-zero solution is $\operatorname{rank} A \le 3$.

Therefore we equate to zero the determinant $D = \det(A)$ and study when identity

$$D \equiv 0 \tag{5.49}$$

is fulfilled. We compute D and obtain

$$D = \frac{\gamma_1\sqrt{-\gamma_1^2 - \gamma_2^2}}{I_2^3 I_3^3}\widehat{D},$$

where

$$\widehat{D} = D_1\sqrt{-\gamma_1^2 - \gamma_2^2} + D_2.$$

The expressions for D_1 and D_2 are polynomials in variables ω_2, ω_3, γ_1 and γ_2.

It is clear that (5.49) is equivalent to $\widehat{D} = 0$, that is $D_1\sqrt{-\gamma_1^2 - \gamma_2^2} + D_2 = 0$. If $D_1 = 0$ identically, $D_2 = 0$ identically too. Let us suppose that $D_1 \neq 0$. Then we have

$$\sqrt{-\gamma_1^2 - \gamma_2^2} = -\frac{D_2}{D_1}. \tag{5.50}$$

Applying Proposition 4.3 to $V = -\gamma_1^2 - \gamma_2^2$ one sees that (5.50) can never occur because $\sqrt{V} \notin \mathbb{C}(\gamma_1, \gamma_2)$. Consequently $D_1 = D_2 = 0$. Thus we require that all the coefficients of D_1 and D_2 be zero. First we consider polynomial D_2. It has 11 monomials and thus 11 coefficients depending on $\mathcal{I}c$. We want to find all values of the parameters $\mathcal{I}c$ for which the 11 coefficients of D_2 are zero, i.e.

$$
\begin{aligned}
&I_2^2(I_1 - I_2)(2I_1 - 2I_2 + I_3)(I_1 - I_2 + I_3)c_3 = 0,\\
&2I_2 I_3(I_2 - I_3)(I_1 - I_2)(I_1 + I_2 - I_3)c_1 = 0,
\end{aligned}
$$

$$2I_3^2(I_2-I_3)(I_1-I_3)(I_1+I_2-I_3)c_1=0, \quad I_2I_3(I_2-I_3)(I_1-I_3)c_1c_2=0,$$
$$I_2I_3(I_2-I_3)(I_1-I_2)c_1c_3=0, \quad I_3^2(I_1-I_3)(I_1+I_2-I_3)(2I_1+I_2-2I_3)c_2=0,$$
$$I_2I_3(2I_1^2-4I_1I_2+3I_1I_3+2I_2^2-2I_2I_3)c_1c_3=0,$$
$$I_3(2I_1^2I_2c_2^2-4I_1I_2I_3c_2^2-2I_1I_3^2c_3^2+2I_1I_2I_3c_1^2+2I_1^2I_3c_3^2+2I_2I_3^2c_3^2-$$
$$2I_1I_2^2c_1^2+I_1I_2^2c_2^2-I_2^2I_3c_2^2-I_2^2I_3c_3^2+2I_2I_3^2c_2^2-I_1I_2I_3c_3^2)=0,$$
$$I_2I_3(2I_1-I_3)(I_1+I_2-I_3)c_1c_2=0,$$
$$I_2I_3(I_1-I_2)(2I_1^2-2I_1I_2+I_1I_3+3I_2I_3-3I_3^2)c_3=0,$$
$$I_2I_3(I_1-I_3)(2I_1^2+I_1I_2-2I_1I_3-3I_2^2+3I_2I_3)c_2=0.$$

After six consecutive simplifications we obtain the reduced system that consists of eight equations, that is

$$(I_2-I_3)c_1=0, \quad (I_2-I_3)c_3c_2=0, \quad (2I_1-I_3)c_3c_1=0,$$
$$(I_2-I_3)(I_1-I_2)c_3=0, \quad (I_1-I_2)(2I_1+2I_2-3I_3)c_3=0, \quad (2I_1-I_3)c_1c_2=0,$$
$$(I_2-I_3)(I_1-I_3)c_2=0, \quad (2I_1-I_3)(I_1-I_3)c_2=0.$$

Solving this system by the MAPLE command `solve` we obtain seven solutions. Removing the solutions that lead to the Euler, Lagrange, Kovalevskaya and kinetic symmetry cases we obtain only one solution

$$I_2=2I_1, \ I_3=2I_1, \ I_1, \ c_1, \ c_2, \ c_3 \text{ are arbitrary.}$$

Thus we should consider only the case

$$I_2=I_3=2I_1. \tag{5.51}$$

Under condition (5.51) we also have $D_1=0$ and therefore vector fields Y_i, $1\le i\le 4$, are linearly dependent (they satisfy equation $4Y_4+Y_2=0$). That is why we compose the Lie bracket $Y_5=[Y_2,Y_3]$ and obtain

$$Y_5=\frac{c_3\omega_2\gamma_2+\omega_3\left(2c_1\gamma_1-c_2\gamma_2\right)+\left(c_2\omega_2+c_3\omega_3\right)\sqrt{-\gamma_1^2-\gamma_2^2}}{4I_1}\frac{\partial}{\partial\omega_2}$$
$$-\frac{\omega_2\left(2c_1\gamma_1+c_2\gamma_2\right)+c_3\omega_3\gamma_2-\left(c_3\omega_2-c_2\omega_3\right)\sqrt{-\gamma_1^2-\gamma_2^2}}{4I_1}\frac{\partial}{\partial\omega_3}$$
$$+\frac{I_1\omega_2^2+I_1\omega_3^2-c_1\gamma_1}{2I_1}\sqrt{-\gamma_1^2-\gamma_2^2}\frac{\partial}{\partial\gamma_2}$$

and consider the following four equations:

$$Y_i(F)=0, \ 1\le i\le 3, \quad Y_5(F)=0. \tag{5.52}$$

As above we equate to zero the determinant $\Delta=\det(B)$, where B is the matrix of the coefficients of system (5.52) and study when identity

$$\Delta\equiv 0$$

is fulfilled. We compute Δ and obtain

$$\Delta=-\frac{\gamma_1^2}{8I_1^2}\widehat{\Delta},$$

where

$$\widehat{\Delta} = \Delta_1\sqrt{-\gamma_1^2 - \gamma_2^2} + \Delta_2.$$

The expressions for Δ_1 and Δ_2 are the following polynomials in variables ω_2, ω_3, γ_1 and γ_2.

$$\Delta_1 = c_1\gamma_2(-c_3\omega_2^2 + 2c_2\omega_2\omega_3 + c_3\omega_3^2), \qquad \Delta_2 = c_1(\gamma_1^2 + \gamma_2^2)(c_2\omega_2^2 + 2c_3\omega_2\omega_3 - c_2\omega_3^2).$$

As $\widehat{\Delta} = 0$, by Proposition 4.3 we have $\Delta_1 = \Delta_2 = 0$. As it is easily seen the last equations can be satisfied only in two cases: when $c_1 = 0$ which together with condition (5.51) leads to the Kovalevskaya case and when $c_2 = c_3 = 0$ that leads to the Lagrange case. The conclusion is that a partial first integral of type 5 does not exist.

5.4. Invariant manifold $\{H_3 = U_3\}$. Here we proceed as in Sec. 5.2. We first eliminate ω_3 from the equation

$$H_3 = U_3. \tag{5.53}$$

Then we study the elimination of γ_3 from (5.53). The results of these investigations are presented in the next two subsections.

5.4.1. Elimination of ω_3. We express ω_3 from (5.53) and obtain

$$\omega_3 = \sqrt{\frac{U_3 - I_1\omega_1^2 - I_2\omega_2^2 - 2c_1\gamma_1 - 2c_2\gamma_2 - 2c_3\gamma_3}{I_3}}. \tag{5.54}$$

ω_3 is now considered as an algebraic function of all its variables.

To shorten the formulas, we denote the square root of (5.54) by Ω_3 so that we have

$$\omega_3 = \Omega_3. \tag{5.55}$$

Now we insert this form of ω_3 in the Euler-Poisson equations (1.1) and remove the third equation. In this way we obtain the following system of five differential equations:

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{(I_2 - I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3}{I_1},\\
\frac{d\omega_2}{dt} &= \frac{(I_3 - I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1}{I_2},\\
\frac{d\gamma_1}{dt} &= \Omega_3\gamma_2 - \omega_2\gamma_3,\\
\frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \Omega_3\gamma_1,\\
\frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - \omega_1\gamma_2.
\end{aligned} \tag{5.56}$$

There are five possible types of first integrals of this system which depend on at most four variables. They are:

1. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, (case (ii))
2. $F(\omega_1, \omega_2, \gamma_1, \gamma_3)$, (case (iii))
3. $F(\omega_1, \omega_2, \gamma_2, \gamma_3)$, (case (iii))
4. $F(\omega_1, \gamma_1, \gamma_2, \gamma_3)$, (case (iv))
5. $F(\omega_2, \gamma_1, \gamma_2, \gamma_3)$. (case (iv))

It is then sufficient to examine here the functions of type 1, 2 and 4. Afterwards, eliminating γ_3, we will be able to study the functions belonging to case (i), the function $F(\omega_1,\omega_2,\omega_3,\gamma_1)$ in this circumstance.

Type 1. Let us look for a first integral of system (5.56) that does not depend on γ_3, i.e. of type 1. Moreover we want this integral to be functionally independent of H_1 and H_2 restricted to the invariant manifold $\{H_3 = U_3\}$. Let us suppose that the function $F(\omega_1,\omega_2,\gamma_1,\gamma_2)$ is such a first integral. Then we have

$$\begin{aligned} I_1I_2\frac{dF}{dt} &= I_2\left[(I_2-I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3\right]\frac{\partial F}{\partial\omega_1} \\ &\quad + I_1\left[(I_3-I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1\right]\frac{\partial F}{\partial\omega_2} \\ &\quad + I_1I_2\left(\Omega_3\gamma_2 - \omega_2\gamma_3\right)\frac{\partial F}{\partial\gamma_1} + I_1I_2\left(\omega_1\gamma_3 - \Omega_3\gamma_1\right)\frac{\partial F}{\partial\gamma_2} = Y_1(F) = 0, \end{aligned} \tag{5.57}$$

where Y_1 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1,\omega_2,\gamma_1,\gamma_2,\gamma_3)$.

Equation (5.57) is an identity with respect to all the five variables. F does not depend on γ_3. Thus if we differentiate this identity with respect to γ_3 we again obtain a linear partial differential equation for F. Let us note that from (5.54) and (5.55) it follows that

$$\frac{\partial\Omega_3}{\partial\gamma_3} = -\frac{c_3}{I_3\Omega_3}.$$

In this way we obtain from (5.57)

$$\begin{aligned} I_3\Omega_3\frac{\partial Y_1(F)}{\partial\gamma_3} &= -I_2(I_3c_2\Omega_3 + I_2c_3\omega_2 - I_3c_3\omega_2)\frac{\partial F}{\partial\omega_1} \\ &\quad + I_1(I_1c_3\omega_1 - I_3c_3\omega_1 + I_3c_1\Omega_3)\frac{\partial F}{\partial\omega_2} - I_1I_2(c_3\gamma_2 + I_3\omega_2\Omega_3)\frac{\partial F}{\partial\gamma_1} \\ &\quad + I_1I_2(I_3\omega_1\Omega_3 + c_3\gamma_1)\frac{\partial F}{\partial\gamma_2} = Y_2(F) = 0, \end{aligned} \tag{5.58}$$

where Y_2 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1,\omega_2,\gamma_1,\gamma_2,\gamma_3)$.

We differentiate (5.58) and obtain

$$\Omega_3\frac{\partial Y_2(F)}{\partial\gamma_3} = c_3\left(I_2c_2\frac{\partial F}{\partial\omega_1} - I_1c_1\frac{\partial F}{\partial\omega_2} + I_1I_2\omega_2\frac{\partial F}{\partial\gamma_1} - I_1I_2\omega_1\frac{\partial F}{\partial\gamma_2}\right) = Y_3(F) = 0, \tag{5.59}$$

where Y_3 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1,\omega_2,\gamma_1,\gamma_2,\gamma_3)$.

When $c_3 = 0$, Y_3 vanishes identically. That is why we consider separately two cases: $c_3 \neq 0$ and $c_3 = 0$.

First let $c_3 \neq 0$. Then we compute the Lie bracket $Y_4 = [Y_2, Y_3]/(I_1I_2c_3^2)$ and obtain

$$\begin{aligned} Y_4(F) &= -c_1(I_2-I_3)\frac{\partial F}{\partial\omega_1} - c_2(I_1-I_3)\frac{\partial F}{\partial\omega_2} \\ &\quad + I_1\omega_1(I_1-I_2-I_3)\frac{\partial F}{\partial\gamma_1} - I_2\omega_2(I_1-I_2+I_3) = 0. \end{aligned} \tag{5.60}$$

Equations (5.57)–(5.60) can be considered as a system of three homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\gamma_1}, \frac{\partial F}{\partial\gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a new first integral F exists, system (5.57)–(5.60) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of equations (5.57)–(5.60) is identically equal to zero. We compute this determinant and obtain

$$D = I_1^2 I_2^2 c_3^3 \widehat{D},$$

where

$$\begin{aligned}\widehat{D} = {} & I_1(I_1 - I_3)(I_1 - I_2 - I_3)\omega_1^3\gamma_2 - I_1(I_2 - I_3)(I_1 - I_2 - I_3)\omega_1^2\omega_2\gamma_1 \\ & - I_2(I_1 - I_3)(I_1 - I_2 + I_3)\omega_1\omega_2^2\gamma_2 + I_1c_2(I_1 - I_2 - I_3)\omega_1\gamma_1^2 \\ & - I_1c_1(I_1 - 2I_3)\omega_1\gamma_1\gamma_2 - I_2c_2(I_1 - I_3)\omega_1\gamma_2^2 + I_2(I_2 - I_3)(I_1 - I_2 + I_3)\omega_2^3\gamma_1 \\ & + I_1c_1(I_2 - I_3)\omega_2\gamma_1^2 + I_2c_2(I_2 - 2I_3)\omega_2\gamma_1\gamma_2 + I_2c_1(I_1 - I_2 + I_3)\omega_2\gamma_2^2.\end{aligned}$$

It is clear that the equation $D = 0$ is equivalent to $\widehat{D} = 0$. $\widehat{D} = 0$ has ten coefficients. The annulation of $\widehat{D} = 0$ means that all of its coefficients should be zeros. In this way we obtain a system of ten equations for the parameters $\mathcal{I}c$.

After three consecutive simplifications we come to the reduced system:

$$c_1 = 0, \quad c_2 = 0, \quad I_2 - I_3 = 0, \quad I_1 - I_3 = 0,$$

which obviously leads to a particular case of the kinetic symmetry case. Thus a new partial first integral of type 1 does not exist when $c_3 \neq 0$.

Let $c_3 = 0$. Now Ω_3 does not depend on γ_3 and $Y_1(F)$ is of the form (see (5.57))

$$Y_1(F) = Z_1(F)\gamma_3 + Z_2(F)\Omega_3, \tag{5.61}$$

where the vector fields Z_1 and Z_2, defined on $\mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_2)$, are given as follows:

$$\begin{aligned} Z_1 &= -I_2c_2\frac{\partial}{\partial\omega_1} + I_1c_1\frac{\partial}{\partial\omega_2} - I_1I_2\omega_2\frac{\partial}{\partial\gamma_1} + I_1I_2\omega_1\frac{\partial}{\partial\gamma_2}, \\ Z_2 &= I_2(I_2 - I_3)\omega_2\frac{\partial}{\partial\omega_1} + I_1(I_3 - I_1)\omega_1\frac{\partial}{\partial\omega_2} + I_1I_2\gamma_2\frac{\partial}{\partial\gamma_1} - I_1I_2\gamma_1\frac{\partial}{\partial\gamma_2}.\end{aligned}$$

Equation (5.61) implies that

$$Z_1(F) = Z_2(F) = 0. \tag{5.62}$$

We compute the Lie brackets $Z_3 = [Z_1, Z_2]/(I_1I_2)$ and $Z_4 = [Z_2, Z_3]$ and obtain

$$\begin{aligned} Z_3 = {} & (I_2 - I_3)c_1\frac{\partial}{\partial\omega_1} + (I_1 - I_3)c_2\frac{\partial}{\partial\omega_2} \\ & - I_1(I_1 - I_2 - I_3)\omega_1\frac{\partial}{\partial\gamma_1} + I_2(I_1 - I_2 + I_3)\omega_2\frac{\partial}{\partial\gamma_2}, \\ Z_4 = {} & -I_2c_2(I_2 - I_3)(I_1 - I_3)\frac{\partial}{\partial\omega_1} + I_1c_1(I_2 - I_3)(I_1 - I_3)\frac{\partial}{\partial\omega_2} \\ & - I_1I_2(2I_1I_2 - I_1I_3 - 2I_2^2 + I_2I_3 + I_3^2)\omega_2\frac{\partial}{\partial\gamma_1} \\ & - I_1I_2(2I_1^2 - 2I_1I_2 - I_1I_3 + I_2I_3 - I_3^2)\omega_1\frac{\partial}{\partial\gamma_2}.\end{aligned}$$

Equations (5.62) imply that

$$Z_3(F) = Z_4(F) = 0. \tag{5.63}$$

As in the case $c_3 \neq 0$, the system of equations (5.62) and (5.63) is a linear homogeneous system that has a non-zero solution. Thus the determinant δ of its coefficients should vanish identically. We compute δ and obtain

$$\delta = I_1^2 I_2^2 \widehat{\delta},$$

where

$$\begin{aligned}\widehat{\delta} = {} & I_1^2(I_1 - I_3)(2I_1 - I_2 - 2I_3)(I_1 - I_2 - I_3)c_2\omega_1^3 \\ & + I_1 I_2(I_2 - I_3)(2I_3 I_2 - 3I_1 I_2 + 3I_1^2 - I_3 I_1 - 2I_3^2)c_1\omega_1^2\omega_2 \\ & + I_1 I_2(I_1 - I_3)(3I_1 I_2 - 2I_1 I_3 + I_2 I_3 - 3I_2^2 + 2I_3^2)c_2\omega_1\omega_2^2 \\ & - I_1(2I_1 - I_2 - 2I_3)(I_1 I_2 c_1^2 + I_1 I_2 c_2^2 - I_1 I_3 c_1^2 - I_2 I_3 c_2^2)\omega_1\gamma_2 \\ & - I_2^2(I_2 - I_3)(I_1 - 2I_2 + 2I_3)(I_1 - I_2 + I_3)c_1\omega_2^3 \\ & - I_2(I_1 - 2I_2 + 2I_3)(I_1 I_2 c_1^2 + I_1 I_2 c_2^2 - I_1 I_3 c_1^2 - I_2 I_3 c_2^2)\omega_2\gamma_1.\end{aligned}$$

Equation $\delta = 0$ is equivalent to the equation $\widehat{\delta} = 0$. Thus the six coefficients of $\widehat{\delta}$ which should be zeros. In this way we have obtained a system of six equations for the parameters $\mathcal{I}c$. We subject it to simplification and after five consecutive simplifications we come to the reduced system consisting of the following five equations:

$$(I_1 - I_2)c_1c_2 = 0, \quad (I_1 - I_3)(I_2 - 2I_3)c_2 = 0, \quad (I_1 - I_3)(I_1 - 2I_3)c_2 = 0,$$
$$(I_2 - I_3)(I_2 - 2I_3)c_1 = 0, \quad (I_2 - I_3)(I_1 - 2I_3)c_1 = 0.$$

We solve them by the Maple command `solve` and obtain five solutions:

$$\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0\}$$
$$\{I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2\}$$
$$\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0\}$$
$$\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2\}$$
$$\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2\}.$$

Taking into account that now $c_3 = 0$ we see that the first of these solutions leads to the Euler case, the second and third ones - to the Lagrange case, the fourth solution leads to the Kovalevskaya case and the last one - to the kinetic symmetry case.

Thus a new partial first integral of type 1 does not exist also when $c_3 = 0$.

Type 2. Let us study now a first integral of type 2, i.e. $F(\omega_1, \omega_2, \gamma_1, \gamma_3)$. We have

$$\begin{aligned}I_1 I_2 \frac{dF}{dt} = {} & I_2\left[(I_2 - I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3\right]\frac{\partial F}{\partial \omega_1} \\ & + I_1\left[(I_3 - I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1\right]\frac{\partial F}{\partial \omega_2} \\ & + I_1 I_2\left(\Omega_3\gamma_2 - \omega_2\gamma_3\right)\frac{\partial F}{\partial \gamma_1} + I_1 I_2\left(\omega_2\gamma_1 - \omega_1\gamma_2\right)\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \qquad (5.64)\end{aligned}$$

where Y_1 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

Equation (5.64) is an identity with respect to all the five variables. F does not depend on γ_2. Thus if we differentiate this identity with respect to γ_2 we again obtain a linear partial differential equation for F. Let us note that

$$\frac{\partial \Omega_3}{\partial \gamma_2} = -\frac{c_2}{I_3 \Omega_3}.$$

In this way we have

$$I_3\Omega_3 \frac{\partial Y_1(F)}{\partial \gamma_2} = I_2(I_3c_3\Omega_3 - I_2c_2\omega_2 + I_3c_2\omega_2)\frac{\partial F}{\partial \omega_1} + I_1(I_1 - I_3)c_2\omega_1\frac{\partial F}{\partial \omega_2} + I_1I_2(-c_2\gamma_2 + I_3\Omega_3^2)\frac{\partial F}{\partial \gamma_1} - I_1I_2I_3\omega_1\Omega_3\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \qquad (5.65)$$

where Y_2 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

We differentiate (5.65) with respect to γ_2 and obtain

$$\frac{\Omega_3}{I_2}\frac{\partial Y_2(F)}{\partial \gamma_2} = -c_2\left(c_3\frac{\partial F}{\partial \omega_1} + 3I_1\Omega_3\frac{\partial F}{\partial \gamma_1} - I_1\omega_1\frac{\partial F}{\partial \gamma_3}\right) = Y_3(F) = 0, \qquad (5.66)$$

where Y_3 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

When $c_2 = 0$, Y_3 vanishes identically. That is why we consider separately two cases: $c_2 \neq 0$ and $c_2 = 0$.

Let first $c_2 \neq 0$. Then we differentiate (5.66) with respect to γ_2 and obtain

$$\frac{I_3\Omega_3}{3I_1c_2^2}\frac{\partial Y_3(F)}{\partial \gamma_2} = \frac{\partial F}{\partial \gamma_1} = Y_4(F) = 0. \qquad (5.67)$$

Equations (5.64)–(5.67) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a fourth integral F exists, system (5.64)–(5.67) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of equations (5.64)–(5.67) is identically equal to zero. We compute this determinant and obtain

$$D = I_1^2 I_2 c_2^2 \omega_1 \widehat{D},$$

where

$$\widehat{D} = c_2(I_1 - I_3)\omega_1\gamma_3 - c_3(I_1 - I_2)\omega_2\gamma_1 - c_1(I_2 - I_3)\omega_2\gamma_3.$$

The equation $D = 0$ is equivalent to $\widehat{D} = 0$. Thus, as $c_2 \neq 0$, the first coefficient of $\widehat{D}$ vanishes identically if and only if $I_1 = I_3$. At this condition the two remaining terms vanish either if $I_2 = I_3$ or if $c_1 = c_3 = 0$. The first possibility leads to the kinetic symmetry case and the second one - to the Lagrange case.

Thus a new partial first integral of type 2 does not exist when $c_2 \neq 0$.

Let $c_2 = 0$. Now Ω_3 does not depend on γ_2 and $Y_1(F)$ is of the form (see (5.64))

$$Y_1(F) = Z_1(F)\gamma_2 + Z_2(F)\Omega_3, \qquad (5.68)$$

where the vector fields Z_1 and Z_2, defined on $\mathbb{C}^4(\omega_1, \omega_2, \gamma_1, \gamma_3)$, are given as follows:

$$Z_1 = I_2c_3\frac{\partial}{\partial \omega_1} + I_1I_2\Omega_3\frac{\partial}{\partial \gamma_1} - I_1I_2\omega_1\frac{\partial}{\partial \gamma_3},$$

$$Z_2 = I_2(I_2 - I_3)\omega_2\Omega_3\frac{\partial}{\partial \omega_1} + I_1\left[(I_3 - I_1)\omega_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1\right]\frac{\partial}{\partial \omega_2}$$

$$- I_1 I_2 \omega_2 \gamma_3 \frac{\partial}{\partial \gamma_1} + I_1 I_2 \omega_2 \gamma_1 \frac{\partial}{\partial \gamma_3}.$$

Equation (5.68) implies that

$$Z_1(F) = Z_2(F) = 0. \tag{5.69}$$

We compute the Lie brackets $Z_3 = [Z_1, Z_2]/(I_1 I_2)$ and $Z_4 = [Z_2, Z_3]/I_2$ and obtain

$$\begin{aligned}
Z_3 &= -I_2(I_2 - I_3)c_1\omega_2 \frac{\partial}{\partial \omega_1} + [I_3 c_3 \Omega_3 (I_3 - 2I_1) + I_1 c_1 \omega_1 (I_1 - 2I_3)] \frac{\partial}{\partial \omega_2} \\
&\quad - I_1 I_2 (I_1 - I_2 - I_3)\omega_1\omega_2 \frac{\partial}{\partial \gamma_1} + I_2 I_3 (I_1 + I_2 - I_3)\omega_2 \Omega_3 \frac{\partial}{\partial \gamma_3}, \\
Z_4 &= a_1 \frac{\partial}{\partial \omega_1} + a_2 \frac{\partial}{\partial \omega_2} - a_3 \frac{\partial}{\partial \gamma_1} - a_4 \frac{\partial}{\partial \gamma_2},
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= (I_2 - I_3)[-I_1(2I_1 - I_3)c_3\omega_1^2 + I_1 I_3 c_1 \omega_1 \Omega_3 - I_2(3I_1 - I_2 - I_3)c_3\omega_2^2 \\
&\quad - (3I_1 - 2I_3)c_1 c_3 \gamma_1 - (I_1 c_1^2 + 4I_1 c_3^2 - 2I_3 c_3^2)\gamma_3 + (2I_1 - I_3)c_3 U_3], \\
a_2 &= -I_1\omega_2[I_1(2I_1 - 2I_2 - I_3)c_3\omega_1 + I_3(I_1 + 2I_2 - 2I_3)c_1\Omega_3], \\
a_3 &= I_1[I_1(I_1 - I_3)(I_1 - I_2 - I_3)\omega_1^2\Omega_3 + I_1(I_1 - I_2 - I_3)c_3\omega_1\gamma_1 + I_1(I_2 - I_3)c_1\omega_1\gamma_3 \\
&\quad - I_2(I_1 I_2 - 2I_1 I_3 - I_2^2 - I_2 I_3 + 2I_3^2)\omega_2^2\Omega_3 - I_3(2I_1 - I_3)c_3\gamma_3\Omega_3], \\
a_4 &= I_1[I_1(I_1 - I_3)(I_1 + I_2 - I_3)\omega_1^3 + I_2(3I_1^2 - 4I_1 I_3 - I_2^2 + I_3^2)\omega_1\omega_2^2 \\
&\quad + (I_1^2 + 2I_1 I_2 - 2I_1 I_3 - 2I_2 I_3 + 2I_3^2)c_1\omega_1\gamma_1 + 2(I_1 - I_3)(I_1 + I_2 - I_3)c_3\omega_1\gamma_3 \\
&\quad - (I_1 - I_3)(I_1 + I_2 - I_3)U_3\omega_1 + I_3(I_1 - I_2)c_3\gamma_1\Omega_3 + I_3(I_1 + I_2 - I_3)c_1\gamma_3\Omega_3].
\end{aligned}$$

Equations (5.69) imply that

$$Z_3(F) = Z_4(F) = 0. \tag{5.70}$$

As in the case $c_2 \neq 0$, the system of equations (5.69) and (5.70) is a linear homogeneous system that has a non-zero solution. Thus the determinant δ of its coefficients should vanish identically. We compute δ and obtain

$$\delta = I_1^2 I_2^3 \omega_2^3 \Omega_3 \widehat{\delta},$$

where the expression for $\widehat{\delta}$ has the following form:

$$\widehat{\delta} = I_3 \Omega_3 b_1 + I_1 \omega_1 b_2.$$

b_1 and b_2 are polynomials of the variables ω_1, ω_2, γ_1 and γ_3 with coefficients that depend on the parameters $\mathcal{I}c$ and U_3. They are given by the following formulas:

$$\begin{aligned}
b_1 &= -2I_1 c_1 (I_2 - I_3)(I_1 - I_3)(2I_1 + I_2 - I_3)\omega_1^2 \\
&\quad - I_2(I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)c_1\omega_2^2 \\
&\quad - (I_1 + 2I_2 - 2I_3)(I_1 I_2 c_1^2 - I_1 I_3 c_1^2 + I_1 I_3 c_3^2 + 2I_2^2 c_1^2 - 4I_2 I_3 c_1^2 - I_2 I_3 c_3^2 + 2I_3^2 c_1^2)\gamma_1 \\
&\quad - 2c_3 c_1 (I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)\gamma_3 \\
&\quad + (I_2 - I_3)(I_1 + I_2 - I_3)(I_1 + 2I_2 - 2I_3)c_1 U_3, \\
b_2 &= 2I_1(I_1 - I_3)(I_1 - I_2)(I_1 - I_2 - 2I_3)c_3\omega_1^2 \\
&\quad + I_2(I_1 - I_2)(2I_1 I_2 - 3I_1 I_3 - 2I_2^2 - I_2 I_3 + 3I_3^2)c_3\omega_2^2
\end{aligned}$$

$$+ 2(I_1 - I_2)(-3I_1I_3 + 2I_1I_2 + 3I_3^2 - I_2I_3 - 2I_2^2)c_1c_3\gamma_1$$
$$+ [(2I_1^2I_2(c_1^2 + 2c_3^2) - 2I_1^2I_3(c_1^2 + 4c_3^2) - 2I_1I_2^2(c_1^2 + 4c_3^2) + I_1I_2I_3(c_1^2 + 8c_3^2)$$
$$+ I_1I_3^2(c_1^2 + 7c_3^2) + 4I_2^3c_3^2 - 7I_2I_3^2c_3^2)]\gamma_3$$
$$- (I_1 - I_2)(2I_1I_2 - 3I_1I_3 - 2I_2^2 - I_2I_3 + 3I_3^2)c_3U_3.$$

The equation $\delta = 0$ is equivalent to $\widehat{\delta} = 0$. $\widehat{\delta}$ depends on function Ω_3 and it is easy to see that $\Omega_3 \notin \mathbb{C}(\omega_1, \omega_2, \gamma_1, \gamma_3)$. Then according to Proposition 4.3, the coefficients b_1 and b_2 of $\widehat{\delta}$ should be zeros. In this way we obtain a system of ten equations for the parameters $\mathcal{I}c$ and U_3. After four consecutive simplifications we come to the reduced system consisting of the following five equations:

$$(I_1 - I_3)c_1c_3 = 0, \quad (2I_2 - I_3)(I_1 - I_2)c_3 = 0, \quad (I_1 - I_2)(2I_1 - 3I_3 + 2I_2)c_3 = 0,$$
$$(2I_2 - I_3)(I_2 - I_3)c_1 = 0, \quad (I_2 - I_3)(I_1 - I_3)c_1 = 0.$$

We solve them by the MAPLE command `solve` and obtain six solutions:

$$\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = 0\}$$
$$\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_3 = c_3\}$$
$$\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = 0\}$$
$$\{I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = 0,\ c_3 = c_3\}$$
$$\{I_1 = I_3,\ I_2 = \frac{I_3}{2},\ I_3 = I_3,\ c_1 = c_1,\ c_3 = c_3\}$$
$$\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_3 = c_3\}.$$

Taking into account that now $c_2 = 0$ we see that the first solution leads to the Euler case, the second and third solutions lead to the Lagrange case, the fourth and fifth ones - to the Kovalevskaya case and the last one - to the kinetic symmetry case.

Thus a new partial first integral of type 2 does not exist also when $c_2 = 0$.

Type 4. Let $F(\omega_1, \gamma_1, \gamma_2, \gamma_3)$ be a new first integral of type 4. Thus we have

$$I_1\frac{dF}{dt} = [(I_2 - I_3)\omega_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3]\,\frac{\partial F}{\partial \omega_1} + I_1\,(\Omega_3\gamma_2 - \omega_2\gamma_3)\,\frac{\partial F}{\partial \gamma_1}$$
$$- I_1\,(\Omega_3\gamma_1 - \omega_1\gamma_3)\,\frac{\partial F}{\partial \gamma_2} + I_1\,(\omega_2\gamma_1 - \omega_1\gamma_2)\,\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \qquad (5.71)$$

where Y_1 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

Equation (5.71) is an identity with respect to all the five variables. F does not depend on ω_2. Thus if we differentiate this identity with respect to ω_2 we again obtain a linear partial differential equation for F. Let us note that

$$\frac{\partial \Omega_3}{\partial \omega_2} = -\frac{I_2\omega_2}{I_3\Omega_3}.$$

In this way we have

$$I_3\Omega_3\frac{\partial Y_1(F)}{\partial \omega_2} = (I_2 - I_3)(I_3\Omega_3^2 - I_2\omega_2^2)\frac{\partial F}{\partial \omega_1} - I_1(I_2\omega_2\gamma_2 + I_3\gamma_3\Omega_3)\frac{\partial F}{\partial \gamma_1}$$
$$+ I_1I_2\omega_2\gamma_1\frac{\partial F}{\partial \gamma_2} + I_1I_3\gamma_1\Omega_3\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \qquad (5.72)$$

where Y_2 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

We differentiate (5.72) with respect to ω_2 and obtain

$$\Omega_3 \frac{\partial Y_2(F)}{\partial \omega_2} = -4I_2(I_2 - I_3)\omega_2\Omega_3 \frac{\partial F}{\partial \omega_1} - I_1 I_2(\gamma_2\Omega_3 - \omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1} + I_1 I_2 \gamma_1 \Omega_3 \frac{\partial F}{\partial \gamma_2} - I_1 I_2 \omega_2 \gamma_1 \frac{\partial F}{\partial \gamma_3} = Y_3(F) = 0, \tag{5.73}$$

where Y_3 is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

Let us note that the first integral H_2 is of type 4, i.e. it satisfies system (5.71)–(5.73). Thus if a new first integral exists, then this system will have two non-zero linearly independent solutions grad H_2 and grad F. This is possible if and only if the 3×4 matrix M of the coefficients of system (5.71)–(5.73) satisfies the condition

$$\operatorname{rank} M \leq 2. \tag{5.74}$$

We compute the determinant M_{124} obtained from matrix M by crossing out its third column and obtain

$$M_{124} = I_1^2 I_2 \gamma_2 \widehat{M}_{124},$$

where

$$\widehat{M}_{124} = \Omega_3 b_1 + b_2.$$

Here the coefficients b_1 and b_2 are polynomials given by the formulas:

$$\begin{aligned} b_1 &= (I_2 - I_3)(-I_1\omega_1^3\gamma_2 + 3I_1\omega_1^2\omega_2\gamma_1 + 2I_2\omega_1\omega_2^2\gamma_2 - 2c_1\omega_1\gamma_1\gamma_2 - 2c_2\omega_1\gamma_2^2 - 2c_3\omega_1\gamma_2\gamma_3 \\ &\quad + U_3\omega_1\gamma_2 + 6c_1\omega_2\gamma_1^2 + 6c_2\omega_2\gamma_1\gamma_2 + 6c_3\omega_2\gamma_1\gamma_3 - 3U_3\omega_2\gamma_1), \\ b_2 &= -3I_1(I_2 - I_3)\omega_1^3\omega_2\gamma_3 - I_1c_3\omega_1^2\gamma_1\gamma_2 + I_1c_2\omega_1^2\gamma_1\gamma_3 - 2I_2(I_2 - I_3)\omega_1\omega_2^3\gamma_3 \\ &\quad - 6(I_2 - I_3)c_1\omega_1\omega_2\gamma_1\gamma_3 - 6(I_2 - I_3)c_2\omega_1\omega_2\gamma_2\gamma_3 - 6(I_2 - I_3)c_3\omega_1\omega_2\gamma_3^2 \\ &\quad + 3(I_2 - I_3)U_3\omega_1\omega_2\gamma_3 - 2c_1c_3\gamma_1^2\gamma_2 + 2c_1c_2\gamma_1^2\gamma_3 - 2c_2c_3\gamma_1\gamma_2^2 + 2(c_2^2 - c_3^2)\gamma_1\gamma_2\gamma_3 \\ &\quad + c_3U_3\gamma_1\gamma_2 + 2c_2c_3\gamma_1\gamma_3^2 - c_2U_3\gamma_1\gamma_3. \end{aligned}$$

Taking into account (5.74) $\widehat{M}_{124}$ should vanish identically. According to Proposition 4.3 the coefficients b_1 and b_2 should be zeros because $\Omega_3 \notin \mathbb{C}(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$. Polynomial b_1 has 11 coefficients and b_2 - 15. We only use $b_2 = 0$. In this way we obtain a system of 15 equations for the parameters $\mathcal{I}c$ and U_3. After two consecutive simplifications we come to the reduced system

$$c_2 = 0, \quad c_3 = 0, \quad I_2 - I_3 = 0.$$

It leads to the Lagrange case, and therefore a new partial first integral of type 4 does not exist.

5.4.2. Elimination of $\boldsymbol{\gamma_3}$. We now study the elimination of γ_3 from the equation (5.53). In this section we suppose that $c_3 \neq 0$ because otherwise the elimination under consideration is not possible. We obtain:

$$\gamma_3 = \frac{U_3 - I_1\omega_1^2 - I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 - 2c_2\gamma_2}{2c_3}. \tag{5.75}$$

To shorten the formulas, we denote the right-side of (5.75) by Γ_3 so that we have $\gamma_3 = \Gamma_3$.

Now we put this value of γ_3 in the Euler-Poisson equations (1.1) and remove the sixth equation. In this way we obtain the following system of five differential equations:

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{(I_2 - I_3)\omega_2\omega_3 + c_3\gamma_2 - c_2\Gamma_3}{I_1},\\
\frac{d\omega_2}{dt} &= \frac{(I_3 - I_1)\omega_1\omega_3 + c_1\Gamma_3 - c_3\gamma_1}{I_2},\\
\frac{d\omega_3}{dt} &= \frac{(I_1 - I_2)\omega_1\omega_3 + c_2\gamma_1 - c_1\gamma_2}{I_3},\\
\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\Gamma_3,\\
\frac{d\gamma_2}{dt} &= \omega_1\Gamma_3 - \omega_3\gamma_1.
\end{aligned} \tag{5.76}$$

There are five possible types of first integrals of this system which depend on at most four variables. They are:

1. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$, (case (i))
2. $F(\omega_1, \omega_2, \omega_3, \gamma_2)$, (case (i))
3. $F(\omega_1, \omega_2, \gamma_1, \gamma_2)$, (case (ii))
4. $F(\omega_1, \omega_3, \gamma_1, \gamma_2)$, (case (iii))
5. $F(\omega_2, \omega_3, \gamma_1, \gamma_2)$. (case (iii))

Considering the fact that the functions belonging to cases (ii) and (iii) have already been examined, it only remains to study a function belonging to case (i).

Type 1. Let us look for a first integral of system (5.76) of type 1, i.e. $F(\omega_1, \omega_2, \omega_3, \gamma_1)$. Moreover we want this integral to be functionally independent of H_1 and H_2 restricted to the invariant manifold $\{H_3 = U_3\}$, but this condition will play no role here. Then we have

$$\begin{aligned}
I_1I_2I_3\frac{dF}{dt} &= I_2I_3\left[(I_2 - I_3)\omega_2\omega_3 + c_3\gamma_2 - c_2\Gamma_3\right]\frac{\partial F}{\partial\omega_1}\\
&\quad + I_1I_3\left[(I_3 - I_1)\omega_1\omega_3 + c_1\Gamma_3 - c_3\gamma_1\right]\frac{\partial F}{\partial\omega_2}\\
&\quad + I_1I_2\left[(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1 - c_1\gamma_2\right]\frac{\partial F}{\partial\omega_3}\\
&\quad + I_1I_2I_3\left(\omega_3\gamma_2 - \omega_2\Gamma_3\right)\frac{\partial F}{\partial\gamma_1} = Z(F) = 0,
\end{aligned} \tag{5.77}$$

where Z is the corresponding vector field, defined on $\mathbb{C}^5(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$.

Vector field Z is of the form $Z = 2Y_1\gamma_2 + Y_2$, where the polynomial vector fields Y_1 and Y_2 are defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$ as follows:

$$Y_1 = I_2I_3(c_2^2 + c_3^2)\frac{\partial}{\partial\omega_1} - I_1I_3c_1c_2\frac{\partial}{\partial\omega_2} - I_1I_2c_1c_3\frac{\partial}{\partial\omega_3} + I_1I_2I_3(c_2\omega_2 + c_3\omega_3)\frac{\partial}{\partial\gamma_1},$$

$$\begin{aligned}
Y_2 &= I_2I_3[c_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) + 2(I_2 - I_3)c_3\omega_2\omega_3]\frac{\partial}{\partial\omega_1}\\
&\quad - I_1I_3[c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) + 2(I_1 - I_3)c_3\omega_1\omega_3 + 2c_3^2\gamma_1]\frac{\partial}{\partial\omega_2}
\end{aligned}$$

$$+ 2I_1I_2c_3[(I_1 - I_2)\omega_1\omega_2 + c_2\gamma_1]\frac{\partial}{\partial\omega_3}$$
$$+ I_1I_2I_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3)\omega_2\frac{\partial}{\partial\gamma_1}.$$

Taking into account that (5.77) should be an identity with respect to all the variables and that F does not depend on γ_2, we conclude that

$$Y_1(F) = Y_2(F) = 0. \tag{5.78}$$

We compute the Lie bracket $Y_3 = [Y_1, Y_2]/(2I_1I_2I_3)$ and obtain

$$\begin{aligned}
Y_3 = {} & [I_2I_3c_2(c_2^2 + c_3^2)\omega_1 - I_2(I_2 - I_3)c_1c_3^2\omega_2 - I_3(I_2 - I_3)c_1c_2c_3\omega_3]\frac{\partial}{\partial\omega_1} \\
& + [I_1c_1(I_1c_3^2 - I_3c_2^2 - 2I_3c_3^2)\omega_1 - I_1I_3c_2c_3^2\omega_2 \\
& - I_3(I_1c_2^2 + 2I_1c_3^2 - I_3c_2^2 - I_3c_3^2)c_3\omega_3]\frac{\partial}{\partial\omega_2} \\
& - c_3[I_1(I_1 - I_2)c_1c_2\omega_1 - I_2(2I_1c_2^2 + I_1c_3^2 - I_2c_2^2 - I_2c_3^2)\omega_2 - I_1I_2c_3c_2\omega_3]\frac{\partial}{\partial\omega_3} \\
& - I_1[I_2(I_1c_3^2 - I_2c_3^2 - I_3c_2^2 - I_3c_3^2)\omega_1\omega_2 \\
& - I_3(I_1 - I_3)c_2c_3\omega_1\omega_3 + (I_2 - I_3)c_2c_3^2\gamma_1]\frac{\partial}{\partial\gamma_1}.
\end{aligned}$$

Then we compute $Y_4 = [Y_2, Y_3]$. Unfortunately the expression for Y_4 is too long to be shown here.

Equations (5.78) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{5.79}$$

System (5.78)–(5.79) can be considered as a homogeneous linear algebraic system with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\omega_3}, \frac{\partial F}{\partial\gamma_1}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a fourth integral F exists, system (5.78)–(5.79) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of this system is identically equal to zero. We compute this determinant and obtain

$$D = I_1^2I_2^2I_3^2c_3^3\widehat{D},$$

where $\widehat{D}$ is a very long expression that we cannot show here. This expression is a polynomial of the variables ω_1, ω_2, ω_3 and γ_1 with 79 coefficients, which are polynomials of the parameters $\mathcal{I}c$ and U_3. As $c_3 \neq 0$ the equation $D = 0$ is equivalent to $\widehat{D} = 0$. Thus all the coefficients of $\widehat{D}$ should be zeros and we have to solve the corresponding system of 79 equations.

After three consecutive simplifications we come to the reduced system consisting of the following six equations:

$$(I_2 - I_3)c_2 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_1 - I_3)c_1 = 0, \quad (2I_2 - I_3)(I_1 - I_2) = 0,$$
$$(I_1 - I_2)(2I_1 + 2I_2 - 3I_3) = 0, \quad (2I_2 - I_3)(I_2 - I_3)c_1 = 0.$$

Solving these equations by the MAPLE command `solve` we obtain four solutions

$$\{U_3 = U_3,\ I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\},$$
$$\{U_3 = U_3,\ I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3\},$$
$$\{U_3 = U_3,\ I_1 = 2I_2,\ I_2 = I_2,\ I_3 = 2I_2,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\},$$
$$\{U_3 = U_3,\ I_1 = I_3,\ I_2 = \frac{I_3}{2},\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = c_3\}.$$

The first solution leads to the Lagrange case, the second one - to the kinetic symmetry case and the remaining two solutions - to the Kovalevskaya case.

Thus a new partial first integral of type 1 does not exist.

6. The gyrostat

6.1. The gyrostat equations. These equations (6.1) are only slightly modified Euler-Poisson equations (1.1)

$$\begin{aligned}
I_1 \frac{d\omega_1}{dt} &= (I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + Mg(c_3\gamma_2 - c_2\gamma_3),\\
I_2 \frac{d\omega_2}{dt} &= (I_3 - I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 + Mg(c_1\gamma_3 - c_3\gamma_1),\\
I_3 \frac{d\omega_3}{dt} &= (I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + Mg(c_2\gamma_1 - c_1\gamma_2),\\
\frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \omega_2\gamma_3,\\
\frac{d\gamma_2}{dt} &= \omega_1\gamma_3 - \omega_3\gamma_1,\\
\frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - \omega_1\gamma_2.
\end{aligned} \tag{6.1}$$

As for the Euler-Poisson equations, we study them in complex domain and without any restriction of generality, we admit that $Mg = 1$.

Like for the Euler-Poisson equations, H_2 and H_3 defined by (1.2) continue to be first integrals of equations of gyrostat (6.1). This is no more true for H_1 defined by (1.2). The area first integral for gyrostat is

$$H_1 = I_1\omega_1\gamma_1 + I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - b_1\gamma_1 - b_2\gamma_2 - b_3\gamma_3. \tag{6.2}$$

Up to the end of Sec. 6, H_1 is defined by (6.2) and such H_1 is a first integral of gyrostat equations (6.1). The first integrals H_1, H_2 and H_3 are always functionally independent.

Formally the definition of permutational symmetries cannot be applied to the gyrostat equations because the number of variables and of parameters does not coincide. But in fact it is easy to see that all permutational symmetries of gyrostat equations, like for Euler-Poisson equations, coincide with symmetric group S_3, where the same permutation is simultaneously applied to variables $\{\omega_1, \omega_2, \omega_3\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ and to parameters $\{I_1, I_2, I_3\}$, $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$. It is easy to verify that property (2.2) remains true. That is:

$$V_k(\sigma(\omega), \sigma(\gamma), \sigma(I), \sigma(b), \sigma(c)) = \varepsilon V_{\sigma(k)}(\omega, \gamma, I, b, c),$$
$$W_k(\sigma(\omega), \sigma(\gamma)) = \varepsilon W_{\sigma(k)}(\omega, \gamma), \qquad 1 \le k \le 3.$$

Here $\{V_k\}_{1\le k\le 3}$ are the right sides of the first three gyrostat equations (6.1), $\{W_k\}_{1\le k\le 3}$ are the remaining three equations (6.1) and $\varepsilon = \pm 1$ only depends on the choice of permutation $\sigma \in S_3$. The same concerns the analogue of the Theorem 2.2. We leave the details to the reader.

The known integrable cases for the real gyrostat equations (6.1) are the same as for the Euler-Poisson equations (1.1) but with some additional restrictions on the constants b_i, $1 \le i \le 3$. Up to permutational symmetry they are the following ones. These cases remains valid also for complex gyrostat equations.

The *Zhukovskii case* which is an extension of the Euler case [21, 24]. It is defined by the condition (1.3) without additional restrictions on b_i, $1 \le i \le 3$. The fourth integral is

$$H_4 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 - 2(I_1b_1\omega_1 + I_2b_2\omega_2 + I_3b_3\omega_3).$$

When $b_1 = b_2 = b_3 = 0$ we recover the fourth integral of Euler case.

The *Lagrange case* for gyrostat [21, 24] is defined by the conditions (1.4) and $b_1 = b_2 = 0$. The fourth integral in this case is the same as for the Euler-Poisson equations, i.e.

$$H_4 = \omega_3.$$

The *Yehia case* [21, 78] which is an extension of the Kovalevskaya case is defined by the conditions (1.5) and $b_1 = b_2 = 0$. The fourth integral in this case is

$$H_4 = \left[I_3(\omega_1^2 - \omega_2^2) - c_1\gamma_1 + c_2\gamma_2\right]^2 + (2I_3\omega_1\omega_2 - c_1\gamma_2 - c_2\gamma_1)^2 + 4b_3\gamma_3(c_1\omega_1 + c_2\omega_2) - 2b_3(\omega_1^2 + \omega_2^2)(I_3\omega_3 + b_3). \tag{6.3}$$

When $b_3 = 0$ we recover Kovalevskaya fourth integral (1.7).

The *kinetic symmetry case* for gyrostat is defined by the conditions (1.6) together with condition that the vectors (c_1, c_2, c_3) and (b_1, b_2, b_3) are proportional, i.e.:

$$b_1c_3 = b_3c_1, \quad b_2c_3 = b_3c_2, \quad b_2c_1 = b_1c_2,$$

and the fourth integral is the same as for the Euler-Poisson equations, i.e.

$$H_4 = c_1\omega_1 + c_2\omega_2 + c_3\omega_3.$$

Let us note that except the Yehia case, in all remaining three cases, the fourth integral can be found along the same lines as in [59], where fourth integrals are computed for integrable cases of the Euler-Poisson equations.

This is not so for the Yehia fourth integral because even if $c_2 = 0$, it depends on all variables. When $c_2 = 0$, this fourth integral can be found in [21] and in [78]. Comparing formula (1.7) of fourth integral in Kovalevskaya case when $c_2 \ne 0$ and when $c_2 = 0$ with formula (6.3) when $c_2 = 0$, it is natural to conjecture that formula (6.3) with an arbitrary c_2 defines a fourth integral in the general Yehia case. Simple MAPLE computation confirms this.

Like for the Euler-Poisson equations, we will call these four cases *classical* integrable cases.

6.2. The Sretenskii case. In 1963 L. N. Sretenskii discovered an extension of the Goryachev-Chaplygin partial first integral (1.8) of the Euler-Poisson equations to the gyrostat case [62, 63].

Now we apply the method used in Sec. 5.2 that led to the successful derivation of the Goryachev-Chaplygin case for the Euler-Poisson equations to the gyrostat equations (6.1). The computations are almost the same, bigger but not so much. That is why we do not give details here.

We express γ_2 from equation $H_1 = U_1$, where H_1 is the function given by (6.2) and obtain

$$\gamma_2 = -\frac{(I_1\omega_1 - b_1)\gamma_1 + (I_3\omega_3 - b_3)\gamma_3 + U_1}{I_2\omega_2 - b_2}.$$

We put this expression for γ_2 in the gyrostat equations (6.1) and remove the fifth equation. We study the obtained system of five equations for the existence of a new first integral $F(\omega_1, \omega_2, \omega_3, \gamma_3)$, i.e. which does not depend on γ_1. For this purpose we compute $\frac{dF}{dt}$ and take only its numerator. It is easily seen that the obtained expression can be represented in the following way:

$$\frac{dF}{dt} = \gamma_1 Y_1(F) + Y_2(F) = 0, \tag{6.4}$$

where Y_1 and Y_2 are vector fields defined in $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_3)$. As (6.4) is an identity with respect to all the variables and as $Y_1(F)$ and $Y_2(F)$ do not depend on γ_1 we have

$$Y_1(F) = Y_2(F) = 0. \tag{6.5}$$

We compute the Lie brackets $Y_3 = [Y_1, Y_2]$ and $Y_4 = [Y_1, Y_3]$. Taking into account equations (6.5) we have that

$$Y_3(F) = Y_4(F) = 0. \tag{6.6}$$

Equations (6.5) and (6.6) can be considered as a system of four homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically on any open subset of domain of definition of F, because F is non-constant on any such open subset.

If a new integral F exists, system (6.5)–(6.6) has at least one non-zero solution. Let us consider the 4×4 matrix A whose columns are the coefficients of vector fields Y_1, Y_2, Y_3 and Y_4. The condition under which system (6.5)–(6.6) has at least one non-zero solution is

$$\operatorname{rank} A \le 3.$$

We equate to zero the determinant D of matrix A and study when it is identically equal to zero.

D is a polynomial of the variables $\omega_1, \omega_2, \omega_3$ and γ_3. We consider the system consisting of the coefficients of polynomial D equated to zero. This system has 226 equations in unknowns U_1, I_i, b_i, c_i, $1 \le i \le 3$. After four consecutive simplifications we obtain the reduced system of 28 equations. Solving these equations by the MAPLE command `solve` we obtain nine solutions. Two of them contain zero values of the moments of inertia, two lead to the Lagrange case and three lead to the kinetic symmetry case. Thus only two essential solutions remain. They are:

1. $I_1 = 4I_3, \quad I_2 = 4I_3, \quad b_1 = 0, \quad b_2 = 0, \quad c_3 = 0;$
2. $c_1 = c_2 = 0.$

Studying them exactly as in Sec. 5.2 we find that the first solution leads to a partial first integral at additional restriction $U_1 = 0$, that is

$$H_4 = (I_3\omega_3 + b_3)(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3,$$

which is the Sretenskii partial first integral of the equations of gyrostat (6.1). When $b_3 = 0$ we recover the Goryachev-Chaplygin partial first integral. As noted in Sec. 1.3 this result was announced already in [16].

The second solution, during the investigations, imposes additional restrictions $I_1 = I_2$ and $b_1 = b_2 = 0$, i.e. leads to the Lagrange case.

6.3. The new complex integrable cases. If we restrict ourselves to the real case, then 1906 E. Husson theorem [32] asserts that for the Euler-Poisson equations only in four classical cases the fourth integral is an algebraic function [3, 17, 20, 54]. The completely analogous assertion for real gyrostat equations was proved in 1992 by L. Gavrilov [21].

The main result of [59] can be formulated as follows. For complex Euler-Poisson equations, the fourth integral that does not depend on all variables, exists only in the four classical cases.

The theorem below proves that for complex gyrostat equations (6.1) the analog of main result of [59] is not true. As consequence, it proves that in complex setting the analog of Gavrilov theorem fails. Indeed, in the proof of this theorem we find two new cases of integrability with not only algebraic but polynomial fourth integrals.

THEOREM 6.1. *Up to permutational symmetry the complex gyrostat equations (6.1) admit exactly two new (non-classical) integrable cases with a fourth integral which does not depend on all variables. These cases are*

$$I_1 = I_2 = 2I_3,\ b_1 = -i\varepsilon b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_3 = 0, \tag{6.7}$$

where $\varepsilon = \pm 1$. *In both cases, the fourth integral can be found as a quadratic polynomial.*

Proof. Let us look for example for a fourth integral F of the gyrostat equations (6.1) that does not depend on ω_3, i.e. $F = F(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$.

We compute the derivative of F with respect to the gyrostat equations and obtain

$$\begin{aligned} I_1 I_2 \frac{dF}{dt} &= I_2\Big[(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3\Big]\frac{\partial F}{\partial \omega_1} \\ &+ I_1\Big[(I_3 - I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 + c_1\gamma_3 - c_3\gamma_1\Big]\frac{\partial F}{\partial \omega_2} \\ &+ I_1 I_2\left[(\omega_3\gamma_2 - \omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1} + (\omega_1\gamma_3 - \omega_3\gamma_1)\frac{\partial F}{\partial \gamma_2} + (\omega_2\gamma_1 - \omega_1\gamma_2)\frac{\partial F}{\partial \gamma_3}\right] = 0. \end{aligned}$$

It is easily seen that

$$I_1 I_2 \frac{dF}{dt} = \omega_3 Y_1(F) + Y_2(F) = 0, \tag{6.8}$$

where Y_1 and Y_2 are the following not depending on ω_3 vector fields, defined on $\mathbb{C}^5 = \mathbb{C}^5(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$:

$$Y_1 = I_2\Big[(I_2 - I_3)\omega_2 - b_2\Big]\frac{\partial}{\partial\omega_1} + I_1\Big[(I_3 - I_1)\omega_1 + b_1\Big]\frac{\partial}{\partial\omega_2} + I_1 I_2 \gamma_2 \frac{\partial}{\partial\gamma_1} - I_1 I_2 \gamma_1 \frac{\partial}{\partial\gamma_2},$$

$$Y_2 = I_2\Big(b_3\omega_2 + c_3\gamma_2 - c_2\gamma_3\Big)\frac{\partial}{\partial\omega_1} + I_1\Big(- b_3\omega_1 + c_1\gamma_3 - c_3\gamma_1\Big)\frac{\partial}{\partial\omega_2}$$
$$+ I_1 I_2 \left[-\omega_2\gamma_3\frac{\partial}{\partial\gamma_1} + \omega_1\gamma_3\frac{\partial}{\partial\gamma_2} + (\omega_2\gamma_1 - \omega_1\gamma_2)\frac{\partial}{\partial\gamma_3}\right].$$

As $Y_1(F)$ and $Y_2(F)$ do not depend on ω_3 then identity (6.8) implies that

$$Y_1(F) = Y_2(F) = 0. \tag{6.9}$$

We consider the Lie brackets $Y_3 = [Y_1, Y_2]/(I_1 I_2)$ and $Y_4 = [Y_1, Y_3]$ and obtain

$$Y_3 = \Big[(I_2 - I_1)b_3\omega_1 - I_3 c_3\gamma_1 + (I_3 - I_2)c_1\gamma_3 + b_1 b_3)\Big]\frac{\partial}{\partial\omega_1}$$
$$+ \Big[(I_1 - I_2)b_3\omega_2 - I_3 c_3\gamma_2 + (I_3 - I_1)c_2\gamma_3 + b_2 b_3)\Big]\frac{\partial}{\partial\omega_2}$$
$$+ I_1\gamma_3\Big[(I_1 - I_2 - I_3)\omega_1 - b_1\Big]\frac{\partial}{\partial\gamma_1} + I_2\gamma_3\Big[(I_2 - I_1 - I_3)\omega_2 - b_2\Big]\frac{\partial}{\partial\gamma_2}$$
$$+ \Big[- I_1(I_1 - I_2 - I_3)\omega_1\gamma_1 - I_2(I_2 - I_1 - I_3)\omega_2\gamma_2 + I_1 b_1\gamma_1 + I_2 b_2\gamma_2\Big]\frac{\partial}{\partial\gamma_3},$$

$$Y_4 = -I_2\Big[2(I_2 - I_3)(I_1 - I_2)b_3\omega_2 + I_3(-I_2 + I_1 + I_3)c_3\gamma_2$$
$$- (I_2 - I_3)(I_1 - I_3)c_2\gamma_3 - (I_1 - 2I_2 + I_3)b_2 b_3\Big]\frac{\partial}{\partial\omega_1}$$
$$- I_1\Big[2(I_1 - I_3)(I_1 - I_2)b_3\omega_1 + I_3(I_1 - I_2 - I_3)c_3\gamma_1$$
$$+ (I_2 - I_3)(I_1 - I_3)c_1\gamma_3 - (2I_1 - I_2 - I_3)b_1 b_3\Big]\frac{\partial}{\partial\omega_2}$$
$$+ I_1 I_2\Big\{\gamma_3\Big[(2I_2 I_1 + I_2 I_3 - 2I_2^2 + I_3^2 - I_1 I_3)\omega_2 - (I_1 - 2I_2 - I_3)b_2\Big]\frac{\partial}{\partial\gamma_1}$$
$$+ \gamma_3\Big[(I_2 I_3 + 2I_1^2 - 2I_2 I_1 - I_3^2 - I_1 I_3)\omega_1 - (2I_1 - I_2 + I_3)b_1\Big]\frac{\partial}{\partial\gamma_2}$$
$$- \Big[(I_2 I_3 + 2I_1^2 - 2I_1 I_2 - I_3^2 - I_1 I_3)\omega_1\gamma_2 + (2I_1 I_2 + I_2 I_3 - 2I_2^2 + I_3^2 - I_1 I_3)\omega_2\gamma_1$$
$$-(I_1 - 2I_2 - I_3)b_2\gamma_1 - (2I_1 - I_2 + I_3)b_1\gamma_2\Big]\frac{\partial}{\partial\gamma_3}\Big\}.$$

Equations (6.9) imply that

$$Y_3(F) = Y_4(F) = 0. \tag{6.10}$$

The system (6.9)–(6.10) is a linear homogeneous system in unknowns $\operatorname{grad} F =$

$\left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically on any open subset of domain of definition of F, because F is non-constant on any such open subset.

As H_2 is a first integral of the sought type, i.e. it does not depend on ω_3, then if a new integral F exists, system (6.9)–(6.10) should have at least two non-zero solutions. Let us consider the 4×5 matrix A whose rows are the coefficients of vector fields Y_1, Y_2, Y_3 and Y_4. The condition under which system (6.9)–(6.10) has at least two non-zero solutions is

$$\operatorname{rank} A \leq 3.$$

We compute all the five 4×4 minors of matrix A and require that they be identically equal to zero. Denoting them by D_{ijkl}, where the index contains the numbers of the included columns of matrix A, we see that $D_{1345} = D_{2345} = 0$. Thus it remains to study when minors D_{1234}, D_{1235} and D_{1245} vanish identically. These three minors are polynomials of ω_1, ω_2, γ_1, γ_2 and γ_3 with coefficients that are polynomials of the parameters I_i, b_i and c_i, $1 \leq i \leq 3$. We cannot write here the expressions for D_{1234}, D_{1235} and D_{1245} because they are too long. They have non-zero factors which we remove by setting

$$d_{1234} = \frac{D_{1234}}{I_1^2 I_2^2 \gamma_3}, \quad d_{1235} = \frac{D_{1235}}{I_1^2 I_2^2 \gamma_2}, \quad d_{1245} = \frac{D_{1245}}{I_1^2 I_2^2 \gamma_1}.$$

After this cancellation of the non-zero factors it turns out that

$$d_{1234} = -d_{1235} = d_{1245}.$$

We can therefore restrict ourselves to considering only the identity $d_{1234} = 0$.

The polynomial d_{1234} has 83 monomials and therefore 83 coefficients which should vanish. We consider the system consisting of the coefficients of d_{1234} equated to zero, i.e. the system of 83 equations in unknowns I_i, b_i and c_i, $1 \leq i \leq 3$. After six consecutive simplifications we obtain the reduced system consisting of 29 equations. Solving that system by the MAPLE command `solve` we obtain the following ten solutions:

1. $\{I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = 0,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0\}$
2. $\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0\}$
3. $\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = 0,\ b_3 = 0,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\}$
4. $\{I_1 = 0,\ I_2 = 0,\ I_3 = 0,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = 0\}$
5. $\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = \dfrac{c_1 b_2}{c_2},\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = 0\}$
6. $\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ b_1 = -i\varepsilon b_2,\ b_2 = b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_2 = c_2,\ c_3 = 0\}$
7. $\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = 0,\ b_2 = 0,\ b_3 = b_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = 0\}$
8. $\{I_1 = -I_3,\ I_2 = -I_3,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\}$
9. $\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ b_1 = 0,\ b_2 = 0,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3\}$
10. $\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = 0,\ b_2 = 0,\ b_3 = b_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\}$,

where $\varepsilon = \pm 1$.

A careful study of this list shows that only three solutions are essential. They are the sixth, seventh and eighth solutions. All other solutions lead to some of the classical cases of integrability of gyrostat equations. Let us stress that the ninth solution implies

$b_1 = b_2 = b_3 = 0$ and therefore the gyrostat equations become the Euler-Poisson equations whose fourth integrals not depending on all variables have been studied in [59]. Note that the sixth solution leads to the condition (6.7). Below we examine these three essential solutions.

Solution 6+: $I_1 = I_2 = 2I_3,\ b_1 = -i\varepsilon b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_3 = 0,\ \varepsilon = 1.$

At these conditions Y_i, $1 \le i \le 4$, are linearly dependent as $d_{1234} = 0$. More precisely we have

$$(I_3\omega_1 - iI_3\omega_2 - ib_2)Y_4 + I_3^2(I_3\omega_1 - iI_3\omega_2 - 4ib_2)Y_2 + 6I_3^2 b_2 Y_3 = 0.$$

We compute $Y_5 = [Y_2, Y_3]/[4I_3^2(I_3\omega_1 - iI_3\omega_2 - ib_2)]$ and obtain

$$Y_5 = -c_2(\gamma_1 + i\gamma_2)\left(\frac{\partial}{\partial\omega_1} - i\frac{\partial}{\partial\omega_2}\right) - 2I_3(\omega_1 + i\omega_2)\left(\gamma_2\frac{\partial}{\partial\gamma_1} - \gamma_1\frac{\partial}{\partial\gamma_2}\right).$$

Like Y_4, Y_5 is also linearly dependent on Y_2 and Y_3. Indeed, we have

$$2I_3(I_3\omega_1 - iI_3\omega_2 - ib_2)\gamma_3 Y_5 - (I_3\omega_1\gamma_1 + I_3\omega_2\gamma_2 - ib_2\gamma_1 + b_2\gamma_2)Y_2$$
$$+2I_3(\omega_1\gamma_2 - \omega_2\gamma_1)Y_3 = 0.$$

Moreover, easy computations show that vector fields Y_i, $1 \le i \le 3$, are linearly independent. Thus system $Y_i(F) = 0$, $1 \le i \le 3$, is in involution and according to the Frobenius Integrability Theorem should have two functionally independent solutions. The first one is function H_2. Finding another one, functionally independent of H_2 is not feasible with crude use of the MAPLE command `pdsolve`. To overcome this difficulty we add the fourth equation $Y_0(F) = 0$, where $Y_0 = \frac{\partial}{\partial\gamma_3}$. We choose such Y_0 because $Y_0(H_2) \ne 0$. The MAPLE command `pdsolve` applied to the system of four equations $Y_i(F) = 0$, $0 \le i \le 3$, gives as an answer the solution:

$$F = G\left[-\frac{(I_3\omega_1^2 + 2iI_3\omega_1\omega_2 - I_3\omega_2^2 - 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 - iI_3\omega_2 + 2ib_2)^2}{2c_2}\right],$$

where G is an arbitrary smooth function. As a second solution of system $Y_i(F) = 0$, $1 \le i \le 3$, we take the function

$$H_{4+} = (I_3\omega_1^2 + 2iI_3\omega_1\omega_2 - I_3\omega_2^2 - 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 - iI_3\omega_2 + 2ib_2)^2,$$

that corresponds to $G(x) = -2c_2x$.

Solution 6-: $I_1 = I_2 = 2I_3,\ b_1 = -i\varepsilon b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_3 = 0,\ \varepsilon = -1.$

In this case in a completely analogous way we find

$$H_{4-} = (I_3\omega_1^2 - 2iI_3\omega_1\omega_2 - I_3\omega_2^2 + 2ic_2\gamma_1 + 2c_2\gamma_2)(I_3\omega_1 + iI_3\omega_2 - 2ib_2)^2.$$

It is easy to verify at the hand that H_{4+} and H_{4-} are functionally independent of the first integrals H_1 (see (6.2)), H_2 and H_3 (see (1.2)) and thus they are fourth integrals of gyrostat equations (6.1), for $\varepsilon = 1$ and $\varepsilon = -1$ respectively.

Solution 7: $I_1 = I_2 = I_3,\ b_1 = b_2 = 0,\ c_3 = 0.$

As in the previous case, due to the equality $d_{1234} = 0$, the Y_i, $1 \le i \le 4$, are linearly dependent. Indeed, we have

$$(\omega_1\gamma_1 + \omega_2\gamma_2)Y_4 - I_3^3(\omega_1^2 + \omega_2^2)\gamma_3 Y_1 - I_3^3(\omega_1\gamma_2 - \omega_2\gamma_1)Y_3 = 0.$$

We compute $Y_5 = [Y_2, Y_3]/I_3^3$ and obtain

$$Y_5 = (\omega_1\gamma_1 + \omega_2\gamma_2)\left(c_2\frac{\partial}{\partial\omega_1} - c_1\frac{\partial}{\partial\omega_2}\right) + \left[I_3(\omega_2^2 + \omega_1^2)\gamma_2 - (b_3\omega_2 - c_2\gamma_3)\gamma_3\right]\frac{\partial}{\partial\gamma_1}$$
$$- \left[I_3(\omega_1^2 + \omega_2^2)\gamma_1 - (b_3\omega_1 - c_1\gamma_3)\gamma_3\right]\frac{\partial}{\partial\gamma_2}$$
$$- \left[(b_3\omega_1 - c_1\gamma_3)\gamma_2 - (b_3\omega_2 - c_2\gamma_3)\gamma_1\right]\frac{\partial}{\partial\gamma_3}.$$

As H_2 is a first integral of the sought type then the existence of a fourth integral of the gyrostat equations requires that the vector fields Y_i, $1 \le i \le 3$, and Y_5 be linearly dependent. We compute the determinant V_{1234} consisting of the first four columns of the matrix of the coefficients of these vector fields and obtain

$$V_{1234} = I_3^5 b_3\gamma_3(\omega_1\gamma_1 + \omega_2\gamma_2)^2(c_2\omega_1 - c_1\omega_2).$$

If the vector fields Y_i, $1 \le i \le 3$, and Y_5 are linearly dependent then V_{1234} should be identically zero. It is clear that this happens either if $b_3 = 0$ which leads to the Euler-Poisson equations or if $c_1 = c_2 = 0$ which leads to the Zhukovskii case. Thus a new integral for Solution 7 does not exists.

Solution 8: $I_1 = I_2 = -I_3,\ b_3 = 0,\ c_1 = c_2 = 0.$

In this case $Y_4 = -Y_2$. We compute $Y_5 = [Y_2, Y_3]/I_3^2$ and obtain

$$Y_5 = c_3\gamma_3(2I_3\omega_2 + b_2)\frac{\partial}{\partial\omega_1} - c_3\gamma_3(2I_3\omega_1 + b_1)\frac{\partial}{\partial\omega_2}$$
$$- I_3(I_3\omega_1^2 + I_3\omega_2^2 + b_1\omega_1 + b_2\omega_2 + 2c_3\gamma_3)\left(\gamma_2\frac{\partial}{\partial\gamma_1} - \gamma_1\frac{\partial}{\partial\gamma_2}\right).$$

As in Solution 7, we compute the determinant W_{1234} consisting of the first four columns of the matrix of the coefficients of vector fields Y_i, $1 \le i \le 3$, and Y_5 and we obtain

$$W_{1234} = I_3^4 c_3(I_3\omega_2^2 + I_3\omega_1^2 + b_1\omega_1 + 3c_3\gamma_3 + b_2\omega_2)\gamma_3 w,$$

where

$$w = I_3(b_2\omega_1\gamma_1^2 - 2b_1\omega_1\gamma_1\gamma_2 - b_2\omega_1\gamma_2^2 + b_1\omega_2\gamma_1^2 + 2b_2\omega_2\gamma_1\gamma_2 - b_1\omega_2\gamma_2^2)$$
$$+ b_2b_1\gamma_1^2 - (b_1^2 - b_2^2)\gamma_1\gamma_2 - b_1b_2\gamma_2^2.$$

If a new first integral exists then $W_{1234} = 0$ should be fulfilled. To avoid the Zhukovskii case we consider that $c_3 \neq 0$. In such a case it is clear that $W_{1234} = 0$ is equivalent to $w = 0$. The last is possible if and only if $b_1 = b_2 = 0$ which leads to the Euler-Poisson equations. Thus a new first integral of the sought type does not exist for Solution 8 too.

All the above considerations lead to the conclusion that the gyrostat equations (6.1) admit a local fourth integral which does not depend on ω_3 either in certain classical cases or else only when the conditions (6.7) are fulfilled.

Now let us look for a fourth integral F of the gyrostat equations (6.1) that does not depend on γ_3, i.e. $F = F(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$.

We compute the derivative of F with respect to the gyrostat equations and obtain

$$I_1I_2I_3\frac{dF}{dt} = I_2I_3\Big[(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3\Big]\frac{\partial F}{\partial\omega_1}$$

$$
\begin{aligned}
&+ I_1I_3\Big[(I_3-I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 + c_1\gamma_3 - c_3\gamma_1\Big]\frac{\partial F}{\partial \omega_2}\\
&+ I_1I_2\Big[(I_1-I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + (c_2\gamma_1 - c_1\gamma_2)\Big]\frac{\partial F}{\partial \omega_3}\\
&+ I_1I_2I_3\left[(\omega_3\gamma_2-\omega_2\gamma_3)\frac{\partial F}{\partial \gamma_1} + (\omega_1\gamma_3-\omega_3\gamma_1)\frac{\partial F}{\partial \gamma_2}\right] = 0.
\end{aligned}
$$

It is easily seen that

$$
I_1I_2I_3\frac{dF}{dt} = I_3\gamma_3Z_1(F) + Z_2(F) = 0, \tag{6.11}
$$

where Z_1 and Z_2 are the following not depending on γ_3 vector fields, defined on $\mathbb{C}^5 = \mathbb{C}^5(\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2)$:

$$
\begin{aligned}
Z_1 &= -I_2c_2\frac{\partial}{\partial\omega_1} + I_1c_1\frac{\partial}{\partial\omega_2} - I_1I_2\Big(\omega_2\frac{\partial}{\partial\gamma_1} - \omega_1\frac{\partial}{\partial\gamma_2}\Big),\\
Z_2 &= I_2I_3\Big[(I_2-I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2\Big]\frac{\partial}{\partial\omega_1}\\
&\quad + I_1I_3\Big[(I_3-I_1)\omega_1\omega_3 + b_1\omega_3 - b_3\omega_1 - c_3\gamma_1\Big]\frac{\partial}{\partial\omega_2}\\
&\quad + I_1I_2\Big[(I_1-I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\gamma_1 - c_1\gamma_2\Big]\frac{\partial}{\partial\omega_3}\\
&\quad + I_1I_2I_3\omega_3\left(\gamma_2\frac{\partial}{\partial\gamma_1} - \gamma_1\frac{\partial}{\partial\gamma_2}\right).
\end{aligned}
$$

As $Z_1(F)$ and $Z_2(F)$ do not depend on γ_3 then identity (6.11) implies that

$$
Z_1(F) = Z_2(F) = 0. \tag{6.12}
$$

We consider the Lie brackets $Z_3 = [Z_1, Z_2]/(I_1I_2)$, $Z_4 = [Z_1, Z_3]$ and $Z_5 = [Z_2, Z_3]/I_3$ and obtain

$$
\begin{aligned}
Z_3 &= I_3\Big[(I_2-I_3)c_1\omega_3 + I_2c_3\omega_1 + c_1b_3\Big]\frac{\partial}{\partial\omega_1} + I_3\Big[(I_1-I_3)c_2\omega_3 + I_1c_3\omega_2 + c_2b_3\Big]\frac{\partial}{\partial\omega_2}\\
&\quad + \Big[I_1(I_1-2I_2)c_1\omega_1 + (I_2-2I_1)I_2c_2\omega_2 - I_2b_2c_2 - I_1b_1c_1\Big]\frac{\partial}{\partial\omega_3}\\
&\quad - I_1I_3\Big[(I_1-I_2-I_3)\omega_1\omega_3 + b_3\omega_1 - b_1\omega_3 + c_3\gamma_1\Big]\frac{\partial}{\partial\gamma_1}\\
&\quad + I_3I_2\Big[(I_1-I_2+I_3)\omega_2\omega_3 - b_3\omega_2 + b_2\omega_3 - c_3\gamma_2\Big]\frac{\partial}{\partial\gamma_2},\\
Z_4 &= -I_2^2I_3c_2c_3\frac{\partial}{\partial\omega_1} + I_1^2I_3c_1c_3\frac{\partial}{\partial\omega_2} - 3I_1I_2(I_1-I_2)c_2c_1\frac{\partial}{\partial\omega_3}\\
&\quad + I_1I_2I_3\Big[(2I_1-I_2-2I_3)c_2\omega_3 + 2I_1c_3\omega_2 + 2b_3c_2\Big]\frac{\partial}{\partial\gamma_1}\\
&\quad + I_1I_2I_3\Big[(I_1-2I_2+2I_3)c_1\omega_3 - 2I_2c_3\omega_1 - 2b_3c_1\Big]\frac{\partial}{\partial\gamma_2},\\
Z_5 &= a_1\frac{\partial}{\partial\omega_1} + a_2\frac{\partial}{\partial\omega_2} + a_3\frac{\partial}{\partial\omega_3} + a_4\frac{\partial}{\partial\gamma_1} + a_5\frac{\partial}{\partial\gamma_2},
\end{aligned}
$$

where

$$\begin{aligned}
a_1 &= I_2\big[I_1I_2(I_2-I_3)c_1\omega_1\omega_2 + I_1(I_1-I_2-I_3)b_2c_1\omega_1 + I_2(I_2-I_3)(2I_1-I_2)c_2\omega_2^2\\
&\quad - I_3(2I_2I_3 - I_1I_3 - 2I_2^2 + 2I_1I_2)c_3\omega_2\omega_3\\
&\quad - (2I_1I_2b_2c_2I_2I_3b_2c_2 + I_1I_3b_3c_3 - 2I_2I_3b_3c_3 - 2I_2^2b_2c_2)\omega_2\\
&\quad - I_3(I_2-I_3)(I_1-I_3)c_2\omega_3^2 - I_3(I_1b_3c_2 - 2I_3b_3c_2 + 2I_2b_2c_3 + I_2b_3c_2)\omega_3\\
&\quad + I_1(I_2-I_3)c_1c_2\gamma_1 + (2I_2I_3c_3^2 - I_1I_2c_1^2 + I_1I_3c_1^2)\gamma_2 - b_2I_1c_1b_1 - I_3b_3^2c_2 - I_2c_2b_2^2\big],\\
a_2 &= I_1\big[I_1(I_1-I_3)(I_1-2I_2)c_1\omega_1^2 - I_1I_2(I_1-I_3)c_2\omega_1\omega_2\\
&\quad - I_3(2I_1^2 - 2I_1I_2 - 2I_1I_3 + I_2I_3)c_3\omega_1\omega_3\\
&\quad - (2I_1^2b_1c_1 - 2I_1I_2b_1c_1 - I_1I_3b_1c_1 + 2I_1I_3b_3c_3 - I_2I_3b_3c_3)\omega_1\\
&\quad + I_2(I_1-I_2+I_3)b_1c_2\omega_2 + I_3(I_2-I_3)(I_1-I_3)c_1\omega_3^2\\
&\quad + I_3(2I_1b_1c_3 + I_1b_3c_1 + I_2b_3c_1 - 2I_3b_3c_1)\omega_3 + (I_1I_2c_2^2 - 2I_1I_3c_3^2 - I_2I_3c_2^2)\gamma_1\\
&\quad - I_2(I_1-I_3)c_1c_2\gamma_2 + I_1b_1^2c_1 + I_2b_2b_1c_2 + I_3b_3^2c_1\big],\\
a_3 &= I_1I_2\big[-(I_1-I_2)(I_1+I_2)c_3\omega_1\omega_2 + I_1(2I_1-I_2-2I_3)c_2\omega_1\omega_3 + (2I_1b_3c_2 - I_2b_2c_3)\omega_1\\
&\quad + I_2(I_1-2I_2+2I_3)c_1\omega_2\omega_3 + (I_1b_1c_3 - 2I_2b_3c_1)\omega_2\\
&\quad + (I_3b_2c_1 - 2I_1b_1c_2 + I_2b_1c_2 - I_1b_2c_1 - I_3b_1c_2 + 2I_2b_2c_1)\omega_3 + c_3(3I_1-I_2)c_2\gamma_1\\
&\quad + (I_1-3I_2)c_1c_3\gamma_2 + b_3(b_1c_2 - b_2c_1)\big],\\
a_4 &= -I_1I_2\big[I_1(I_1-I_2)(I_1-I_2-I_3)\omega_1^2\omega_2 + I_1(I_1-I_2-I_3)b_2\omega_1^2\\
&\quad - I_1(2I_1-I_3-2I_2)b_1\omega_1\omega_2 + I_1(I_1-I_2-I_3)c_2\omega_1\gamma_1 - I_1(I_2-I_3)c_1\omega_1\gamma_2\\
&\quad - I_1b_2b_1\omega_1 + I_3(I_3^2 + I_2I_3 - I_1I_3 + 2I_1I_2 - 2I_2^2)\omega_2\omega_3^2 + I_3(I_1-I_2-2I_3)b_3\omega_2\omega_3\\
&\quad - I_2(2I_1-I_2)c_2\omega_2\gamma_2 + (I_1b_1^2 + I_3b_3^2)\omega_2 - I_3(I_1-2I_2-I_3)b_2\omega_3^2\\
&\quad + I_3(2I_1-2I_2-I_3)c_3\omega_3\gamma_2 - I_3b_2b_3\omega_3 - I_1b_1c_2\gamma_1 + (I_3b_3c_3 - I_2b_2c_2)\gamma_2\big],\\
a_5 &= I_1I_2\big[I_2(I_1-I_2)(I_1-I_2+I_3)\omega_1\omega_2^2 + I_2(2I_1-2I_2+I_3)b_2\omega_1\omega_2\\
&\quad - I_3(2I_1^2 - 2I_1I_2 - I_1I_3 + I_2I_3 - I_3^2)\omega_1\omega_3^2 - I_3(I_1-I_2+2I_3)b_3\omega_1\omega_3\\
&\quad + I_1(I_1-2I_2)c_1\omega_1\gamma_1 + (I_2b_2^2 + I_3b_3^2)\omega_1 - I_2(I_1-I_2+I_3)b_1\omega_2^2 - c_2I_2(I_1-I_3)\omega_2\gamma_1\\
&\quad - I_2(I_1-I_2+I_3)c_1\omega_2\gamma_2 - I_2b_1b_2\omega_2 + I_3(2I_1 + -I_2I_3)b_1\omega_3^2\\
&\quad - I_3(2I_1-2I_2+I_3)c_3\omega_3\gamma_1 - I_3b_1b_3\omega_3 - (I_1b_1c_1 - I_3b_3c_3)\gamma_1 - I_2c_1b_2\gamma_2\big].
\end{aligned}$$

Equations (6.12) imply that

$$Z_3(F) = Z_4(F) = Z_5(F) = 0. \tag{6.13}$$

If a first integral $F = F(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$ exists then the system of five equations (6.12) and (6.13) should have a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}\right)$. This is possible if and only if the determinant D of the coefficients of that system vanishes identically.

We compute D and obtain a very long polynomial whose presentation here is impossible. But D has a factor $I_1^3 I_2^3 I_3^2 c_3$. Thus we should consider two cases: $c_3 \neq 0$ and $c_3 = 0$.

Let us start with $c_3 \neq 0$. We remove the non-zero factor of D and note

$$\widehat{D} = \frac{D}{I_1^3 I_2^3 I_3^2 c_3}.$$

The equation $D = 0$ is equivalent to $\widehat{D} = 0$. Polynomial $\widehat{D}$ has 253 coefficients depending on the parameters I_i, b_i and c_i, $1 \leq i \leq 3$. To satisfy the equation $\widehat{D} = 0$ we should consider the system consisting of the coefficients of $\widehat{D}$ equated to zero, i.e. the system of 253 equations for the parameters. After three consecutive simplifications we obtain the reduced system consisting of seven equations. Solving that system by the MAPLE command `solve` we obtain the following two solutions:

$$\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = b_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\}$$

$$\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = b_3,\ c_1 = \frac{b_1 c_2}{b_2},\ c_2 = c_2,\ c_3 = \frac{b_3 c_2}{b_2}\}.$$

The second solution leads to the kinetic symmetry case. Thus only the first solution should be studied. For this, let $I_1 = I_2$, $c_1 = c_2 = 0$. Under these conditions we have $Z_4 + 2I_2 I_3 c_3 Z_1 = 0$. We compute $Z_6 = [Z_3, Z_5]/(I_2^3 I_3 c_3)$ and the determinant M of the coefficients of the vector fields Z_i, $1 \leq i \leq 3$, Z_5 and Z_6. We know that if the sought first integral exists then $M = 0$. We have

$$M = -I_2^8 I_3^2 c_3^2 (b_2\omega_1 - b_1\omega_2)^2 \widehat{M},$$

where

$$\begin{aligned}\widehat{M} = &-3I_3(I_2 - 3I_3)\omega_1^4\omega_3 - 9I_3 b_3\omega_1^4 - 2(I_2 - 11I_3)b_1\omega_1^3\omega_3 - 15I_3 c_3\omega_1^3\gamma_1 - 7b_1 b_3\omega_1^3\\ &- 6I_3(I_2 - 3I_3)\omega_1^2\omega_2^2\omega_3 - 18I_3 b_3\omega_1^2\omega_2^2 - 2(I_2 - 11I_3)b_2\omega_1^2\omega_2\omega_3 - 15I_3 c_3\omega_1^2\omega_2\gamma_2\\ &- 7b_2 b_3\omega_1^2\omega_2 + 12b_1^2\omega_1^2\omega_3 - 12b_1 c_3\omega_1^2\gamma_1 - 2(I_2 - 11I_3)b_1\omega_1\omega_2^2\omega_3 - 15I_3 c_3\omega_1\omega_2^2\gamma_1\\ &- 7b_1 b_3\omega_1\omega_2^2 + 24b_1 b_2\omega_1\omega_2\omega_3 - 12b_2 c_3\omega_1\omega_2\gamma_1 - 12b_1 c_3\omega_1\omega_2\gamma_2 - 3I_3(I_2 - 3I_3)\omega_2^4\omega_3\\ &- 9I_3 b_3\omega_2^4 - 2(I_2 - 11I_3)b_2\omega_2^3\omega_3 - 15I_3 c_3\omega_2^3\gamma_2 - 7b_2 b_3\omega_2^3 + 12b_2^2\omega_2^2\omega_3 - 12b_2 c_3\omega_2^2\gamma_2.\end{aligned}$$

Let us consider, for example, the coefficient of $\widehat{M}$ in front of $\omega_1^3\gamma_1$, that is $-15I_3c_3$. However $-15I_3c_3 \neq 0$, therefore $\widehat{M}$ never vanishes either. Thus the only possibility to satisfy the equation $M = 0$ is to put $b_1 = b_2 = 0$. Taking into account that now $I_1 = I_2$, $c_1 = c_2 = 0$ we come to the Lagrange case. Thus a new first integral $F = F(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$ does not exist when $c_3 \neq 0$.

We now study the case $c_3 = 0$. Under this condition, the first integral H_3 does not depend on γ_3, i.e. it is of the type sought. If a fourth integral $F = F(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2)$ exists then system (6.12)–(6.13) should have at least two non-zero solutions. This is possible if and only if the matrix B of its coefficients satisfies the condition $\operatorname{rank} B \leq 3$. This condition means that all four equations of system (6.12)–(6.13) should be linearly dependent. We chose to consider the system

$$Z_i(F) = 0,\ 1 \leq i \leq 3, \quad Z_5(F) = 0. \tag{6.14}$$

This choice is appropriate because if we choose, for example, $Z_i(F) = 0$, $1 \leq i \leq 4$, then we come to a great number of cases which should be studied.

We compute all the five 4×4 minors of the 4×5 matrix consisting of the coefficients of system (6.14) and require that they be identically equal to zero. These five minors are polynomials of ω_1, ω_2, γ_1, γ_2 and γ_3 with coefficients that are polynomials of the parameters I_i, b_i, $1\le i\le 3$, c_1 and c_2. Like before we note them by Δ_{ijkl} and see that they have some non-zero factors. To remove these factors we introduce the following notations:

$$\delta_{1234}=\frac{\Delta_{1234}}{I_1^2I_2^2I_3},\quad \delta_{1235}=\frac{\Delta_{1235}}{I_1^2I_2^2I_3},\quad \delta_{1245}=\frac{\Delta_{1245}}{I_1^2I_2^2I_3^2\omega_2},$$
$$\delta_{1345}=\frac{\Delta_{1345}}{I_1^2I_2^3I_3\omega_2},\quad \delta_{2345}=\frac{\Delta_{2345}}{I_1^3I_2^2I_3\omega_1}.$$

Let us note that δ_{1234} has a factor c_2 and δ_{1235} has a factor c_1. We have left them intentionally at the above cancellation because we do not know whether c_1 or c_2 is zero. But, as we consider now the case $c_3=0$, we know that $(c_1,c_2)\neq(0,0)$ because otherwise we come to the Zhukovskii case.

It turns out that

$$\delta_{1245}=-\delta_{1345}=\delta_{2345}. \tag{6.15}$$

Moreover if $c_1\neq 0$

$$\frac{\delta_{1235}}{c_1}=-\delta_{1245}, \tag{6.16}$$

independently of the value of c_2. If $c_2\neq 0$

$$\frac{\delta_{1234}}{c_2}=\delta_{1245}, \tag{6.17}$$

independently of the value of c_1. If $c_1\neq 0$ and $c_2\neq 0$ we have

$$\frac{\delta_{1234}}{c_2}=-\frac{\delta_{1235}}{c_1}. \tag{6.18}$$

Thus if the identity $\delta_{1245}=0$ is satisfied equations (6.14) will be linearly dependent. Indeed, from (6.15) it follows that $\delta_{1345}=\delta_{2345}=0$ independently of the values of c_1 and c_2.

Let $c_1\neq 0$ and $c_2=0$. Then $\delta_{1234}=0$ because it has a factor c_2 and $\delta_{1235}=0$ follows from (6.16).

Let $c_1\neq 0$ and $c_2\neq 0$. Then $\delta_{1235}=0$ follows from (6.16) and $\delta_{1234}=0$ - from (6.18).

Let $c_1=0$ and $c_2\neq 0$. Then $\delta_{1235}=0$ because it has a factor c_1 and $\delta_{1234}=0$ follows from (6.17).

The polynomial δ_{1245} has 84 monomials and therefore 84 coefficients which should vanish. We consider the system consisting of the coefficients of δ_{1245} equated to zero, i.e. the system of 84 equations in unknowns I_i, b_i, $1\le i\le 3$, c_1 and c_2. After eight consecutive simplifications we obtain the reduced system consisting of 15 equations. Solving that system by the MAPLE command `solve` we obtain the following seven solutions:

$$\{I_1=2I_3,\ I_2=2I_3,\ I_3=I_3,\ b_1=0,\ b_2=0,\ b_3=0,\ c_1=c_1,\ c_2=c_2\}$$
$$\{I_1=I_3,\ I_2=I_2,\ I_3=I_3,\ b_1=0,\ b_2=b_2,\ b_3=0,\ c_1=0,\ c_2=c_2\}$$
$$\{I_1=I_1,\ I_2=I_2,\ I_3=I_3,\ b_1=b_1,\ b_2=b_2,\ b_3=0,\ c_1=0,\ c_2=0\}$$
$$\{I_1=I_1,\ I_2=I_3,\ I_3=I_3,\ b_1=b_1,\ b_2=0,\ b_3=0,\ c_1=c_1,\ c_2=0\}$$

$$\{I_1 = 0,\ I_2 = 0,\ I_3 = 0,\ b_1 = b_1,\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2\}$$

$$\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ b_1 = \frac{c_1 b_2}{c_2},\ b_2 = b_2,\ b_3 = 0,\ c_1 = c_1,\ c_2 = c_2\}$$

$$\{I_1 = 2I_3,\ I_2 = 2I_3,\ I_3 = I_3,\ b_1 = -i\varepsilon b_2,\ b_2 = b_2,\ b_3 = 0,\ c_1 = i\varepsilon c_2,\ c_2 = c_2\},$$

where $\varepsilon = \pm 1$.

Examining this list we see that only the last solution is essential. All the other solutions lead either to the classical cases of gyrostat equations, or to the Euler-Poisson equations or to the excluded cases with values zero of the moments of inertia.

Taking into account that now $c_3 = 0$ we see that this last solution determines the conditions (6.7).

At these conditions the vector field Z_4 vanishes identically and Z_5 is linearly dependent on Z_i, $1 \le i \le 3$. Moreover the vector fields Z_i, $1 \le i \le 3$, are linearly independent. Thus system $Z_i(F) = 0$, $1 \le i \le 3$, is in involution and according to the Frobenius Integrability Theorem it has two functionally independent solutions. The first one is H_3 and the second one is the fourth integral we look for. But it is not necessary to look for this fourth integral. We should only notice that the fourth integrals H_{4+} from Case 6+ and H_{4-} from Case 6- not only do not depend of ω_3, they do not depend on γ_3 too. ∎

Thus the problem of characterization of all cases when complex gyrostat equations have a fourth integral that does not depend on all variables is solved.

7. Domain of the Sretenskii partial first integral

7.1. Definition of the domain. Given that the gyrostat equations (6.1) are polynomial with polynomial first integrals, in this section we will restrict ourselves exclusively to polynomial systems of ordinary differential equations and their polynomial first integrals, although this is not necessary.

Let us consider a polynomial system of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(x),\ 1 \le i \le n, \tag{7.1}$$

$x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $f_1, \ldots, f_n \in \mathbb{C}[x]$. To consider system (7.1) and its solutions is equivalent to consider the associate polynomial vector field $V(x) = (f_1, \ldots, f_n)$ and its orbits.

A subset of $\mathbb{C}^n$ is *V-invariant* if it is filled with whole orbits of vector field V.

A differentiable function Φ such that the set $\{x;\ \Phi(x) = 0\}$ is filled by the whole orbits of system (7.1) is called *invariant relation* ([20], [38, Chapt. X, §4]).

Let $F \in \mathbb{C}[x] \setminus \mathbb{C}$ be some non-constant polynomial that is not a first integral of system (7.1), or equivalently that is not V-invariant. Let $M \subset \mathbb{C}^n$ be a V-invariant subset such that $F|_M$ (F restricted to M) is non-constant on any open subset of M. When $F|_M$ is a local first integral of system (7.1) (or vector field V) restricted to M?

For this aim let us compute

$$\frac{dF}{dt}(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x)\frac{dx_i}{dt}(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x) f_i(x) \stackrel{def}{=} A(x).$$

$A(x)$ is a polynomial, because F and $\{f_i\}_{1\leq i\leq n}$ are polynomials. F is not a first integral of system (7.1) on $\mathbb{C}^n$, then A does not vanishes identically, but $A(x) = 0$ for $x \in M$.

Let us note $\widehat{A} = \{x \in \mathbb{C}^n; A(x) = 0\}$, $\widehat{A} \subsetneq \mathbb{C}^n$. Thus $M \subset \widehat{A}$ and the problem is reduced to the study of V-invariant subsets of the algebraic subset $\widehat{A}$ of $\mathbb{C}^n$.

Let $M \subseteq \widehat{A}$ be a V-invariant smooth submanifold such that $F|_M$ is a partial first integral of system (7.1) restricted to M. Any such submanifold will be called a *domain* of F. In what follows we will consider exclusively the case when submanifold M is of codimension one. Nevertheless the case of smaller codimension also deserves the study. In what follows we will be interested of the identification of maximal domains of F.

7.2. Determination of the maximal domain. Now we shall determine codimension one maximal domains of the Sretenskii partial first integral

$$F = (I_3\omega_3 + b_3)(\omega_1^2 + \omega_2^2) - (c_1\omega_1 + c_2\omega_2)\gamma_3. \tag{7.2}$$

To avoid the Zhukovskii and Lagrange cases, we will suppose that $(c_1, c_2) \neq (0,0)$.

Let $U \in \mathbb{C}^6$ be an open subset. We want to find all invariant manifolds

$$\widehat{S} = \{(\omega,\gamma) \in U; S(\omega,\gamma) = 0\}, \tag{7.3}$$

where S is a C^1 smooth function defined on U, such that on $\widehat{S}$, F is a partial first integral of equations (6.1).

We will consider five distinct cases.

1) Let us suppose that $\frac{\partial S}{\partial \gamma_1} \neq 0$ in some point of U, and by continuity also in some open subset of U. We express γ_1 from equation (7.3) and obtain that locally

$$\gamma_1 = \Gamma_1(\omega_1,\omega_2,\omega_3,\gamma_2,\gamma_3). \tag{7.4}$$

We compute $\frac{dF}{dt}$ and replace γ_1 with Γ_1 everywhere.

$$\begin{aligned}\frac{dF}{dt} &= \frac{[2(I_3\omega_3 + b_3)\omega_1 - c_1\gamma_3]\,[(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3]}{I_1}\\ &+ \frac{[2(I_3\omega_3 + b_3)\omega_2 - c_2\gamma_3]\,[(I_3 - I_1)\omega_1\omega_3 - b_3\omega_1 + b_1\omega_3 + c_1\gamma_3 - c_3\Gamma_1]}{I_2}\\ &+ (\omega_1^2 + \omega_2^2)\,[(I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\Gamma_1 - c_1\gamma_2]\\ &- (c_1\omega_1 + c_2\omega_2)(\omega_2\Gamma_1 - \omega_1\gamma_2).\end{aligned}$$

As we suppose that F is a partial first integral on invariant manifold (7.3) we have

$$\frac{dF}{dt} = 0. \tag{7.5}$$

We solve equation (7.5) with respect to Γ_1 and obtain a rational function depending on ω_1, ω_2, ω_3, γ_2 and γ_3:

$$\Gamma_1 = \frac{1}{I_1(-I_2c_2\omega_1^2 + I_2c_1\omega_1\omega_2 + 2b_3c_3\omega_2 + 2I_3c_3\omega_2\omega_3 - c_2c_3\gamma_3)}\Big[I_1I_2(I_1 - I_2)\omega_1^3\omega_2$$

$$+ I_1I_2b_2\omega_1^3 - I_1I_2b_1\omega_1^2\omega_2 + I_1I_2(I_1 - I_2)\omega_1\omega_2^3 + I_1I_2b_2\omega_1\omega_2^2$$
$$- 2I_3(I_1 - I_2)(I_1 + I_2 - I_3)\omega_1\omega_2\omega_3^2 - 2(I_1^2 - I_2^2)b_3\omega_1\omega_2\omega_3 + I_1I_2c_2\omega_1\omega_2\gamma_2$$
$$- 2(I_1 - I_2)b_3^2\omega_1\omega_2 - 2I_2I_3b_2\omega_1\omega_3^2 + 2I_2I_3c_3\omega_1\omega_3\gamma_2$$
$$+ (I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\omega_3\gamma_3 - 2I_2b_2b_3\omega_1\omega_3 + 2I_2b_3c_3\omega_1\gamma_2$$
$$+ (I_1 - 2I_2)b_3c_2\omega_1\gamma_3 - I_1I_2b_1\omega_2^3 - I_1I_2c_1\omega_2^2\gamma_2 + 2I_1I_3b_1\omega_2\omega_3^2$$
$$+ (2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_2\omega_3\gamma_3 + 2I_1b_1b_3\omega_2\omega_3 + (2I_1 - I_2)b_3c_1\omega_2\gamma_3$$
$$- (I_1b_1c_2 - I_2b_2c_1)\omega_3\gamma_3 - I_2c_1c_3\gamma_2\gamma_3 - (I_1 - I_2)c_1c_2\gamma_3^2\Big]. \qquad (7.6)$$

As $\Gamma_1 = \gamma_1$ then

$$W = \frac{d\Gamma_1}{dt} - \frac{d\gamma_1}{dt} = 0. \qquad (7.7)$$

Function W depends on γ_1 linearly. Indeed, as the expression for Γ_1 from (7.6) does not depend on γ_1 then its derivative $\frac{d\Gamma_1}{dt}$ is a linear function of γ_1 which is easily seen taking into account that the right-hand sides of Euler-Poisson equations (1.1) are linear with respect to γ_1. It may happen that the coefficient of γ_1 in $\frac{d\Gamma_1}{dt}$ is identically zero. This only occurs in the following three cases:

Case 1.

$$I_1 = \frac{I_2(I_2 - I_3)}{2I_3 + I_2}, \quad b_2 = 0, \quad b_3 = 0, \quad c_2 = 0, \quad c_3 = 0,$$

Case 2.

$$I_1 = 2I_3, \quad I_2 = 4I_3, \quad b_2 = 0, \quad c_2 = 0, \quad c_3 = 0,$$

Case 3.

$$I_1 = 4RootOf(8Z^2 + 78 - 51Z)I_3 - 12I_3, \quad I_2 = RootOf(8Z^2 + 78 - 51Z)I_3,$$
$$b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \quad c_1 = 0, \quad c_3 = 0.$$

In the last case the gyrostat equations (6.1) are reduced to the Euler-Poisson equations (1.1).

The two values of $RootOf(8Z^2 + 78 - 51Z)$ are $\frac{51\pm\sqrt{105}}{16}$. In fact Case 3 presents two different solutions, where in I_1 and I_2 the same sign + or - appears and so it will be in what follows.

Let F be the Sretenskii partial first integral (7.2).

Let us compute $\frac{dF}{dt}$ along the orbits of gyrostat equations (6.1). In above three cases we obtain:

Case 1.

$$\frac{dF}{dt} = -\frac{3I_2I_3\omega_1^3\omega_2}{I_2 + 2I_3} - b_1\omega_1^2\omega_2 - c_1\omega_1\omega_2\gamma_1 + \frac{12I_3^2(I_2 + I_3)\omega_1\omega_2\omega_3^2}{I_2(I_2 + 2I_3)}$$
$$- \frac{3I_2I_3\omega_1\omega_2^3}{I_2 + 2I_3} - b_1\omega_2^3 - c_1\omega_2^2\gamma_2 + \frac{2I_3b_1\omega_2\omega_3^2}{I_2} - c_1\omega_2\omega_3\gamma_3,$$

Case 2.

$$\frac{dF}{dt} = -2I_3\omega_1^3\omega_2 - b_1\omega_1^2\omega_2 - 2I_3\omega_1\omega_2^3 + \frac{5I_3\omega_1\omega_2\omega_3^2}{2} + 3b_3\omega_1\omega_2\omega_3 + \frac{b_3^2\omega_1\omega_2}{2I_3}$$

$$- c_1\omega_1\omega_2\gamma_1 - b_1\omega_2^3 - c_1\omega_2^2\gamma_2 + \frac{b_1\omega_2\omega_3^2}{2} - c_1\omega_2\omega_3\gamma_3 + \frac{b_1b_3\omega_2\omega_3}{2I_3},$$

Case 3.

$$\frac{dF}{dt} = 3\left(\frac{51 \pm \sqrt{105}}{16} - 4\right) I_3\omega_1^3\omega_2 + c_2\omega_1^2\gamma_1 + 3\left(\frac{51 \pm \sqrt{105}}{16} - 4\right) I_3\omega_1\omega_2^3$$
$$+ c_2\omega_1\omega_2\gamma_2 + \frac{I_3\omega_1\omega_2\omega_3^2}{2}.$$

In all these cases, $\frac{dF}{dt}$ does not vanish identically and thus these cases are outside of the domain of the Sretenskii partial first integral and we will ignore them.

Outside of these three cases the MAPLE command `degree` gives that the degree of W with respect to γ_1 is 1.

We solve equation (7.7) with respect to γ_1 and obtain

$$\gamma_1 = \widehat{\Gamma}_1(\omega_1, \omega_2, \omega_3, \gamma_2, \gamma_3).$$

The expression for $\widehat{\Gamma}_1$ is too long to be written here and we skip it. Let us note however that W is a rational function of all variables (ω, γ), whose numerator has 296 monomials and its denominator is

$$I_1^2 I_3 (I_2c_2\omega_1^2 - I_2c_1\omega_1\omega_2 - 2b_3c - 3\omega_2 - 2I_3c_3\omega_2\omega_3 + c_2c_3\gamma_3)^2.$$

Thus $\Gamma_1 - \widehat{\Gamma}_1 = 0$. Function $\Gamma_1 - \widehat{\Gamma}_1$ is a rational function of variables ω_1, ω_2, ω_3, γ_2 and γ_3. We only take its numerator which we denote by D. We want to know when D is identically equal to zero with respect to all the variables ω_1, ω_2, ω_3, γ_2 and γ_3. For the purpose we compute the coefficients of polynomial D. They are 513. We should find the conditions on the parameters $\mathcal{I}c$ at which all of these 513 coefficients are zero.

We apply simplification to the obtained system of 513 equations. After three consecutive simplifications we come to the reduced system consisting of nine equations:

$$\begin{aligned} &c_3 = 0, \quad b_2c_2 = 0, \quad b_1c_2 = 0, \quad (I_2 - 4I_3)c_2 = 0, \quad (I_1 - 4I_3)c_2 = 0, \\ &b_2c_1 = 0, \quad b_1c_1 = 0, \quad (I_2 - 4I_3)c_1 = 0, \quad (I_1 - 4I_3)c_1 = 0. \end{aligned} \tag{7.8}$$

We solve it by the MAPLE command `solve` and obtain two solutions. The first one is $c_1 = c_2 = c_3 = 0$ and we remove it because it leads to the Zhukovskii case. The second solution is

$$I_1 = I_2 = 4I_3, \quad b_1 = b_2 = 0, \quad c_3 = 0. \tag{7.9}$$

Now D vanishes identically. Taking into account (7.4) we compute γ_1 from (7.6) at condition (7.9) and obtain

$$\gamma_1 = -\frac{4I_3\omega_2\gamma_2 + I_3\omega_3\gamma_3 - b_3\gamma_3}{4I_3\omega_1},$$

that is

$$4I_3\omega_1\gamma_1 + 4I_3\omega_2\gamma_2 + I_3\omega_3\gamma_3 - b_3\gamma_3 = 0.$$

Let us note that the last equation is actually nothing but $H_1 = 0$ (see (6.2)) when $I_1 = I_2 = 4I_3$, $b_1 = b_2 = 0$. $\{H_1 = 0\}$ is an invariant manifold. Finally we conclude that when $\frac{\partial S}{\partial \gamma_1} \neq 0$ in some point of U, when $I_1 = I_2 = 4I_3$, $b_1 = b_2 = 0$, $(c_1, c_2) \neq (0, 0)$,

$c_3 = 0$, $\{H_1 = 0\}$ is the searched maximal invariant manifold. Thus we remain in the framework of the Sretenskii case.

The gyrostat equations (6.1) admit permutational symmetry (see Sec. 2)

$$\sigma_4 = \{(2,1,3),(2,1,3)\}.$$

Function F (see (7.2)) and also first integral H_1 are σ_4 invariant. Thus the solution of the problem about maximal invariant manifold S when $\frac{\partial S}{\partial \gamma_2} \neq 0$ in some point is exactly the same as in the just studied case when $\frac{\partial S}{\partial \gamma_1} \neq 0$ in some point of U.

2) Let us suppose now that $\frac{\partial S}{\partial \gamma_1}$ and $\frac{\partial S}{\partial \gamma_2}$ vanish identically on U that means that S does not depend on γ_1 and γ_2. Thus

$$S = S(\omega_1,\omega_2,\omega_3,\gamma_3).$$

Let us suppose that $\frac{\partial S}{\partial \gamma_3} \neq 0$ in some point of U, and by continuity also in some open subset of U. Like before we express γ_3 from equation (7.3) and obtain

$$\gamma_3 = \Gamma_3(\omega_1,\omega_2,\omega_3).$$

We compute $\frac{dF}{dt}$, replace γ_3 with Γ_3 everywhere and obtain

$$\begin{aligned}\frac{dF}{dt} &= \frac{[2(I_3\omega_3+b_3)\omega_1 - c_1\Gamma_3]\,[(I_2-I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\Gamma_3]}{I_1}\\ &\quad + \frac{[2(I_3\omega_3+b_3)\omega_2 - c_2\Gamma_3]\,[(I_3-I_1)\omega_1\omega_3 - b_3\omega_1 + b_1\omega_3 + c_1\Gamma_3 - c_3\gamma_1]}{I_2}\\ &\quad + (\omega_1^2+\omega_2^2)\,[(I_1-I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\gamma_1 - c_1\gamma_2]\\ &\quad - (c_1\omega_1 + c_2\omega_2)(\omega_2\gamma_1 - \omega_1\gamma_2).\end{aligned}$$

As we suppose that F is a partial first integral on invariant manifold $\{S = 0\}$ we have $\frac{dF}{dt} = 0$. Let us denote the numerator of $\frac{dF}{dt}$ by J. In this way we obtain the following equation for Γ_3

$$\begin{aligned}J &= -(I_1-I_2)c_1c_2\Gamma_3^2 + \big[(I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\omega_3 + (I_1-2I_2)b_3c_2\omega_1\\ &\quad + (2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_2\omega_3 + (2I_1-I_2)b_3c_1\omega_2 + (I_2c_1b_2 - I_1b_1c_2)\omega_3\\ &\quad + I_1c_2c_3\gamma_1 - I_2c_1c_3\gamma_2\big]\Gamma_3 + I_1I_2(I_1-I_2)\omega_1^3\omega_2 + I_1I_2b_2\omega_1^3 - I_1I_2b_1\omega_1^2\omega_2\\ &\quad + I_1I_2c_2\omega_1^2\gamma_1 + I_1I_2(I_1-I_2)\omega_1\omega_2^3 + I_1I_2b_2\omega_1\omega_2^2\\ &\quad - 2I_3(I_1-I_2)(I_1+I_2-I_3)\omega_1\omega_2\omega_3^2 - 2(I_1^2-I_2^2)b_3\omega_1\omega_2\omega_3 + I_1I_2c_2\omega_1\omega_2\gamma_2\\ &\quad + (-I_1I_2c_1\gamma_1 + 2I_2b_3^2 - 2I_1b_3^2)\omega_1\omega_2 - 2I_2I_3b_2\omega_1\omega_3^2 + 2I_2I_3c_3\omega_1\omega_3\gamma_2\\ &\quad - 2I_2b_2b_3\omega_1\omega_3 + 2I_2b_3c_2\omega_1\gamma_2 - I_1I_2b_1\omega_2^3 - I_1I_2c_1\omega_2^2\gamma_2 + 2I_1I_3b_1\omega_2\omega_3^2\\ &\quad - 2I_1(I_3c_3\gamma_1 - b_1b_3)\omega_2\omega_3 - 2I_1b_3c_3\omega_2\gamma_1 = 0.\end{aligned}$$

As $\Gamma_3 = \Gamma_3(\omega_1,\omega_2,\omega_3)$, after differentiation of J with respect to γ_1 and γ_2 we have

$$\begin{aligned}\frac{\partial J}{\partial \gamma_1} &= I_1I_2c_2\omega_1^2 - I_1I_2c_1\omega_1\omega_2 - 2I_1I_3c_3\omega_2\omega_3 - 2I_1b_3c_3\omega_2 + I_1c_2c_3\Gamma_3 = 0,\\ \frac{\partial J}{\partial \gamma_2} &= I_1I_2c_2\omega_1\omega_2 + 2I_2I_3c_3\omega_1\omega_3 + 2I_2b_3c_3\omega_1 - I_1I_2c_1\omega_2^2 - I_2c_1c_3\Gamma_3 = 0.\end{aligned} \tag{7.10}$$

Let us first suppose that c_1, c_2 and c_3 are all different from zero. Then excluding Γ_3 from (7.10) we obtain

$$(c_1\omega_2 - c_2\omega_1)(I_2c_1\omega_1 + I_1c_2\omega_2 + 2I_3c_3\omega_3 + 2b_3c_3) = 0,$$

which is obviously impossible.

Let now $c_1 = 0$. Then from second equation (7.10) we have the following identity

$$I_1I_2c_2\omega_1\omega_2 + 2I_2I_3c_3\omega_1\omega_3 + 2I_2b_3c_3\omega_1 = 0,$$

which is possible only when $c_2 = c_3 = 0$. But this is the Zhukovskii case.

Let now $c_2 = 0$. From first equation (7.10) we have

$$-I_1I_2c_1\omega_1\omega_2 - 2I_1I_3c_3\omega_2\omega_3 - 2I_1b_3c_3\omega_2 = 0,$$

which is possible only when $c_1 = c_3 = 0$, i.e. again we are in the Zhukovskii case.

Finally let $c_3 = 0$. Equations (7.10) give

$$I_1I_2(c_2\omega_1 - c_1\omega_2)\omega_1 = 0, \qquad I_1I_2(c_2\omega_1 - I_1I_2c_1\omega_2)\omega_2 = 0.$$

The above two conditions can be fulfilled only when $c_1 = c_2 = 0$, i.e. only in the Zhukovskii case.

The conclusion is that the sought function $S(\omega_1, \omega_2, \omega_3, \gamma_3)$ does not exist.

3) Let us suppose now that all $\frac{\partial S}{\partial \gamma_i}$, $i = 1, 2, 3$, vanish identically on U, that means that S does not depend on γ_1, γ_2 and γ_3. Thus

$$S = S(\omega_1, \omega_2, \omega_3).$$

Let us suppose that $\frac{\partial S}{\partial \omega_3} \neq 0$ in some point of U, and thus also in some open subset of U. We express ω_3 from equation (7.3) and obtain

$$\omega_3 = \Omega_3(\omega_1, \omega_2). \tag{7.11}$$

We compute $\frac{dF}{dt}$ and replace ω_3 with Ω_3 everywhere.

$$\begin{aligned}\frac{dF}{dt} = {} & \frac{[2(I_3\Omega_3 + b_3)\omega_1 - c_1\gamma_3]\,[(I_2 - I_3)\omega_2\Omega_3 + b_3\omega_2 - b_2\Omega_3 + c_3\gamma_2 - c_2\gamma_3]}{I_1} \\ & + \frac{[2(I_3\Omega_3 + b_3)\omega_2 - c_2\gamma_3]\,[(I_3 - I_1)\omega_1\Omega_3 - b_3\omega_1 + b_1\Omega_3 + c_1\gamma_3 - c_3\gamma_1]}{I_2} \\ & + (\omega_1^2 + \omega_2^2)\,[(I_1 - I_2)\omega_1\omega_2 + b_2\omega_1 - b_1\omega_2 + c_2\gamma_1 - c_1\gamma_2] \\ & - (c_1\omega_1 + c_2\omega_2)(\omega_2\gamma_1 - \omega_1\gamma_2).\end{aligned}$$

As above $\frac{dF}{dt} = 0$. Denote the numerator of $\frac{dF}{dt}$ by K. In this way we obtain the following equation for Ω_3

$$\begin{aligned}K = {} & \big[-2I_3(I_1 - I_2)(I_1 + I_2 - I_3)\omega_1\omega_2 - 2I_2I_3b_2\omega_1 + 2I_1I_3b_1\omega_2\big]\Omega_3^2 \\ & + \big[-2b_3(I_1^2 - I_2^2)\omega_1\omega_2 + 2I_2I_3c_3\omega_1\gamma_2 + (I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\gamma_3 - 2I_2b_2b_3\omega_1 \\ & - 2I_1I_3c_3\omega_2\gamma_1 + (2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_2\gamma_3 + 2I_1b_1b_3\omega_2 + (I_2b_2c_1 - I_1b_1c_2)\gamma_3\big]\Omega_3 \\ & + I_1I_2(I_1 - I_2)\omega_1^3\omega_2 + I_1I_2b_1\omega_1^3 - I_1I_2b_1\omega_1^2\omega_2 + I_1I_2c_2\omega_1^2\gamma_1 + I_1I_2(I_1 - I_2)\omega_1\omega_2^3 \\ & + I_1I_2b_2\omega_1\omega_2^2 - I_1I_2c_1\omega_1\omega_2\gamma_1 + I_1I_2c_2\omega_1\omega_2\gamma_2 - 2(I_1 - I_2)b_3^2\omega_1\omega_2 + 2I_2b_3c_3\omega_1\gamma_2 \\ & + (I_1 - 2I_2)b_3c_2\omega_1\gamma_3 - I_1I_2b_1\omega_2^3 - I_1I_2c_1\omega_2^2\gamma_2 - 2I_1b_3c_3\omega_2\gamma_1 + (2I_1 - I_2)b_3c_1\omega_2\gamma_3 \\ & + I_1c_2c_3\gamma_1\gamma_3 - I_2c_1c_3\gamma_2\gamma_3 - (I_1 - I_2)c_1c_2\gamma_3^2 = 0.\end{aligned}$$

As $\Omega_3 = \Omega_3(\omega_1, \omega_2)$, then after differentiation of K with respect to γ_1 and γ_2 we have

$$\begin{aligned}\frac{\partial K}{\partial \gamma_1} &= I_1I_2c_2\omega_1^2 - I_1I_2c_1\omega_1\omega_2 - 2I_1b_3c_3\omega_2 + I_1c_2c_3\gamma_3 - 2I_1I_3c_3\omega_2\Omega_3 = 0,\\ \frac{\partial K}{\partial \gamma_2} &= I_1I_2c_2\omega_1\omega_2 - I_1I_2c_1\omega_2^2 + 2I_2b_3c_3\omega_1 - I_2c_1c_3\gamma_3 + 2I_2I_3c_3\omega_1\Omega_3 = 0.\end{aligned} \tag{7.12}$$

If we suppose that $c_3 \neq 0$ then we come to the Lagrange case. Indeed, as Ω_3 depends only on ω_1 and ω_2, then from first equation of (7.12) it follows that $c_2 = 0$. This is so because otherwise Ω_3 would depend on γ_3 too. For the same reason second equation of (7.12) gives that $c_1 = 0$. However at the condition $c_1 = c_2 = 0$ and $c_3 \neq 0$ both first and second equations of (7.12) lead to the conclusion that $\Omega_3 = -b_3/I_3$, i.e., according to (7.11), $\omega_3 = -b_3/I_3$. In such a case we obtain from the third equation of the gyrostat equations (6.1) that $I_1 = I_2$, $b_1 = b_2 = 0$ and $c_1 = c_2 = 0$. Thus we come to the Lagrange case.

Let now $c_3 = 0$. Then the first and second equations of (7.12) become:

$$I_1I_2(c_2\omega_1 - c_1\omega_2)\omega_1 = 0, \qquad I_1I_2(c_2\omega_1 - c_1\omega_2)\omega_2 = 0.$$

The above two equations should be identities which is possible only when $c_1 = c_2 = 0$, i.e. we come to the Zhukovskii case.

The conclusion is that the sought function $S(\omega_1, \omega_2, \omega_3)$ does not exist.

4) Let now the function S be

$$S = S(\omega_1, \omega_2)$$

and $\frac{\partial S}{\partial \omega_2} \neq 0$ in some point of U and therefore in some open subset of U. We express ω_2 from equation (7.3) and obtain

$$\omega_2 = \Omega_2(\omega_1). \tag{7.13}$$

We compute $\frac{dF}{dt}$, replace ω_2 with Ω_2 everywhere and obtain

$$\begin{aligned}\frac{dF}{dt} = &\frac{[2(I_3\omega_3 + b_3)\omega_1 - c_1\gamma_3]\,[(I_2 - I_3)\Omega_2\omega_3 + b_3\Omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3]}{I_1}\\ &+ \frac{[2(I_3\omega_3 + b_3)\Omega_2 - c_2\gamma_3]\,[(I_3 - I_1)\omega_1\omega_3 - b_3\omega_1 + b_1\omega_3 + c_1\gamma_3 - c_3\gamma_1]}{I_2}\\ &+ (\omega_1^2 + \Omega_2^2)\,[(I_1 - I_2)\omega_1\Omega_2 + b_2\omega_1 - b_1\Omega_2 + c_2\gamma_1 - c_1\gamma_2]\\ &- (c_1\omega_1 + c_2\Omega_2)(\Omega_2\gamma_1 - \omega_1\gamma_2).\end{aligned}$$

As above $\frac{dF}{dt} = 0$. Denote the numerator of $\frac{dF}{dt}$ by L. In this way we obtain the following equation for Ω_2

$$\begin{aligned}L = {}&I_1I_2\big[(I_1 - I_2)\omega_1 - b_1\big]\Omega_2^3 + I_1I_2(b_2\omega_1 - c_1\gamma_2)\Omega_2^2\\ &+ \big[I_1I_2(I_1 - I_2)\omega_1^3 - I_1I_2b_1\omega_1^2 - 2I_3(I_1 - I_2)(I_1 + I_2 - I_3)\omega_1\omega_3^2 - 2(I_1^2 - I_2^2)b_3\omega_1\omega_3\\ &- I_1I_2c_1\omega_1\gamma_1 + I_1I_2c_2\omega_1\gamma_2 - 2(I_1 - I_2)b_3^2\omega_1 + 2I_1I_3b_1\omega_3^2 - 2I_1I_3c_3\omega_3\gamma_1\\ &+ (2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_3\gamma_3 + 2I_1b_3b_1\omega_3 - 2I_1b_3c_3\gamma_1 + (2I_1 - I_2)b_3c_1\gamma_3\big]\Omega_2\\ &+ I_1I_2b_2\omega_1^3 + I_1I_2c_2\omega_1^2\gamma_1 - 2I_2I_3b_2\omega_1\omega_3^2 + 2I_2I_3c_3\omega_1\omega_3\gamma_2\\ &+ (I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\omega_3\gamma_3 - 2I_2b_2b_3\omega_1\omega_3 + 2I_2b_3c_3\omega_1\gamma_2 + (I_1 - 2I_2)b_3c_2\omega_1\gamma_3\end{aligned}$$

$$+ (I_2b_2c_1 - I_1b_1c_2)\omega_3\gamma_3 + I_1c_2c_3\gamma_1\gamma_3 - I_2c_1c_3\gamma_2\gamma_3 - (I_1 - I_2)c_1c_2\gamma_3^2 = 0.$$

As $\Omega_2 = \Omega_2(\omega_1)$, then after differentiation of L with respect to γ_1, γ_2 and γ_3 we have

$$\begin{aligned}
\frac{\partial L}{\partial \gamma_1} &= I_1c_2(I_2\omega_1^2 + c_3\gamma_3) - I_1(I_2c_1\omega_1 + 2I_3c_3\omega_3 + 2b_3c_3)\Omega_2 = 0,\\
\frac{\partial L}{\partial \gamma_2} &= I_2c_3(2I_3\omega_1\omega_3 + 2b_3\omega_1 - c_1\gamma_3) + I_1I_2c_2\omega_1\Omega_2 - I_1I_2c_1\Omega_2^2 = 0,\\
\frac{\partial L}{\partial \gamma_3} &= (I_1^2 - I_1I_3 - 2I_2I_3)c_2\omega_1\omega_3 + (I_1 - 2I_2)b_3c_2\omega_1 - I_1c_2b_1\omega_3\\
&\quad + I_2c_1b_2\omega_3 + I_1c_2c_3\gamma_1 - I_2c_1c_3\gamma_2 - 2(I_1 - I_2)c_1c_2\gamma_3\\
&\quad + \big[(2I_1I_3 - I_2^2 + I_2I_3)c_1\omega_3 + (2I_1 - I_2)b_3\big]\Omega_2 = 0.
\end{aligned} \tag{7.14}$$

From the first equation of (7.14) it follows that $c_2 \neq 0$. Indeed, if $c_2 = 0$ then either $c_1 = c_3 = 0$ that is the Zhukovskii case or Ω_2 vanishes identically, that contradicts (7.13). From the same equation it is seen that $c_3 = 0$ because if $c_3 \neq 0$ then this equation contains only one monomial depending on γ_3 which cannot be canceled because Ω_2 depends on ω_1 only. Thus $c_3 = 0$.

In such a case the first equation of (7.14) is rewritten as follows:

$$I_1I_2\omega_1(c_2\omega_1 - c_1\Omega_2) = 0. \tag{7.15}$$

Equation (7.15) imposes the restriction $c_1 \neq 0$ because otherwise $I_1I_2c_2\omega_1^2 = 0$ which is impossible. We solve (7.15) with respect to Ω_2 and obtain

$$\Omega_2 = \Omega_2(\omega_1) = \frac{c_2\omega_1}{c_1}.$$

At this condition the second equation of (7.14) is satisfied and the third one becomes

$$(I_1-I_2)(I_1+I_3+I_2)c_2\omega_1\omega_3+3(I_1-I_2)b_3c_2\omega_1+(I_2b_2c_1-I_1b_1c_2)\omega_3-2(I_1-I_2)c_1c_2\gamma_3 = 0.$$

Taking into account that $c_1 \neq 0$ and $c_2 \neq 0$, the last item of the above equation leads to $I_1 = I_2$. Thus

$$I_2(b_2c_1 - b_1c_2)\omega_3 = 0.$$

In this way we come to the case

$$I_1 = I_2, \quad c_1 \neq 0, \quad c_2 \neq 0, \quad c_3 = 0, \quad b_2c_1 - b_1c_2 = 0.$$

In this case L vanishes identically and therefore we can take as function S the following function

$$S = c_1\omega_2 - c_2\omega_1.$$

So far we have not yet examined when $\{S = 0\} = \{c_1\omega_2 - c_2\omega_1 = 0\}$ is an invariant manifold. To do this we compute $\frac{dS}{dt}$ and obtain

$$\frac{dS}{dt} = \frac{(c_1\omega_1 + c_2\omega_2)\big[(I_3 - I_2)c_2\omega_3 - b_3c_2\big] + (c_1^2 + c_2^2)(b_2\omega_3 + c_2\gamma_3)}{I_2c_2}.$$

It is seen that in generic case $\{S = 0\}$ is not an invariant manifold. But let us put

$$c_1^2 + c_2^2 = 0,$$

i.e. either $c_1 = ic_2$ or $c_1 = -ic_2$. Let us first consider the case $c_1 = ic_2$. We have

$$S = c_2 S_1, \quad S_1 = i\omega_2 - \omega_1$$

and

$$\frac{dS_1(t)}{dt} = \frac{i\big[(I_2 - I_3)\omega_3(t) + b_3\big](i\omega_2(t) - \omega_1(t))}{I_2} = S_1(t)\frac{i\big[(I_2 - I_3)\omega_3(t) + b_3\big]}{I_2}.$$

This equation admits the zero solution $S_1(t) = 0$ for all t. Thus from the unicity of solutions for this equation one obtains that if $S_1(t_0) = 0$ for some t_0, then $S_1(t) = 0$ for all t. In other words, if for some t_0, $i\omega_2(t_0) - \omega_1(t_0) = 0$, then $i\omega_2(t) - \omega_1(t) = 0$ for all t. But this is precisely the invariance of manifold $\widehat{S}_1 = \{S_1 = 0\}$. In this way we come to the conclusion that when

$$I_1 = I_2, \quad c_1 = ic_2 \neq 0, \quad c_3 = 0, \quad b_2c_1 - b_1c_2 = 0.$$

the gyrostat equations (6.1) have an invariant manifold $\{i\omega_2 - \omega_1 = 0\}$.

The case $c_1 = -ic_2$ is considered in the same way. The difference is that now $S = c_2 S_2$, where $S_2 = -i\omega_2 - \omega_1$ and the invariant manifold is $\{i\omega_2 + \omega_1\}$.

But F is not a partial first integral of the gyrostat equations (6.1) on the invariant manifolds $\widehat{S}_1 = \{S_1 = 0\}$ and $\widehat{S}_2 = \{S_2 = 0\}$. In fact it is easy to see that on them F vanishes identically.

Finally, we conclude that the codimension one maximal domain of the Sretenskii partial first integral (7.2) coincides with the manifold $\{H_1 = 0\}$ under the conditions

$$I_1 = I_2 = 4I_3, \quad b_1 = b_2 = 0, \quad (c_1, c_2) \neq (0,0), \quad c_3 = 0.$$

In addition, when

$$I_1 = I_2, \quad b_2c_1 - b_1c_2 = 0, \quad c_1^2 + c_2^2 = 0, \quad c_3 = 0,$$

we found two invariant relations for gyrostat equations (6.1):

$$S_1 = i\omega_2 - \omega_1 \text{ when } c_1 = ic_2 \qquad \text{and} \qquad S_2 = -i\omega_2 - \omega_1 \text{ when } c_1 = -ic_2.$$

We will prove that S_1 is functionally independent of first integrals H_1, H_2 and H_3. For this purpose we consider the Jacobi matrix J of functions H_1, H_2, H_3 and S_1. We prove that $\operatorname{rank} J = 4$. Indeed, computing the determinant J_{14} composed from matrix J by crossing out its first and fourth columns we obtain

$$J_{14} = -4iI_3(I_2\omega_2\omega_3\gamma_3 - I_3\omega_3^2\gamma_2 - c_2\gamma_3^2 - b_2\omega_3\gamma_3 + b_3\omega_3\gamma_2).$$

It is clearly seen that J_{14} never vanishes identically. Thus functions H_1, H_2, H_3 and S_1 are functionally independent.

The study of the functional independence of H_1, H_2, H_3 and S_2 is the same. The only difference is that now the value of the determinant is $-J_{14}$.

5) Finally let $S = S(\omega, \gamma) = S(\omega_1)$, where S does not vanish identically. It is easy to see that then $\widehat{S} = \{(\omega, \gamma) \in U; S(\omega_1) = 0\}$ is a five-dimensional submanifold if and only if $\widehat{S} = \{(\omega, \gamma) \in U;\ \omega_1 \in \Omega_1\}$, where Ω_1 is a set of zeros of S. In complex case Ω_1 is at most countable. In real case Ω_1 is a subset of $\mathbb{R}$ that does not contains an open interval. In both cases $\frac{d\omega_1}{dt} = 0$ and from the first of the gyrostat equations (6.1) one obtains that

$$(I_2 - I_3)\omega_2\omega_3 + b_3\omega_2 - b_2\omega_3 + c_3\gamma_2 - c_2\gamma_3 = 0$$

for all ω_2, ω_3, γ_2 $\gamma_3 \in \mathbb{C}$. Thus $I_2 = I_3$, $b_2 = b_3 = 0$ and $c_2 = c_3 = 0$ and we recover the Lagrange case. Thus in both cases, complex and real, the sought function $S(\omega_1)$ does not exist.

Let us recall that when in gyrostat equations (6.1) $b_1 = b_2 = b_3 = 0$, we recover the Euler-Poisson equations (1.1). The Sretenskii case of partial integrability becomes the Goryachev-Chaplygin case of partial integrability and the Sretenskii partial first integral (7.2) becomes the Goryachev-Chaplygin partial first integral (1.8).

Consequently, from the above, one deduces immediately that the maximal domain of the Goryachev-Chaplygin partial first integral (1.8) is

$$\{H_1 = I_1\omega_1\gamma_1 + I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 = 0\},$$

where $I_1 = I_2 = 4I_3$, $(c_1, c_2) \neq (0, 0)$, $c_3 = 0$.

Finally let us note that the invariant relations are the same in Sretenskii and in Goryachev-Chaplygin cases.

8. Four-dimensional invariant manifolds. New integrals on $\{H_i{=}U_i,\ H_j{=}U_j\}$, $1 \leq i < j \leq 3$

8.1. Extraction procedure. In this section we study the existence of a local partial first integral of the Euler-Poisson equations (1.1) restricted to the invariant complex four-dimensional level manifold $\{H_i = U_i,\ H_j = U_j\}$, $1 \leq i < j \leq 3$. We study when on each of them there exists a local partial first integral that depends on at most three variables and such that on $\{H_i = U_i,\ H_j = U_j\}$ it is functionally independent of H_k, $k \neq i, j$.

Let us fix i and j, $1 \leq i < j \leq 3$. According to (2.5)

$$M(U_0, U_i, U_j, \mathcal{I}c) = \{x \in \mathbb{C}^6;\ H_i((\omega,\gamma), \mathcal{I}c) = U_i,\ H_j((\omega,\gamma), \mathcal{I}c) = U_j\},$$

where $(\omega, \gamma) = (\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$.

We search all functions F of three variables $F = F(s_1, s_2, s_3)$ where $(s_1, s_2, s_3) \in (\omega, \gamma)$, of class $\mathbb{C}^1$, such that $\operatorname{grad} F$ does not vanish identically on each open subset of $M(U_0, U_i, U_j, \mathcal{I}c)$, which are local partial first integrals of the Euler-Poisson equations (1.1) restricted to $M(U_0, U_i, U_j, \mathcal{I}c)$. Like in Sec. 5 the unique intrinsic property of $\mathbb{C}^1$ function F that is a local partial first integral is that $\operatorname{grad} F$ does not vanish identically on any open subset of its domain of definition. This implies that some of the partial derivatives of F may be identically zero. Thus the results of Sec. 8 also remain valid for the functions of at most three variables.

We follow the same way as in Sec. 5.1. As in Sec. 5.1 the order of variables s_i, $1 \leq i \leq 3$, in $F(s_1, s_2, s_3)$ is irrelevant for F to be a first integral.

We have exactly 20 different three elements subsets of (ω, γ) and thus 20 cases of functions of three elements to examine. We will describe now an extraction procedure based on permutational symmetries which reduces the above 20 cases to only six.

These 20 functions of three variables (up to the order of variables) are shown in Table 8.1.

Table 8.1

Functions	Case
$F(\omega_1,\omega_2,\omega_3)$	(i)
$F(\omega_1,\omega_2,\gamma_3)$, $F(\omega_1,\omega_3,\gamma_2)$, $F(\omega_2,\omega_3,\gamma_1)$	(ii)
$F(\omega_1,\omega_2,\gamma_1)$, $F(\omega_1,\omega_3,\gamma_1)$, $F(\omega_2,\omega_3,\gamma_2)$, $F(\omega_1,\omega_2,\gamma_2)$, $F(\omega_1,\omega_3,\gamma_3)$, $F(\omega_2,\omega_3,\gamma_3)$	(iii)
$F(\omega_1,\gamma_1,\gamma_2)$, $F(\omega_1,\gamma_1,\gamma_3)$, $F(\omega_2,\gamma_2,\gamma_3)$, $F(\omega_2,\gamma_1,\gamma_2)$, $F(\omega_3,\gamma_1,\gamma_3)$, $F(\omega_3,\gamma_2,\gamma_3)$	(iv)
$F(\omega_3,\gamma_1,\gamma_2)$, $F(\omega_2,\gamma_1,\gamma_3)$, $F(\omega_1,\gamma_2,\gamma_3)$	(v)
$F(\gamma_1,\gamma_2,\gamma_3)$	(vi)

It is easy to see that under the group of permutational symmetries (2.3) of the Euler-Poisson equations for every case (i)–(vi) from Table 8.1 each function from the fixed case is consequently transformed into all remaining functions from the same case.

Thus in virtue of Theorem 2.2 we can restrict ourselves to the study of only six functions where every one belongs to a different case from Table 8.1 and is chosen arbitrary from the functions of this case.

We will call such six functions F_i, $1 \le i \le 6$, (up to the order of variables) a basis.

8.2. Invariant manifold $\{H_1{=}U_1,\ H_2{=}U_2\}$. Here we continue the study of the existence of a partial first integral of the Euler-Poisson equations (1.1) restricted to the complex four-dimensional level manifold

$$\{H_1 = U_1,\ H_2 = U_2\}, \tag{8.1}$$

supposing that this first integral depends on at most three variables and that is functionally independent of H_3. For this aim we shall use the same approach as in Sec. 5.

In the future when we refer to "some suitable open set" in space $\mathbb{C}^4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ in Sec. 8 where $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} \in \{\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2,\gamma_3\}$ or in space $\mathbb{C}^3(\alpha_1,\alpha_2,\alpha_3)$ in Sec. 9 where $\{\alpha_1,\alpha_2,\alpha_3\} \in \{\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2,\gamma_3\}$, we mean an open set such that all functions of the above variables never vanish on it when this is necessary for a proof. For example if such a function appears in some denominator or when we need to have a holomorphic branch of roots of some of these functions. In the future, we will use this terminology without any further discussion.

8.2.1. Elimination of γ_2 and γ_3. Using the MAPLE command `solve` we express γ_2 and γ_3 from the equations $H_1 = U_1$ and $H_2 = U_2$ and obtain the following solution:

$$\gamma_2 = \frac{I_1\omega_1\gamma_1 + I_3\omega_3 R - U_1}{I_2\omega_2}, \qquad \gamma_3 = R, \tag{8.2}$$

where R is a root of equation

$$Q(x) = Ax^2 + Bx + C = 0,$$

that is

$$Q(R) = AR^2 + BR + C = 0 \tag{8.3}$$

and $A = A(\omega_2, \omega_3)$, $B = B(\omega_1, \omega_3, \gamma_1)$ and $C = C(\omega_1, \omega_2, \gamma_1)$ are the following polynomials:

$$\begin{aligned} A &= I_2^2\omega_2^2 + I_3^2\omega_3^2, \qquad B = 2I_3\omega_3(I_1\omega_1\gamma_1 - U_1), \\ C &= I_1^2\omega_1^2\gamma_1^2 - 2I_1U_1\omega_1\gamma_1 - I_2^2U_2\omega_2^2 + I_2^2\omega_2^2\gamma_1^2 + U_1^2. \end{aligned} \tag{8.4}$$

Here MAPLE does not give an explicit formula for R but expresses R as a root of the following quadratic polynomial:

$$\begin{aligned} &RootOf((I_3^2\omega_3^2 + I_2^2\omega_2^2)Z^2 + 2I_3\omega_3(I_1\omega_1\gamma_1 - U_1)Z \\ &+ I_1^2\omega_1^2\gamma_1^2 - 2I_1U_1\omega_1\gamma_1 - I_2^2U_2\omega_2^2 + I_2^2\omega_2^2\gamma_1^2 + U_1^2) \\ &= RootOf(AZ^2 + BZ + C), \end{aligned}$$

Thus we can say that R is a root of equation (8.3) where the coefficients A, B and C are defined by (8.4).

Let us consider the four-dimensional vector space $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$ and a point $(\omega_1, \omega_2, \omega_3, \gamma_1) \in \mathbb{C}^4$ with $\omega_i \neq 0$, $i = 1, 2, 3$, $\gamma_1 \neq 0$.

All our considerations are local. Thus from the beginning we can restrict ourselves to some suitable open set Ω in the space $\mathbb{C}^4 = \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

By their very definition the first integrals are not constant on any open subset of their domain of definition. As we consider C^1 first integrals, this means that their gradients are non-zero on any open subset of their domain of definition.

We put the values of γ_2 and γ_3 from (8.2) in the Euler-Poisson equations (1.1) and remove the fifth and sixth equations. In this way we have the following system of four equations in unknowns ω_1, ω_2, ω_3 and γ_1:

$$\begin{aligned} \frac{d\omega_1}{dt} &= \frac{1}{I_1I_2\omega_2}\big[I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1c_3\omega_1\gamma_1 - (I_2c_2\omega_2 + I_3c_3\omega_3)R + c_3U_1\big], \\ \frac{d\omega_2}{dt} &= \frac{1}{I_2}\big[(I_3 - I_1)\omega_1\omega_3 + c_1R - c_3\gamma_1\big], \\ \frac{d\omega_3}{dt} &= \frac{1}{I_2I_3\omega_2}\big[I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1c_1\omega_1\gamma_1 + I_2c_2\omega_2\gamma_1 + I_3c_1\omega_3R - c_1U_1\big], \\ \frac{d\gamma_1}{dt} &= \frac{1}{I_2\omega_2}\big[-I_1\omega_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)R + \omega_3U_1\big]. \end{aligned} \tag{8.5}$$

Here we study whether system (8.5) has a first integral that depends on at most three variables among the variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$ and that is functionally independent of H_3 restricted to invariant manifold (8.1). Thus we should investigate the following four types of a new first integral:

1. $F(\omega_1, \omega_2, \omega_3)$, (case (i))
2. $F(\omega_1, \omega_2, \gamma_1)$, (case (iii))
3. $F(\omega_1, \omega_3, \gamma_1)$, (case (iii))
4. $F(\omega_2, \omega_3, \gamma_1)$. (case (ii))

Then, like in Sec. 5 it suffices to examine the functions of types 1, 2 and 4 respectively.

Type 1. Let us suppose that the sought first integral F is of type 1, i.e. $F = F(\omega_1, \omega_2, \omega_3)$. As F is a first integral of system (8.5) we have

$$\frac{dF}{dt} = \frac{1}{I_1 I_2 \omega_2}\big[I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1 c_3 \omega_1 \gamma_1 - (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)R + c_3 U_1\big]\frac{\partial F}{\partial \omega_1} + \frac{1}{I_2}\big[(I_3 - I_1)\omega_1\omega_3 + c_1 R - c_3\gamma_1\big]\frac{\partial F}{\partial \omega_2} + \frac{1}{I_2 I_3 \omega_2}\big[I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1 c_1 \omega_1 \gamma_1 + I_2 c_2 \omega_2 \gamma_1 + I_3 c_1 \omega_3 R - c_1 U_1\big]\frac{\partial F}{\partial \omega_3} = 0,$$

or equivalently

$$I_1 I_2 I_3 \omega_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.6}$$

where Y_1 is the corresponding vector field defined on Ω.

Equation (8.6) should be an identity with respect to all the four variables ω_1, ω_2, ω_3 and γ_1. As function F does not depend on γ_1 then its partial derivatives will not depend on γ_1 too. Thus if we differentiate identity (8.6) with respect to γ_1 we shall obtain again a linear partial differential equation for function F. We obtain

$$\frac{\partial Y_1(F)}{\partial \gamma_1} = -I_3\left[I_1 c_3 \omega_1 + (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial R}{\partial \gamma_1}\right]\frac{\partial F}{\partial \omega_1} + I_1 I_3 \omega_2\left(c_1 \frac{\partial R}{\partial \gamma_1} - c_3\right)\frac{\partial F}{\partial \omega_2} + I_1\left(I_1 c_1 \omega_1 + I_2 c_2 \omega_2 + I_3 c_1 \omega_3 \frac{\partial R}{\partial \gamma_1}\right)\frac{\partial F}{\partial \omega_3} = 0,$$

i.e.

$$\frac{\partial Y_1(F)}{\partial \gamma_1} = Y_2(F) = 0, \tag{8.7}$$

where Y_2 is the corresponding vector field defined on Ω.

We differentiate one time more identity (8.7) with respect to γ_1 and obtain

$$\frac{\partial Y_2(F)}{\partial \gamma_1} = I_3 \frac{\partial^2 R}{\partial \gamma_1^2}\left[-(I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial F}{\partial \omega_1} + I_2 c_1 \omega_2 \frac{\partial F}{\partial \omega_2} + I_1 c_1 \omega_3 \frac{\partial F}{\partial \omega_3}\right] = 0,$$

i.e.

$$\frac{1}{I_3}\frac{\partial Y_2(F)}{\partial \gamma_1} = Y_3(F) = 0, \tag{8.8}$$

where Y_3 is the corresponding vector field defined on Ω.

Let us suppose first that

$$\frac{\partial^2 R}{\partial \gamma_1^2} \neq 0. \tag{8.9}$$

Equations (8.6)–(8.8) can be considered as a system of three homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of Ω.

Thus, if a fourth integral F exists, system (8.6)–(8.8) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of this system is identically equal to zero. We compute this determinant and obtain

$$D = I_1^2 I_2 I_3 \omega_2^2 \frac{\partial^2 R}{\partial \gamma_1^2} D_1 D_2,$$

where

$$\begin{aligned} D_1 &= (I_1c_1\omega_1 + I_2c_2\omega_2 + I_3c_3\omega_3),\\ D_2 &= (I_2 - I_1)c_3\omega_1\omega_2 + (I_1 - I_3)c_2\omega_1\omega_3 + (I_3 - I_2)c_1\omega_2\omega_3. \end{aligned} \tag{8.10}$$

Note that D depends neither on R nor on $\frac{\partial R}{\partial \gamma_1}$. Taking into account (8.9) it is clear that $D \equiv 0$ if and only if at least one of the expressions (8.10) is identically equal to zero. It is easily seen that this happens only in the Euler, Lagrange and kinetic symmetry cases.

Thus the restriction (8.9) leads to nothing new and we suppose now that

$$\frac{\partial^2 R}{\partial \gamma_1^2} = 0. \tag{8.11}$$

In such a case only equations (8.6) and (8.7) remain because $Y_3 \equiv 0$.

Let us study whether there are such values of parameters $\mathcal{I}c$, U_1 and U_2 at which (8.11) is fulfilled. For this purpose we differentiate (8.3) twice with respect to γ_1. Taking into account that polynomial A from (8.4) does not depend on γ_1 we have

$$\frac{\partial Q}{\partial \gamma_1} = \frac{\partial B}{\partial \gamma_1}R + \frac{\partial C}{\partial \gamma_1} + \frac{dQ}{dR}\frac{\partial R}{\partial \gamma_1} = 0 \tag{8.12}$$

and

$$\frac{\partial^2 Q}{\partial \gamma_1^2} = \frac{\partial^2 B}{\partial \gamma_1^2}R + \frac{\partial B}{\partial \gamma_1}\frac{\partial R}{\partial \gamma_1} + \frac{\partial^2 C}{\partial \gamma_1^2} + \frac{\partial}{\partial \gamma_1}\left(\frac{dQ}{dR}\right)\frac{\partial R}{\partial \gamma_1} + \frac{dQ}{dR}\frac{\partial^2 R}{\partial \gamma_1^2} = 0. \tag{8.13}$$

First we prove that if R is a root of equation (8.3), then $\frac{dQ}{dR} \neq 0$. For this purpose we apply Proposition 4.1 to polynomial Q. We consider the resultant ρ of polynomials $Q(R)$ and $\frac{dQ}{dR}$ and prove that $\rho \neq 0$. Indeed, we have

$$\rho = A(4AC - B^2).$$

As we are interested in cases where ρ vanishes identically with respect to ω_1, ω_2, ω_3 and γ_1 only and as A never vanishes identically we do not consider ρ but $\widehat{\rho} = 4AC - B^2$ instead. Putting in $\widehat{\rho}$ the expressions for A, B and C (see (8.4)) we obtain

$$\widehat{\rho} = 4I_2^2\omega_2^2(I_1^2\omega_1^2\gamma_1^2 + I_2^2\omega_2^2\gamma_1^2 + I_3^2\omega_3^2\gamma_1^2 - 2I_1\omega_1\gamma_1U_1 - I_2^2\omega_2^2U_2 - I_3^2\omega_3^2U_2 + U_1^2), \tag{8.14}$$

which, as one can easily see, never vanishes identically. Thus we can express $\frac{\partial R}{\partial \gamma_1}$ from equation (8.12) and then determine $\frac{\partial^2 R}{\partial \gamma_1^2}$ from (8.13) as follows:

$$\frac{\partial^2 R}{\partial \gamma_1^2} = \frac{I_2^2\omega_2^2 S}{[(I_2^2\omega_2^2 + I_3^2\omega_3^2)R + I_3\omega_3(I_1\omega_1\gamma_1 - U_1)]^3}, \tag{8.15}$$

where

$$\begin{aligned} S = &-(I_2^2\omega_2^2 + I_3^2\omega_3^2)(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2)R^2\\ &+ 2I_3\omega_3(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2)(U_1 - I_1\omega_1\gamma_1)R\\ &- I_1^4\omega_1^4\gamma_1^2 + 2I_1^3\omega_1^3\gamma_1U_1 - 2I_1^2I_2^2\omega_1^2\omega_2^2\gamma_1^2 - I_1^2I_3^2\omega_1^2\omega_3^2\gamma_1^2 - I_1^2\omega_1^2U_1^2\\ &+ 2I_1I_2^2\omega_1\omega_2^2\gamma_1U_1 + 2I_1I_3^2\omega_1\omega_3^2\gamma_1U_1 - I_2^4\omega_2^4\gamma_1^2 - I_2^2I_3^2\omega_2^2\omega_3^2\gamma_1^2 - I_3^2\omega_3^2U_1^2. \end{aligned}$$

Equations (8.15) and (8.11) imply that $S = 0$. Taking into account that $Q = 0$ (see (8.3)) we can assert that

$$(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2)Q + S = I_2^2\omega_2^2[U_1^2 - U_2(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2)]$$

is also zero. This is possible if and only if

$$U_1 = U_2 = 0. \tag{8.16}$$

Let us consider the case defined by (8.16). In this case R can be presented in the form $R = P\gamma_1$, where P is a root of equation

$$(I_2^2\omega_2^2 + I_3^2\omega_3^2)P^2 + 2I_1I_3\omega_1\omega_3P + I_1^2\omega_1^2 + I_2^2\omega_2^2 = 0. \tag{8.17}$$

This fact is obtained very easily from (8.3) if we take into account condition (8.16). Indeed, it suffices to divide equation (8.3) by γ_1^2 and denote $\frac{R}{\gamma_1}$ by P. Moreover, being a root of equation (8.17), P is a homogeneous function of degree zero and depends only on ω_1, ω_2 and ω_3.

Thus vector field Y_1 (see (8.6)) is linear with respect to γ_1 and can be presented as follows

$$Y_1 = K_1\gamma_1 + K_2,$$

where K_1 and K_2 are vector fields defined on Ω by the formulas

$$\begin{aligned}
K_1 &= -I_3(I_1c_3\omega_1 + I_2c_2\omega_2P + I_3c_3\omega_3P)\frac{\partial}{\partial\omega_1} + I_1I_3\omega_2(c_1P - c_3)\frac{\partial}{\partial\omega_2} \\
&\quad + I_1(I_1c_1\omega_1 + I_2c_2\omega_2 + I_3c_1\omega_3P)\frac{\partial}{\partial\omega_3}, \\
K_2 &= \omega_2\Big[I_2I_3\omega_2\omega_3(I_2 - I_3)\frac{\partial}{\partial\omega_1} - I_1I_3\omega_1\omega_3(I_1 - I_3)\frac{\partial}{\partial\omega_2} \\
&\quad + I_1I_2\omega_1\omega_2(I_1 - I_2)\frac{\partial}{\partial\omega_3}\Big].
\end{aligned}$$

Equation (8.6) and the fact that first integral F does not depend on γ_1 imply

$$K_1(F) = K_2(F) = 0. \tag{8.18}$$

Function $I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$ is non-constant on all open subsets of $\mathbb{C}_4(\omega_1, \omega_2, \omega_3, \gamma_1)$. Thus without any restriction of generality one can suppose that on our suitable open set the function $I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \neq 0$.

In order to simplify the formulas let us put

$$\alpha = I_2^2\omega_2^2 + I_3^2\omega_3^2, \quad \beta = \sqrt{-I_1^2\omega_1^2 - I_2^2\omega_2^2 - I_3^2\omega_3^2}, \quad \mathrm{Re}(\beta) > 0. \tag{8.19}$$

Equation (8.17) has two roots

$$P = \frac{-I_1I_3\omega_1\omega_3 + \varepsilon I_2\omega_2\beta}{\alpha},$$

where $\varepsilon = \pm 1$. Substituting this value of P in the expression for K_1 we obtain

$$\begin{aligned}
\frac{\alpha K_1}{\omega_2} &= -I_2I_3\Big[I_1\omega_1(I_2c_3\omega_2 - I_3c_2\omega_3) + \varepsilon\beta(I_2c_2\omega_2 + I_3c_3\omega_3)\Big]\frac{\partial}{\partial\omega_1} \\
&\quad + I_1I_3\Big[-I_1I_3c_1\omega_1\omega_3 - c_3\alpha + \varepsilon\beta I_2c_1\omega_2\Big]\frac{\partial}{\partial\omega_2}
\end{aligned}$$

$$+\, I_1 I_2 \Big[I_1 I_2 c_1 \omega_1 \omega_2 + c_2 \alpha + \varepsilon\beta I_3 c_1 \omega_3 \Big] \frac{\partial}{\partial \omega_3}.$$

As we are interested in equations (8.18) we remove all the non-zero factors from K_1 and K_2, i.e. we shall work with the vector fields

$$Z_1 = \frac{\alpha K_1}{\omega_2} \quad \text{and} \quad Z_2 = \frac{K_2}{\omega_2}.$$

Therefore instead of (8.18) we consider the following equations:

$$Z_1(F) = Z_2(F) = 0. \tag{8.20}$$

We compute the commutator $K_3 = [Z_1, Z_2]$ and obtain

$$\begin{aligned}
\frac{K_3}{I_1 I_2 I_3} = \Big[&-(I_1 - I_2)\, I_1 I_2 I_3 c_2 \omega_1^2 \omega_2 - (I_1 - I_3)\, I_1 I_2 I_3 c_3 \omega_1^2 \omega_3 \\
&+ (I_2 - I_3)\, I_1 I_2^2 c_1 \omega_1 \omega_2^2 - (I_2 - I_3)\, I_1 I_3^2 c_1 \omega_1 \omega_3^2 \\
&+ \varepsilon\beta\, (I_1 - I_2)\, I_2 I_3 c_3 \omega_1 \omega_2 - \varepsilon\beta\, (I_1 - I_3)\, I_2 I_3 c_2 \omega_1 \omega_3 \\
&+ (I_2 - I_3)\, I_2^3 c_2 \omega_2^3 + 2\, \varepsilon\beta\, (I_2 - I_3)\, I_2 I_3 c_1 \omega_2 \omega_3 - (I_2 - I_3)\, I_3^3 c_3 \omega_3^3 \Big] \frac{\partial}{\partial \omega_1} \\
&+ \Big[-(I_2 - I_3)\, I_1^3 c_1 \omega_1^2 \omega_2 - (I_1 - I_3)\, I_1 I_2^2 c_2 \omega_1 \omega_2^2 \\
&- (I_1 I_2 - 2\, I_1 I_3 + I_2 I_3)\, I_1 I_3 c_3 \omega_1 \omega_2 \omega_3 - 2\, (I_1 - I_3)\, I_1 I_3^2 c_2 \omega_1 \omega_3^2 \\
&+ (I_2 - I_3)\, I_1 I_3^2 c_1 \omega_2 \omega_3^2 + \varepsilon\beta\, (I_1 - I_3)\, I_2 I_3 c_2 \omega_2 \omega_3 \\
&+ \varepsilon\beta\, (I_1 - I_3)\, I_3^2 c_3 \omega_3^2 \Big] \frac{\partial}{\partial \omega_2} \\
&+ \Big[(I_2 - I_3)\, I_1^3 c_1 \omega_1^2 \omega_3 - 2\, (I_1 - I_2)\, I_1 I_2^2 c_3 \omega_1 \omega_2^2 \\
&+ (2\, I_1 I_2 - I_1 I_3 - I_2 I_3)\, I_1 I_2 c_2 \omega_1 \omega_2 \omega_3 - (I_1 - I_2)\, I_1 I_3^2 c_3 \omega_1 \omega_3^2 \\
&- (I_2 - I_3)\, I_1 I_2^2 c_1 \omega_2^2 \omega_3 - \varepsilon\beta\, (I_1 - I_2)\, I_2^2 c_2 \omega_2^2 - \varepsilon\beta\, (I_1 - I_2)\, I_2 I_3 c_3 \omega_2 \omega_3 \Big] \frac{\partial}{\partial \omega_3}.
\end{aligned}$$

We consider vector field

$$Z_3 = \frac{K_3}{I_1 I_2 I_3}$$

instead of K_3.

Equations (8.20) imply that $Z_3(F) = 0$. In this way we obtain the following three equations for determining function F

$$Z_1(F) = Z_2(F) = Z_3(F) = 0. \tag{8.21}$$

Equations (8.21) can be considered as a system of three homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3} \right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if a fourth integral F exists, system (8.21) has a non-zero solution $\operatorname{grad} F$. We know that this is possible if and only if the determinant D of the coefficients of equations (8.21) is identically equal to zero. We compute this determinant and obtain that on Ω

$$D = I_1 I_2 I_3 (f_1 \beta + f_2), \tag{8.22}$$

where f_1 and f_2 are the following polynomials in ω_1, ω_2 and ω_3:

$$\begin{aligned}
\frac{f_1}{\varepsilon} = &- (I_1 - I_2)(I_1 - I_3) I_1^2 I_2^3 c_1 c_2 \omega_1^3 \omega_2^3 - (I_1 - I_2)(I_1 - I_3) I_1^2 I_2^2 I_3 c_1 c_3 \omega_1^3 \omega_2^2 \omega_3 \\
&- (I_1 - I_2)(I_1 - I_3) I_1^2 I_2 I_3^2 c_1 c_2 \omega_1^3 \omega_2 \omega_3^2 - (I_1 - I_2)(I_1 - I_3) I_1^2 I_3^3 c_1 c_3 \omega_1^3 \omega_3^3 \\
&+ (I_1 - I_2)\left[(I_2 - I_3) I_1 c_1^2 - (I_1 - I_3) I_2 c_2^2 - (I_1 - I_2) I_3 c_3^2\right] I_1 I_2^3 \omega_1^2 \omega_2^4 \\
&- 2\,(I_1 - I_2)(I_2 - I_3) I_1^2 I_2^2 I_3 c_2 c_3 \omega_1^2 \omega_2^3 \omega_3 \\
&+ I_1 I_2 I_3 \Big[(I_2 - I_3)^2 I_1^2 c_1^2 + (I_1 - I_3)(I_1 I_2 - 3\, I_1 I_3 + 2\, I_2 I_3) I_2 c_2^2 \\
&- (I_1 - I_2)(3\, I_1 I_2 - I_1 I_3 - 2\, I_2 I_3) I_3 c_3^2\Big] \omega_1^2 \omega_2^2 \omega_3^2 \\
&+ 2\,(I_1 - I_3)(I_2 - I_3) I_1^2 I_2 I_3^2 c_2 c_3 \omega_1^2 \omega_2 \omega_3^3 \\
&- (I_1 - I_3)\left[(I_2 - I_3) I_1 c_1^2 + (I_1 - I_3) I_2 c_2^2 + (I_1 - I_2) I_3 c_3^2\right] I_1 I_3^3 \omega_1^2 \omega_3^4 \\
&+ (I_1 - I_2)(I_2 - I_3) I_1 I_2^4 c_1 c_2 \omega_1 \omega_2^5 + (I_1 - I_2)(I_2 - I_3) I_1 I_2^2 I_3^2 c_1 c_2 \omega_1 \omega_2^3 \omega_3^2 \\
&- (I_1 - I_3)(I_2 - I_3) I_1 I_2^2 I_3^2 c_1 c_3 \omega_1 \omega_2^2 \omega_3^3 - (I_1 - I_3)(I_2 - I_3) I_1 I_3^4 c_1 c_3 \omega_1 \omega_3^5 \\
&- (I_1 - I_2)(I_2 - I_3) I_2^4 I_3 c_2 c_3 \omega_2^5 \omega_3 \\
&+ (I_2 - I_3)\left[(I_2 - I_3) I_1 c_1^2 + (I_1 - I_3) I_2 c_2^2 - (I_1 - I_2) I_3 c_3^2\right] I_2^3 I_3 \omega_2^4 \omega_3^2 \\
&+ (I_2 - I_3)^2 I_1 I_2^2 I_3^2 c_2 c_3 \omega_2^3 \omega_3^3 \\
&+ (I_2 - I_3)\left[(I_2 - I_3) I_1 c_1^2 + (I_1 - I_3) I_2 c_2^2 - (I_1 - I_2) I_3 c_3^2\right] I_2 I_3^3 \omega_2^2 \omega_3^4 \\
&+ (I_1 - I_3)(I_2 - I_3) I_2 I_3^4 c_2 c_3 \omega_2 \omega_3^5,
\end{aligned}$$

$$\begin{aligned}
\frac{f_2}{\beta^2} = &-(I_1 - I_2) I_2^2 I_1 c_3 c_1 (-I_1 I_3 + I_1 I_2 + \varepsilon^2 I_1 I_3 - \varepsilon^2 I_2 I_3) \omega_1^2 \omega_2^3 \\
&+ \varepsilon^2 (I_1 - I_2)(-I_3 + I_1) I_1 I_2^2 I_3 c_2 c_1 \omega_1^2 \omega_2^2 \omega_3 \\
&- \varepsilon^2 (I_1 - I_2)(I_1 - I_3) I_1 I_2 I_3^2 c_3 c_1 \omega_1^2 \omega_2 \omega_3^2 \\
&+ (I_1 - I_3)(I_1 I_3 - I_2 I_1 + \varepsilon^2 I_1 I_2 - \varepsilon^2 I_2 I_3) I_1 I_3^2 c_2 c_1 \omega_1^2 \omega_3^3 \\
&- (I_1 - I_2)(I_2 - I_3) I_1 I_2^3 c_2 c_3 \omega_1 \omega_2^4 \\
&- 2(I_1 - I_2)(I_2 - I_3) I_1 I_2^2 I_3 (\varepsilon^2 c_1^2 + c_3^2) \omega_1 \omega_2^3 \omega_3 \\
&+ I_1 (I_2 - I_3)(I_1 I_2 + I_1 I_3 - 2 I_2 I_3) I_2 I_3 c_2 c_3 \omega_1 \omega_2^2 \omega_3^2 \\
&- 2(I_2 - I_3)(I_1 - I_3) I_1 I_2 I_3^2 (\varepsilon^2 c_1^2 + c_2^2) \omega_1 \omega_2 \omega_3^3 \\
&- (I_1 - I_3)(I_2 - I_3) I_1 I_3^3 c_3 c_2 \omega_1 \omega_3^4 - \varepsilon^2 (I_1 - I_2)(I_2 - I_3) I_2^3 I_3 c_2 c_1 \omega_2^4 \omega_3 \\
&- (I_2 - I_3)(I_1 I_2 - I_1 I_3 + \varepsilon^2 I_1 I_3 - \varepsilon^2 I_2 I_3) I_2^2 I_3 c_1 c_3 \omega_2^3 \omega_3^2 \\
&- (I_2 - I_3)(I_1 I_3 - I_1 I_2 + \varepsilon^2 I_1 I_2 - \varepsilon^2 I_2 I_3) I_2 I_3^2 c_1 c_2 \omega_2^2 \omega_3^3 \\
&- \varepsilon^2 (I_1 - I_3)(I_2 - I_3) I_2 I_3^3 c_1 c_3 \omega_2 \omega_3^4.
\end{aligned}$$

As $I_i \neq 0$, $1 \leq i \leq 3$, then from (8.22) one deduces that $f_1 \beta + f_2 = 0$. If $f_1 = 0$ identically, $f_2 = 0$ identically too. Let us suppose that $f_1 \neq 0$. (8.22) is then equivalent to

$$\beta = -\frac{f_2}{f_1}. \tag{8.23}$$

Applying Proposition 4.3 to $\beta^2 = -I_1^2 \omega_1^2 - I_2^2 \omega_2^2 - I_3^2 \omega_3^2$ one sees that (8.23) can never occur because $\beta \notin \mathbb{C}(\omega_1, \omega_2, \omega_3)$. Consequently

$$f_1 = f_2 = 0.$$

Note that ε appears in f_1 as a factor and in f_2 only as ε^2. We replace ε^2 with 1 everywhere in f_2. As β^2 can never be identically equal to zero, we require that all the 18 coefficients of f_1/ε and all the 13 coefficients of f_2/β^2 be zero, i.e. we obtain a system of 31 equations for the parameters $\mathcal{I}c$.

After three consecutive simplifications we obtain the reduced system that contains eight equations:

$$\begin{aligned}
&(I_2 - I_3)c_2c_3 = 0, \quad (I_1 - I_3)c_1c_3 = 0, \quad (I_1 - I_2)(I_1 - I_3)(I_2 - I_3)c_3 = 0,\\
&(I_1 - I_2)(I_1 - I_3)(c_2^2 + c_3^2) = 0, \quad (I_1 - I_2)(I_2 - I_3)c_1c_2 = 0,\\
&(I_1 - I_2)(I_1 + I_2 - 2I_3)c_1c_2 = 0,\\
&(I_2 - I_3)\big[(I_2 - I_3)c_1^2 + (I_1 - I_3)c_2^2 + (I_2 - I_1)c_3^2\big] = 0,\\
&(I_1 - I_3)(I_2 - I_3)(c_1^2 + c_2^2) = 0.
\end{aligned}$$

We solve these eight equations by the Maple command `solve` and obtain a set of eight solutions. We remove the solutions that lead to the Euler, Lagrange and kinetic symmetry cases and obtain only three new solutions. They are:

I. $I_1 = I_2$, $c_1 = \pm i c_2$, $c_3 = 0$,
II. $I_1 = I_3$, $c_1 = \pm i c_3$, $c_2 = 0$,
III. $I_2 = I_3$, $c_1 = 0$, $c_2 = \pm i c_3$.

We describe here only solution I because solutions II and III are obtained from it by permutational symmetries σ_2 and σ_3, respectively.

We consider separately two cases:

1. $c_1 = i c_2$ with $\varepsilon = 1$ and with $\varepsilon = -1$.
2. $c_1 = -i c_2$ with $\varepsilon = 1$ and with $\varepsilon = -1$.

Let us remark that the above situation is exactly the one we met in Sec. 7.2 when finding invariant manifolds.

Case 1. Let $I_1 = I_2$, $c_1 = i c_2$, $c_3 = 0$ and $\varepsilon = 1$. Now vector field Z_3 is linearly dependent on Z_1 and Z_2 and therefore the local solvability of system (8.21) around any point $(\omega_1, \omega_2, \omega_3) \neq (0, 0, 0)$ follows from the Frobenius Integrability Theorem. Moreover, system (8.5) is quasi-homogeneous (we recall that $R = P\gamma_1$ and P is a homogeneous function of degree zero). Thus from [41] it follows that the searched first integral F can be chosen as a homogeneous function of the variables $(\omega_1, \omega_2, \omega_3)$. But in fact we shall compute F by a crude computation, without any use of the Frobenius Integrability Theorem nor the results of [41]. Nevertheless the above facts guide our approach to the problem.

Let us add to equations (8.20) the Euler "homogeneity equation"

$$\omega_1 \frac{\partial F}{\partial \omega_1} + \omega_2 \frac{\partial F}{\partial \omega_2} + \omega_3 \frac{\partial F}{\partial \omega_3} = F. \tag{8.24}$$

Dividing equations (8.20) and (8.24) by F we obtain a system of three linear partial differential equations for determining function $V = \log F$. We solve this system as a linear inhomogeneous algebraic system with respect to partial derivatives of V and obtain

$$\frac{\partial V}{\partial \omega_1} = \frac{\left(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta\right)\omega_1}{\left(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta\right)(\omega_1 - i\omega_2)},$$
$$\frac{\partial V}{\partial \omega_2} = \frac{\left(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta\right)\omega_2}{\left(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta\right)(\omega_1 - i\omega_2)},$$
$$\frac{\partial V}{\partial \omega_3} = \frac{iI_3\left(I_3\omega_1\omega_3 - \omega_2\beta\right)}{I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta}.$$

Now, by the standard procedure, we find function V.

We integrate $\frac{\partial V}{\partial \omega_i}$, $1 \le i \le 3$, with respect to ω_i and in this way we obtain three expressions for the function V.

$$V = \int \frac{\left(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta\right)\omega_1}{\left(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta\right)(\omega_1 - i\omega_2)} d\omega_1 + G_1(\omega_2, \omega_3),$$
$$V = \int \frac{\left(I_2^2\omega_1\omega_2 - iI_2^2\omega_2^2 - iI_3^2\omega_3^2 + I_3\omega_3\beta\right)\omega_2}{\left(I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta\right)(\omega_1 - i\omega_2)} d\omega_2 + G_2(\omega_1, \omega_3),$$
$$V = \int \frac{iI_3\left(I_3\omega_1\omega_3 - \omega_2\beta\right)}{I_2^2\omega_1^2\omega_2 + I_2^2\omega_2^3 + I_3^2\omega_2\omega_3^2 + I_3\omega_1\omega_3\beta} d\omega_3 + G_3(\omega_1, \omega_2),$$

where G_1, G_2 and G_3 are arbitrary functions of the corresponding variables.

The first two expressions for V are too complicated and we do not use them. We only use the third expression which is rewritten as follows:

$$V = \operatorname{csgn}(I_3)i \arctan \frac{\operatorname{csgn}(I_3)I_3\omega_3}{\beta} + G_3(\omega_1, \omega_2),$$

where the function $\operatorname{csgn}(z)$ is used to determine in which half-plane ("left" or "right") the complex-valued number z lies. It is defined by

$$\operatorname{csgn}(z) = \begin{cases} 1 & \text{if } \operatorname{Re}(z) > 0, \\ -1 & \text{if } \operatorname{Re}(z) < 0, \\ \operatorname{sgn}(\operatorname{Im}(z)) & \text{if } \operatorname{Re}(z) = 0. \end{cases}$$

As arctan is an odd function we can write

$$V = i \arctan \frac{I_3\omega_3}{\beta} + G_3(\omega_1, \omega_2). \tag{8.25}$$

In order to determine function $G_3(\omega_1, \omega_2)$ we differentiate (8.25) with respect to ω_i, $1 \le i \le 3$, and obtain

$$\frac{\partial V}{\partial \omega_1} = -\frac{iI_3\omega_1\omega_3}{(\omega_1^2 + \omega_2^2)\beta} + \frac{\partial G_3}{\partial \omega_1},$$
$$\frac{\partial V}{\partial \omega_2} = -\frac{iI_3\omega_2\omega_3}{(\omega_1^2 + \omega_2^2)\beta} + \frac{\partial G_3}{\partial \omega_2},$$
$$\frac{\partial V}{\partial \omega_3} = \frac{iI_3}{\beta}.$$

We know that function V satisfies system (8.20) so that we have

$$Z_1(V) = Z_2(V) = 0. \tag{8.26}$$

System (8.26) is a system of two linear partial differential equations with respect to function $G_3(\omega_1, \omega_2)$. We solve it as a linear inhomogeneous algebraic system in unknowns $\frac{\partial G_3}{\partial \omega_1}$ and $\frac{\partial G_3}{\partial \omega_2}$ and obtain

$$\frac{\partial G_3}{\partial \omega_1} = \frac{\omega_1}{\omega_1^2 + \omega_2^2}, \qquad \frac{\partial G_3}{\partial \omega_2} = \frac{\omega_2}{\omega_1^2 + \omega_2^2}.$$

After integration these expressions lead to

$$G_3(\omega_1, \omega_2) = \frac{1}{2} \log{(\omega_1^2 + \omega_2^2)} + C,$$

where C is a constant which can be considered as a zero because it is added to a first integral. Thus from (8.25) we have

$$V = i \arctan \frac{I_3 \omega_3}{\beta} + \frac{1}{2} \log{(\omega_1^2 + \omega_2^2)}.$$

As $V = \log F$ we have

$$F = \exp V = \frac{-I_3 \omega_3 + i\beta}{\pm I_2}.$$

We remove the constant denominator and change the sign of the function. Let us note the obtained function by F_1. We have

$$F_1 = I_3 \omega_3 - i\beta.$$

F_1 satisfies equations (8.20) that means that in Case 1 with $\varepsilon = 1$, F_1 is a first integral of system (8.5).

Let now $\varepsilon = -1$. The considerations are exactly the same as above however now

$$V = -i \arctan \frac{I_3 \omega_3}{\beta} + \frac{1}{2} \log{(\omega_1^2 + \omega_2^2)}$$

and $F = \exp V$ is

$$F = \frac{I_2(\omega_1^2 + \omega_2^2)}{I_3 \omega_3 - i\beta}.$$

Function F can be simplified. Indeed, as according to (8.19)

$$\beta^2 = -I_2^2(\omega_1^2 + \omega_2^2) - I_3^2 \omega_3^2,$$

then

$$\omega_1^2 + \omega_2^2 = -\frac{I_3^2 \omega_3^2 + \beta^2}{I_2^2} = -\frac{(I_3 \omega_3 + i\beta)(I_3 \omega_3 - i\beta)}{I_2^2}.$$

We put the obtained value for $\omega_1^2 + \omega_2^2$ in the above expression for F and obtain

$$F = -\frac{I_3 \omega_3 + i\beta}{I_2}.$$

By removing the constant denominator I_2 and changing the sign of F we obtain a new function noted by F_2

$$F_2 = I_3 \omega_3 + i\beta.$$

F_2 satisfies equations (8.20). Thus in Case 1 with $\varepsilon = -1$, F_2 is a first integral of system (8.5).

Case 2. Let $I_1 = I_2$, $c_1 = -ic_2$, $c_3 = 0$. The only difference in this case in comparison with Case 1 is that when $\varepsilon = 1$, F_2 is a first integral of system (8.5) and when $\varepsilon = -1$ the first integral is F_1.

The functional independence of these partial first integrals with H_3 follows from the fact that they do not depend on γ_1 while H_3 does.

Let us note that the partial first integrals F_1 and F_2 are algebraic without being rational. This is a new fact. Up to now, the known first integrals or partial first integrals not depending on all variables, have been polynomials.

Type 2. Let us study the existence of a first integral of type 2, i.e. $F = F(\omega_1, \omega_2, \gamma_1)$. Now we have

$$\begin{aligned}\frac{dF}{dt} = &\frac{1}{I_1 I_2 \omega_2}\big[I_2(I_2 - I_3)\omega_2^2\omega_3 - I_1 c_3 \omega_1 \gamma_1 - (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)R + c_3 U_1\big]\frac{\partial F}{\partial \omega_1}\\ &+ \frac{1}{I_2}\big[(I_3 - I_1)\omega_1\omega_3 + c_1 R - c_3\gamma_1\big]\frac{\partial F}{\partial \omega_2}\\ &+ \frac{1}{I_2\omega_2}\big[- I_1\omega_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)R + \omega_3 U_1\big]\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

or equivalently

$$I_1 I_2 \omega_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.27}$$

where Y_1 is the corresponding vector field defined on Ω.

Equation (8.27) should be an identity with respect to all four variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$. Similarly to the consideration of a first integral of type 1 if we differentiate identity (8.27) with respect to ω_3 we will again obtain a linear partial differential equation for function F. We obtain

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \omega_3} = &\left[I_2(I_2 - I_3)\omega_2^2 - I_3 c_3 R - (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial R}{\partial \omega_3}\right]\frac{\partial F}{\partial \omega_1}\\ &- I_1\omega_2\left[(I_1 - I_3)\omega_1 - c_1\frac{\partial R}{\partial \omega_3}\right]\frac{\partial F}{\partial \omega_2}\\ &- I_1\left[I_1\omega_1\gamma_1 + 2I_3\omega_3 R + (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial R}{\partial \omega_3} - U\right]\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

i.e.

$$\frac{\partial Y_1(F)}{\partial \omega_3} = Y_2(F) = 0, \tag{8.28}$$

where Y_2 is the corresponding vector field defined on Ω.

After differentiating identity (8.28) with respect to ω_3 we obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial \omega_3} = &-\left[2I_3 c_3\frac{\partial R}{\partial \omega_3} + (I_2 c_2 \omega_2 + I_3 c_3 \omega_3)\frac{\partial^2 R}{\partial \omega_3^2}\right]\frac{\partial F}{\partial \omega_1} + I_1 c_1 \omega_2 \frac{\partial^2 R}{\partial \omega_3^2}\frac{\partial F}{\partial \omega_2}\\ &- I_1\left[2I_3 R + 4I_3\omega_3\frac{\partial R}{\partial \omega_3} + (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial^2 R}{\partial \omega_3^2}\right]\frac{\partial F}{\partial \gamma_1} = 0,\end{aligned}$$

i.e.

$$\frac{\partial Y_2(F)}{\partial \omega_3} = Y_3(F) = 0, \tag{8.29}$$

where Y_3 is the corresponding vector field defined on Ω.

We already know that if a fourth integral F exists, system (8.27)–(8.29) has a non-zero solution $\operatorname{grad} F$ and it is possible if and only if the determinant D of the coefficients of this system is identically equal to zero.

We compute D and obtain a long expression which we do not write here. Let us note that D has a non-zero factor $I_1^2\omega_2$ and we note

$$\widehat{D}(R) = \frac{D(R)}{I_1^2\omega_2}.$$

It is clear that the equation $D(R) = 0$ is equivalent to $\widehat{D}(R) = 0$.

Derivatives $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$ appear in $\widehat{D}(R)$. To determine them we differentiate equation (8.3) with respect to ω_3 two times. Taking into account that polynomial C from (8.4) does not depend on ω_3 we obtain

$$\frac{\partial Q}{\partial \omega_3} = \frac{\partial A}{\partial \omega_3}R^2 + \frac{\partial B}{\partial \omega_3}R + \frac{dQ}{dR}\frac{\partial R}{\partial \omega_3} = 0. \tag{8.30}$$

and

$$\begin{aligned}\frac{\partial^2 Q}{\partial \omega_3^2} = \frac{\partial^2 A}{\partial \omega_3^2}R^2 + 2R\frac{\partial A}{\partial \omega_3}\frac{\partial R}{\partial \omega_3} + \frac{\partial^2 B}{\partial \omega_3^2}R + \frac{\partial B}{\partial \omega_3}\frac{\partial R}{\partial \omega_3}\\ + \frac{\partial}{\partial \omega_3}\left(\frac{dQ}{dR}\right)\frac{\partial R}{\partial \omega_3} + \frac{dQ}{dR}\frac{\partial^2 R}{\partial \omega_3^2} = 0.\end{aligned} \tag{8.31}$$

Like in the investigation for a first integral of type 1, by Proposition 4.1 we prove that if R is a root of equation (8.3), then $\frac{dQ}{dR} \neq 0$. Of course, the resultant ρ of polynomials $Q(R)$ and $\frac{dQ}{dR}$ is the same and we make the conclusion that $\frac{dQ}{dR} \neq 0$ (see formula (8.14)).

Thus we can correctly determine $\frac{\partial R}{\partial \omega_3}$ from equation (8.30) and put its value in (8.31). Then we easily determine $\frac{\partial^2 R}{\partial \omega_3^2}$ from (8.31). This determination is also correct because the coefficient in front of $\frac{\partial^2 R}{\partial \omega_3^2}$ is also $\frac{dQ}{dR}$.

Then we put the obtained values for the derivatives of R in the expression for $\widehat{D}(R)$ and obtain that $\widehat{D}(R)$ has a non-zero factor $8I_3R$ and denominator $\left(\frac{dQ}{dR}\right)^3$. We note

$$\delta(R) = \frac{\left(\frac{dQ}{dR}\right)^3}{8I_3R}\widehat{D}(R),$$

where $\delta(R)$ is a polynomial of R of degree five with coefficients which are polynomials of ω_1, ω_2, ω_3 and γ_1. It is clear that the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$.

We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_1, \omega_2, \gamma_1)$ of system (8.5) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_1, \omega_2, \omega_3, \gamma_1)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using Maple command `rem` we divide δ by Q and obtain a remainder which is a polynomial r of the form:

$$r(x) = \frac{I_2^4\omega_2^4}{I_2^2\omega_2^2 + I_3^2\omega_3^2}(a_0x + a_1),$$

where a_0 and a_1 are polynomials of ω_1, ω_2, ω_3 and γ_1.

According to Proposition 4.2 we know that if R is a root of equation (8.3), then a_0 and a_1 should be identically equal to zero. We use a_0 only because it turns out to be sufficient for our purposes.

Polynomial a_0 has 81 coefficients. Thus we should equate to zero all of them. In this way we obtain a system of 81 equations for parameters $\mathcal{I}c$, U_1 and U_2. After two consecutive simplifications we obtain the reduced system which is very simple:

$$c_3 = 0, \qquad (I_1 - I_3)c_2 = 0, \qquad (I_2 - I_3)c_1 = 0.$$

The solutions are obvious:

$$\begin{aligned}
&\{c_1 = 0,\ c_2 = 0,\ c_3 = 0\},\ U_1,\ U_2,\ I_1,\ I_2 \text{ and } I_3 \text{ are parameters},\\
&\{I_2 = I_3,\ c_2 = 0,\ c_3 = 0\},\ U_1,\ U_2,\ I_1,\ I_3 \text{ and } c_1 \text{ are parameters},\\
&\{I_1 = I_3,\ c_1 = 0,\ c_3 = 0\},\ U_1,\ U_2,\ I_2,\ I_3 \text{ and } c_2 \text{ are parameters},\\
&\{I_1 = I_3,\ I_2 = I_3,\ c_3 = 0\},\ U_1,\ U_2,\ I_3,\ c_1 \text{ and } c_2 \text{ are parameters}.
\end{aligned}$$

It is easily seen that these solutions lead to the Euler, Lagrange and kinetic symmetry cases, respectively. Thus the sought partial first integral of type 2 cannot exist.

Type 4. Finally let us investigate the possibilities for the existence of a first integral of type 4, $F(\omega_2, \omega_3, \gamma_1)$, i.e. when it does not depend on ω_1.

As F is a first integral of system (8.5) we have

$$\begin{aligned}
\frac{dF}{dt} &= \frac{1}{I_2}\big[(I_3 - I_1)\omega_1\omega_3 + c_1 R - c_3\gamma_1\big]\frac{\partial F}{\partial \omega_2}\\
&+ \frac{1}{I_2 I_3 \omega_2}\big[I_2(I_1 - I_2)\omega_1\omega_2^2 + I_1 c_1 \omega_1\gamma_1 + I_2 c_2 \omega_2 \gamma_1 + I_3 c_1 \omega_3 R - c_1 U_1\big]\frac{\partial F}{\partial \omega_3}\\
&+ \frac{1}{I_2\omega_2}\big[- I_1\omega_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)R + \omega_3 U_1\big]\frac{\partial F}{\partial \gamma_1} = 0,
\end{aligned}$$

or equivalently

$$I_2 I_3 \omega_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.32}$$

where Y_1 is the corresponding vector field defined on Ω.

Equation (8.32) should be an identity with respect to all four variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$. As in the previous considerations taking into account that F does not depend on ω_1, differentiating identity (8.32) with respect to ω_1 we obtain again a linear partial differential equation for function F

$$\begin{aligned}
\frac{\partial Y_1(F)}{\partial \omega_1} &= I_3\omega_2\left[(I_3 - I_1)\omega_3 + c_1\frac{\partial R}{\partial \omega_1}\right]\frac{\partial F}{\partial \omega_2}\\
&\quad + \left[I_2(I_1 - I_2)\omega_2^2 + I_1 c_1 \gamma_1 + I_3 c_1 \omega_3 \frac{\partial R}{\partial \omega_1}\right]\frac{\partial F}{\partial \omega_3}\\
&\quad + I_3\left[-I_1\omega_3\gamma_1 - (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial R}{\partial \omega_1}\right]\frac{\partial F}{\partial \gamma_1} = Y_2(F) = 0,
\end{aligned} \tag{8.33}$$

where Y_2 is the corresponding vector field defined on Ω.

We differentiate identity (8.33) with respect to ω_1 and obtain

$$\frac{1}{I_3}\frac{\partial Y_2(F)}{\partial \omega_1} = \frac{\partial^2 R}{\partial \omega_1^2}\left[c_1\omega_2\frac{\partial F}{\partial \omega_2} + c_1\omega_3\frac{\partial F}{\partial \omega_3} - (I_2\omega_2^2 + I_3\omega_3^2)\frac{\partial F}{\partial \gamma_1}\right] = Y_3(F) = 0, \qquad (8.34)$$

where Y_3 is the corresponding vector field defined on Ω.

Let us first suppose that $\frac{\partial^2 R}{\partial \omega_1^2} \neq 0$. In such a case, if first integral F exists then the system (8.32)–(8.34) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if the determinant D of the coefficients of this system is identically equal to zero.

Let us compute D. We have

$$D = I_2 I_3 \omega_2^2 \frac{\partial^2 R}{\partial \omega_1^2}\widehat{D},$$

where

$$\begin{aligned}\widehat{D} &= I_2(I_1 - I_2)c_3\omega_2^3\gamma_1 - I_2(I_1 - I_3)c_2\omega_2^2\omega_3\gamma_1 + I_3(I_1 - I_2)c_3\omega_2\omega_3^2\gamma_1 \\ &\quad + (I_2 - I_3)c_1U_1\omega_2\omega_3 + I_1c_1c_3\omega_2\gamma_1^2 - I_3(I_1 - I_3)c_2\omega_3^3\gamma_1 - I_1c_1c_2\gamma_1^2\omega_3.\end{aligned}$$

As $I_2I_3\omega_2^2\frac{\partial^2 R}{\partial \omega_1^2} \neq 0$ we use $\widehat{D}$ instead D.

Polynomial $\widehat{D}$ has 7 coefficients. Equating to zero all of them we obtain a system of 7 equations for parameters $\mathcal{I}c$ and U_1. After two consecutive simplifications we obtain the reduction (see Sec. 3) of this system that consists of 6 equations as follows:

$$\begin{aligned}&c_1c_3 = 0, \quad (I_1 - I_2)c_3 = 0, \quad c_1c_2 = 0, \quad (I_1 - I_3)c_2 = 0, \\ &(I_2 - I_3)c_2c_3 = 0, \quad (I_2 - I_3)c_1U_1 = 0.\end{aligned} \qquad (8.35)$$

A simple case analysis leads to a set of six solutions that the Maple command `solve` gives in the following way:

$$\begin{aligned}&\{U_1 = 0,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\},\\ &\{U_1 = U_1,\ I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\},\\ &\{U_1 = U_1,\ I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = c_3\},\\ &\{U_1 = U_1,\ I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0\},\\ &\{U_1 = U_1,\ I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\},\\ &\{U_1 = U_1,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0\}.\end{aligned}$$

This list should be understood as follows. If an equation $U_i = U_i$ or $I_i = I_i$ or $c_i = c_i$, $1 \le i \le 3$, appears we should consider the corresponding parameter as an arbitrary complex number. For example, let us consider the third row. There U_1, I_3, c_2 and c_3 are arbitrary complex numbers but I_1, I_2 and c_1 have fixed values. Some of these fixed values can depend on the chosen value of some arbitrary parameter like in this example I_1 and I_2 depend on the arbitrary fixed value of I_3.

We remove the solutions that lead to the Euler, Lagrange and kinetic symmetry cases and obtain only one new solution

$$\{U_1 = 0,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\}.$$

Let us study this solution. Therefore we have $U_1 = 0$, $c_2 = c_3 = 0$ and $I_i \neq 0$, $1 \le i \le 3$, and c_1 are arbitrary parameters. In this case $\widehat{D} = 0$. Thus vector fields

$Y_i \neq 0$, $1 \leq i \leq 3$, (see (8.32)–(8.34)) are linearly dependent. More precisely the following equation

$$Y_1 = \omega_1 Y_2 + I_3 \left(R - \omega_1 \frac{\partial R}{\partial \omega_1} \right) Y_3$$

holds.

We remove vector field Y_1 because it is a linear combination of Y_2 and Y_3 and compute the Lie bracket $Z = [Y_2, Y_3]$

$$\begin{aligned} Z = I_3 c_1 \omega_2 \Big[(I_2\omega_2^2 + I_3\omega_3^2) \frac{\partial^2 R}{\partial \omega_1 \partial \gamma_1} - c_1\omega_2 \frac{\partial^2 R}{\partial \omega_1 \partial \omega_2} - c_1\omega_3 \frac{\partial^2 R}{\partial \omega_1 \partial \omega_3} + (I_1 - I_3)\omega_3 \Big] \frac{\partial}{\partial \omega_2} \\ + c_1 \Bigg\{ I_3\omega_3 \Big[(I_2\omega_2^2 + I_3\omega_3^2) \frac{\partial^2 R}{\partial \omega_1 \partial \gamma_1} - c_1\omega_2 \frac{\partial^2 R}{\partial \omega_1 \partial \omega_2} - c_1\omega_3 \frac{\partial^2 R}{\partial \omega_1 \partial \omega_3} \Big] \\ + I_2^2\omega_2^2 + I_1 I_3 \omega_3^2 + I_1 c_1 \gamma_1 \Bigg\} \frac{\partial}{\partial \omega_3} \\ - I_3 \Bigg\{ (I_2\omega_2^2 + I_3\omega_3^2) \Big[(I_2\omega_2^2 + I_3\omega_3^2) \frac{\partial^2 R}{\partial \omega_1 \partial \gamma_1} - c_1\omega_2 \frac{\partial^2 R}{\partial \omega_1 \partial \omega_2} - c_1\omega_3 \frac{\partial^2 R}{\partial \omega_1 \partial \omega_3} \Big] \\ + I_2(I_1 - 2I_2 + 2I_3)\omega_2^2\omega_3 + I_1 I_3 \omega_3^3 + I_1 c_1 \omega_3 \gamma_1 \Bigg\} \frac{\partial}{\partial \gamma_1}. \end{aligned}$$

As we have already known if a first integral F of the sought type exists, then it should satisfy the following system

$$Y_2(F) = Y_3(F) = Z(F) = 0$$

and the determinant of the coefficients of that system should be identically equal to zero. We compute the determinant and obtain the following expression

$$-I_2 I_3 (I_2 - I_3) c_1 \omega_2^3 \omega_3 \big[I_2\omega_2^2(3I_1 - 2I_2) + I_3\omega_3^2(3I_1 - 2I_3) + 4I_1 c_1 \gamma_1 \big].$$

It is easily seen that this expression can be identically equal to zero only if $I_2 = I_3$ or if $c_1 = 0$. The first possibility leads to the Lagrange case and the second one - to the Euler case.

Thus if we suppose that $\frac{\partial^2 R}{\partial \omega_1^2} \neq 0$, then a first integral of type 4, i.e. $F(\omega_2, \omega_3, \gamma_1)$ does not exist.

Let us suppose now that $\frac{\partial^2 R}{\partial \omega_1^2} = 0$. In such a case we have

$$R = f(\omega_2, \omega_3, \gamma_1)\omega_1 + g(\omega_2, \omega_3, \gamma_1), \tag{8.36}$$

where f and g are arbitrary smooth functions not depending on ω_1.

We put the value of R from (8.36) in (8.3) and obtain

$$\begin{aligned} Q = \big[(I_2^2\omega_2^2 + I_3^2\omega_3^2) f^2 + 2I_1 I_3 \omega_3 \gamma_1 f + I_1^2 \gamma_1^2 \big] \omega_1^2 \\ + 2\big[(I_2^2\omega_2^2 + I_3^2\omega_3^2) g f - I_3 U_1 \omega_3 f + I_1 I_3 \omega_3 \gamma_1 g - I_1 U_1 \gamma_1 \big] \omega_1 \\ + (I_2^2\omega_2^2 + I_3^2\omega_3^2) g^2 + I_2^2\omega_2^2\gamma_1^2 - U_2\omega_2^2 - 2I_3 U_1 \omega_3 g + I_2^2 U_1^2 = 0, \end{aligned}$$

that is Q is a polynomial of second degree of ω_1 with coefficients depending on ω_2, ω_3 and γ_1. As $Q = 0$, then its three coefficients should be zeros.

We equate to zero the coefficient of Q in front of ω_1^2 and determine f from the obtained equation. We have

$$f = \frac{I_1(i\varepsilon I_2\omega_2 - I_3\omega_3)\gamma_1}{I_2^2\omega_2^2 + I_3^2\omega_3^2},$$

where $\varepsilon = \pm 1$.

With this value of f we equate to zero the coefficient of Q in front of ω_1 and determine g as follows

$$g = -\frac{iU_1(I_2\omega_2 + iI_3\varepsilon\omega_3)\gamma_1}{\varepsilon(I_2^2\omega_2^2 + I_3^2\omega_3^2)}.$$

Using these values of f and g and having in mind that $\varepsilon^2 = 1$, we equate to zero the constant term of polynomial Q developed in powers of ω_1, that is the value of polynomial Q when $\omega_1 = 0$ and obtain

$$\frac{I_2^2\omega_2^2(I_2^2\omega_2^2 + I_3^2\omega_3^2)(\gamma_1^2 - U_2)}{I_2^2\omega_2^2 + I_3^2\omega_3^2} = 0.$$

Taking into account that $I_2^2\omega_2^2$, $I_2^2\omega_2^2 + I_3^2\omega_3^2$ and $\gamma_1^2 - U_2$ never vanish identically, we conclude that the last equality cannot be fulfilled.

Thus a first integral of type 4, i.e. $F(\omega_2, \omega_3, \gamma_1)$ does not exist also in the case when $\frac{\partial^2 R}{\partial \omega_1^2} = 0$.

8.2.2. Elimination of $\boldsymbol{\omega_1}$ and $\boldsymbol{\gamma_1}$. We eliminate variables ω_1 and γ_1 from equations $H_1 = U_1,\ H_2 = U_2$ and obtain the following solution:

$$\omega_1 = -\frac{I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1\sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}}, \qquad \gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}. \tag{8.37}$$

Further to simplify the formulas we note

$$\Gamma = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}.$$

As all our considerations are local we can restrict ourselves to some suitable open set $\Omega \subseteq \mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

We put the values of ω_1 and γ_1 from (8.37) in the Euler-Poisson equations (1.1) and remove the first and fourth equations. In this way we obtain the following system of four equations in unknowns $\omega_2,\ \omega_3,\ \gamma_2$ and γ_3:

$$\begin{aligned}
\frac{d\omega_2}{dt} &= \frac{(I_1 - I_3)(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\omega_3 - I_1c_3\Gamma^2 + I_1c_1\gamma_3\Gamma}{I_1I_2\Gamma},\\
\frac{d\omega_3}{dt} &= -\frac{(I_1 - I_2)(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\omega_2 - I_1c_2\Gamma^2 + I_1c_1\gamma_2\Gamma}{I_1I_3\Gamma},\\
\frac{d\gamma_2}{dt} &= -\frac{(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\Gamma^2}{I_1\Gamma},\\
\frac{d\gamma_3}{dt} &= \frac{(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_2 + I_1\omega_2\Gamma^2}{I_1\Gamma}.
\end{aligned} \tag{8.38}$$

Here we study whether system (8.38) has a first integral that depends on at most three variables among the variables $(\omega_2, \omega_3, \gamma_2, \gamma_3)$ and that is functionally independent of H_3 restricted to invariant manifold (8.1). Thus we should investigate the following four types of a new first integral:

1. $F(\omega_2, \omega_3, \gamma_2)$, (case (iii))
2. $F(\omega_2, \omega_3, \gamma_3)$, (case (iii))
3. $F(\omega_2, \gamma_2, \gamma_3)$, (case (iv))
4. $F(\omega_3, \gamma_2, \gamma_3)$. (case (iv))

As local partial first integrals belonging to case (iii) were already excluded in Sec. 8.2.1, we will now study if the function of type 3, belonging to case (iv) can be a local partial first integral of the Euler-Poisson equations (1.1).

Type 3. Let us study the existence of a first integral F of type 3, i.e. $F = F(\omega_2, \gamma_2, \gamma_3)$. F being a first integral of system (8.38) satisfies the following equation

$$\begin{aligned}\frac{dF}{dt} &= \frac{(I_1 - I_3)\omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1) - I_1c_3\Gamma^2 + I_1c_1\gamma_3\Gamma}{I_1I_2\Gamma}\frac{\partial F}{\partial\omega_2}\\ &\quad - \frac{\gamma_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1) + I_1\omega_3\Gamma^2}{I_1\Gamma}\frac{\partial F}{\partial\gamma_2}\\ &\quad + \frac{\gamma_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1) + I_1\omega_2\Gamma^2}{I_1\Gamma}\frac{\partial F}{\partial\gamma_3} = 0,\end{aligned}$$

or equivalently

$$I_1I_2\Gamma\frac{dF}{dt} = Y(F) = 0, \tag{8.39}$$

where Y is the corresponding vector field defined on Ω.

The left hand side of equation (8.39), i.e. $Y(F)$, is a polynomial of ω_3 of degree two with coefficients depending on parameters $\mathcal{I}c$, U_1, U_2 and variables $(\omega_2, \gamma_2, \gamma_3)$.

Let us write $Y(F)$ as follows

$$Y(F) = Y_1(F)\omega_3^2 + Y_2(F)\omega_3 + Y_3(F),$$

where Y_i, $1 \le i \le 3$, are the following vector fields:

$$\begin{aligned}Y_1 &= I_3\gamma_3(I_1 - I_3)\frac{\partial}{\partial\omega_2},\\ Y_2 &= (I_1 - I_3)(I_2\omega_2\gamma_2 - U_1)\frac{\partial}{\partial\omega_2} + I_2\Big[I_1\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1U_2\Big]\frac{\partial}{\partial\gamma_2} + I_2I_3\gamma_2\gamma_3\frac{\partial}{\partial\gamma_3},\\ Y_3 &= I_1(c_3\gamma_2^2 + c_3\gamma_3^2 + c_1\gamma_3\Gamma - c_3U_2)\frac{\partial}{\partial\omega_2} - I_2\gamma_3(I_2\omega_2\gamma_2 - U_1)\frac{\partial}{\partial\gamma_2}\\ &\quad - I_2\Big[(I_1 - I_2)\omega_2\gamma_2^2 + I_1\omega_2\gamma_3^2 - I_1U_2\omega_2 + U_1\gamma_2\Big]\frac{\partial}{\partial\gamma_3},\end{aligned}$$

defined on Ω.

$Y(F)$ should be identically equal to zero with respect to all four variables ω_2, ω_3, γ_2 and γ_3. As $Y_i(F)$, $1 \le i \le 3$, do not depend on ω_3 we have the following three equations:

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{8.40}$$

If a first integral $F = F(\omega_2, \gamma_2, \gamma_3)$ exists then system (8.40) has a non-zero solution $\operatorname{grad} F$. We know that this is possible if and only if the determinant D of system (8.40) is identically equal to zero. We compute D and obtain

$$D = I_1 I_2^2 I_3 (I_1 - I_3)\gamma_3 \Gamma^2 \Big[(I_1 - I_2)\omega_2\gamma_2^2 + (I_1 - I_3)\omega_2\gamma_3^2 - I_1 U_2 \omega_2 + \gamma_2 U_1\Big] = 0.$$

One can easily see that $D \equiv 0$ if and only if

$$I_1 = I_3. \tag{8.41}$$

Let us study this case. Now $Y_1 = 0$ but simple computations show that vector fields Y_2 and Y_3 are always linearly independent. We compute Lie bracket $Z = [Y_2, Y_3]/(I_2 I_3)$ and obtain

$$\begin{aligned} Z = {} & 2I_3\gamma_2\Gamma(c_1\gamma_3 - c_3\Gamma)\frac{\partial}{\partial\omega_2} + I_2(I_2U_2\omega_2 - U_1\gamma_2)\gamma_3\frac{\partial}{\partial\gamma_2} \\ & + I_2\Big\{\big[(I_3 - I_2)\Gamma^2 - I_2U_2\big]\omega_2\gamma_2 - U_1\gamma_3^2 + U_1U_2\Big]\Big\}\frac{\partial}{\partial\gamma_3}. \end{aligned}$$

Second and third equations (8.40) imply that $Z(F) = 0$ and we come to the following system for first integral F:

$$Y_2(F) = Y_3(F) = Z(F) = 0. \tag{8.42}$$

As above we should study when determinant $\widehat{D}$ of system (8.42) is identically equal to zero. We compute $\widehat{D}$ and obtain $\widehat{D} = d_1 d_2 d_3$, where

$$\begin{aligned} d_1 &= I_2^3 I_3^3 \Gamma^2, \\ d_2 &= (c_1\gamma_3 - c_3\Gamma)\Gamma, \\ d_3 &= (I_3 - I_2)\omega_2\gamma_2^3 - (2I_2 + I_3)U_2\omega_2\gamma_2 + 2U_1\gamma_2^2 + U_1U_2. \end{aligned}$$

It is clear that d_1 never vanishes identically. If d_2 vanishes then it follows that $c_1 = c_3 = 0$ which together with condition (8.41) leads to the Lagrange case. The third factor $d_3 = 0$ if and only if $I_2 = I_3$ and $U_1 = U_2 = 0$. But taking into account (8.41) this is a particular case of the kinetic symmetry case. Thus a partial first integral of type 3 does not exist.

It only remains to study the functions belonging to the cases (v) and (vi).

8.2.3. Elimination of $\boldsymbol{\omega_1}$ and $\boldsymbol{\gamma_2}$. Solving equations $H_1 = U_1,\ H_2 = U_2$ with respect to ω_1 and γ_2 we obtain:

$$\omega_1 = -\frac{I_2\omega_2\sqrt{-\gamma_1^2 - \gamma_3^2 + U_2} + I_3\omega_3\gamma_3 - U_1}{I_1\gamma_1}, \qquad \gamma_1 = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}. \tag{8.43}$$

To simplify the formulas we note

$$\Gamma = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}.$$

As till now we restrict ourselves to some suitable open set $\Omega \subseteq \mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

We put the values of ω_1 and γ_2 from (8.43) in the Euler-Poisson equations (1.1) and remove the first and fifth equations. In this way we obtain the following system of four equations in unknowns ω_2, ω_3, γ_1 and γ_3:

$$\begin{aligned}
\frac{d\omega_2}{dt} &= \frac{(I_1-I_3)(I_3\omega_3\gamma_3+I_2\omega_2\Gamma-U_1)\omega_3-(I_1c_3\gamma_1-I_1c_1\gamma_3)\gamma_1}{I_1I_2\gamma_1},\\
\frac{d\omega_3}{dt} &= -\frac{(I_1-I_2)(I_3\omega_3\gamma_3+I_2\omega_2\Gamma-U_1)\omega_2-(I_1c_2\gamma_1-I_1c_1\Gamma)\gamma_1}{I_1I_3\gamma_1},\\
\frac{d\gamma_1}{dt} &= \omega_3\Gamma-\omega_2\gamma_3,\\
\frac{d\gamma_3}{dt} &= \frac{(I_1\gamma_1^2+I_2\Gamma^2)\omega_2+I_3\omega_3\gamma_3\Gamma-U_1\Gamma}{I_1\gamma_1}.
\end{aligned} \tag{8.44}$$

Here we look for first integrals of system (8.44) of the following four types:

1. $F(\omega_2,\omega_3,\gamma_1)$, case(ii)
2. $F(\omega_2,\omega_3,\gamma_3)$, case(iii)
3. $F(\omega_2,\gamma_1,\gamma_3)$, case(v)
4. $F(\omega_3,\gamma_1,\gamma_3)$, case(iv)

requiring in addition that they are functionally independent of H_3 restricted to invariant manifold (8.1). The functions from cases (ii), (iii) and (iv) was already examined. There remains only to examine case (v).

Type 3. Let us study the existence of a partial first integral of type 3 $F(\omega_2,\gamma_1,\gamma_3)$ belonging to case (v). Then we have

$$\begin{aligned}
\frac{dF}{dt} &= \frac{(I_1-I_3)(I_3\omega_3\gamma_3+I_2\omega_2\Gamma-U_1)\omega_3-(I_1c_3\gamma_1-I_1c_1\gamma_3)\gamma_1}{I_1I_2\gamma_1}\frac{\partial F}{\partial\omega_2}\\
&\quad+(\omega_3\Gamma-\omega_2\gamma_3)\frac{\partial F}{\partial\gamma_1}+\frac{(I_1\gamma_1^2+I_2\Gamma^2)\omega_2+I_3\omega_3\gamma_3\Gamma-U_1\Gamma}{I_1\gamma_1}\frac{\partial F}{\partial\gamma_3}=0,
\end{aligned}$$

or equivalently

$$I_1I_2\gamma_1\frac{dF}{dt}=Y(F)=0,$$

where Y is the corresponding vector field defined on Ω.

$Y(F)$ is a polynomial of ω_3 of degree two with coefficients depending on parameters $\mathcal{I}c$, U_1, U_2 and variables ω_2, γ_1 and γ_3.

Let us write $Y(F)$ in the following way

$$Y(F)=Y_1(F)\omega_3^2+Y_2(F)\omega_3+Y_3(F),$$

where Y_i, $1\le i\le 3$, are:

$$\begin{aligned}
Y_1 &= I_3(I_1-I_3)\gamma_3\frac{\partial}{\partial\omega_2},\\
Y_2 &= (I_1-I_3)(I_2\omega_2\Gamma-U_1)\frac{\partial}{\partial\omega_2}+I_1I_2\gamma_1\Gamma\frac{\partial}{\partial\gamma_1}+I_2I_3\gamma_3\Gamma\frac{\partial}{\partial\gamma_3},\\
Y_3 &= I_1\gamma_1(c_1\gamma_3-c_3\gamma_1)\frac{\partial}{\partial\omega_2}-I_1I_2\omega_2\gamma_1\gamma_3\frac{\partial}{\partial\gamma_1}+I_2\Big[(I_2\Gamma^2+I_1\gamma_1^2)\omega_2-U_1\Gamma\Big]\frac{\partial}{\partial\gamma_3}.
\end{aligned}$$

$Y(F)$ should be identically equal to zero with respect to all four variables ω_2, ω_3, γ_2

and γ_3. As $Y_i(F)$, $1 \le i \le 3$, do not depend on ω_3 we have

$$Y_1(F) = Y_2(F) = Y_3(F) = 0. \tag{8.45}$$

If a first integral $F = F(\omega_2, \gamma_2, \gamma_3)$ exists then the determinant D of system (8.45) is identically equal to zero. We compute D and obtain

$$D = I_1 I_2^2 I_3 (I_1 - I_3)\gamma_1\gamma_3\Gamma\Big\{\omega_2[(I_1 - I_2)\gamma_1^2 + (I_3 - I_2)\gamma_3^2 + I_2U_2] - U_1\Gamma\Big\} = 0.$$

It is easily seen that D vanishes identically if either

$$I_1 = I_3 \tag{8.46}$$

or the expression in the curly brackets vanishes. This expression is a linear function of ω_2 and we should require that its two coefficients vanish. But this leads to the kinetic symmetry case with the additional restriction $U_1 = U_2 = 0$.

Thus we study only the case (8.46). Now $Y_1 = 0$ but simple computations show that outside of the particular case ($U_1 = U_2 = 0$) of the kinetic symmetry vector fields Y_2 and Y_3 are always linearly independent. We compute Lie bracket $Z = [Y_2, Y_3]$ and obtain

$$\frac{Z\Gamma}{I_2I_3} = 2I_3\gamma_1(c_1\gamma_3 - c_3\gamma_1)\Gamma^2\frac{\partial}{\partial\omega_2} + I_2\gamma_1\gamma_3\Gamma\big[(I_2 - I_3)\omega_2\Gamma - U_1\big]\frac{\partial}{\partial\gamma_1}$$
$$+ I_2\Gamma\big[\omega_2\Gamma(I_3\gamma_1^2 - I_2\gamma_1^2 - I_2U_2) - U_1\gamma_3^2 + U_1U_2\big]\frac{\partial}{\partial\gamma_3}.$$

Second and third equations (8.45) imply that $Z(F) = 0$ and thus

$$\widehat{Z}(F) = \frac{Z\Gamma}{I_2I_3} = 0.$$

In this way we come to the following system for first integral F:

$$Y_2(F) = Y_3(F) = \widehat{Z}(F) = 0. \tag{8.47}$$

As above we should find the cases when determinant $\widehat{D}$ of system (8.47) is identically equal to zero. We compute $\widehat{D}$ and obtain $\widehat{D} = d_1 d_2$, where

$$d_1 = I_2^2 I_3^2 \gamma_1^2 (c_1\gamma_3 - c_3\gamma_1)\Gamma^2,$$
$$d_2 = [(I_3 - I_2)\gamma_1^2 + (I_3 - I_2)\gamma_3^2 + 3I_2U_2]\omega_2\Gamma - U_1(2\Gamma^2 + U_2).$$

It is clear that d_1 vanishes identically only when $c_1 = c_3 = 0$ which together with condition (8.46) leads to the Lagrange case. As d_2 is a linear function of ω_2 equation $d_2 = 0$ is fulfilled if and only if its two coefficients with respect to ω_2 vanish identically. Thus $I_2 = I_3$ and $U_1 = U_2 = 0$. Taking into account (8.46) this is a particular case of the kinetic symmetry case. Thus a partial first integral of type 3 does not exist.

8.2.4. First integrals $F(\gamma_1, \gamma_2, \gamma_3)$. Finally it remains to study the existence of the partial first integral $F(\gamma_1, \gamma_2, \gamma_3)$, that cannot be studied by elimination of variables like above.

We have $H_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = U_2$, thus $\gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}$ and then

$$F(\gamma_1, \gamma_2, \gamma_3) = F(\sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}, \gamma_2, \gamma_3) = \widetilde{F}(\gamma_2, \gamma_3).$$

Our problem now is reduced to the study of partial first integrals of the form $\widetilde{F} = \widetilde{F}(\gamma_2, \gamma_3)$ on the submanifold $\{H_1 = U_1\}$. Absence of these partial first integrals follows from Sec. 8.2.2 where the absence of partial first integrals of more general form $F(\omega_i, \gamma_2, \gamma_3)$, $i = 2, 3$, is proved for all U_1 and U_2.

8.3. Invariant manifold $\{H_1{=}U_1,\ H_3{=}U_3\}$. Here we study the existence of a partial first integral of the Euler-Poisson equations (1.1) restricted to the complex four-dimensional level manifold

$$\{H_1 = U_1,\ H_3 = U_3\}, \tag{8.48}$$

supposing that this partial first integral depends on at most three variables and that is functionally independent of H_2.

8.3.1. Elimination of ω_1 and ω_2. In the same way as in Sec. 8.2.1 we express ω_1 and ω_2 from the equations $H_1 = U_1$ and $H_3 = U_3$ and obtain the following solution:

$$\omega_1 = -\frac{I_2 R\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1\gamma_1}, \qquad \omega_2 = R, \tag{8.49}$$

where R is a root of equation

$$Q(x) = Ax^2 + Bx + C = 0,$$

that is

$$Q(R) = AR^2 + BR + C = 0. \tag{8.50}$$

Here the functions $A = A(\gamma_1, \gamma_2)$, $B = B(\omega_3, \gamma_2, \gamma_3)$ and $C = C(\omega_3, \gamma_1, \gamma_2, \gamma_3)$ are the following polynomials:

$$\begin{aligned} A &= I_2(I_1\gamma_1^2 + I_2\gamma_2^2), \qquad B = 2I_2\gamma_2(I_3\omega_3\gamma_3 - U_1),\\ C &= I_1I_3\omega_3^2\gamma_1^2 + I_3^2\omega_3^2\gamma_3^2 - 2I_3U_1\omega_3\gamma_3 + 2I_1c_1\gamma_1^3 + 2I_1c_2\gamma_1^2\gamma_2\\ &\quad + 2I_1c_3\gamma_1^2\gamma_3 - I_1U_3\gamma_1^2 + U_1^2. \end{aligned} \tag{8.51}$$

We put the values of ω_1 and ω_2 from (8.49) in the Euler-Poisson equations (1.1) and remove the first and second equations. In this way we have the following system of four equations in unknowns ω_3, γ_1, γ_2 and γ_3:

$$\begin{aligned} \frac{d\omega_3}{dt} &= -\frac{(I_1 - I_2)(I_2\gamma_2 R + I_3\omega_3\gamma_3 - U_1)R - I_1(\gamma_1c_2 - c_1\gamma_2)\gamma_1}{I_1I_3\gamma_1},\\ \frac{d\gamma_1}{dt} &= \omega_3\gamma_2 - \gamma_3 R,\\ \frac{d\gamma_2}{dt} &= -\frac{(I_2R\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\gamma_1^2}{I_1\gamma_1},\\ \frac{d\gamma_3}{dt} &= \frac{(I_1\gamma_1^2 + I_2\gamma_2^2)R + (I_3\omega_3\gamma_3 - U_1)\gamma_2}{I_1\gamma_1}. \end{aligned} \tag{8.52}$$

Now we study the existence of a first integral of system (8.52) that depends on at most three variables among the variables $(\omega_3, \gamma_1, \gamma_2, \gamma_3)$ and that is functionally independent of H_2 restricted to invariant manifold (8.48). Thus we should investigate the following four types of a first integral:

1. $F(\omega_3, \gamma_1, \gamma_2)$, (case (iv))

2. $F(\omega_3, \gamma_1, \gamma_3)$, (case (iv))
3. $F(\omega_3, \gamma_2, \gamma_3)$, (case (iv))
4. $F(\gamma_1, \gamma_2, \gamma_3)$. (case (vi))

Then, like in Sec. 5 it suffices to examine the functions of types 1 and 4 respectively.

Type 1. Let us consider the existence of a first integral F of system (8.52) which is of type 1, i.e. $F = F(\omega_3, \gamma_1, \gamma_2)$. Thus

$$\frac{dF}{dt} = -\frac{(I_1 - I_2)(I_2\gamma_2 R + I_3\omega_3\gamma_3 - U_1)R - I_1(\gamma_1 c_2 - c_1\gamma_2)\gamma_1}{I_1 I_3 \gamma_1}\frac{\partial F}{\partial \omega_3} + (\omega_3\gamma_2 - \gamma_3 R)\frac{\partial F}{\partial \gamma_1} - \frac{(I_2 R\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\gamma_1^2}{I_1\gamma_1}\frac{\partial F}{\partial \gamma_2} = 0.$$

We rewrite the above equation as follows

$$I_1 I_3 \gamma_1 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.53}$$

where Y_1 is the corresponding vector field defined on some suitable open set $\Omega \subseteq \mathbb{C}^4(\omega_3, \gamma_1, \gamma_2, \gamma_3)$.

We differentiate identity (8.53) with respect to γ_3 and obtain a linear partial differential equation for function F

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \gamma_3} &= (I_1 - I_2)\left[-2I_2\gamma_2 R\frac{\partial R}{\partial \gamma_3} - I_3\omega_3 R - (I_3\omega_3\gamma_3 - U_1)\frac{\partial R}{\partial \gamma_3}\right]\frac{\partial F}{\partial \omega_3}\\ &- I_1 I_3 \gamma_1\left(R + \gamma_3\frac{\partial R}{\partial \gamma_3}\right)\frac{\partial F}{\partial \gamma_1}\\ &- I_3\left[2I_3\omega_3\gamma_3 + I_2\gamma_2\left(R + \gamma_3\frac{\partial R}{\partial \gamma_3}\right) - U_1\right]\frac{\partial F}{\partial \gamma_2} = Y_2(F) = 0,\end{aligned} \tag{8.54}$$

where Y_2 is the corresponding vector field defined on Ω.

After differentiating identity (8.54) with respect to γ_3 we obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial \gamma_3} &= (I_1 - I_2)\Big[-2I_2\gamma_2 R\frac{\partial^2 R}{\partial \gamma_3^2} - 2I_2\gamma_2\left(\frac{\partial R}{\partial \gamma_3}\right)^2\\ &- 2I_3\omega_3\frac{\partial R}{\partial \gamma_3} - (I_3\omega_3\gamma_3 - U_1)\frac{\partial^2 R}{\partial \gamma_3^2}\Big]\frac{\partial F}{\partial \omega_3}\\ &- I_1 I_3\gamma_1\left(2\frac{\partial R}{\partial \gamma_3} + \gamma_3\frac{\partial^2 R}{\partial \gamma_3^2}\right)\frac{\partial F}{\partial \gamma_1}\\ &- I_3\left(2I_2\gamma_2\frac{\partial R}{\partial \gamma_3} + I_2\gamma_2\gamma_3\frac{\partial^2 R}{\partial \gamma_3^2} + 2I_3\omega_3\right)\frac{\partial F}{\partial \gamma_2} = Y_3(F) = 0,\end{aligned} \tag{8.55}$$

where Y_3 is the corresponding vector field defined on Ω.

If a first integral F exists, then the linear system (8.53)–(8.55) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}\right)$, which is possible if and only if the determinant D of the coefficients of this system is identically equal to zero.

We compute D. It has a non-zero factor $I_1 I_3^2 \gamma_1$ and that is why we work with

$$\widehat{D} = \frac{D}{I_1 I_3^2 \gamma_1}.$$

The expression for $\widehat{D}$ is

$$\widehat{D} = a_1R^3 + a_2R^2\frac{\partial R}{\partial\gamma_3} + a_3R^2\frac{\partial^2 R}{\partial\gamma_3^2} + a_4R^2 + a_5R\left(\frac{\partial R}{\partial\gamma_3}\right)^2 + a_6R\frac{\partial R}{\partial\gamma_3} + a_7R\frac{\partial^2 R}{\partial\gamma_3^2} + a_8R + a_9\left(\frac{\partial R}{\partial\gamma_3}\right)^3 + a_{10}\left(\frac{\partial R}{\partial\gamma_3}\right)^2 + a_{11}\frac{\partial R}{\partial\gamma_3} + a_{12}\frac{\partial^2 R}{\partial\gamma_3^2}, \tag{8.56}$$

where

$$\begin{aligned}
a_1 &= -2I_2I_3(I_1-I_2)\omega_3\gamma_2, \quad a_2 = -2I_2(I_1-I_2)\gamma_2(U_1-3I_3\omega_3\gamma_3),\\
a_3 &= -I_2(I_1-I_2)\gamma_2(-2\omega_3\gamma_1^2I_1-2\omega_3\gamma_2^2I_2+\gamma_3U_1), \quad a_4 = 2I_3(I_1-I_2)U_1\omega_3,\\
a_5 &= 2(I_1-I_2)I_2\gamma_2(-\omega_3\gamma_1^2I_1-\omega_3\gamma_2^2I_2-3I_3\omega_3\gamma_3^2+2\gamma_3U_1),\\
a_6 &= 2(I_1-I_2)(-2I_2I_3\omega_3^2\gamma_2^2-2I_3U_1\omega_3\gamma_3+U_1^2),\\
a_7 &= (I_1-I_2)(-U_1\omega_3\gamma_1^2I_1-3\omega_3\gamma_2^2I_2U_1-I_3\omega_3\gamma_3^2U_1+U_1^2\gamma_3+4I_3\omega_3^2\gamma_3\gamma_2^2I_2),\\
a_8 &= -2I_3\omega_3(-I_3\omega_3^2I_2\gamma_2+I_3\omega_3^2I_1\gamma_2-I_1\gamma_1^2c_2+c_1\gamma_2I_1\gamma_1),\\
a_9 &= -2I_2(I_1-I_2)\gamma_2\gamma_3(-\omega_3\gamma_1^2I_1-\omega_3\gamma_2^2I_2-I_3\omega_3\gamma_3^2+\gamma_3U_1),\\
a_{10} &= -2(I_1-I_2)(-U_1\omega_3\gamma_1^2I_1-2I_3\omega_3^2\gamma_3\gamma_2^2I_2-I_3\omega_3\gamma_3^2U_1+U_1^2\gamma_3),\\
a_{11} &= 2(-I_1\gamma_1^2c_2I_3\gamma_3\omega_3+I_1I_3^2\omega_3^3\gamma_3\gamma_2-c_1\gamma_2I_1\gamma_1U_1+c_1\gamma_2I_1\gamma_1I_3\gamma_3\omega_3\\
&\quad +I_1\gamma_1^2c_2U_1-I_2I_3^2\omega_3^3\gamma_3\gamma_2),\\
a_{12} &= (U_1-2I_3\omega_3\gamma_3)(-U_1I_2\omega_3\gamma_2+U_1I_1\omega_3\gamma_2-c_1\gamma_2I_1\gamma_1\gamma_3-I_3\omega_3^2I_1\gamma_3\gamma_2\\
&\quad +I_1\gamma_1^2c_2\gamma_3+I_3\omega_3^2I_2\gamma_3\gamma_2).
\end{aligned}$$

To determine the derivatives $\frac{\partial R}{\partial\gamma_3}$ and $\frac{\partial^2 R}{\partial\gamma_3^2}$ we differentiate equation (8.50) with respect to γ_3 two times. Taking into account that polynomial A from (8.51) does not depend on γ_3 we obtain

$$\frac{\partial Q}{\partial\gamma_3} = \frac{\partial B}{\partial\gamma_3}R + \frac{\partial C}{\partial\gamma_3} + \frac{dQ}{dR}\frac{\partial R}{\partial\gamma_3} = 0. \tag{8.57}$$

and

$$\frac{\partial^2 Q}{\partial\gamma_3^2} = \frac{\partial^2 B}{\partial\gamma_3^2}R + \frac{\partial B}{\partial\gamma_3}\frac{\partial R}{\partial\gamma_3} + \frac{\partial^2 C}{\partial\gamma_3^2} + \frac{\partial}{\partial\gamma_3}\left(\frac{dQ}{dR}\right)\frac{\partial R}{\partial\gamma_3} + \frac{dQ}{dR}\frac{\partial^2 R}{\partial\gamma_3^2} = 0. \tag{8.58}$$

By Proposition 4.1 we prove that if R is a root of equation (8.50), then $\frac{dQ}{dR} \neq 0$. For the purpose we consider the resultant $\rho = A(4AC-B^2)$ of polynomials $Q(R)$ and $\frac{dQ}{dR}$ and prove that $\rho \neq 0$. As A never vanishes identically we do not consider ρ but $\widehat{\rho} = 4AC-B^2$. Putting in $\widehat{\rho}$ the expressions for A, B and C (see (8.51)) we obtain

$$\begin{aligned}
\widehat{\rho} = 4I_1I_2\gamma_1^2\bigl[&I_3\omega_3^2(I_1\gamma_1^2+I_2\gamma_2^2+I_3\gamma_3^2)+2(I_1\gamma_1^2+I_2\gamma_2^2)(c_1\gamma_1+c_2\gamma_2+c_3\gamma_3)\\
&-2I_3U_1\omega_3\gamma_3-U_3I_1\gamma_1^2-I_2U_3\gamma_2^2+U_1^2\bigr],
\end{aligned}$$

which never vanishes identically at least because contains a monomial $4I_1^2I_2I_3\omega_3^2\gamma_1^4$.

Thus $\frac{dQ}{dR} \neq 0$ and $\frac{\partial R}{\partial\gamma_3}$ can be correctly determined from equation (8.57). Then by equation (8.58) we determine $\frac{\partial^2 R}{\partial\gamma_3^2}$ and put the obtained values for the derivatives of R

in the expression for $\widehat{D}$ (see (8.56)). In this way we obtain

$$\widehat{D}(R) = \frac{8I_2\delta(R)}{\left(\frac{dQ}{dR}\right)^3},$$

where $\delta(R)$ is a polynomial of R of degree six with coefficients depending on ω_3, γ_1, γ_2 and γ_3.

It is clear that the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_3, \gamma_1, \gamma_2)$ of system (8.52) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_3, \gamma_1, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

By the MAPLE command `rem` we compute remainder r from the division of polynomial $\delta(x)$ by polynomial $Q(x)$. The remainder is of the form:

$$r(R) = \frac{I_1^2\gamma_1^4}{I_1\gamma_1^2 + I_2\gamma_2^2}(a_0x + a_1),$$

where a_0 and a_1 are polynomials of ω_3, γ_1, γ_2 and γ_3.

According to Proposition 4.2 if R is a root of equation (8.3), then a_0 and a_1 should be identically equal to zero. We shall use a_1 only.

Polynomial a_1 has 210 coefficients. Thus we should equate to zero all of them. In this way we obtain a system of 210 equations for parameters $\mathcal{I}c$, U_1 and U_3. The reduced system (see Sec. 3) that is obtained after two consecutive simplifications is very simple:

$$c_1 = 0, \quad c_2 = 0, \quad I_1 - I_2 = 0.$$

These equations lead to the Lagrange case. Thus the sought partial first integral of type 1 does not exist.

Type 4. The study of the existence of a first integral of type 4 is considerably different. Indeed, let us suppose that $F = F(\gamma_1, \gamma_2, \gamma_3)$ is a first integral of system (8.52). Then we have

$$\frac{dF}{dt} = (\omega_3\gamma_2 - \gamma_3 R)\frac{\partial F}{\partial \gamma_1} - \frac{(I_2R\gamma_2 + I_3\omega_3\gamma_3 - U_1)\gamma_3 + I_1\omega_3\gamma_1^2}{I_1\gamma_1}\frac{\partial F}{\partial \gamma_2} + \frac{(I_1\gamma_1^2 + I_2\gamma_2^2)R + (I_3\omega_3\gamma_3 - U_1)\gamma_2}{I_1\gamma_1}\frac{\partial F}{\partial \gamma_3} = 0,$$

which we rewrite as follows

$$I_1\gamma_1\frac{dF}{dt} = Y_1(F) = 0, \tag{8.59}$$

where Y_1 is the corresponding vector field defined on Ω.

After differentiation identity (8.59) with respect to ω_3 one obtains again a linear partial differential equation for function F

$$\frac{\partial Y_1(F)}{\partial \omega_3} = I_1\gamma_1\left(\gamma_2 - \gamma_3\frac{\partial R}{\partial \omega_3}\right)\frac{\partial F}{\partial \gamma_1} - \left(I_2\gamma_2\gamma_3\frac{\partial R}{\partial \omega_3} + I_3\gamma_3^2 + I_1\gamma_1^2\right)\frac{\partial F}{\partial \gamma_2} + \left[\left(I_1\gamma_1^2 + I_2\gamma_2^2\right)\frac{\partial R}{\partial \omega_3} + I_3\gamma_2\gamma_3\right]\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \tag{8.60}$$

where Y_2 is defined on Ω.

System (8.59)–(8.60) has one solution. This is first integral H_2. In order to have one more solution this system should consist of dependent equations. Let us study when this is possible.

We compute determinant D_{23} corresponding to the square matrix obtained from the 2×3 matrix of the coefficients of system (8.59)–(8.60) by crossing out its first column. The result is

$$D_{23} = I_1\gamma_1^2 \left[(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2) \left(R - \omega_3 \frac{\partial R}{\partial \omega_3} \right) - U_1 \left(\gamma_2 - \gamma_3 \frac{\partial R}{\partial \omega_3} \right) \right].$$

The expression for D_{23} depends on derivative $\frac{\partial R}{\partial \omega_3}$. We determine it by differentiating equation (8.50) with respect to ω_3. Polynomial A does not depend on ω_3 (see (8.51)) and therefore

$$\frac{\partial Q}{\partial \omega_3} = \frac{\partial B}{\partial \omega_3} R + \frac{\partial C}{\partial \omega_3} + \frac{dQ}{dR} \frac{\partial R}{\partial \omega_3} = 0.$$

As in the studying of a first integral of type 1, using Proposition 4.1, we prove that if R is a root of equation (8.50) then $\frac{dQ}{dR} \neq 0$ and obtain from the above equation

$$\frac{\partial R}{\partial \omega_3} = -\frac{I_3 \big[\gamma_3 (I_2 R \gamma_2 - U_1) + \omega_3 (I_1\gamma_1^2 + I_3\gamma_3^2) \big]}{I_2 \big[R(I_1\gamma_1^2 + I_2\gamma_2^2) + \gamma_2 (I_3\omega_3\gamma_3 - U_1) \big]}.$$

We put this value of $\frac{\partial R}{\partial \omega_3}$ in the expression for D_{23} and obtain

$$D_{23} = \frac{I_1\gamma_1^2}{I_2 \big[R(I_1\gamma_1^2 + I_2\gamma_2^2) + \gamma_2 (I_3\omega_3\gamma_3 - U_1) \big]} \widehat{D}_{23},$$

where $\widehat{D}_{23}$ is

$$\begin{aligned} \widehat{D}_{23} = {} & I_2 (I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2) \big[(I_1\gamma_1^2 + I_2\gamma_2^2) R^2 + 2\gamma_2 (I_3\omega_3\gamma_3 - U_1) R \big] \\ & + I_3 (I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2) \big[(I_1\gamma_1^2 + I_3\gamma_3^2)\omega_3^2 - 2U_1\gamma_3\omega_3 \big] + U_1^2 (I_2\gamma_2^2 + I_3\gamma_3^2). \end{aligned}$$

It is clear that $D_{23} = 0$ is equivalent to

$$\widehat{D}_{23} = 0. \tag{8.61}$$

If first integral $F = F(\gamma_1, \gamma_2, \gamma_3)$ exists then equation (8.61) is fulfilled. Thus R is simultaneously a root of equations (8.50) and (8.61). In such a case $\widehat{D}_{23}$ and Q as polynomials of R should have a zero resultant.

Let us denote the resultant with ρ and compute it. We obtain

$$\rho = I_1^2 I_2^2 \gamma_1^4 (I_1\gamma_1^2 + I_2\gamma_2^2)^2 \widehat{\rho}^2,$$

where

$$\widehat{\rho} = (2c_1\gamma_1 + 2c_2\gamma_2 + 2c_3\gamma_3 + U_3)(I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2) + U_1^2.$$

Equation $\rho = 0$ implies $\widehat{\rho} = 0$. It is easily seen that this happens only if $c_1 = c_2 = c_3 = 0$ and $U_1 = U_3 = 0$, i.e. a particular case of the Euler case. Thus a new first integral of type 4 does not exist.

8.3.2. Elimination of ω_1 and γ_1. Here we should study two cases: when $c_1 \neq 0$ and when $c_1 = 0$. This is necessary because when we express γ_1 from equation $H_1 = U_1$ we obtain

$$\gamma_1 = -\frac{I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1\omega_1} \tag{8.62}$$

independently of c_1. But putting γ_1 from (8.62) in equation $H_3 = U_3$, two different case for determining ω_1 arises. When $c_1 \neq 0$ the equation for ω_1 is of degree three while when $c_1 = 0$ it is of degree two. That is why, to avoid any confusion, we consider separately two cases.

Case A. $c_1 \neq 0$. In the same way as in Sec. 8.2.1, taking into account the value of γ_1 from (8.62) we solve equation $H_3 = U_3$ with respect to ω_1 and obtain

$$\omega_1 = R,$$

where R is a root of equation

$$Q(x) = I_1^2x^3 + Ax + B = 0,$$

that is

$$Q(R) = I_1^2R^3 + AR + B = 0 \tag{8.63}$$

and $A = A(\omega_2, \omega_3, \gamma_2, \gamma_3)$ and $B = B(\omega_2, \omega_3, \gamma_2, \gamma_3)$ are the following polynomials:

$$\begin{aligned} A &= I_1(I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 + 2c_3\gamma_3 - U_3), \\ B &= -c_1(2I_2\omega_2\gamma_2 + 2I_3\omega_3\gamma_3 - 2U_1). \end{aligned} \tag{8.64}$$

In this way we come to the following values of the eliminated variables:

$$\omega_1 = R, \quad \gamma_1 = -\frac{I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1}{I_1R}. \tag{8.65}$$

We put the values of ω_1 and γ_1 from (8.65) in the Euler-Poisson equations (1.1), remove the first and fourth equations and obtain the following system of four equations in unknowns ω_2, ω_3, γ_2 and γ_3:

$$\begin{aligned} \frac{d\omega_2}{dt} &= -\frac{I_1(I_1 - I_3)\omega_3R^2 - I_1c_1\gamma_3R - c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1I_2R}, \\ \frac{d\omega_3}{dt} &= \frac{I_1(I_1 - I_2)\omega_2R^2 - I_1c_1\gamma_2R - c_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1I_3R}, \\ \frac{d\gamma_2}{dt} &= \frac{I_1\gamma_3R^2 + \omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1R}, \\ \frac{d\gamma_3}{dt} &= -\frac{I_1\gamma_2R^2 + \omega_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1R}. \end{aligned} \tag{8.66}$$

Case B. $c_1 = 0$. We solve equation $H_3 = U_3$ with respect to ω_1 at the value of γ_1 given from (8.62) and obtain $\omega_1 = R$, where R is a root of equation

$$Q(x) = I_1x^2 + B,$$

that is

$$Q(R) = I_1R^2 + B, \tag{8.67}$$

and $B(\omega_2, \omega_3, \gamma_2, \gamma_3)$ is the following polynomial:

$$B = I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 + 2c_3\gamma_3 - U_3.$$

In fact the values of the eliminated variables are determined as in Case A, i.e. by formula (8.65) but R is a root of different equation.

The restricted Euler-Poisson equations are

$$\begin{aligned}
\frac{d\omega_2}{dt} &= -\frac{I_1(I_1-I_3)\omega_3 R^2 - c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_2 R},\\
\frac{d\omega_3}{dt} &= \frac{I_1(I_1-I_2)\omega_2 R^2 - c_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_3 R},\\
\frac{d\gamma_2}{dt} &= \frac{I_1\gamma_3 R^2 + \omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R},\\
\frac{d\gamma_3}{dt} &= -\frac{I_1\gamma_2 R^2 + \omega_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R}.
\end{aligned} \tag{8.68}$$

To study the existence of a first integrals of systems (8.66) and (8.68) that depend on at most three variables among the variables ω_2, ω_3, γ_2 and γ_3 and that are functionally independent of H_2 restricted to invariant manifold (8.48) we should consider the following four types of a first integral:

1. $F(\omega_2, \omega_3, \gamma_2)$, (case (iii))
2. $F(\omega_2, \omega_3, \gamma_3)$, (case (iii))
3. $F(\omega_2, \gamma_2, \gamma_3)$, (case (iv))
4. $F(\omega_3, \gamma_2, \gamma_3)$. (case (iv))

Thus we should study the first integral of type 1 only.

Case A.1. We consider a first integral of type 1, i.e. $F = F(\omega_2, \omega_3, \gamma_2)$. Thus

$$\begin{aligned}
\frac{dF}{dt} = &-\frac{I_1(I_1-I_3)\omega_3 R^2 - I_1c_1\gamma_3 R - c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_2 R}\frac{\partial F}{\partial \omega_2}\\
&+ \frac{I_1(I_1-I_2)\omega_2 R^2 - I_1c_1\gamma_2 R - c_2(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 I_3 R}\frac{\partial F}{\partial \omega_3}\\
&+ \frac{I_1\gamma_3 R^2 + \omega_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1 R}\frac{\partial F}{\partial \gamma_2} = 0,
\end{aligned}$$

which is equivalent to

$$I_1 I_2 I_3 R\frac{dF}{dt} = Y_1(F) = 0. \tag{8.69}$$

Y_1 from (8.69) is the corresponding vector field defined on some suitable open set $\Omega \subseteq \mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

We differentiate identity (8.69) with respect to γ_3 and obtain again a linear partial differential equation for function F

$$\begin{aligned}
\frac{\partial Y_1(F)}{\partial \gamma_3} = &I_3\left[-2I_1(I_1-I_3)\omega_3 R\frac{\partial R}{\partial \gamma_3} + I_1c_1R + I_1c_1\gamma_3\frac{\partial R}{\partial \gamma_3} + I_3c_3\omega_3\right]\frac{\partial F}{\partial \omega_2}\\
&+ I_2\left[2I_1(I_1-I_2)\omega_2 R\frac{\partial R}{\partial \gamma_3} - I_1c_1\gamma_2\frac{\partial R}{\partial \gamma_3} - I_3c_2\omega_3\right]\frac{\partial F}{\partial \omega_3}
\end{aligned}$$

$$+ I_2 I_3 \left(I_1 R^2 + 2I_1\gamma_3 R \frac{\partial R}{\partial \gamma_3} + I_3\omega_3^2 \right) \frac{\partial F}{\partial \gamma_2} = Y_2(F) = 0, \tag{8.70}$$

where Y_2 is the corresponding vector field defined on Ω.

After differentiating identity (8.70) with respect to γ_3 we obtain

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial \gamma_3} &= I_1 I_3 \Big[-2(I_1 - I_3)\omega_3 R \frac{\partial^2 R}{\partial \gamma_3^2} - 2(I_1 - I_3)\omega_3 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 \\ &\quad + 2c_1 \frac{\partial R}{\partial \gamma_3} + c_1\gamma_3 \frac{\partial^2 R}{\partial \gamma_3^2} \Big] \frac{\partial F}{\partial \omega_2} \\ &\quad + I_1 I_2 \Big[2(I_1 - I_2)\omega_2 R \frac{\partial^2 R}{\partial \gamma_3^2} + 2(I_1 - I_2)\omega_2 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 - c_1\gamma_2 \frac{\partial^2 R}{\partial \gamma_3^2} \Big] \frac{\partial F}{\partial \omega_3} \\ &\quad + 2I_1 I_2 I_3 \Big[\gamma_3 R \frac{\partial^2 R}{\partial \gamma_3^2} + \gamma_3 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + 2R \frac{\partial R}{\partial \gamma_3} \Big] \frac{\partial F}{\partial \gamma_2} = Y_3(F) = 0, \end{aligned} \tag{8.71}$$

where Y_3 is the corresponding vector field defined on Ω.

If a first integral F exists, system (8.69)–(8.71) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_2}\right)$, which is possible if and only if the determinant D of its coefficients is identically equal to zero.

We compute D. It has a non-zero factor $I_1^2 I_2^2 I_3^2$ and that is why we note

$$\widehat{D} = \frac{D}{I_1^2 I_2^2 I_3^2}.$$

The expression for $\widehat{D}$ is

$$\begin{aligned}\widehat{D} &= a_1 R^4 \frac{\partial R}{\partial \gamma_3} + a_2 R^4 \frac{\partial^2 R}{\partial \gamma_3^2} + a_3 R^3 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + a_4 R^3 \frac{\partial R}{\partial \gamma_3} + a_5 R^3 \frac{\partial^2 R}{\partial \gamma_3^2} \\ &\quad + a_6 R^2 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + a_7 R^2 \frac{\partial R}{\partial \gamma_3} + a_8 R^2 \frac{\partial^2 R}{\partial \gamma_3^2} + a_9 R \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + a_{10} R \frac{\partial R}{\partial \gamma_3} \\ &\quad + a_{11} R \frac{\partial^2 R}{\partial \gamma_3^2} + a_{12} \left(\frac{\partial R}{\partial \gamma_3}\right)^3 + a_{13} \left(\frac{\partial R}{\partial \gamma_3}\right)^2,\end{aligned}$$

where

$$\begin{aligned} a_1 &= -2I_1 c_1 \omega_2 (I_1 - I_2), \qquad a_2 = I_1 c_1 [(-I_1 + I_2)\omega_2\gamma_3 + (I_1 - I_3)\omega_3\gamma_2], \\ a_3 &= -2I_1 c_1 [(-I_1 + I_2)\omega_2\gamma_3 + (I_1 - I_3)\omega_3\gamma_2], \\ a_4 &= 2[2I_3(I_2 - I_1)c_3\omega_2\omega_3 + 2I_3(I_1 - I_3)c_2\omega_3^2 + I_1 c_1^2\gamma_2], \\ a_5 &= -2I_2(I_1 - I_2)c_3\omega_2^2\gamma_2 + 2I_2(-I_3 + I_1)c_2\omega_2\omega_3\gamma_2 - 2I_3(I_1 - I_2)c_3\omega_2\omega_3\gamma_3 \\ &\quad + 2(I_1 - I_2)c_3 U_1\omega_2 + 2I_3(-I_3 + I_1)c_2\omega_3^2\gamma_3 - 2(-I_3 + I_1)c_2 U_1\omega_3 + I_1 c_1^2\gamma_2\gamma_3, \\ a_6 &= 6[(I_2 - I_1)c_3\omega_2 + (I_1 - I_3)c_2\omega_3](-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1), \\ a_7 &= -2c_1(-I_3(I_1 - I_2)\omega_2\omega_3^2 - c_2 I_2\omega_2\gamma_2 - 2c_3 I_3\omega_3\gamma_2 + c_2 U_1), \\ a_8 &= -c_1[-2I_2(I_1 - I_2)\omega_2^2\omega_3\gamma_2 - I_3(I_1 - I_2)\omega_2\omega_3^2\gamma_3 + 2(I_1 - I_2)U_1\omega_2\omega_3 \\ &\quad - I_2 c_3\omega_2\gamma_2^2 - I_2 c_2\omega_2\gamma_2\gamma_3 - I_3(-I_3 + I_1)\omega_3^3\gamma_2 - 2I_3 c_3\omega_3\gamma_2\gamma_3 + c_3 U_1\gamma_2 + c_2 U_1\gamma_3], \\ a_9 &= 2c_1[-I_2(I_1 - I_2)\omega_2^2\omega_3\gamma_2 - 2I_3(I_1 - I_2)\omega_2\omega_3^2\gamma_3 + U_1(I_1 - I_2)\omega_2\omega_3 - 2c_3 I_2\omega_2\gamma_2^2 \end{aligned}$$

$$+ I_2c_2\omega_2\gamma_2\gamma_3 + I_3(-I_3 + I_1)\omega_3^3\gamma_2 - I_3c_3\omega_3\gamma_2\gamma_3 + 2c_3U_1\gamma_2 - c_2U_1\gamma_3],$$

$$a_{10} = -2I_3c_1^2\omega_3^2\gamma_2, \qquad a_{11} = c_1^2\omega_3\gamma_2(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1),$$

$$a_{12} = 2c_1[(I_2 - I_1)\omega_2\omega_3\gamma_3 + (-I_3 + I_1)\omega_3^2\gamma_2 + c_3\gamma_2\gamma_3 - c_2\gamma_3^2](-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1),$$

$$a_{13} = -2c_1^2\omega_3\gamma_2(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1).$$

The first and second derivatives of R with respect to γ_3 appear in D. To determine them we differentiate equation (8.63) with respect to γ_3 two times and obtain

$$\frac{\partial Q}{\partial \gamma_3} = \frac{\partial A}{\partial \gamma_3}R + \frac{\partial B}{\partial \gamma_3} + \frac{dQ}{dR}\frac{\partial R}{\partial \gamma_3} = 0 \tag{8.72}$$

and

$$\frac{\partial^2 Q}{\partial \gamma_3^2} = \frac{\partial^2 A}{\partial \gamma_3^2}R + \frac{\partial A}{\partial \gamma_3}\frac{\partial R}{\partial \gamma_3} + \frac{\partial^2 B}{\partial \gamma_3^2} + \frac{\partial}{\partial \gamma_3}\left(\frac{dQ}{dR}\right)\frac{\partial R}{\partial \gamma_3} + \frac{dQ}{dR}\frac{\partial^2 R}{\partial \gamma_3^2} = 0. \tag{8.73}$$

We prove that if R is a root of equation (8.63), then $\frac{dQ}{dR} \neq 0$. For the purpose we consider the resultant $\rho = I_1^4(4A^3 + 27I_1^2B^2)$ of polynomials $Q(R)$ and $\frac{dQ}{dR} = 3I_1^2R^2 + A$ and prove that $\rho \neq 0$. Indeed, putting in ρ the expressions for A and B from (8.64) we obtain a polynomial which we do not give here but it never vanishes identically as it has a monomial $4I_1^3I_2^3\omega_2^6$.

Thus, by Proposition 4.1, $\frac{dQ}{dR} \neq 0$ and $\frac{\partial R}{\partial \gamma_3}$ can be correctly determined from equation (8.72). Then by equation (8.73) we determine $\frac{\partial^2 R}{\partial \gamma_3^2}$, put the obtained values for the derivatives of R in the expression for $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{L\delta(R)}{\left(\frac{dQ}{dR}\right)^3},$$

where $L = 4(I_1c_3R - I_3c_1\omega_3)$ and $\delta(R)$ is a polynomial of R of degree eight with coefficients depending on ω_2, ω_3, γ_2 and γ_3.

We prove that expression L does not vanish identically provided that R is a root of (8.63). Indeed, if $c_3 = 0$ then $L = -4I_3c_1\omega_3$ and as $c_1 \neq 0$ L could not vanish identically. If $c_3 \neq 0$ and we suppose that $L = 0$ then we have $R = I_3c_1\omega_3/(I_1c_3)$. Simple computations show that this value of R cannot be a root of (8.63). Thus we can work with $\widehat{D}$ instead of D because L and $\frac{dQ}{dR}$ are non-zeros.

Thus the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \omega_3, \gamma_2)$ of system (8.66) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

By the MAPLE command `rem` we compute remainder r from the division of polynomial $\delta(x)$ by polynomial $Q(x)$. The remainder is of the form:

$$r(x) = 2I_1c_1a_0x^2 + a_1x + 4c_1(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1)a_2,$$

where the coefficients a_0, a_1 and a_2 are polynomials of ω_2, ω_3, γ_2 and γ_3.

According to Proposition 4.2 if R is a root of equation (8.63), then, as $2I_1c_1$ is not zero and $4c_1(-I_2\omega_2\gamma_2 - I_3\omega_3\gamma_3 + U_1)$ does not vanish identically, we have $a_0 = a_1 = a_2 = 0$

identically. We use only the last equation $a_2 = 0$.

a_2 is a polynomial with 51 monomials and therefore with 51 coefficients. We equate them to zeros and apply simplification on the obtained system. After three consecutive simplifications we come to the reduced system that consists of only one equation 1=0. Thus a first integral of type 1, $F = F(\omega_2, \omega_3, \gamma_2)$ does not exist when $c_1 \neq 0$.

Case B.1. Here we consider Case B, i.e. $c_1 = 0$ and study the existence of a first integral of system (8.68) which is of type 1, $F(\omega_2, \omega_3, \gamma_2)$.

In the same way as in Case A we obtain a system

$$Y_1(F) = Y_2(F) = Y_3(F) = 0, \tag{8.74}$$

where vector fields Y_i, $1 \leq i \leq 3$, are defined on Ω.

System (8.74) coincides with system (8.69)–(8.71) if we substitute in the last one $c_1 = 0$. As we know the existence of the sought first integral is possible if and only if the determinant D of the coefficients of system (8.74) is identically equal to zero.

Let us compute this determinant. We obtain

$$D(R) = 2I_1^2 I_2^2 I_3^2 R^2 \big[(I_2 - I_1)c_3\omega_2 + (I_1 - I_3)c_2\omega_3\big]\widehat{D}(R), \tag{8.75}$$

where

$$\widehat{D}(R) = (I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)\left[R\frac{\partial^2 R}{\partial\gamma_3^2} - 3\left(\frac{\partial R}{\partial\gamma_3}\right)^2\right] + 2I_3\omega_3 R\frac{\partial R}{\partial\gamma_3}.$$

Taking into account that now $R = 0$ cannot be a root of equation (8.67) and that $c_1 = 0$ we easily see that the factor before $\widehat{D}(R)$ in (8.75) can vanish identically only in the Euler, Lagrange and kinetic symmetry cases. Thus the equation $D(R) = 0$ is equivalent to the equation $\widehat{D}(R) = 0$.

We compute the derivatives of R (see (8.67)) with respect to γ_3 and obtain

$$\frac{\partial R}{\partial\gamma_3} = -\frac{c_3}{I_1 R}, \qquad \frac{\partial^2 R}{\partial\gamma_3^2} = -\frac{c_3^2}{I_1^2 R^3}$$

and determine $\widehat{D}(R)$ as follows

$$\widehat{D}(R) = -2c_3\frac{I_1 I_3\omega_3 R^2 + 2c_3(I_2\omega_2\gamma_2 + I_3\omega_3\gamma_3 - U_1)}{I_1^2 R^2}.$$

Let us suppose that $c_3 \neq 0$. It is clear that now $\widehat{D}(R)$ never vanishes identically and consequently the sought partial first integral does not exist.

Let $c_3 = 0$. In such a case $\widehat{D} = 0$ and therefore equations (8.74) are linearly dependent. More precisely $Y_3 \equiv 0$ because when $c_3 = 0$ then R does not depend on γ_3 but every item of Y_3 contains either $\frac{\partial R}{\partial\gamma_3}$ or $\frac{\partial^2 R}{\partial\gamma_3^2}$ (see (8.71) under condition $c_1 = 0$).

Thus we have only two partial differential equations for first integral F. They are

$$Y_1(F) = 0, \quad Y_2(F) = 0. \tag{8.76}$$

Easy computations show that these equations are independent unless $c_2 = 0$ which leads to the Euler case or $I_1 = I_3$ - Lagrange case.

We compose the Lie bracket $Z = [Y_1, Y_2]$. Equations (8.76) imply that

$$\begin{aligned} Z(F) = {} & 3(I_1 - I_3)I_2 I_3^2 c_2 \omega_3 (I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 - U_3)\frac{\partial F}{\partial \omega_2} \\ & - I_2^2 I_3 c_2 \big[I_2 I_1 \omega_2^3 + I_3(I_1 - I_2)\omega_2\omega_3^2 \\ & + c_2(2I_1 - I_2)\omega_2\gamma_2 - I_1 U_3 \omega_2 + c_2 U_1 \big] \frac{\partial F}{\partial \omega_3} \\ & + I_2^2 I_3^2 \omega_3 \big[- I_2(2I_1 - I_2 - 2I_3)\omega_2^3 - 2I_3(I_1 - I_3)\omega_2\omega_3^2 \\ & - c_2(4I_1 - I_2 - 4I_3)\omega_2\gamma_2 + (2I_1 - I_2 - 2I_3)U_3\omega_2 + c_2 U_1 \big] \frac{\partial F}{\partial \gamma_2} = 0. \end{aligned} \tag{8.77}$$

Determinant δ composed from the coefficients of equations (8.76) and (8.77) should be identically equal to zero. We compute it and obtain

$$\delta = \delta_1 \delta_2,$$

where

$$\begin{aligned} \delta_1 &= I_2^3 I_3^3 (I_1 - I_3) c_2 \omega_3 (I_2\omega_2^2 + I_3\omega_3^2 + 2c_2\gamma_2 - U_3)^2, \\ \delta_2 &= -I_2(2I_1 - 3I_2)\omega_2^3 - 2I_3(I_1 - I_3)\omega_2\omega_3^2 - 2c_2(2I_1 - I_2)\omega_2\gamma_2 \\ & \quad + U_3(2I_1 - 3I_2)\omega_2 + 4c_2 U_1. \end{aligned}$$

It is easily seen that δ_1 can vanish identically only in the Euler and Lagrange cases. The expression for δ_2 contains a monomial $2I_3(I_1 - I_3)\omega_2\omega_3^2$ and therefore the minimal requirement for δ_2 to vanish identically is $I_1 = I_3$ - the Lagrange case.

Thus a new first integral of type 1 does not exist also when $c_1 = 0$.

8.3.3. Elimination of ω_1 and γ_2. In the same way as in Sec. 8.2.1 we solve equations $H_1 = U_1$ and $H_3 = U_3$ with respect to ω_1 and γ_2 and obtain

$$\omega_1 = R, \qquad \gamma_2 = -\frac{I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1}{I_2\omega_2}, \tag{8.78}$$

where R is a root of equation

$$Q(x) = Ax^2 + Bx + C = 0,$$

that is

$$Q(R) = AR^2 + BR + C = 0. \tag{8.79}$$

Functions $A = A(\omega_2)$, $B = B(\gamma_1)$ and $C = C(\omega_2, \omega_3, \gamma_1, \gamma_3)$ are given by the following polynomials:

$$\begin{aligned} A &= I_1 I_2 \omega_2, \qquad B = -2I_1 c_2 \gamma_1, \\ C &= I_2^2\omega_2^3 + I_2 I_3 \omega_2 \omega_3^2 + 2I_2 c_1 \omega_2 \gamma_1 + 2I_2 c_3 \omega_2 \gamma_3 \\ & \quad - I_2 U_3 \omega_2 - 2I_3 c_2 \omega_3 \gamma_3 + 2c_2 U_1. \end{aligned} \tag{8.80}$$

We put the values of ω_1 and γ_2 from (8.78) in the Euler-Poisson equations (1.1) and remove its first and fifth equations. In this way we obtain the following system of four equations in unknowns $\omega_2,\ \omega_3,\ \gamma_1$ and γ_3:

$$\begin{aligned}\frac{d\omega_2}{dt} &= -\frac{(I_1-I_3)\omega_3 R - c_1\gamma_3 + c_3\gamma_1}{I_2},\\ \frac{d\omega_3}{dt} &= \frac{I_2(I_1-I_2)\omega_2^2 R + I_2 c_2\omega_2\gamma_1 + c_1(I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1)}{I_2 I_3\omega_2},\\ \frac{d\gamma_1}{dt} &= -\frac{I_1\omega_3\gamma_1 R + I_3\omega_3^2\gamma_3 - U_1\omega_3 + I_2\omega_2^2\gamma_3}{I_2\omega_2},\\ \frac{d\gamma_3}{dt} &= \frac{I_2\omega_2^2\gamma_1 + (I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1)R}{I_2\omega_2}\end{aligned} \tag{8.81}$$

We consider the following four possible types of a first integral of system (8.81) that depends on at most three variables among the variables $\omega_2,\ \omega_3,\ \gamma_1$ and γ_3:

1. $F(\omega_2,\omega_3,\gamma_1)$, (case (ii))
2. $F(\omega_2,\omega_3,\gamma_3)$, (case (iii))
3. $F(\omega_2,\gamma_1,\gamma_3)$, (case (v))
4. $F(\omega_3,\gamma_1,\gamma_3)$. (case (iv))

As we have already studied cases (iii) and (iv) now we should consider the first integrals of types 1 and 3.

We suppose that the studied partial first integral is functionally independent of H_2 restricted to invariant manifold (8.48).

Type 1. Let us start with a first integral of type 1, $F = F(\omega_2,\omega_3,\gamma_1)$. We have

$$\begin{aligned}\frac{dF}{dt} = &-\frac{(I_1-I_3)\omega_3 R - c_1\gamma_3 + c_3\gamma_1}{I_2}\frac{\partial F}{\partial\omega_2}\\ &+ \frac{I_2(I_1-I_2)\omega_2^2 R + I_2 c_2\omega_2\gamma_1 + c_1(I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1)}{I_2 I_3\omega_2}\frac{\partial F}{\partial\omega_3}\\ &- \frac{I_1\omega_3\gamma_1 R + I_3\omega_3^2\gamma_3 - U_1\omega_3 + I_2\omega_2^2\gamma_3}{I_2\omega_2}\frac{\partial F}{\partial\gamma_1} = 0,\end{aligned}$$

which is equivalent to

$$I_2 I_3\omega_2\frac{dF}{dt} = Y_1(F) = 0. \tag{8.82}$$

Y_1 from the above equation is the corresponding vector field defined on some suitable open set $\Omega \subseteq \mathbb{C}^4(\omega_2,\omega_3,\gamma_1,\gamma_3)$.

We differentiate identity (8.82) with respect to γ_3 and obtain again a linear partial differential equation for function F

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial\gamma_3} = &I_3\left[(I_3-I_1)\omega_2\omega_3\frac{\partial R}{\partial\gamma_3} + c_1\omega_2\right]\frac{\partial F}{\partial\omega_2}\\ &+\left[I_2(I_1-I_2)\omega_2^2\frac{\partial R}{\partial\gamma_3} + I_1c_1\gamma_1\frac{\partial R}{\partial\gamma_3} + I_3c_1\omega_3\right]\frac{\partial F}{\partial\omega_3}\\ &- I_3\left(I_1\omega_3\gamma_1\frac{\partial R}{\partial\gamma_3} + I_2\omega_2^2 + I_3\omega_3^2\right)\frac{\partial F}{\partial\gamma_1} = Y_2(F) = 0.\end{aligned} \tag{8.83}$$

We differentiate $Y_2(F)$ with respect to γ_3 and obtain

$$\frac{\partial Y_2(F)}{\partial \gamma_3} = \frac{\partial^2 R}{\partial \gamma_3^2}\Big[I_3(I_3-I_1)\omega_2\omega_3\frac{\partial F}{\partial \omega_2} + (I_1I_2\omega_2^2 - I_2^2\omega_2^2 + I_1c_1\gamma_1)\frac{\partial F}{\partial \omega_3} - I_1I_3\omega_3\gamma_1\frac{\partial F}{\partial \gamma_1}\Big] = Y_3(F) = 0, \tag{8.84}$$

where Y_2 and Y_3 are the corresponding vector fields defined on Ω.

If a first integral $F = F(\omega_2, \omega_3, \gamma_1)$ exists, system (8.82)–(8.84) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_1}\right)$. This is possible if and only if the determinant D of its coefficients is identically equal to zero.

We compute D and obtain

$$D(R) = -I_2I_3^2\omega_2^2\frac{\partial^2 R}{\partial \gamma_3^2}\Delta,$$

where

$$\begin{aligned}\Delta = {} & (I_2-I_3)c_1U_1\omega_2\omega_3 + I_1c_1\gamma_1^2(c_3\omega_2 - c_2\omega_3)\\ & + (I_2\omega_2^2 + I_3\omega_3^2)[(I_1-I_2)c_3\omega_2 - (I_1-I_3)c_2\omega_3]\gamma_1.\end{aligned}$$

Let us first suppose that

$$\frac{\partial^2 R}{\partial \gamma_3^2} \neq 0. \tag{8.85}$$

In such a case $D(R) = 0$ if and only if $\Delta = 0$. Polynomial Δ has seven coefficients and they should be zeros, i.e. we have a system of seven equations for the parameters $\mathcal{I}c$ and U_1.

After two consecutive simplifications we come to the reduced system that consists of the following six equations:

$$\begin{aligned}&c_3c_1 = 0, \quad (I_1-I_2)c_3 = 0, \quad c_1c_2 = 0, \quad (I_1-I_3)c_2 = 0,\\ &(I_2-I_3)c_2c_3 = 0, \quad (I_2-I_3)c_1U_1 = 0.\end{aligned}$$

The system obtained coincides with system (8.35) and therefore has the same six solutions. After removing the solutions that lead to the Euler, Lagrange and kinetic symmetry cases we obtain only one new solution

$$\{U_1 = 0,\ I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\},$$

which is impossible because the condition $c_2 = c_3 = 0$ contradicts to (8.85). Indeed, equation (8.79) has no root that depends on γ_3 because the three its coefficients A, B and C (see (8.80)) do not depend on γ_3 when $c_2 = c_3 = 0$ and therefore R does not depend too.

Thus a first integral of type 1 does not exist when condition (8.85) is fulfilled.

Let us study what happens when

$$\frac{\partial^2 R}{\partial \gamma_3^2} = 0. \tag{8.86}$$

Equation (8.86) implies that $R = M(\omega_2, \omega_3, \gamma_1)\gamma_3 + N(\omega_2, \omega_3, \gamma_1)$, where M and N are some functions not depending on γ_3. Now equation (8.79) gives

$$Q = AM^2\gamma_3^2 + M\gamma_3(2AN + B) + AN^2 + BN + C = 0.$$

Thus Q is a polynomial of γ_3 of degree two. The coefficient of γ_3^2 is AM^2. As $Q = 0$ then $AM^2 = 0$. As A cannot be zero (see (8.80)) it remains the only possibility $M = 0$ and equation (8.79) is transform in the form

$$Q = AN^2 + BN + C = 0,$$

i.e. Q is already a polynomial of γ_3 of degree one. Its leading coefficient is $2(I_2c_3\omega_2 - I_3c_2\omega_3)$ (see the expression for C in (8.80)). This coefficient should be identically equal to zero. Thus $c_2 = c_3 = 0$.

As function B vanishes at this condition then R takes the following simple form

$$R = \frac{\sqrt{I_1(-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)}}{I_1}. \tag{8.87}$$

Now $Y_3(F) \equiv 0$ and equations $Y_i(F) = 0$, $1 \le i \le 2$, are obtained from (8.82) and (8.83) where we put $c_2 = c_3 = 0$ and $\frac{\partial R}{\partial \gamma_3} = 0$, i.e.

$$\begin{aligned} Y_1(F) = {} & -I_3\omega_2\left[(I_1 - I_3)\omega_3 R - c_1\gamma_3\right]\frac{\partial F}{\partial \omega_2} \\ & + \left[I_2(I_1 - I_2)\omega_2^2 R + c_1(I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1)\right]\frac{\partial F}{\partial \omega_3} \\ & - I_3\left[(I_2\omega_2^2 + I_3\omega_3^2)\gamma_3 + (I_1\gamma_1 R - U_1)\omega_3\right]\frac{\partial F}{\partial \gamma_1} = 0, \\ Y_2(F) = {} & I_3c_1\omega_2\frac{\partial F}{\partial \omega_2} + I_3c_1\omega_3\frac{\partial F}{\partial \omega_3} - I_3\left(I_2\omega_2^2 + I_3\omega_3^2\right)\frac{\partial F}{\partial \gamma_1} = 0, \end{aligned}$$

where R is taken from (8.87).

Further we work with vector fields

$$Z_1 = Y_1 - \gamma_3 Y_2, \quad Z_2 = Y_2, \quad Z_3 = \frac{[Z_1, Z_2]}{I_3}.$$

We compute Z_3

$$\begin{aligned} Z_3 = {} & I_3(I_1 - I_3)c_1\omega_2\omega_3 R\frac{\partial}{\partial \omega_2} + c_1\left[(I_2^2\omega_2^2 + I_1I_3\omega_3^2 + I_1c_1\gamma_1)R - c_1U_1\right]\frac{\partial}{\partial \omega_3} \\ & + I_3\omega_3\left[(2I_2^2\omega_2^2 - 2I_2I_3\omega_2^2 - I_1I_2\omega_2^2 - I_1I_3\omega_3^2 - I_1c_1\gamma_1)R + c_1U_1\right]\frac{\partial}{\partial \gamma_1}. \end{aligned}$$

In this way we obtain the following system of three equations for function F:

$$Z_i(F) = 0, \ 1 \le i \le 3,$$

and we know that determinant δ of the coefficients of this system should vanish identically with respect to ω_2, ω_3 and γ_1. We compute δ and obtain

$$I_1^2\delta = \delta_1\delta_2,$$

where

$$\delta_1 = I_1I_2I_3^2(I_2 - I_3)c_1\omega_2^3\omega_3$$

$$\delta_2 = \left[I_2(2I_2 - 3I_1)\omega_2^2 + I_3(2I_3 - 3I_1)\omega_3^2 - 4I_1c_1\gamma_1\right](-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3) + 4c_1U_1\sqrt{I_1(-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)}$$

δ_1 vanishes identically if $I_2 = I_3$ which leads to the Lagrange case or if $c_1 = 0$ which leads to the Euler case. Thus we suppose that $I_2 \neq I_3$ and $c_1 \neq 0$ and should study when δ_2 vanishes.

According to Proposition 4.3 applied to $V = I_1(-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)$ we conclude that $U_1 = 0$. Thus

$$\delta_2 = \left[I_2(2I_2 - 3I_1)\omega_2^2 + I_3(2I_3 - 3I_1)\omega_3^2 - 4I_1c_1\gamma_1\right](-I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 + U_3)$$

and it is easily seen that if $c_1 \neq 0$ then neither the first factor in the square brackets nor the second one vanishes independently of the values of the moments of inertia.

Thus a new first integral of type 1, $F(\omega_2, \omega_3, \gamma_1)$ does not exist.

Type 3. We go to the consideration of a first integral of type 3, $F(\omega_2, \gamma_1, \gamma_3)$. Thus

$$\begin{aligned}\frac{dF}{dt} = &-\frac{(I_1 - I_3)\omega_3 R - c_1\gamma_3 + c_3\gamma_1}{I_2}\frac{\partial F}{\partial \omega_2} \\ &- \frac{I_1\omega_3\gamma_1 R + I_2\omega_2^2\gamma_3 + I_3\omega_3^2\gamma_3 - U_1\omega_3}{I_2\omega_2}\frac{\partial F}{\partial \gamma_1} \\ &+ \frac{I_2\omega_2^2\gamma_1 + (I_1\gamma_1 R + I_3\omega_3\gamma_3 - U_1)R}{I_2\omega_2}\frac{\partial F}{\partial \gamma_3} = 0,\end{aligned}$$

which is equivalent to

$$I_2\omega_2\frac{dF}{dt} = Y_1(F) = 0, \tag{8.88}$$

where Y_1 is the corresponding vector field defined on a Ω.

We differentiate (8.88) with respect to ω_3 two times and obtain

$$\begin{aligned}\frac{\partial Y_1(F)}{\partial \omega_3} = &\ \omega_2(I_3 - I_1)\left(\omega_3\frac{\partial R}{\partial \omega_3} + R\right)\frac{\partial F}{\partial \omega_2} \\ &- \left[I_1\gamma_1\left(\omega_3\frac{\partial R}{\partial \omega_3} + R\right) + 2I_3\omega_3\gamma_3 - U_1\right]\frac{\partial F}{\partial \gamma_1} \\ &+ \left[I_3\gamma_3\left(\omega_3\frac{\partial R}{\partial \omega_3} + R\right) + 2I_1\gamma_1 R\frac{\partial R}{\partial \omega_3} - U_1\frac{\partial R}{\partial \omega_3}\right]\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0\end{aligned} \tag{8.89}$$

and

$$\begin{aligned}\frac{\partial Y_2(F)}{\partial \omega_3} = &\ \omega_2(I_3 - I_1)\left(\omega_3\frac{\partial^2 R}{\partial \omega_3^2} + 2\frac{\partial R}{\partial \omega_3}\right)\frac{\partial F}{\partial \omega_2} \\ &- \left[I_1\gamma_1\left(\omega_3\frac{\partial^2 R}{\partial \omega_3^2} + 2\frac{\partial R}{\partial \omega_3}\right) + 2I_3\gamma_3\right]\frac{\partial F}{\partial \gamma_1} \\ &+ \Bigg[I_3\gamma_3\left(\omega_3\frac{\partial^2 R}{\partial \omega_3^2} + 2\frac{\partial R}{\partial \omega_3}\right) + 2I_1\gamma_1\left(\frac{\partial R}{\partial \omega_3}\right)^2 \\ &+ (2I_1\gamma_1 R - U_1)\frac{\partial^2 R}{\partial \omega_3^2}\Bigg]\frac{\partial F}{\partial \gamma_3} = Y_3(F) = 0,\end{aligned} \tag{8.90}$$

where Y_2 and Y_3 are the corresponding vector fields defined on Ω.

If a first integral $F(\omega_2, \gamma_1, \gamma_3)$ exists, system (8.88)–(8.90) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_3}\right)$. This is possible if and only if the determinant $D(R)$ of its coefficients is identically equal to zero.

$D(R)$ has a factor ω_2 and we note

$$\widehat{D}(R) = \frac{D(R)}{\omega_2}.$$

The expression for $\widehat{D}(R)$ is

$$\begin{aligned}\widehat{D}(R) &= a_1R^3 + a_2R^2\frac{\partial R}{\partial \omega_3} + a_3R^2\frac{\partial^2 R}{\partial \omega_3^2} + a_4R^2 + a_5R\left(\frac{\partial R}{\partial \omega_3}\right)^2 + a_6R\frac{\partial^2 R}{\partial \omega_3^2}\\ &\quad + a_7R\frac{\partial^2 R}{\partial \omega_3^2} + a_8R + a_9\left(\frac{\partial R}{\partial \omega_3}\right)^3 + a_{10}\left(\frac{\partial R}{\partial \omega_3}\right)^2 + a_{11}\frac{\partial R}{\partial \omega_3} + a_{12}\frac{\partial^2 R}{\partial \omega_3^2}, \qquad (8.91)\end{aligned}$$

where

$$\begin{aligned}
a_1 &= 2I_1I_3(I_1 - I_3)\gamma_1\gamma_3, \qquad a_2 = 2I_1(I_1 - I_3)(-3I_3\omega_3\gamma_3 + U_1)\gamma_1,\\
a_3 &= I_1[-2I_2(I_1 - I_3)\omega_2^2\gamma_3 + U_1(I_1 - I_3)\omega_3 + 2I_1c_3\gamma_1^2 - 2I_1c_1\gamma_1\gamma_3]\gamma_1,\\
a_4 &= -2I_3(I_1 - I_3)U_1\gamma_3\\
a_5 &= 2I_1[I_2(I_1 - I_3)\omega_2^2\gamma_3 + 3I_3(I_1 - I_3)\omega_3^2\gamma_3 - 2U_1(I_1 - I_3)\omega_3 - I_1c_3\gamma_1^2 + I_1c_1\gamma_1\gamma_3]\gamma_1\\
a_6 &= 4I_3(I_1 - I_3)U_1\omega_3\gamma_3 - 4I_1I_3c_3\gamma_1^2\gamma_3 + 4I_1I_3c_1\gamma_1\gamma_3^2 - 2(I_1 - I_3)U_1^2,\\
a_7 &= I_2(I_1 - I_3)U_1\omega_2^2\gamma_3 + I_3(I_1 - I_3)U_1\omega_3^2\gamma_3 + 4I_1I_3c_3\omega_3\gamma_1^2\gamma_3 - 4I_1I_3c_1\omega_3\gamma_1\gamma_3^2\\
&\quad - (I_1 - I_3)U_1^2\omega_3 - 3I_1c_3U_1\gamma_1^2 + 3I_1c_1U_1\gamma_1\gamma_3,\\
a_8 &= 2I_3\gamma_3[I_2(I_1 - I_3)\omega_2^2\gamma_1 - I_3c_3\gamma_1\gamma_3 + I_3c_1\gamma_3^2],\\
a_9 &= 2I_1\omega_3\gamma_1[-I_2(I_1 - I_3)\omega_2^2\gamma_3 - I_3(I_1 - I_3)\omega_3^2\gamma_3\\
&\quad + (I_1 - I_3)U_1\omega_3 + I_1c_3\gamma_1^2 - I_1c_1\gamma_1\gamma_3],\\
a_{10} &= -2I_2(I_1 - I_3)U_1\omega_2^2\gamma_3 - 2I_3(I_1 - I_3)U_1\omega_3^2\gamma_3 + 4I_1I_3c_3\omega_3\gamma_1^2\gamma_3\\
&\quad - 4I_1I_3c_1\omega_3\gamma_1\gamma_3^2 + 2(I_1 - I_3)U_1^2\omega_3,\\
a_{11} &= -2I_2I_3(I_1 - I_3)\omega_2^2\omega_3\gamma_1\gamma_3 + 2I_2(I_1 - I_3)U_1\omega_2^2\gamma_1 + 2I_3^2c_3\omega_3\gamma_1\gamma_3^2 - 2I_3^2c_1\omega_3\gamma_3^3,\\
a_{12} &= (U_1 - 2I_3\omega_3\gamma_3)[I_2(I_1 - I_3)\omega_2^2\omega_3\gamma_1 - I_3c_3\omega_3\gamma_1\gamma_3 + I_3c_1\omega_3\gamma_3^2 + c_3U_1\gamma_1 - c_1U_1\gamma_3].
\end{aligned}$$

$\widehat{D}(R)$ contains $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$. We determine them by the same method we used for a first integral of type 2 and obtain

$$\begin{aligned}
\frac{\partial R}{\partial \gamma_1} &= -\frac{I_3(I_2\omega_2\omega_3 - c_2\gamma_3)}{I_1(I_2\omega_2R - c_2\gamma_1)},\\
\frac{\partial^2 R}{\partial \gamma_1^2} &= -\frac{I_2I_3\omega_2}{I_1^2(I_2\omega_2R - c_2\gamma_1)^3}\big(I_2^2I_3\omega_2^2\omega_3^2 - 2I_2I_3c_2\omega_2\omega_3\gamma_3 + I_1I_2^2R^2\omega_2^2\\
&\quad - 2I_1I_2c_2\omega_2\gamma_1R + I_1c_2^2\gamma_1^2 + I_3c_2^2\gamma_3^2\big).
\end{aligned}$$

We put these values of the derivatives of R in the expression (8.91) and obtain

$$\widehat{D}(R) = \frac{I_3}{I_1^2(I_2\omega_2R - c_2\gamma_1)^3}\delta(R),$$

where $\delta(R)$ is a polynomial of R of degree six with coefficients that are polynomials of ω_2, ω_3, γ_1, γ_3 and parameters $\mathcal{I}c$ and U_1.

It is clear that the equation $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \gamma_1, \gamma_3)$ of system (8.81) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_1, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

By the MAPLE command `rem` we compute remainder $r(x)$ from the division of polynomial $\delta(x)$ by polynomial $Q(x)$. It is of the form:

$$r(x) = \frac{1}{I_2^2 \omega_2^2}(a_0 x + a_1),$$

where the coefficients a_0 and a_1 are polynomials of ω_2, ω_3, γ_1 and γ_3 and parameters $\mathcal{I}c$, U_1 and U_3.

According to Proposition 4.2 if R is a root of equation (8.79), then we have $a_0 = a_1 = 0$ identically. Although polynomial a_0 has 84 coefficients we use a_1 which has 160 ones. This is because if we use $a_0 = 0$ then the reduced system has one solution $U_1 = 0$, $c_2 = 0$ which should be studied separately whereas only two consecutive simplifications on the system with 160 equations coming from $a_1 = 0$ lead to the reduced system

$$c_1 = 0, \quad c_3 = 0, \quad I_1 - I_3 = 0,$$

which immediately implies the Lagrange case and leads to the conclusion that a new first integral of type 3 does not exist.

8.3.4. Elimination of γ_2 and γ_3. Like in Sec. 8.2.1 we solve equations $H_1 = U_1$ and $H_3 = U_3$ with respect to γ_2 and γ_3 and obtain

$$\begin{aligned} \gamma_2 &= \frac{I_3\omega_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) - 2I_1c_3\omega_1\gamma_1 + 2c_3U_1}{2(I_2c_3\omega_2 - I_3c_2\omega_3)}, \\ \gamma_3 &= -\frac{I_2\omega_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) - 2I_1c_2\omega_1\gamma_1 + 2c_2U_1}{2(I_2c_3\omega_2 - I_3c_2\omega_3)}. \end{aligned} \tag{8.92}$$

Let us note that the elimination of γ_2 and γ_3 from equations $H_1 = U_1$ and $H_3 = U_3$ is possible only if

$$(c_2, c_3) \neq (0, 0). \tag{8.93}$$

Further we suppose that this condition is fulfilled.

We put the values of γ_2 and γ_3 from (8.92) in the Euler-Poisson equations (1.1) and remove its fifth and sixth equations. In this way we obtain the following system of four equations in unknowns ω_1, ω_2, ω_3 and γ_1:

$$\begin{aligned} \frac{d\omega_1}{dt} = \frac{1}{2I_1(I_2c_3\omega_2 - I_3c_2\omega_3)} \big[& I_1I_2c_2\omega_1^2\omega_2 + I_1I_3c_3\omega_1^2\omega_3 + I_2^2c_2\omega_2^3 \\ & + I_3^2c_3\omega_3^3 + 2I_2c_1c_2\omega_2\gamma_1 + I_2(2I_2 - I_3)c_3\omega_2^2\omega_3 + 2I_3c_1c_3\omega_3\gamma_1 \\ & - I_3(I_2 - 2I_3)c_2\omega_2\omega_3^2 - I_2c_2U_3\omega_2 - I_3c_3U_3\omega_3 \\ & - 2I_1(c_2^2 + c_3^2)\omega_1\gamma_1 + 2(c_2^2 + c_3^2)U_1 \big], \end{aligned}$$

$$
\begin{aligned}
\frac{d\omega_2}{dt} &= \frac{1}{2I_2(I_2c_3\omega_2 - I_3c_2\omega_3)}\big[- I_1I_2c_1\omega_1^2\omega_2 - I_2^2c_1\omega_2^3 - I_2I_3c_1\omega_2\omega_3^2 \\
&\quad + 2I_1c_1c_2\omega_1\gamma_1 - 2I_2(I_1 - I_3)c_3\omega_1\omega_2\omega_3 + 2I_3c_2c_3\omega_3\gamma_1 \\
&\quad + 2I_3(I_1 - I_3)c_2\omega_1\omega_3^2 + I_2c_1U_3\omega_2 - 2I_2(c_1^2 + c_3^2)\omega_2\gamma_1 - 2c_1c_2U_1\big], \qquad (8.94)\\
\frac{d\omega_3}{dt} &= \frac{1}{2I_3(I_2c_3\omega_2 - I_3c_2\omega_3)}\big[- I_1I_3c_1\omega_1^2\omega_3 - I_2I_3c_1\omega_2^2\omega_3 - I_3^2c_1\omega_3^3 \\
&\quad + 2I_1c_1c_3\omega_1\gamma_1 + 2I_2c_2c_3\omega_2\gamma_1 + 2I_2(I_1 - I_2)c_3\omega_1\omega_2^2 \\
&\quad - 2I_3(I_1 - I_2)c_2\omega_1\omega_2\omega_3 + I_3c_1U_3\omega_3 - 2I_3(c_1^2 + c_2^2)\omega_3\gamma_1 - 2c_1c_3U_1\big],\\
\frac{d\gamma_1}{dt} &= \frac{1}{2(I_2c_3\omega_2 - I_3c_2\omega_3)}\big[I_1I_2\omega_1^2\omega_2^2 + I_1I_3\omega_1^2\omega_3^2 + I_2^2\omega_2^4 + 2I_2I_3\omega_2^2\omega_3^2 \\
&\quad + I_3^2\omega_3^4 - 2I_1c_2\omega_1\omega_2\gamma_1 - 2I_1c_3\omega_1\omega_3\gamma_1 + 2I_2c_1\omega_2^2\gamma_1 + 2I_3c_1\omega_3^2\gamma_1 \\
&\quad - I_2U_3\omega_2^2 - I_3U_3\omega_3^2 + 2c_2U_1\omega_2 + 2c_3U_1\omega_3\big].
\end{aligned}
$$

We consider the following four possible types of a first integral of system (8.94) that depends on at most three variables among the variables $\omega_1,\ \omega_2,\ \omega_3$ and γ_1:

1. $F(\omega_1, \omega_2, \omega_3)$, (case (i))
2. $F(\omega_1, \omega_2, \gamma_1)$, (case (iii))
3. $F(\omega_1, \omega_3, \gamma_1)$, (case (iii))
4. $F(\omega_2, \omega_3, \gamma_1)$. (case (ii))

The only not yet studied case for the invariant manifold (8.48) is case (i). Thus here we should study the existence of a first integral of type 1 only.

We suppose that the studied partial first integral is functionally independent of H_2 restricted to invariant manifold (8.48).

Type 1. Let us consider a first integral of type 1, $F(\omega_1, \omega_2, \omega_3)$. Thus

$$
2I_1I_2I_3(I_2c_3\omega_2 - I_3c_2\omega_3)\frac{dF}{dt} = Z(F) = 0, \qquad (8.95)
$$

where Z is the corresponding vector field defined on some suitable open set $\Omega \subseteq \mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

Note that the right-hand sides of equations (8.94) are linear functions of γ_1. Thus, as F does not depend on γ_1, $Z(F)$ is also a linear function of γ_1, i.e. $Z(F) = \gamma_1 Y_1(F) + Y_2(F)$. Equation (8.95) which is an identity with respect to variables ω_1, ω_2, ω_3 and γ_1 implies that coefficients $Y_1(F)$ and $Y_2(F)$ should vanish. The vector fields Y_1 and Y_2 are given by the following expressions:

$$
\begin{aligned}
Y_1 &= 2I_2I_3\big[I_2c_1c_2\omega_2 + I_3c_1c_3\omega_3 - I_1(c_2^2 + c_3^2)\omega_1\big]\frac{\partial}{\partial\omega_1} \\
&\quad + 2I_1I_3\big[I_1c_1c_2\omega_1 + I_3c_2c_3\omega_3 - I_2(c_1^2 + c_3^2)\omega_2\big]\frac{\partial}{\partial\omega_2} \\
&\quad + 2I_1I_2\big[2I_1c_1c_3\omega_1 + 2I_2c_2c_3\omega_2 - 2I_3(c_1^2 + c_2^2)\omega_3\big]\frac{\partial}{\partial\omega_3},\\
Y_2 &= I_2I_3\big[I_2c_2\omega_2(I_1\omega_1^2 + I_2\omega_2^2) + I_3c_3\omega_3(I_1\omega_1^2 + I_3\omega_3^2) + I_2(2I_2 - I_3)c_3\omega_2^2\omega_3
\end{aligned}
$$

$$- I_3(I_2 - 2I_3)c_2\omega_2\omega_3^2 - I_2c_2U_3\omega_2 - I_3c_3U_3\omega_3 + 2(c_2^2 + c_3^2)U_1\big]\frac{\partial}{\partial\omega_1}$$
$$+ I_1I_3\big[- I_2c_1\omega_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) - 2I_2(I_1 - I_3)c_3\omega_1\omega_2\omega_3$$
$$+ 2I_3(I_1 - I_3)c_2\omega_1\omega_3^2 + I_2c_1U_3\omega_2 - 2c_1c_2U_1\big]\frac{\partial}{\partial\omega_2}$$
$$+ I_1I_2\big[- I_3c_1\omega_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2I_2(I_1 - I_2)c_3\omega_1\omega_2^2$$
$$- 2I_3(I_1 - I_2)c_2\omega_1\omega_2\omega_3 + I_3c_1U_3\omega_3 - 2c_1c_3U_1\big]\frac{\partial}{\partial\omega_3}.$$

We compose the Lie bracket $Y_3 = -[Y_1, Y_2]/(2I_1I_2I_3)$. The expression for Y_3 is long and we do not write it here.

We consider equations

$$Y_i(F) = 0,\ 1 \leq i \leq 3. \tag{8.96}$$

If a first integral $F(\omega_1, \omega_2, \omega_3)$ exists, system (8.96) has a non-zero solution grad $F = \left(\frac{\partial F}{\partial\omega_1}, \frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\omega_3}\right)$. This is possible if and only if the determinant D of its coefficients is identically equal to zero.

The expression for D is very long to be given here but our computations show that

$$D = 4I_1I_2I_3(I_2c_3\omega_2 - I_3c_2\omega_3)^2\widehat{D}.$$

As the factor in front of $\widehat{D}$ never vanishes identically because of the condition (8.93), equation $D = 0$ is equivalent to $\widehat{D} = 0$. $\widehat{D}$ is a polynomial of ω_1, ω_2 and ω_3 with 37 coefficients depending on parameters $\mathcal{I}c$, U_1 and U_3.

Thus we should solve the system obtained by equating to zero the 37 coefficients of $\widehat{D}$. After four consecutive simplifications we obtain the following simple reduced system:

$$c_3(I_1 - I_2) = 0, \quad c_2(I_1 - I_3) = 0, \quad c_1(I_2 - I_3) = 0, \quad c_2c_3(I_2 - I_3) = 0.$$

We solve this system by the MAPLE command `solve` and obtain the following five solutions with arbitrary values of U_1 and U_3:

$$\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0\}$$
$$\{I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0\}$$
$$\{I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3\}$$
$$\{I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0\}$$
$$\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3\}.$$

Taking into account condition (8.93), we remove the first and second solutions. The remaining three solutions lead either to the Lagrange case or to the kinetic symmetry.

Thus the sought first integral of type 1 does not exist.

8.4. Invariant manifold $\{H_2{=}U_2,\ H_3{=}U_3\}$. Here we study the existence of a partial first integral of the Euler-Poisson equations (1.1) restricted to the complex four-dimensional level manifold

$$\{H_2 = U_2,\ H_3 = U_3\}, \tag{8.97}$$

supposing that this first integral depends on at most three variables and that is functionally independent of H_1. U_2 and U_3 are arbitrary complex numbers, fixed once and for all.

Let us stress that the elimination of ω_1 and ω_2 is impossible on invariant manifold (8.97).

8.4.1. Elimination of ω_1 and γ_1. We express γ_1 from the equation $H_2 = U_2$ and obtain

$$\gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}. \tag{8.98}$$

Then we put γ_1 from (8.98) in the equation $H_3 = U_3$ and like in Sec. 8.2.1 solve it by the Maple command `solve`. In this way we obtain

$$\omega_1 = R, \tag{8.99}$$

where R is a root of equation

$$Q(x) = I_1 x^2 + B = 0, \tag{8.100}$$

that is

$$Q(R) = I_1 R^2 + B = 0, \tag{8.101}$$

where $B = B(\omega_2, \omega_3, \gamma_2, \gamma_3)$ is the following function:

$$B = I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\sqrt{-\gamma_2^2 - \gamma_3^2 + U_2} + 2c_2\gamma_2 + 2c_3\gamma_3 - U_3. \tag{8.102}$$

R and B are algebraic functions defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$. The equation (8.100) has only simple roots because the function B does not vanish identically.

Further, to simplify the notations, we put

$$\Gamma = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}.$$

We put the values of γ_1 and ω_1 from (8.98) and (8.99) in the Euler-Poisson equations (1.1), remove the first and fourth equations and obtain the following system of four differential equations in unknowns ω_2, ω_3, γ_2 and γ_3:

$$\begin{aligned} \frac{d\omega_2}{dt} &= \frac{1}{I_2}\left[(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\Gamma\right], & \frac{d\gamma_2}{dt} &= \gamma_3 R - \omega_3\Gamma, \\ \frac{d\omega_3}{dt} &= \frac{1}{I_3}\left[(I_1 - I_2)\omega_2 R - c_1\gamma_2 + c_2\Gamma\right], & \frac{d\gamma_3}{dt} &= -\gamma_2 R + \omega_2\Gamma. \end{aligned} \tag{8.103}$$

We want to study the existence of a first integral of system (8.103) that depends on at most three variables among the variables ω_2, ω_3, γ_2 and γ_3 and that is functionally independent of H_1 restricted to invariant manifold (8.97). The following four types of a first integral are possible:

1. $F(\omega_2, \omega_3, \gamma_2)$, (case (iii))
2. $F(\omega_2, \omega_3, \gamma_3)$, (case (iii))
3. $F(\omega_2, \gamma_2, \gamma_3)$, (case (iv))
4. $F(\omega_3, \gamma_2, \gamma_3)$. (case (iv))

Like in Sec. 5 we consider here only types 1 and 3.

Type 1. Let us suppose that there exists a first integral of type 1, $F(\omega_2, \omega_3, \gamma_2)$. Then

$$I_2 I_3 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.104}$$

where Y_1 is the vector field

$$\begin{aligned} Y_1 = I_3 \left[(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\Gamma\right] \frac{\partial}{\partial \omega_2} \\ + I_2 \left[(I_1 - I_2)\omega_2 R - c_1\gamma_2 + c_2\Gamma\right] \frac{\partial}{\partial \omega_3} \\ + I_2 I_3 \left[\gamma_3 R - \omega_3 \Gamma\right] \frac{\partial}{\partial \gamma_2}, \end{aligned}$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

As function F does not depend on γ_3, then if we differentiate identity (8.104) with respect to γ_3 we obtain again a linear partial differential equation for function F

$$\begin{aligned} \Gamma \frac{\partial Y_1(F)}{\partial \gamma_3} &= I_3 \left[\frac{\partial R}{\partial \gamma_3}(I_3 - I_1)\omega_3\Gamma + c_1\Gamma + c_3\gamma_3\right] \frac{\partial F}{\partial \omega_2} \\ &+ I_2 \left[\frac{\partial R}{\partial \gamma_3}(I_1 - I_2)\omega_2\Gamma - c_2\gamma_3\right] \frac{\partial F}{\partial \omega_3} \\ &+ I_2 I_3 \left[\frac{\partial R}{\partial \gamma_3}\gamma_3\Gamma + R\Gamma + \omega_3\gamma_3\right] \frac{\partial F}{\partial \gamma_2} = Y_2(F) = 0, \end{aligned} \tag{8.105}$$

where Y_2 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

After differentiating identity (8.105) with respect to γ_3 we obtain

$$\begin{aligned} \Gamma \frac{\partial Y_2(F)}{\partial \gamma_3} &= I_3 \left[\frac{\partial^2 R}{\partial \gamma_3^2}(I_3 - I_1)\omega_3\Gamma^2 - \frac{\partial R}{\partial \gamma_3}(I_3 - I_1)\omega_3\gamma_3 - c_1\gamma_3 + c_3\Gamma\right] \frac{\partial F}{\partial \omega_2} \\ &+ I_2 \left[\frac{\partial^2 R}{\partial \gamma_3^2}(I_1 - I_2)\omega_2\Gamma^2 - \frac{\partial R}{\partial \gamma_3}(I_1 - I_2)\omega_2\gamma_3 - c_2\Gamma\right] \frac{\partial F}{\partial \omega_3} \\ &+ I_2 I_3 \left[\frac{\partial^2 R}{\partial \gamma_3^2}\gamma_3\Gamma^2 + \frac{\partial R}{\partial \gamma_3}\left(2\Gamma^2 - \gamma_3^2\right) - R\gamma_3 + \omega_3\Gamma\right] \frac{\partial F}{\partial \gamma_2} \\ &= Y_3(F) = 0, \end{aligned} \tag{8.106}$$

where Y_3 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

If a first integral F exists, linear system (8.104)–(8.106) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_2}\right)$. This is possible if and only if the determinant $D(R)$ of its coefficients is identically equal to zero on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

The expression for $D(R)$ is too long to be shown here. $D(R)$ has a non-zero factor $I_2^2 I_3^2$ so we remove it and note

$$\widehat{D}(R) = \frac{D(R)}{I_2^2 I_3^2}.$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial \gamma_3}$ and $\frac{\partial^2 R}{\partial \gamma_3^2}$ as well. To determine them we use equation (8.101) which we differentiate with respect to γ_3 two times and obtain two equations for the derivatives of R:

$$\frac{\partial Q(R)}{\partial \gamma_3} = 2I_1 R \frac{\partial R}{\partial \gamma_3} + \frac{\partial B}{\partial \gamma_3} = 0,$$
$$\frac{\partial^2 Q(R)}{\partial \gamma_3^2} = 2I_1 \left(\frac{\partial R}{\partial \gamma_3}\right)^2 + 2I_1 R \frac{\partial^2 R}{\partial \gamma_3^2} + \frac{\partial^2 B}{\partial \gamma_3^2} = 0.$$

The determination of $\frac{\partial R}{\partial \gamma_3}$ and $\frac{\partial^2 R}{\partial \gamma_3^2}$ from these two equations is possible if and only if $2I_1 R \neq 0$. It is clear that this is always so because $R = 0$ is not a root of equation (8.101). Thus the derivatives of R can be found. We obtain the following expressions

$$\frac{\partial R}{\partial \gamma_3} = \frac{c_1\gamma_3 - c_3\Gamma}{I_1 R\Gamma},$$
$$\frac{\partial^2 R}{\partial \gamma_3^2} = -\frac{(c_1^2\gamma_3^2 + c_3^2\Gamma^2)\Gamma - 2c_1c_3\gamma_3\Gamma^2 + I_1c_1R^2(\gamma_2^2 - U_2)}{I_1^2R^3\Gamma^3}.$$

We put the obtained values for the derivatives of R in the expression for determinant $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{\delta(R)}{I_1^2 R^3},$$

where δ is a huge polynomial of R of degree five, whose coefficients are algebraic functions of $(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

It is clear that $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \omega_3, \gamma_2)$ of system (8.103) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using the MAPLE command `rem` we compute the remainder r from the division of polynomial $\delta(x)$ by polynomial $Q(x)$. We obtain

$$r(x) = a_0 x + a_1,$$

where $a_i = a_i(\omega_2, \omega_3, \gamma_2, \gamma_3)$, $i = 0, 1$, depend linearly on Γ.

According to Proposition 4.2, a_0 and a_1 should vanish identically with respect to ω_2, ω_3, γ_2 and γ_3. We use only a_0 which suffices for our aims. We have

$$a_0 = b_0\Gamma + b_1,$$

where b_0 and b_1 are polynomials of variables ω_2, ω_3, γ_2 and γ_3.

According to Proposition 4.3, the coefficients b_0 and b_1 should vanish identically because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. Polynomial b_0 has 30 coefficients and b_1 has 68. Equating to zero all of them we obtain 98 equations for the parameters $\mathcal{I}c$, U_2 and U_3.

After three consecutive simplifications we come to the reduced system that consists of the following four equations:

$$c_1 = 0, \quad c_3(I_1 - I_2) = 0, \quad c_2(I_1 - I_3) = 0, \quad c_2c_3(I_2 - I_3) = 0.$$

We solve this system by the MAPLE command `solve` and obtain four solutions all of them with arbitrary values of U_2 and U_3:

$$I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0;$$

$$I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3;$$
$$I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0;$$
$$I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = c_3.$$

The first solution is the Euler case, the second and third ones are the Lagrange case and the fourth solution is a particular cases of the kinetic symmetry case.

Thus a sought partial first integral of type 1 $F(\omega_2, \omega_3, \gamma_2)$ does not exist.

Type 3. The study of a new first integral of type 3, $F(\omega_2, \gamma_2, \gamma_3)$ follows the algorithm already described in the considerations concerning a first integral of type 1. There are some differences of course. For example, the computations of vector fields Y_2 and Y_3 require differentiation with respect to ω_3 instead of γ_3. By the way, as it is seen below, this considerably simplifies the computations because the differentiation does not affect the function Γ.

Let us suppose that there exists a first integral of type 3, $F(\omega_2, \gamma_2, \gamma_3)$. Then we have

$$I_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.107}$$

where Y_1 is the vector field

$$Y_1 = [(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\Gamma] \frac{\partial}{\partial \omega_2} + I_2 (\gamma_3 R - \omega_3 \Gamma) \frac{\partial}{\partial \gamma_2} - I_2 (\gamma_2 R - \omega_2 \Gamma) \frac{\partial}{\partial \gamma_3},$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

As in the study of a first integral of type 1, we differentiate identity (8.107) with respect to ω_3 and obtain

$$\begin{aligned} \frac{\partial Y_1(F)}{\partial \omega_3} &= (I_3 - I_1)\left(\frac{\partial R}{\partial \omega_3}\omega_3 + R\right)\frac{\partial F}{\partial \omega_2} + I_2\left(\frac{\partial R}{\partial \omega_3}\gamma_3 - \Gamma\right)\frac{\partial F}{\partial \gamma_2} \\ &- I_2 \frac{\partial R}{\partial \omega_3}\gamma_2 \frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \end{aligned} \tag{8.108}$$

where Y_2 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

After differentiating identity (8.108) with respect to ω_3 we obtain

$$\begin{aligned} \frac{\partial Y_2(F)}{\partial \omega_3} &= (I_3 - I_1)\left[\frac{\partial^2 R}{\partial \omega_3^2}\omega_3 + 2\frac{\partial R}{\partial \omega_3}\right]\frac{\partial F}{\partial \omega_2} + I_2 \frac{\partial^2 R}{\partial \omega_3^2}\gamma_3 \frac{\partial F}{\partial \gamma_2} \\ &- I_2 \frac{\partial^2 R}{\partial \omega_3^2}\gamma_2 \frac{\partial F}{\partial \gamma_3} = Y_3(F) = 0, \end{aligned} \tag{8.109}$$

where Y_3 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_2, \gamma_3)$.

As in the investigation of a first integral of type 1, we require that the determinant $D(R)$ of the coefficients of system (8.107)–(8.109) be identically equal to zero. Computing it we see that it has a non-zero factor $I_2^2\Gamma$. We remove it and note

$$\widehat{D}(R) = \frac{D(R)}{I_2^2 \Gamma}.$$

In this way we obtain

$$\widehat{D}(R) = -\left\{[(I_1 - I_3)\omega_2\omega_3 + c_3\gamma_2] \frac{\partial^2 R}{\partial \omega_3^2} + 2\omega_2(I_1 - I_3)\frac{\partial R}{\partial \omega_3}\right\}\Gamma + c_1\gamma_2\gamma_3 \frac{\partial^2 R}{\partial \omega_3^2}$$

$$+ 2\gamma_2(I_1 - I_3)R\frac{\partial R}{\partial \omega_3} + (\omega_2\gamma_3 - \omega_3\gamma_2)(I_1 - I_3)\left[2\left(\frac{\partial R}{\partial \omega_3}\right)^2 - \frac{\partial^2 R}{\partial \omega_3^2}R\right]$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$. We use equation (8.101) to determine them. For this aim we differentiate (8.101) with respect to ω_3 two times and obtain two equations for the sought derivatives of R:

$$\frac{\partial Q}{\partial \omega_3} = 2I_1 R\frac{\partial R}{\partial \omega_3} + \frac{\partial B}{\partial \omega_3} = 0,$$
$$\frac{\partial^2 Q}{\partial \omega_3^2} = 2I_1\left(\frac{\partial R}{\partial \omega_3}\right)^2 + 2I_1 R\frac{\partial^2 R}{\partial \omega_3^2} + \frac{\partial^2 B}{\partial \omega_3^2} = 0.$$

As we have mentioned studying the first integral of type 1, $R = 0$ cannot be a root of equation (8.101) and therefore the partial derivatives of R can be correctly determined from the above equations. We put the value of B taken from (8.102) in these equations and solve them. The solution is

$$\frac{\partial R}{\partial \omega_3} = -\frac{I_3\omega_3}{I_1 R}, \qquad \frac{\partial^2 R}{\partial \omega_3^2} = -\frac{I_3(I_3\omega_3^2 + I_1 R^2)}{I_1^2 R^3}.$$

We put the above values of $\frac{\partial R}{\partial \omega_3}$ and $\frac{\partial^2 R}{\partial \omega_3^2}$ in the expression for determinant $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{I_3\delta(R)}{I_1^2 R^3},$$

where $\delta(R)$ is the following polynomial of R of degree three:

$$\delta(R) = -I_1(I_1 - I_3)(3\omega_3\gamma_2 - \omega_2\gamma_3)R^3 + I_1\left[3(I_1 - I_3)\omega_2\omega_3\Gamma + (c_3\Gamma - c_1\gamma_3)\gamma_2\right]R^2$$
$$- 3I_3(I_1 - I_3)\omega_3^2(\omega_3\gamma_2 - \omega_2\gamma_3)R + I_3\omega_3^2\left[(I_1 - I_3)\omega_2\omega_3\Gamma + (c_3\Gamma - c_1\gamma_3)\gamma_2\right].$$

It is clear that $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2, \gamma_2, \gamma_3)$ of system (8.103) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2, \omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using the Maple command `rem` we compute the remainder r from the division of polynomial $\delta(x)$ by polynomial $Q(x)$. We obtain

$$r(x) = (a_0 + b_0\Gamma)x + a_1 + b_1\Gamma,$$

where

$$a_0 = (I_3 - I_1)(I_2\omega_2^3\gamma_3 - 3I_2\omega_2^2\omega_3\gamma_2 - 2I_3\omega_2\omega_3^2\gamma_3 + 2c_2\omega_2\gamma_3\gamma_2 + 2c_3\omega_2\gamma_3^2 - U_3\omega_2\gamma_3$$
$$- 6c_2\omega_3\gamma_2^2 - 6\omega_3 c_3\gamma_2\gamma_3 + 3U_3\omega_3\gamma_2),$$
$$b_0 = 2(I_3 - I_1)c_1(\omega_2\gamma_3 - 3\omega_3\gamma_2),$$
$$a_1 = c_1[I_2\omega_2^2\gamma_2\gamma_3 - 6c_1(I_1 - I_3)\omega_2\omega_3\Gamma^2 + 2c_3\gamma_2^3 + 2c_2\gamma_2^2\gamma_3 + 4c_3\gamma_2\gamma_3^2$$
$$- U_3\gamma_2\gamma_3 - 2c_3U_2\gamma_2],$$
$$b_1 = -I_2c_3\omega_2^2\gamma_2 - (I_1 - I_3)(3I_2\omega_2^2 + 2I_3\omega_3^2 + 6c_2\gamma_2 + 6c_3\gamma_3 - 3U_3)\omega_2\omega_3$$
$$- 2c_2c_3\gamma_2^2 + 2(c_1^2 - c_3^2)\gamma_2\gamma_3 + c_3U_3\gamma_2.$$

According to Proposition 4.2, all the coefficients of the remainder r should vanish identically with respect to ω_2, ω_3, γ_2 and γ_3. We use only the coefficient $a_1 + b_1\Gamma$ which is sufficient for our aims.

According to Proposition 4.3, the coefficients a_1 and b_1 should vanish identically because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. We use only b_1. It has nine coefficients. Equating to zero all of them we obtain nine equations for the parameters $\mathcal{I}c$, U_2 and U_3 as follows:

$$2I_3(I_3 - I_1) = 0, \quad 3I_2(I_3 - I_1) = 0, \quad -I_2c_3 = 0, \quad 3(I_1 - I_3)U_3, \quad c_3U_3 = 0,$$
$$2c_2c_3 = 0, \quad 2(c_1^2 - c_3^2) = 0, \quad 6(I_3 - I_1)c_2 = 0, \quad 6(I_3 - I_1)c_3 = 0.$$

It is very easy to see that this equations imply that

$$c_1 = 0, \quad c_3 = 0, \quad I_1 - I_3 = 0,$$

which obviously leads to the Lagrange case.

Thus a sought partial first integral of type 3 does not exist.

8.4.2. Elimination of ω_1 and γ_2. Like in Sec. 8.4.1, we express γ_2 from the equations $H_2 = U_2$ and obtain

$$\gamma_2 = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}. \tag{8.110}$$

Then we put γ_2 from (8.110) in the equation $H_3 = U_3$ and like in Sec. 8.2.1 solve it by the MAPLE command `solve`. In this way we obtain

$$\omega_1 = R, \tag{8.111}$$

where R is a root of equation

$$Q(x) = I_1x^2 + B = 0, \tag{8.112}$$

that is

$$Q(R) = I_1R^2 + B = 0, \tag{8.113}$$

and $B = B(\omega_2, \omega_3, \gamma_1, \gamma_3)$ is the following function:

$$B = I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 + 2c_2\sqrt{-\gamma_1^2 - \gamma_3^2 + U_2} + 2c_3\gamma_3 - U_3.$$

R and B are algebraic functions defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$. The equation (8.112) has only simple roots because the function B does not vanish identically.

Further, to simplify the notations, we put

$$\Gamma = \sqrt{-\gamma_1^2 - \gamma_3^2 + U_2}.$$

We put the values of γ_2 and ω_1 from (8.110) and (8.111) in the Euler-Poisson equations (1.1), remove the first and fifth equations and obtain the following system of four differential equations in unknowns ω_2, ω_3, γ_1 and γ_3:

$$\begin{aligned} \frac{d\omega_2}{dt} &= \frac{1}{I_2}\left[(I_3 - I_1)\omega_3R + c_1\gamma_3 - c_3\gamma_1\right], & \frac{d\gamma_1}{dt} &= \omega_3\Gamma - \omega_2\gamma_3, \\ \frac{d\omega_3}{dt} &= \frac{1}{I_3}\left[(I_1 - I_2)\omega_2R + c_1\Gamma - c_2\gamma_1\right], & \frac{d\gamma_3}{dt} &= \omega_2\gamma_1 - R\Gamma. \end{aligned} \tag{8.114}$$

We want to study the existence of a first integral of system (8.114) that depends on at most three variables among the variables ω_2, ω_3, γ_1 and γ_3 and that is functionally independent of H_1 restricted to invariant manifold (8.97). The following four types of a first integral are possible:

1. $F(\omega_2, \omega_3, \gamma_1)$, (case (ii))
2. $F(\omega_2, \omega_3, \gamma_3)$, (case (iii))
3. $F(\omega_2, \gamma_1, \gamma_3)$, (case (v))
4. $F(\omega_3, \gamma_1, \gamma_3)$. (case (iv))

As the cases (iii) and (iv) were already examined, there remains only to examine cases (ii) and (v).

Type 1. Let us suppose that there exists a first integral of type 1, $F(\omega_2, \omega_3, \gamma_1)$. Then

$$I_2 I_3 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.115}$$

where Y_1 is the vector field

$$\begin{aligned} Y_1 = I_3 \left[(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\gamma_1\right] \frac{\partial}{\partial\omega_2} \\ + I_2 \left[(I_1 - I_2)\omega_2 R + c_1\Gamma - c_2\gamma_1\right] \frac{\partial}{\partial\omega_3} + I_2 I_3 \left(\omega_3\Gamma - \omega_2\gamma_3\right) \frac{\partial}{\partial\gamma_1}, \end{aligned}$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

As function F does not depend on γ_3, then if we differentiate identity (8.115) with respect to γ_3 we obtain again a linear partial differential equation for function F

$$\begin{aligned} \Gamma \frac{\partial Y_1(F)}{\partial\gamma_3} = I_3 \left[\frac{\partial R}{\partial\gamma_3}(I_3 - I_1)\omega_3 + c_1\right] \Gamma \frac{\partial F}{\partial\omega_2} + I_2 \left[\frac{\partial R}{\partial\gamma_3}(I_1 - I_2)\omega_2\Gamma + c_1\gamma_3\right] \frac{\partial F}{\partial\omega_3} \\ - I_2 I_3 \left[\omega_2\Gamma + \omega_3\gamma_3\right] \frac{\partial F}{\partial\gamma_1} = Y_2(F) = 0, \end{aligned} \tag{8.116}$$

where Y_2 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

After differentiating identity (8.116) with respect to γ_3 we obtain

$$\begin{aligned} \Gamma \frac{\partial Y_2(F)}{\partial\gamma_3} = I_3 \left[\frac{\partial^2 R}{\partial\gamma_3^2}(I_3 - I_1)\omega_3\Gamma^2 - \frac{\partial R}{\partial\gamma_3}(I_3 - I_1)\omega_3\gamma_3 - c_1\gamma_3\right] \frac{\partial F}{\partial\omega_2} \\ + I_2 \left[\frac{\partial^2 R}{\partial\gamma_3^2}(I_1 - I_2)\omega_2\Gamma^2 - \frac{\partial R}{\partial\gamma_3}(I_1 - I_2)\omega_2\gamma_3 + c_1\Gamma\right] \frac{\partial F}{\partial\omega_3} \\ + I_2 I_3 \left(\omega_2\gamma_3 - \omega_3\Gamma\right) \frac{\partial F}{\partial\gamma_1} = Y_3(F) = 0, \end{aligned} \tag{8.117}$$

where Y_3 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

If a first integral F exists, linear system (8.115)–(8.117) has a non-zero solution $\operatorname{grad} F = \left(\frac{\partial F}{\partial\omega_2}, \frac{\partial F}{\partial\omega_3}, \frac{\partial F}{\partial\gamma_1}\right)$. This is possible if and only if the determinant $D(R)$ of its coefficients is identically equal to zero on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

The expression for $D(R)$ has a non-zero factor $I_2^2 I_3^2$ so we remove it and note

$$\widehat{D}(R) = \frac{D(R)}{I_2^2 I_3^2}.$$

In this way we obtain

$$\begin{aligned}\widehat{D}(R) = &-\frac{\partial^2 R}{\partial \gamma_3^2}\big[c_3(I_1-I_2)\omega_2^2\gamma_1\Gamma - c_1(I_2-I_3)\omega_2\omega_3\gamma_1^2 + c_3(I_1-I_2)\omega_2\omega_3\gamma_1\gamma_3 \\ &- c_2(I_1-I_3)\omega_2\omega_3\gamma_1\Gamma + (I_2-I_3)c_1U_2\omega_2\omega_3 - c_2(I_1-I_3)\omega_3^2\gamma_1\gamma_3\big]\Gamma^2 \\ &- \frac{\partial R}{\partial \gamma_3}\omega_3\big[c_3(I_2-I_1)\omega_2\gamma_1 - c_1(I_2-I_3)\omega_2\gamma_3 + c_2(I_1-I_3)\omega_3\gamma_1\big](U_2-\gamma_1^2) \\ &- c_1\big[R(I_2-I_3)\omega_2\omega_3 + c_3\omega_2\gamma_1 - c_2\omega_3\gamma_1\big](U_2-\gamma_1^2)\end{aligned}$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial \gamma_3}$ and $\frac{\partial^2 R}{\partial \gamma_3^2}$. To determine them we use equation (8.113) which we differentiate with respect to γ_3 two times and in the same way as in Sec. 8.4.1 obtain

$$\begin{aligned}\frac{\partial R}{\partial \gamma_3} &= \frac{c_2\gamma_3 - c_3\Gamma}{I_1 R\Gamma}, \\ \frac{\partial^2 R}{\partial \gamma_3^2} &= -\frac{(c_2^2\gamma_3^2 + c_3^2\Gamma^2)\Gamma - 2c_2c_3\gamma_3\Gamma^2 + I_1c_2R^2(\gamma_1^2 - U_2)}{I_1^2R^3\Gamma^3}.\end{aligned}$$

We put the obtained values for the derivatives of R in the expression for determinant $\widehat{D}(R)$ and obtain

$$\widehat{D}(R) = \frac{\delta(R)}{I_1^2R^3},$$

where δ is a long polynomial of R of degree four, whose coefficients are algebraic functions of $(\omega_2,\omega_3,\gamma_1,\gamma_3)$.

The identity $\widehat{D}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_2,\omega_3,\gamma_1)$ of system (8.114) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_2,\omega_3,\gamma_1,\gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using the Maple command `rem` we compute the remainder r from the division of polynomial $\delta(x)$ by polynomial $Q(x)$. We obtain:

$$r(x) = a_0x + a_1, \tag{8.118}$$

where $a_i = a_i(\omega_2,\omega_3,\gamma_1,\gamma_3)$, $i = 0,1$, depend linearly on Γ.

According to Proposition 4.2, a_0 and a_1 should vanish identically with respect to ω_2, ω_3, γ_1 and γ_3. We use only a_1 which suffices for our aims. We have

$$a_1 = b_0\Gamma + b_1,$$

where b_0 and b_1 are polynomials of variables ω_2, ω_3, γ_1 and γ_3.

According to Proposition 4.3, the coefficients b_0 and b_1 should vanish identically because $\Gamma \notin \mathbb{C}(\gamma_1,\gamma_3)$.

We use only polynomial b_1. It has 48 coefficients. Equating to zero all of them we obtain a system of 48 equations for the parameters $\mathcal{I}c$, U_2 and U_3.

After four consecutive simplifications we come to the reduced system that consists of the following four equations:

$$c_1(I_2-I_3) = 0, \quad c_2(I_1-I_3) = 0 \quad c_3(I_1-I_2) = 0, \quad c_2c_3(I_2-I_3) = 0.$$

We solve this system by the MAPLE command `solve` and obtain five solutions all of them with arbitrary values of U_2 and U_3:

$$I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0;$$
$$I_1 = I_1,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = 0,\ c_3 = 0;$$
$$I_1 = I_2,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = 0,\ c_3 = c_3;$$
$$I_1 = I_3,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = 0;$$
$$I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = c_1,\ c_2 = c_2,\ c_3 = c_3.$$

The first solution is the Euler case, the next three are the Lagrange case and the last one is the kinetic symmetry case.

Thus a partial first integral of type 1 $F(\omega_2, \omega_3, \gamma_1)$ does not exist.

Type 3. Let us suppose that there exists a first integral of type 3, $F(\omega_2, \gamma_1, \gamma_3)$. The independence of the first integral of ω_3 considerably simplifies the computations because there is not need of differentiation of the function Γ.

So, let $F(\omega_2, \gamma_1, \gamma_3)$ be a first integral of system (8.114). Then we have

$$I_2 \frac{dF}{dt} = Y_1(F) = 0, \tag{8.119}$$

where Y_1 is the vector field

$$Y_1 = [(I_3 - I_1)\omega_3 R + c_1\gamma_3 - c_3\gamma_1] \frac{\partial}{\partial \omega_2} + I_2(\omega_3\Gamma - \omega_2\gamma_3) \frac{\partial}{\partial \gamma_1} + I_2(\omega_2\gamma_1 - R\Gamma) \frac{\partial}{\partial \gamma_3},$$

defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

We differentiate identity (8.119) with respect to ω_3 and obtain

$$\frac{\partial Y_1(F)}{\partial \omega_3} = (I_3 - I_1)\left(\frac{\partial R}{\partial \omega_3}\omega_3 + R\right)\frac{\partial F}{\partial \omega_2} + I_2\Gamma\frac{\partial F}{\partial \gamma_1} - I_2\frac{\partial R}{\partial \omega_3}\Gamma\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \tag{8.120}$$

where Y_2 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

After differentiating identity (8.120) with respect to ω_3 we obtain

$$\frac{\partial Y_2(F)}{\partial \omega_3} = (I_3 - I_1)\left(\frac{\partial^2 R}{\partial \omega_3^2}\omega_3 + 2\frac{\partial R}{\partial \omega_3}\right)\frac{\partial F}{\partial \omega_2} - I_2\frac{\partial^2 R}{\partial \omega_3^2}\Gamma\frac{\partial F}{\partial \gamma_3} = Y_3(F) = 0, \tag{8.121}$$

where Y_3 is the corresponding vector field defined on $\mathbb{C}^4(\omega_2, \omega_3, \gamma_1, \gamma_3)$.

The existence of a first integral $F(\omega_2, \gamma_1, \gamma_3)$ implies that the determinant $D(R)$ of the coefficients of system (8.119)–(8.121) is identically equal to zero. Computing $D(R)$ we see that it has a non-zero factor $I_2^2\Gamma$. We remove it and note

$$\widehat{D}(R) = \frac{D(R)}{I_2^2\Gamma}.$$

In this way we obtain

$$\widehat{D}(R) = -\left\{[(I_1 - I_3)\omega_3 R - c_3\gamma_1 + c_1\gamma_3]\frac{\partial^2 R}{\partial \omega_3^2} \right.$$
$$\left. - 2\omega_3(I_1 - I_3)\left(\frac{\partial R}{\partial \omega_3}\right)^2 + 2(I_1 - I_3)R\frac{\partial R}{\partial \omega_3}\right\}\Gamma$$

$$+\,\omega_2(I_1-I_3)\left[(\omega_3\gamma_1+\gamma_3R)\frac{\partial^2 R}{\partial\omega_3^2}-2\gamma_3\left(\frac{\partial R}{\partial\omega_3}\right)^2+2\gamma_1\frac{\partial R}{\partial\omega_3}\right]$$

$\widehat{D}(R)$ contains the partial derivatives $\frac{\partial R}{\partial\omega_3}$ and $\frac{\partial^2 R}{\partial\omega_3^2}$. We use equation (8.113) to determine them. We differentiate (8.113) with respect to ω_3 two times and in the same way as in Sec. 8.4.1 obtain

$$\frac{\partial R}{\partial\omega_3}=-\frac{I_3\omega_3}{I_1R},\qquad \frac{\partial^2 R}{\partial\omega_3^2}=-\frac{I_3(I_3\omega_3^2+I_1R^2)}{I_1^2R^3}.$$

We put the above values of $\frac{\partial R}{\partial\omega_3}$ and $\frac{\partial^2 R}{\partial\omega_3^2}$ in the expression for determinant $\widehat{D}(R)$ and obtain

$$\widehat{D}(R)=\frac{I_3\delta(R)}{I_1^2R^3},$$

where δ is the following polynomial of R of degree three

$$\begin{aligned}\delta(R)&=I_1(I_1-I_3)(3\omega_3\Gamma-\omega_2\gamma_3)R^3+I_1\left[3(I_3-I_1)\omega_2\omega_3\gamma_1-(c_3\gamma_1-c_1\gamma_3)\Gamma\right]R^2\\&\quad+3I_3(I_1-I_3)\omega_3^2(\omega_3\Gamma-\omega_2\gamma_3)R+I_3\omega_3^2\left[(I_3-I_1)\omega_2\omega_3\gamma_1-\Gamma(c_3\gamma_1-c_1\gamma_3)\right],\end{aligned}$$

whose coefficients are algebraic functions of $(\omega_2,\omega_3,\gamma_1,\gamma_3)$.

The identity $\widehat{D}(R)=0$ is equivalent to $\delta(R)=0$. We know that if $Q(R)=0$, then if in addition some supplementary first integral $F(\omega_2,\gamma_1,\gamma_3)$ of system (8.114) exists, then also $\delta(R)=0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K}=\mathbb{A}\mathrm{lg}(\omega_2,\omega_3,\gamma_1,\gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using Maple we divide δ by Q and obtain a remainder which is a polynomial r of the form (8.118) and coefficients $a_i=a_i(\omega_2,\omega_3,\gamma_1,\gamma_3)$, $i=0,1$, which depend linearly on Γ.

According to Proposition 4.2, a_0 and a_1 should vanish identically with respect to ω_2, ω_3, γ_1 and γ_3. We use only a_1 which suffices for our aims. We have

$$a_1=b_0\Gamma+b_1,$$

where b_0 and b_1 are polynomials of variables ω_2, ω_3, γ_1 and γ_3.

According to Proposition 4.3, the coefficients b_0 and b_1 should vanish identically because $\Gamma\notin\mathbb{C}(\gamma_1,\gamma_3)$.

We use only polynomial b_0. It has eight coefficients. Equating to zero all of them we obtain a system of eight equations for the parameters $\mathcal{I}c$, U_2 and U_3. These eight equations are:

$$\begin{aligned}&c_1^2-c_3^2=0,\quad I_2c_3=0,\quad I_2c_1=0\quad (I_1-I_3)c_2=0,\\&c_3U_3=0,\quad c_3c_1=0,\quad c_1U_3=0,\quad c_1c_3=0.\end{aligned}$$

After two consecutive simplifications we come to the reduced system that is

$$c_1=0,\quad c_3=0,\quad (I_1-I_3)c_2=0$$

and leads either to the Euler case or the Lagrange case.

Thus a partial first integral of type 3, $F(\omega_2,\gamma_1,\gamma_3)$ does not exist.

8.4.3. Elimination of γ_2 and γ_3. Let us note that the elimination of γ_2 and γ_3 from equations $H_2 = U_2$ and $H_3 = U_3$ is possible only if

$$(c_2, c_3) \neq (0, 0). \tag{8.122}$$

Further we suppose that this condition is always fulfilled.

We start with the case $c_2 \neq 0$ and c_3 is arbitrary. The elimination is made in a similar way like in Sec. 8.4.1. First we express γ_2 from equation $H_3 = U_3$ and put the obtained value of γ_2 in equation $H_2 = U_2$ from where we find γ_3. In this way we have:

$$\gamma_2 = -\frac{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3 + 2c_3R}{2c_2}, \qquad \gamma_3 = R, \tag{8.123}$$

where, if $c_2^2 + c_3^2 \neq 0$, R is a root of equation

$$Q(x) = 4(c_2^2 + c_3^2)x^2 + Bx + C = 0,$$

that is

$$Q(R) = 4(c_2^2 + c_3^2)R^2 + BR + C = 0. \tag{8.124}$$

If $c_2^2 + c_3^2 = 0$ R is a root of equation

$$Q(x) = Bx + C = 0,$$

that is

$$Q(R) = BR + C = 0. \tag{8.125}$$

Functions $B = B(\omega_2, \omega_3, \gamma_1, \gamma_3)$ and $C = C(\omega_2, \omega_3, \gamma_1, \gamma_3)$ are the following polynomials:

$$\begin{aligned} B &= 4c_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3) \\ C &= (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)^2 + 4c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)\gamma_1 \\ &\quad - 2U_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 4(c_1^2 + c_2^2)\gamma_1^2 - 4c_1U_3\gamma_1 - 4c_2^2U_2 + U_3^2. \end{aligned} \tag{8.126}$$

Let us note that if $c_2^2 + c_3^2 = 0$ then $c_3 \neq 0$ because if $c_3 = 0$ the condition (8.122) will not be satisfied. Consequently $B \neq 0$ and therefore (8.125) is well defined.

We put the values of γ_2 and γ_3 from (8.123) in the Euler-Poisson equations (1.1) and remove its fifth and sixth equations. In this way we obtain the following system of four equations in unknowns $\omega_1,\ \omega_2,\ \omega_3$ and γ_1:

$$\begin{aligned} \frac{d\omega_1}{dt} &= \frac{1}{2I_1c_2}\big[-c_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2(I_2 - I_3)c_2\omega_2\omega_3 \\ &\qquad - 2c_1c_3\gamma_1 - 2(c_2^2 + c_3^2)R + c_3U_3\big], \\ \frac{d\omega_2}{dt} &= \frac{1}{I_2}\big[(I_3 - I_1)\omega_1\omega_3 + c_1R - c_3\gamma_1\big], \\ \frac{d\omega_3}{dt} &= \frac{1}{2I_3c_2}\big[c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2(I_1 - I_2)c_2\omega_1\omega_2 \\ &\qquad + c_1(2c_3R - U_3) + 2(c_1^2 + c_2^2)\gamma_1\big], \\ \frac{d\gamma_1}{dt} &= \frac{1}{2c_2}\big[(-I_1\omega_1^2 - I_2\omega_2^2 - I_3\omega_3^2 - 2c_1\gamma_1 - 2c_3R + U_3)\omega_3 - 2c_2\omega_2R\big]. \end{aligned} \tag{8.127}$$

We consider the following four possible types of a first integral of system (8.127) that depends on at most three variables among the variables $\omega_1,\ \omega_2,\ \omega_3$ and γ_1:

1. $F(\omega_1, \omega_2, \omega_3)$, (case(i))
2. $F(\omega_1, \omega_2, \gamma_1)$, (case(iii))
3. $F(\omega_1, \omega_3, \gamma_1)$, (case(iii))
4. $F(\omega_2, \omega_3, \gamma_1)$. (case(iii))

We suppose that the sought first integral is functionally independent of H_1 restricted to invariant manifold (8.97). As the case (iii) was already examined, there remains only to examine case (i).

Type 1. Let us consider a first integral of type 1, i.e. $F(\omega_1, \omega_2, \omega_3)$. We have

$$2I_1I_2I_3c_2\frac{dF}{dt} = Y_1(F) = 0, \tag{8.128}$$

where the vector field Y_1, defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$, is:

$$\begin{aligned} Y_1 = I_2I_3\Big[&- c_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2(I_2 - I_3)c_2\omega_2\omega_3 - 2c_1c_3\gamma_1 \\ &- 2(c_2^2 + c_3^2)R + c_3U_3\Big]\frac{\partial}{\partial\omega_1} + 2I_1I_3c_2\Big[(I_3 - I_1)\omega_1\omega_3 + c_1R - c_3\gamma_1\Big]\frac{\partial}{\partial\omega_2} \\ &+ I_1I_2\Big[c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2(I_1 - I_2)c_2\omega_1\omega_2 \\ &+ c_1(2c_3R - U_3) + 2(c_1^2 + c_2^2)\gamma_1\Big]\frac{\partial}{\partial\omega_3}. \end{aligned}$$

We differentiate identity (8.128) with respect to γ_1 and obtain again a linear partial differential equation for function F

$$\begin{aligned} \frac{1}{2}\frac{\partial Y_1(F)}{\partial\gamma_1} &= -I_2I_3\left[(c_2^2 + c_3^2)\frac{\partial R}{\partial\gamma_1} + c_1c_3\right]\frac{\partial F}{\partial\omega_1} + I_1I_3c_2\left(c_1\frac{\partial R}{\partial\gamma_1} - c_3\right)\frac{\partial F}{\partial\omega_2} \\ &+ I_1I_2\left(c_1c_3\frac{\partial R}{\partial\gamma_1} + c_1^2 + c_2^2\right)\frac{\partial F}{\partial\omega_3} = Y_2(F) = 0, \end{aligned} \tag{8.129}$$

where Y_2 is the corresponding vector field defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

The derivative of $Y_2(F)$ with respect to γ_1 has a factor $\frac{\partial^2 R}{\partial\gamma_1^2}$. Crude computations show that for the two roots of equation (8.124), i.e. when $c_2^2 + c_3^2 \neq 0$, and also for the single root of equation (8.125), i.e. when $c_2^2 + c_3^2 = 0$, one has

$$\frac{\partial^2 R}{\partial\gamma_1^2} \neq 0.$$

In this way differentiating identity (8.129) with respect to γ_1 we obtain

$$\begin{aligned} \left(\frac{\partial^2 R}{\partial\gamma_1^2}\right)^{-1}\frac{\partial Y_2(F)}{\partial\gamma_1} &= -I_2I_3(c_2^2 + c_3^2)\frac{\partial F}{\partial\omega_1} + I_1I_3c_1c_2\frac{\partial F}{\partial\omega_2} + I_1I_2c_1c_3\frac{\partial F}{\partial\omega_3} \\ &= Y_3(F) = 0, \end{aligned} \tag{8.130}$$

where Y_3 is the corresponding vector field defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

Instead of vector field Y_1 we consider $Y_4 = Y_1 - 2RY_3$ which implies that $Y_4(F) = 0$. We obtain

$$Y_4(F) = I_2I_3\Big[- c_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2(I_2 - I_3)c_2\omega_2\omega_3 - 2c_1c_3\gamma_1 + c_3U_3\Big]\frac{\partial F}{\partial\omega_1}$$

$$+ 2I_1I_3c_2\Big[(I_3 - I_1)\omega_1\omega_3 - c_3\gamma_1\Big]\frac{\partial F}{\partial \omega_2} + I_1I_2\Big[c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$
$$+ 2(I_1 - I_2)c_2\omega_1\omega_2 - c_1U_3 + 2(c_1^2 + c_2^2)\gamma_1\Big]\frac{\partial F}{\partial \omega_3} = 0. \tag{8.131}$$

Note that Y_4 does not depend on R.

Instead of vector field Y_2 we consider $Y_5 = Y_2 - Y_3\frac{\partial R}{\partial \gamma_1}$ which also does not depend on R. We have

$$Y_5(F) = -I_2I_3c_1c_3\frac{\partial F}{\partial \omega_1} - I_1I_3c_2c_3\frac{\partial F}{\partial \omega_2} + I_1I_2(c_1^2 + c_2^2)\frac{\partial F}{\partial \omega_3} = 0. \tag{8.132}$$

We compute the Lie bracket $Y_6 = [Y_3, Y_4]/(2I_1I_2I_3)$. We know that $Y_6(F) = 0$ so we have

$$Y_6(F) = \Big[I_2I_3(c_3^2 + c_2^2)c_3\omega_1 + I_2(I_2 - 2I_3)c_1c_2c_3\omega_2 + I_3(I_2c_2^2 - I_3c_2^2 - I_2c_3^2)c_1\omega_3\Big]\frac{\partial F}{\partial \omega_1}$$
$$- (I_1 - I_3)c_2\Big[I_1c_1c_3\omega_1 - I_3(c_3^2 + c_2^2)\omega_3\Big]\frac{\partial F}{\partial \omega_2} + \Big[I_1(I_1c_2^2 - 2I_2c_2^2 - I_2c_3^2)c_1\omega_1$$
$$+ I_2(I_1c_1^2 + I_2c_2^2 - I_1c_2^2 + I_2c_3^2 - I_1c_3^2)c_2\omega_2 + I_2I_1c_1^2c_3\omega_3\Big]\frac{\partial F}{\partial \omega_3} = 0. \tag{8.133}$$

Thus we have obtained four linear homogeneous equations in unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right)$, that is system (8.130)–(8.133). If a first integral F exists, system (8.130)–(8.133) has a non-zero solution. This is possible if and only if

$$\operatorname{rank} M < 3, \tag{8.134}$$

where M is the (4×3) matrix composed from the coefficients of system (8.130)–(8.133).

Let us compute the determinant D_{345} that consists of the coefficients of Y_3, Y_4 and Y_5. It should be identically zero because of requirement (8.134).

We compute D_{345} and obtain

$$D_{345} = -2I_1^2I_2^2I_3^2c_2^2\big[c_3(I_1 - I_2)\omega_1\omega_2 + c_2(I_3 - I_1)\omega_1\omega_3 + c_1(I_2 - I_3)\omega_2\omega_3\big]\delta_{345},$$

where

$$\delta_{345} = c_1^2 + c_2^2 + c_3^2.$$

The expression in the square brackets vanishes identically only in the kinetic symmetry case and in the Lagrange case $I_1 = I_3,\ c_1 = c_3 = 0$. The factor $-2I_1^2I_2^2I_3^2c_2^2 \neq 0$. Thus $D_{345} = 0$ is equivalent to $\delta_{345} = 0$.

Now we compute the determinant D_{346} that consists of the coefficients of Y_3, Y_4 and Y_6. It should be identically equal to zero too (see (8.134)). We have $D_{346} = I_1I_2I_3c_2^2\delta_{346}$, where

$$\begin{aligned}\delta_{346} = &-I_1^3c_1^2c_2c_3(I_2 - I_3)\omega_1^3\\ &- I_1I_2c_1c_3\big[(I_1 - I_2)(2I_1 - 3I_3)(c_2^2 + c_3^2) - I_1(I_2 - I_3)c_1^2\big]\omega_1^2\omega_2\\ &+ I_1I_3c_1c_2\big[(I_1 - I_3)(2I_1 - 3I_2)(c_2^2 + c_3^2) + I_1(I_2 - I_3)c_1^2\big]\omega_1^2\omega_3\\ &+ I_1I_2c_1^2c_2c_3(I_1I_2 - 3I_1I_3 - 2I_2^2 + 4I_2I_3)\omega_1\omega_2^2\\ &+ 2I_1I_2I_3c_1^2\big[(-I_1 + 2I_2 - I_3)c_3^2 + (I_1 + I_2 - 2I_3)c_2^2\big]\omega_1\omega_2\omega_3\\ &+ I_1I_3c_1^2c_2c_3(3I_1I_2 - I_1I_3 - 4I_2I_3 + 2I_3^2)\omega_1\omega_3^2\end{aligned}$$

$$
\begin{aligned}
&- 2I_1^2c_1c_2c_3(c_1^2+c_2^2+c_3^2)(I_2-I_3)\omega_1\gamma_1 + I_1^2(I_2-I_3)c_1^2c_2c_3U_3\omega_1\\
&- I_2^2c_1c_3\big[I_3(I_1-I_2)(c_2^2+c_3^2) - I_1(I_2-I_3)c_1^2\big]\omega_2^3\\
&- I_2I_3c_1c_2\big[(-3I_1I_2+2I_1I_3+2I_2^2-I_2I_3)(c_2^2+c_3^2) + I_1(I_2-I_3)c_1^2\big]\omega_3\omega_2^2\\
&- I_2I_3c_1c_3\big[(3I_1I_3-2I_1I_2+I_2I_3-2I_3^2)(c_2^2+c_3^2) + I_1(I_2-I_3)c_1^2\big]\omega_3^2\omega_2\\
&- 2I_2c_3(c_3^2+c_1^2+c_2^2)\big[I_3(I_1-I_2)(c_2^2+c_3^2) - I_1(I_2-I_3)c_1^2\big]\omega_2\gamma_1\\
&+ I_2c_1c_3U_3\big[I_3(I_1-I_2)(c_2^2+c_3^2) - I_1(I_2-I_3)c_1^2\big]\omega_2\\
&+ I_3^2c_1c_2\big[I_2(I_1-I_3)(c_2^2+c_3^2) + I_1(I_2-I_3)c_1^2\big]\omega_3^3\\
&+ 2I_3c_2(c_1^2+c_2^2+c_3^2)\big[I_2(I_1-I_3)(c_2^2+c_3^2) + I_1(I_2-I_3)c_1^2\big]\omega_3\gamma_1\\
&- I_3c_1c_2U_3\big[I_2(I_1-I_3)(c_2^2+c_3^2) + I_1(I_2-I_3)c_1^2\big]\omega_3.
\end{aligned}
$$

As $I_1I_2I_3c_2^2 \neq 0$ then $D_{346} = 0$ is equivalent to $\delta_{346} = 0$.

Thus we should find the conditions at which polynomials δ_{345} and δ_{346} vanish identically with respect to variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$. This means to find the values of the parameters $\mathcal{I}c$ and U_3 at which all the coefficients of δ_{345} and δ_{346} are zero.

Polynomial δ_{345} has only one coefficient and δ_{346} has 16 coefficients. In this way we obtain a system of 17 equations. To solve it we apply a simplification. At the fourth consecutive simplification we obtain the reduced system:

$$
\begin{aligned}
&c_3^2+c_1^2+c_2^2 = 0, \quad (I_2-I_3)c_1 = 0, \quad (I_1-I_3)c_1 = 0,\\
&(I_2-I_3)(c_2^2+c_3^2) = 0, \quad (I_1-I_3)(c_2^2+c_3^2) = 0.
\end{aligned}
$$

The Maple command `solve` gives two solutions at an arbitrary value of U_3:

$$
\begin{aligned}
&\{I_1 = I_1,\ I_2 = I_2,\ I_3 = I_3,\ c_1 = 0,\ c_2 = c_2,\ c_3 = \mathit{RootOf}(Z^2+1)c_2\}\\
&\{I_1 = I_3,\ I_2 = I_3,\ I_3 = I_3,\ c_1 = \mathit{RootOf}(Z^2+c_2^2+c_3^2),\ c_2 = c_2,\ c_3 = c_3\}.
\end{aligned}
$$

The second solution is a particular case of the kinetic symmetry case so we remove it. We have to consider the first solution. Thus

$$c_1 = 0, \quad c_3 = \varepsilon i c_2,$$

where $\varepsilon = \pm 1$. We consider here only the case $\varepsilon = 1$ because the final result is the same also when $\varepsilon = -1$.

Let us study this case. Now $Y_3(F)$ and $Y_6(F)$ are identically zeros and therefore condition (8.134) is fulfilled. $Y_4(F)$ and $Y_5(F)$ are:

$$
\begin{aligned}
Y_4(F) = &-iI_2I_3c_2\Big[I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2+2i(I_2-I_3)\omega_2\omega_3 - U_3\Big]\frac{\partial F}{\partial \omega_1}\\
&- 2I_1I_3c_2\Big[(I_1-I_3)\omega_1\omega_3 + ic_2\gamma_1\Big]\frac{\partial F}{\partial \omega_2} + 2I_1I_2c_2\Big[(I_1-I_2)\omega_1\omega_2 + c_2\gamma_1\Big]\frac{\partial F}{\partial \omega_3},\\
Y_5(F) = &-I_1c_2^2\left(iI_3\frac{\partial F}{\partial \omega_2} - I_2\frac{\partial F}{\partial \omega_3}\right).
\end{aligned}
$$

We compute

$$Y_7(F) = \frac{[Y_4(F), Y_5(F)]}{2I_1I_2I_3c_2^3} = \Big[I_2(I_2-2I_3)\omega_2 - iI_3(I_3-2I_2)\omega_3\Big]\frac{\partial F}{\partial \omega_1}$$

$$- I_1(I_1 - I_3)\omega_1 \frac{\partial F}{\partial \omega_2} + iI_1(I_1 - I_2)\omega_1 \frac{\partial F}{\partial \omega_3}.$$

We compute the determinant Δ of the coefficients of equations $Y_4(F)$, $Y_5(F)$ and $Y_7(F)$ and obtain $\Delta = I_1^2 I_2 I_3 c_2^3 \omega_1 \widetilde{\Delta}$ where

$$\widetilde{\Delta} = iI_1^2(I_2 - I_3)\omega_1^2 + iI_2(2I_2^2 - 4I_2I_3 - I_1I_2 + 3I_3I_1)\omega_2^2$$
$$- 2I_2I_3(2I_1 - I_2 - I_3)\omega_2\omega_3 - iI_3(2I_3^2 - 4I_2I_3 - I_3I_1 + 3I_1I_2)\omega_3^2 - iI_1(I_2 - I_3)U_3.$$

As the factor $I_1^2 I_2 I_3 c_2^3 \omega_1 \neq 0$, we require that $\widetilde{\Delta} = 0$. Looking at the coefficient of ω_1^2 in the expression for $\widetilde{\Delta}$ we see that $I_2 = I_3$ should be fulfilled. At this condition we obtain

$$\widetilde{\Delta} = 2I_3^2(I_1 - I_3)(\omega_2 + i\omega_3)^2.$$

Thus $\widetilde{\Delta} = 0$ only if $I_1 = I_2 = I_3$, i.e. we come to the kinetic symmetry case. Consequently the sought integral of type 1 does not exist when $c_2 \neq 0$.

Let us consider the case $c_2 = 0$. In this case, according to condition (8.122), $c_3 \neq 0$. First we express γ_3 from equation $H_3 = U_3$ and put the obtained value of γ_3 in equation $H_2 = U_2$ from where we find γ_2. In this way we have:

$$\gamma_2 = \frac{\widehat{R}}{2c_3}, \qquad \gamma_3 = -\frac{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3}{2c_3}, \tag{8.135}$$

where $\widehat{R}$ is a root of equation

$$Q(x) = x^2 + \widehat{C} = 0,$$

that is

$$Q(\widehat{R}) = \widehat{R}^2 + \widehat{C} = 0. \tag{8.136}$$

Function $\widehat{C} = \widehat{C}(\omega_2, \omega_3, \gamma_1, \gamma_3)$ is the following polynomial:

$$\widehat{C} = (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)^2 + 4c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)\gamma_1$$
$$- 2U_3(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 4(c_1^2 + c_3^2)\gamma_1^2 - 4c_1U_3\gamma_1 - 4c_3^2U_2 + U_3^2.$$

We put the values of γ_2 and γ_3 from (8.135) in the Euler-Poisson equations (1.1) and remove its fifth and sixth equations. In this way we obtain the following system of four equations in unknowns ω_1, ω_2, ω_3 and γ_1:

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{2(I_2 - I_3)\omega_2\omega_3 + \widehat{R}}{2I_1}, \\
\frac{d\omega_2}{dt} &= -\frac{1}{2I_2c_3}\big[c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2(I_1 - I_3)c_3\omega_1\omega_3 \\
&\qquad + 2(c_1^2 + c_3^2)\gamma_1 - c_1U_3\big], \\
\frac{d\omega_3}{dt} &= \frac{2(I_1 - I_2)c_3\omega_1\omega_2 - c_1\widehat{R}}{2I_3c_3}, \\
\frac{d\gamma_1}{dt} &= \frac{1}{2c_3}\big[(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + 2c_1\gamma_1 - U_3)\omega_2 + \omega_3\widehat{R}\big].
\end{aligned} \tag{8.137}$$

Let $c_2 = 0$ and function $F(\omega_1, \omega_2, \omega_3)$ be a first integral of type 1. We have

$$2I_1I_2I_3c_3\frac{dF}{dt} = Y_1(F) = 0, \tag{8.138}$$

where the vector field Y_1, defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$, is:

$$\begin{aligned} Y_1 = I_2 I_3 c_3 \Big[2(I_2 - I_3)\omega_2\omega_3 + \widehat{R} \Big] \frac{\partial}{\partial \omega_1} \\ + I_1 I_3 \Big[- c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + 2c_3(I_3 - I_1)\omega_1\omega_3 - 2(c_1^2 + c_3^2)\gamma_1 + c_1 U_3 \Big] \frac{\partial}{\partial \omega_2} \\ + I_1 I_2 \Big[2(I_1 - I_2)c_3\omega_1\omega_2 - c_1 \widehat{R} \Big] \frac{\partial}{\partial \omega_3}. \end{aligned}$$

We differentiate identity (8.138) with respect to γ_1 and obtain again a linear partial differential equation for function F

$$\frac{\partial Y_1(F)}{\partial \gamma_1} = I_2 I_3 c_3 \frac{\partial \widehat{R}}{\partial \gamma_1} \frac{\partial F}{\partial \omega_1} - 2 I_1 I_3 (c_1^2 + c_3^2) \frac{\partial F}{\partial \omega_2} - I_1 I_2 c_1 \frac{\partial \widehat{R}}{\partial \gamma_1} \frac{\partial F}{\partial \omega_3} = Y_2(F) = 0,$$

where Y_2 is the corresponding vector field defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

The derivative of $Y_2(F)$ with respect to γ_1 has a factor $I_2 \frac{\partial^2 \widehat{R}}{\partial \gamma_1^2}$. We have verified that for the two roots of equation (8.136) this derivative is not zero. Thus differentiating identity (8.129) with respect to γ_1 we obtain

$$\frac{1}{I_2} \left(\frac{\partial^2 \widehat{R}}{\partial \gamma_1^2} \right)^{-1} \frac{\partial Y_2(F)}{\partial \gamma_1} = I_3 c_3 \frac{\partial F}{\partial \omega_1} - I_1 c_1 \frac{\partial F}{\partial \omega_3} = Y_3(F) = 0, \tag{8.139}$$

where Y_3 is the corresponding vector field defined on $\mathbb{C}^4(\omega_1, \omega_2, \omega_3, \gamma_1)$.

Instead of vector field Y_1 we consider $Y_4 = Y_1 - I_2 R Y_3$ which implies that $Y_4(F) = 0$. We obtain

$$\begin{aligned} Y_4(F) = 2 I_2 I_3 (I_2 - I_3) c_3 \omega_2 \omega_3 \frac{\partial F}{\partial \omega_1} - I_1 I_3 \Big[c_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \\ + 2(I_1 - I_3)c_3\omega_1\omega_3 + 2(c_1^2 + c_3^2)\gamma_1 - c_1 U_3 \Big] \frac{\partial F}{\partial \omega_2} \\ + 2 I_1 I_2 (I_1 - I_2) c_3 \omega_1 \omega_2 \frac{\partial F}{\partial \omega_3} = 0. \end{aligned} \tag{8.140}$$

Note that Y_4 does not depend on $\widehat{R}$.

Instead of vector field Y_2 we consider

$$Y_5 = \frac{I_2 Y_3 \frac{\partial \widehat{R}}{\partial \gamma_1} - Y_2}{2 I_1 I_3},$$

which also does not depend on $\widehat{R}$. We have

$$Y_5(F) = (c_1^2 + c_3^2) \frac{\partial F}{\partial \omega_2} = 0. \tag{8.141}$$

We compute the Lie bracket $Y_6 = \frac{[Y_3, Y_4]}{2 I_1 I_3}$. We know that $Y_6(F) = 0$ so we have

$$\begin{aligned} Y_6(F) = -I_2(I_2 - I_3)c_1 c_3 \omega_2 \frac{\partial F}{\partial \omega_1} \\ + \Big[I_1(I_1 - 2I_3)c_1 c_3 \omega_1 + I_3(I_1 c_1^2 - I_1 c_3^2 + I_3 c_3^2)\omega_3 \Big] \frac{\partial F}{\partial \omega_2} \\ + I_2(I_1 - I_2)c_3^2 \omega_2 \frac{\partial F}{\partial \omega_3} = 0. \end{aligned} \tag{8.142}$$

Thus we have obtained four linear homogeneous equations in unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}, \frac{\partial F}{\partial \omega_3}\right)$, that is system (8.139)–(8.142). If a first integral F exists, system (8.139)–(8.142) has a non-zero solution. This is possible if and only if

$$\operatorname{rank} M < 3, \tag{8.143}$$

where M is the (4×3) matrix composed from the coefficients of system (8.139)–(8.142).

Let us compute the determinant D_{345} that consists of the coefficients of Y_3, Y_4 and Y_5. It should be identically zero because of requirement (8.143).

We compute D_{345} and obtain

$$D_{345} = -2I_1I_2I_3c_3\omega_2\big[c_3(I_1 - I_2)\omega_1 + c_1(I_2 - I_3)\omega_3\big]\delta_{345},$$

where

$$\delta_{345} = c_1^2 + c_3^2.$$

As now $c_2 = 0$ then according to (8.122), $c_3 \neq 0$. Thus the expression in the square brackets vanishes identically only in the kinetic symmetry case and in the Lagrange case $I_1 = I_3$, $c_1 = c_2 = 0$. The factor $-2I_1I_2I_3c_3\omega_2 \neq 0$. Thus $D_{345} = 0$ is equivalent to $\delta_{345} = 0$.

We compute also the determinant D_{346} that consists of the coefficients of Y_3, Y_4 and Y_6. It should be identically equal to zero too (see (8.134)). We have $D_{346} = I_1I_2I_3c_3\omega_2\delta_{346}$, where

$$\begin{aligned}\delta_{346} = {} & I_1\big[I_1(I_2 - I_3)c_1^2 - (2I_1 - 3I_3)(I_1 - I_2)c_3^2\big]c_1\omega_1^2 - 2I_1I_3(I_1 + I_3 - 2I_2)c_1^2c_3\omega_1\omega_3 \\ & + I_2\big[I_3(I_2 - I_1)c_3^2 + I_1(I_2 - I_3)c_1^2\big]c_1\omega_2^2 - I_3\big[I_1(I_2 - I_3)c_1^2 + I_3(3I_1 - 2I_3)c_3^2 \\ & + I_2(I_3 - 2I_1)c_3^2\big]c_1\omega_3^2 + 2(c_1^2 + c_3^2)\big[I_1(I_2 - I_3)c_1^2 + I_3(I_2 - I_1)c_3^2\big]\gamma_1 \\ & - c_1\big[I_1(I_2 - I_3)c_1^2 + I_3(I_2 - I_1)c_3^2\big]U_3.\end{aligned}$$

As $I_1I_2I_3c_3\omega_2 \neq 0$ then $D_{346} = 0$ is equivalent to $\delta_{346} = 0$.

Thus we should find the conditions at which polynomials δ_{345} and δ_{346} vanish identically with respect to variables $(\omega_1, \omega_2, \omega_3, \gamma_1)$. This means to find the values of the parameters $\mathcal{I}c$ and U_3 at which all the coefficients of δ_{345} and δ_{346} are zero.

Polynomial δ_{345} has only one coefficient and δ_{346} has six coefficients. In this way we obtain a system of seven equations. To solve it we apply a simplification. At the third consecutive simplification we obtain the reduced system:

$$I_2 - I_3 = 0, \quad I_1 - I_3 = 0 \quad c_1^2 + c_3^2 = 0.$$

This system obviously lead to the kinetic symmetry case. Thus the sought integral of type 1 does not exist also when $c_2 = 0$.

8.4.4. First integrals $\boldsymbol{F(\gamma_1, \gamma_2, \gamma_3)}$. Finally it remains to study the existence of the partial first integral $F(\gamma_1, \gamma_2, \gamma_3)$, that cannot be studied by elimination of variables like above.

We proceed here in the same way as in Sec. 8.2.4.

$F(\gamma_1, \gamma_2, \gamma_3) = \widetilde{F}(\gamma_2, \gamma_3)$. Our problem now is reduced to the study of partial first integrals of the form $\widetilde{F} = \widetilde{F}(\gamma_2, \gamma_3)$ on the submanifold $\{H_3 = U_3\}$. Absence of these partial first integrals follows from Sec. 8.4.1 where the absence of partial first integrals of more general form $F(\omega_i, \gamma_2, \gamma_3)$, $i = 2, 3$, is proved for all U_2 and U_3.

This concludes the description of the four-dimensional invariant manifolds.

9. Three-dimensional invariant manifold $\{H_1{=}U_1,\ H_2{=}U_2,\ H_3{=}U_3\}$

9.1. Extraction procedure. In this section we study the existence of a local partial first integral of the Euler-Poisson equations (1.1) restricted to the invariant complex three-dimensional level manifold

$$\{H_1 = U_1,\ H_2 = U_2,\ H_3 = U_3\},$$

which depends on at most two variables.

According to (2.5)

$$M(U_0, U_1, U_2, U_3, \mathcal{I}c) =$$
$$= \{x \in \mathbb{C}^6;\ H_1((\omega, \gamma), \mathcal{I}c) = U_1,\ H_2((\omega, \gamma), \mathcal{I}c) = U_2,\ H_3((\omega, \gamma), \mathcal{I}c) = U_3\},$$

where $(\omega, \gamma) = (\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3)$.

We search all functions F of two variables $F = F(s_1, s_2)$ where $(s_1, s_2) \in (\omega, \gamma)$, of class $\mathbb{C}^1$, such that $\operatorname{grad} F$ does not vanish identically on each open subset of the manifold $M(U_0, U_1, U_2, U_3, \mathcal{I}c)$, which are local partial first integrals of the Euler-Poisson equations (1.1) restricted to this manifold.

As in Sec. 5.1 the order of variables s_i, $1 \le i \le 2$, in $F(s_1, s_2)$ is irrelevant for F to be a first integral.

We have exactly 15 different two elements subsets of (ω, γ) and thus 15 cases of functions of two elements to examine. We will describe now an extraction procedure based on permutational symmetries which reduces the above 15 cases to only four.

These 15 functions of two variables (up to the order of variables) are shown in Table 9.1. This Table can be easily obtained directly like Table 5.1. But it can be also easily deduced from Table 5.1 and reciprocally.

Table 9.1

Functions	Case
$F(\gamma_i, \gamma_j)$, $1 \le i < j \le 3$	(i)
$F(\omega_1, \gamma_1)$, $F(\omega_2, \gamma_2)$, $F(\omega_3, \gamma_3)$	(ii)
$F(\omega_3, \gamma_2)$, $F(\omega_2, \gamma_3)$, $F(\omega_1, \gamma_3)$, $F(\omega_3, \gamma_1)$, $F(\omega_2, \gamma_1)$, $F(\omega_1, \gamma_2)$	(iii)
$F(\omega_i, \omega_j)$, $1 \le i < j \le 3$	(iv)

Like in Sec. 8, let us stress that the permutational symmetries act on variables

(ω, γ) and parameters $\mathcal{I}c$ but not on the constants U_1, U_2, U_3 that define the manifold $M(U_0, U_1, U_2, U_3, \mathcal{I}c)$.

It is easy to see that under the group of permutational symmetries (2.3) of the Euler-Poisson equations for every case (i)–(iv) from Table 9.1 the first function from the case is consequently transformed into all remaining functions from the same case.

Thus in virtue of Theorem 2.2 we can restrict ourselves to the study of only four functions where every one belongs to a different case from Table 9.1 and is chosen arbitrary from the functions of this case.

Like in Secs. 5 and 8, we will call such four functions F_i, $1 \le i \le 4$, (up to the order of variables) a basis.

9.2. Elimination of ω_1,ω_2,γ_1. Here we study the existence of a partial first integral of the Euler-Poisson equations (1.1) after expressing variables ω_1, ω_2 and γ_1 from equations

$$H_i = U_i, \quad 1 \le i \le 3. \tag{9.1}$$

First we express γ_1 from second equation of (9.1) and obtain

$$\gamma_1 = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}. \tag{9.2}$$

Further, to simplify the notations, we put

$$\Gamma = \sqrt{-\gamma_2^2 - \gamma_3^2 + U_2}.$$

Then, using the Maple command `solve`, we express ω_1 and ω_2 from first and third equations of (9.1) and obtain the following solution:

$$\omega_1 = R, \quad \omega_2 = -\frac{I_1 R\Gamma + I_3\omega_3\gamma_3 - U_1}{I_2\gamma_2}, \tag{9.3}$$

where $R = R(\omega_3, \gamma_2, \gamma_3)$ is a root of equation

$$Q(x) = Ax^2 + Bx + C = 0,$$

that is

$$Q(R) = AR^2 + BR + C = 0. \tag{9.4}$$

Here $A = A(\gamma_2, \gamma_3)$, $B = B(\omega_3, \gamma_2, \gamma_3)$ and $C = C(\omega_3, \gamma_2, \gamma_3)$ are the following functions:

$$\begin{aligned} A &= I_1\left[(I_2 - I_1)\gamma_2^2 - I_1\gamma_3^2 + I_1 U_2\right], \\ B &= 2I_1\Gamma(I_3\omega_3\gamma_3 - U_1), \\ C &= I_3\omega_3^2(I_2\gamma_2^2 + I_3\gamma_3^2) - 2I_3\omega_3\gamma_3 U_1 \\ &\quad + I_2\gamma_2^2(2c_2\gamma_2 + 2c_3\gamma_3 + 2c_1\Gamma - U_3) + U_1^2. \end{aligned} \tag{9.5}$$

R, A, B and C are algebraic functions defined on $\mathbb{C}^3(\omega_3, \gamma_2, \gamma_3)$.

We put the values of γ_1, ω_1 and ω_2 from (9.2) and (9.3) in the Euler-Poisson equations (1.1) and remove the first, second and fourth equations. In this way we have the following system of three equations in unknowns ω_3, γ_2 and γ_3:

$$\begin{aligned}
\frac{d\omega_3}{dt} &= \frac{1}{I_2 I_3 \gamma_2}\Big[I_1(I_2 - I_1)\Gamma R^2 + (I_2 - I_1)(I_3\omega_3\gamma_3 - U_1)R + I_2\gamma_2(c_2\Gamma - c_1\gamma_2)\Big],\\
\frac{d\gamma_2}{dt} &= \gamma_3 R - \omega_3\Gamma,\\
\frac{d\gamma_3}{dt} &= \frac{1}{I_2\gamma_2}\Big[(I_1\gamma_2^2 - I_2\gamma_2^2 + I_1\gamma_3^2 - I_1U_2)R - \Gamma(I_3\omega_3\gamma_3 - U_1)\Big].
\end{aligned} \tag{9.6}$$

Now we study whether system (9.6) has a first integral that depends on at most two variables among the variables $(\omega_3, \gamma_2, \gamma_3)$. Thus we should investigate the following three types of a first integral:

1. $F(\gamma_2, \gamma_3)$, (case (i))
2. $F(\omega_3, \gamma_3)$, (case (ii))
3. $F(\omega_3, \gamma_2)$. (case (iii))

Then, like in Secs. 5 and 8 we should examine the three types given above because they belong to different cases (see Table 9.1).

Let us fix $U_2 \in \mathbb{C}$. Let us consider some suitable open set $\Omega \subseteq \mathbb{C}^3(\omega_3, \gamma_2, \gamma_3)$ belonging to the domain of definition of F.

From the now we consider system (9.6) and the first integral F only on Ω. System (9.6) restricted to Ω has C^1 right-hand sides.

We always suppose that the considered first integrals are not constant on any open subset of their domain of definition. As we consider C^1 first integrals, this means that their gradients do not vanish identically on any open subset of their domain of definition.

Type 1. Let us consider the existence of a first integral F of system (9.6) which is of type 1, i.e. $F = F(\gamma_2, \gamma_3)$. Thus we have

$$\frac{dF}{dt} = \frac{d\gamma_2}{dt}\frac{\partial F}{\partial \gamma_2} + \frac{d\gamma_3}{dt}\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \tag{9.7}$$

where Y_1 is the corresponding vector field defined on Ω.

Equation (9.7) should be an identity with respect to all the three variables $(\omega_3, \gamma_2, \gamma_3)$. As function F does not depend on ω_3 then its partial derivatives will not depend on ω_3 too. Thus if we differentiate identity (9.7) with respect to ω_3 we shall obtain again a linear partial differential equation for function F.

$$\frac{\partial Y_1(F)}{\partial \omega_3} = \frac{\partial}{\partial \omega_3}\left(\frac{d\gamma_2}{dt}\right)\frac{\partial F}{\partial \gamma_2} + \frac{\partial}{\partial \omega_3}\left(\frac{d\gamma_3}{dt}\right)\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \tag{9.8}$$

where Y_2 is the corresponding vector field defined on Ω.

Equations (9.7) and (9.8) can be considered as a system of two homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \gamma_2}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if first integral F exists, system (9.7)–(9.8) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if determinant Δ of this linear system satisfies identity $\Delta \equiv 0$ provided that R is a root of equation (9.4).

We compute this determinant and obtain

$$\Delta(R) = \frac{d\gamma_2}{dt}\frac{\partial}{\partial\omega_3}\left(\frac{d\gamma_3}{dt}\right) - \frac{d\gamma_3}{dt}\frac{\partial}{\partial\omega_3}\left(\frac{d\gamma_2}{dt}\right)$$
$$= \frac{\Gamma}{I_2\gamma_2}\left\{\left[(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1U_2\right]\left(R - \omega_3\frac{\partial R}{\partial\omega_3}\right) + U_1\gamma_3\frac{\partial R}{\partial\omega_3} - U_1\Gamma\right\}.$$

As we are interested in case $\Delta = 0$ we remove the denominator $I_2\gamma_2$ and the non-zero factor Γ and note

$$\Delta(R) = \frac{\Gamma}{I_2\gamma_2}\delta(R),$$

where

$$\delta(R) = \left[(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1U_2\right]\left(R - \omega_3\frac{\partial R}{\partial\omega_3}\right) - U_1\gamma_3\frac{\partial R}{\partial\omega_3} + U_1\Gamma. \quad (9.9)$$

As $\delta(R)$ contains $\frac{\partial R}{\partial\omega_3}$ we should determine this derivative. For the purpose we use equation (9.4). We differentiate it with respect to ω_3 and, as A does not depend on ω_3 (see (9.5)), obtain

$$\frac{\partial Q}{\partial\omega_3} = (2AR + B)\frac{\partial R}{\partial\omega_3} + \frac{\partial B}{\partial\omega_3}R + \frac{\partial C}{\partial\omega_3} = 0. \quad (9.10)$$

The determination of $\frac{\partial R}{\partial\omega_3}$ from the last equation is possible only if $\frac{dQ(R)}{dR} = 2AR+B \neq 0$. Using Proposition 4.1 we prove that if R is a root of equation (9.4) then $2AR + B = 0$ only in a very particular case

$$I_1 = I_2 = I_3, \quad c_1 = c_2 = c_3 = 0, \quad U_1 = U_2 = U_3 = 0. \quad (9.11)$$

Indeed, let us compute the resultant ρ of $Q(R)$ and $2AR + B$ with respect to R. We obtain

$$\rho = A(4AC - B^2).$$

As we are interested only in the cases when ρ vanishes identically with respect to ω_3, γ_2 and γ_3 and as A never vanishes identically we consider $\widehat{\rho} = 4AC - B^2$ instead of ρ. We compute $\widehat{\rho}$ with values of A, B and C from (9.5) and obtain

$$\widehat{\rho} = 4I_1I_2\gamma_2^2\left(a_0\Gamma + a_1\right),$$

where

$$\begin{aligned} a_0 &= 2c_1(I_2\gamma_2^2 - I_1\gamma_2^2 - I_1\gamma_3^2 + I_1U_2), \\ a_1 &= -I_3(I_1 - I_2)\omega_3^2\gamma_2^2 - I_3(I_1 - I_3)\omega_3^2\gamma_3^2 + I_1I_3U_2\omega_3^2 - 2I_3U_1\omega_3\gamma_3 \\ &\quad - 2(I_1 - I_2)c_2\gamma_2^3 - 2(I_1 - I_2)c_3\gamma_2^2\gamma_3 + (I_1 - I_2)U_3\gamma_2^2 - 2I_1c_2\gamma_2\gamma_3^2 \\ &\quad + 2I_1c_2U_2\gamma_2 - 2I_1c_3\gamma_3^3 + I_1U_3\gamma_3^2 + 2I_1c_3U_2\gamma_3 + U_1^2 - I_1U_2U_3. \end{aligned}$$

According to Proposition 4.3, if $\widehat{\rho} = 0$ then $a_0 = a_1 = 0$ because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. $a_0 = 0$ is possible if and only if $c_1 = 0$. One immediately sees that $a_1 = 0$ will be true if and only if $I_1 = I_2 = I_3$, $c_2 = c_3 = 0$ and $U_1 = U_2 = U_3 = 0$, i.e. we come to condition (9.11). Thus out of this case equations $Q(R) = 0$ and $2AR + B = 0$ have no common roots, i.e. if $Q(R) = 0$ then $2AR + B \neq 0$.

Thus the determination of $\frac{\partial R}{\partial \omega_3}$ from (9.10) is possible and we obtain

$$\frac{\partial R}{\partial \omega_3} = -\frac{\frac{\partial B}{\partial \omega_3} R + \frac{\partial C}{\partial \omega_3}}{2AR + B}$$

and put it in the expression (9.9) for $\delta(R)$. The non-zero expression $2AR + B$ appears as a denominator of $\delta(R)$ and we note

$$\delta(R) = \frac{\widetilde{\delta}(R)}{2AR + B},$$

where

$$\widetilde{\delta}(R) = \left[(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1 U_2\right] \left[(2AR + B)R + \omega_3 \left(R\frac{\partial B}{\partial \omega_3} + \frac{\partial C}{\partial \omega_3}\right)\right] + U_1\gamma_3 \left(R\frac{\partial B}{\partial \omega_3} + \frac{\partial C}{\partial \omega_3}\right) + U_1(2AR + B)\Gamma.$$

After substituting A, B and C with their values from (9.5) we obtain

$$\begin{aligned} \widetilde{\delta}(R) = 2\Big\{ & \left[(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1 U_2\right] \Big[I_1(I_1U_2 + I_2\gamma_2^2 - I_1\gamma_2^2 - I_1\gamma_3^2)R^2 \\ & + 2I_1(I_3\omega_3\gamma_3 - U_1)R\Gamma\Big] + I_2I_3(I_1 - I_2)\omega_3^2\gamma_2^4 + I_3(I_1I_2 + I_1I_3 - 2I_2I_3)\omega_3^2\gamma_2^2\gamma_3^2 \\ & - I_1I_2I_3U_2\omega_3^2\gamma_2^2 + I_3^2(I_1 - I_3)\omega_3^2\gamma_3^4 - I_1I_3^2U_2\omega_3^2\gamma_3^2 - 2I_3(I_1 - I_2)U_1\omega_3\gamma_2^2\gamma_3 \\ & - 2I_3(I_1 - I_3)U_1\omega_3\gamma_3^3 + 2I_1I_3U_1U_2\omega_3\gamma_3 + I_1U_1^2\gamma_2^2 + (I_1 - I_3)U_1^2\gamma_3^2 - I_1U_1^2U_2\Big\}. \end{aligned}$$

Let us note the following observation. The expression in square brackets above, i.e.

$$I_1(I_1U_2 + I_2\gamma_2^2 - I_1\gamma_2^2 - I_1\gamma_3^2)R^2 + 2I_1(I_3\omega_3\gamma_3 - U_1)R\Gamma = AR^2 + BR = Q(R) - C$$

and, as $Q(R) = 0$ (cf. (9.4)), we replace this expression with $-C$. In this way we obtain $\widetilde{\delta}$ as a function that does not depend on R as follows:

$$\widetilde{\delta} = b_0\Gamma + b_1,$$

where

$$\begin{aligned} b_0 &= -4I_2c_1\gamma_2^2\Big[(I_1 - I_2)\gamma_2^2 + (I_1 - I_3)\gamma_3^2 - I_1U_2\Big], \\ b_1 &= 2I_2\gamma_2^2\Big[2c_2(I_2 - I_1)\gamma_2^3 + 2c_3(I_2 - I_1)\gamma_2^2\gamma_3 + 2c_2(I_3 - I_1)\gamma_2\gamma_3^2 + 2c_3(I_3 - I_1)\gamma_3^3 \\ &\quad + (I_1 - I_2)U_3\gamma_2^2 + (I_1 - I_3)U_3\gamma_3^2 + 2I_1c_2U_2\gamma_2 + 2I_1c_3U_2\gamma_3 + U_1^2 - I_1U_2U_3\Big]. \end{aligned}$$

According to Proposition 4.3, if $\widetilde{\delta} = 0$ then $b_0 = b_1 = 0$ because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. $b_0 = 0$ is possible either if $I_1 = I_2 = I_3$ and $U_2 = 0$ which is a particular case of the kinetic symmetry or when $c_1 = 0$.

Let $c_1 = 0$. We consider $b_1 = 0$. As $c_1 = 0$ we should have $(c_2, c_3) \neq (0, 0)$ to avoid the Euler case. First let us suppose that $c_2 \neq 0$. Then the annulment of the coefficients of γ_2^3 and $\gamma_2\gamma_3^2$ of b_1 leads to the kinetic symmetry case. Let us suppose now that $c_3 \neq 0$. Then the coefficients of γ_3^3 and $\gamma_2^2\gamma_3$ lead to the same case. Consequently the sought partial first integral of type 1 does not exist.

Type 2. Let us study the existence of a first integral of type 2. That means to look for a first integral of system (9.6) which does not depend on γ_2, i.e. $F(\omega_3, \gamma_3)$.

In fact the investigations go along the same lines but, of course, the expressions are different. Now we have

$$\frac{dF}{dt} = \frac{d\omega_3}{dt}\frac{\partial F}{\partial \omega_3} + \frac{d\gamma_3}{dt}\frac{\partial F}{\partial \gamma_3} = Y_1(F) = 0, \tag{9.12}$$

where Y_1 is the corresponding vector field defined on Ω.

Equation (9.12) should be an identity with respect to all three variables $(\omega_3, \gamma_2, \gamma_3)$. As function F does not depend on γ_2 then its partial derivatives will not depend on γ_2 too. Thus if we differentiate identity (9.12) with respect to γ_2 we shall obtain again a linear partial differential equation for function F.

$$\frac{\partial Y_1(F)}{\partial \gamma_2} = \frac{\partial}{\partial \gamma_2}\left(\frac{d\omega_3}{dt}\right)\frac{\partial F}{\partial \omega_3} + \frac{\partial}{\partial \gamma_2}\left(\frac{d\gamma_3}{dt}\right)\frac{\partial F}{\partial \gamma_3} = Y_2(F) = 0, \tag{9.13}$$

where Y_2 is the corresponding vector field defined on Ω.

Equations (9.12) and (9.13) can be considered as a system of two homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_3}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if integral F exists, system (9.12)–(9.13) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if determinant $\Delta(R)$ composed of the coefficients of this system satisfies identity $\Delta(R) \equiv 0$ provided that R is a root of equation (9.4).

We compute this determinant and obtain a long expression which we do not show here. We only mention that $\Delta(R)$ has a non-zero denominator $I_2^2 I_3 \gamma_2^2 \Gamma$ and we note

$$\Delta(R) = \frac{\widehat{\Delta}(R)}{I_2^2 I_3 \gamma_2^2 \Gamma}.$$

Thus $\Delta(R) = 0$ is equivalent to $\widehat{\Delta}(R) = 0$.

$\widehat{\Delta}(R)$ depends on $\frac{\partial R}{\partial \gamma_2}$. To determine this derivative we use the same steps as in the case of the first integral of type 1 and obtain

$$\frac{\partial R}{\partial \gamma_2} = -\frac{\frac{\partial A}{\partial \gamma_2}R^2 + \frac{\partial B}{\partial \gamma_2}R + \frac{\partial C}{\partial \gamma_2}}{2AR + B}.$$

We put it in the expression for $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR + B$ appears as a denominator of $\widehat{\Delta}(R)$ and we note

$$\widehat{\Delta}(R) = \frac{\widetilde{\Delta}(R)}{2AR + B}.$$

The identity $\widehat{\Delta}(R) = 0$ is equivalent to $\widetilde{\Delta}(R) = 0$ but $\widetilde{\Delta}(R)$ depends on the functions A, B and C from (9.5) and their derivatives with respect to γ_2. We put these functions in the expression for $\widetilde{\Delta}(R)$ and obtain that $\widetilde{\Delta}(R)$ has a denominator Γ. We note

$$\widetilde{\Delta}(R) = \frac{\delta(R)}{\Gamma}.$$

The identity $\widetilde{\Delta}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_3, \gamma_3)$ of system (9.6) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_3, \gamma_2, \gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using MAPLE we divide δ by Q and obtain a remainder which is a polynomial r of the form

$$r = \frac{r_0 x + r_1}{[(I_2 - I_1)\gamma_2^2 - I_1\gamma_3^2 + I_1 U_2]^2},$$

where

$$r_0 = r_{01}\Gamma + r_{02} \text{ and } r_1 = r_{11}\Gamma + r_{12},$$

Here r_{01}, r_{02}, r_{11} and r_{12} are polynomials of variables ω_3, γ_2, γ_3 and parameters $\mathcal{I}c$ and U_i, $1 \le i \le 3$.

According to Propositions 4.2 we have $r_0 = r_1 = 0$. Then by Propositions 4.3 we conclude that $r_{01} = r_{02} = r_{11} = r_{12} = 0$ because $\Gamma \notin \mathbb{C}(\gamma_2, \gamma_3)$. It turns out that for our aims equation $r_{11} = 0$ is sufficient. Equation $r_{11} = 0$ will be identically satisfied if and only if all the coefficients of polynomial r_{11} are zero. The coefficients of r_{11} are 109. We should find all values of the parameters $\mathcal{I}c$ and U_i, $1 \le i \le 3$, for which the 109 coefficients are zero. At the fourth consecutive simplification we obtain the reduced system of only three very simple equations:

$$I_1 - I_2 = 0, \quad c_1 = 0, \quad c_2 = 0$$

and the values of U_i, $1 \le i \le 3$, I_2, I_3 and c_3 are arbitrary. It is clear that this is the Lagrange case.

Thus the sought partial first integral of type 2, $F(\omega_3, \gamma_3)$ does not exist.

Type 3. Let us consider the existence of a first integral of type 3, i.e. $F(\omega_3, \gamma_2)$. Now we have

$$\frac{dF}{dt} = \frac{d\omega_3}{dt}\frac{\partial F}{\partial \omega_3} + \frac{d\gamma_2}{dt}\frac{\partial F}{\partial \gamma_2} = Y_1(F) = 0, \tag{9.14}$$

where Y_1 is the corresponding vector field defined on Ω.

Equation (9.14) should be an identity with respect to all three variables $(\omega_3, \gamma_2, \gamma_3)$. As function F does not depend on γ_3 then its partial derivatives will not depend on γ_3 too. Thus if we differentiate identity (9.14) with respect to γ_3 we shall obtain again a linear partial differential equation for function F.

$$\frac{\partial Y_1(F)}{\partial \gamma_2} = \frac{\partial}{\partial \gamma_3}\left(\frac{d\omega_3}{dt}\right)\frac{\partial F}{\partial \omega_3} + \frac{\partial}{\partial \gamma_3}\left(\frac{d\gamma_2}{dt}\right)\frac{\partial F}{\partial \gamma_2} = Y_2(F) = 0, \tag{9.15}$$

where Y_2 is the corresponding vector field defined on Ω.

Equations (9.14) and (9.15) can be considered as a system of two homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_3}, \frac{\partial F}{\partial \gamma_2}\right)$, which do not vanish identically, because F is non-constant on any open subset of its domain of definition.

Thus, if integral F exists, system (9.14)–(9.15) has a non-zero solution $\operatorname{grad} F$. This is possible if and only if determinant $\Delta(R)$ composed of the coefficients of this system satisfies identity $\Delta(R) \equiv 0$ provided that R is a root of equation (9.4).

We compute this determinant and obtain

$$\Delta(R) = \frac{\widehat{\Delta}(R)}{I_2 I_3 \gamma_2 \Gamma},$$

where

$$\begin{aligned}\widehat{\Delta}(R) = &\left[-2I_1(I_1-I_2)\omega_3\Gamma^2\frac{\partial R}{\partial\gamma_3}R - I_2c_1\gamma_2^2\gamma_3\frac{\partial R}{\partial\gamma_3} + (I_1-I_2)U_1R^2 - I_2c_1\gamma_2^2R\right]\Gamma\\ &+ I_1(I_1-I_2)\gamma_3\Gamma^2\frac{\partial R}{\partial\gamma_3}R^2 + \Gamma^2\left[I_3(I_2-I_1)\omega_3^2\gamma_3 + (I_1-I_2)U_1\omega_3 + I_2c_2\gamma_2\gamma_3\right]\frac{\partial R}{\partial\gamma_3}\\ &+ \left[I_3(I_1-I_2)\omega_3^2\gamma_2^2 - I_3(I_1-I_2)U_2\omega_3^2 + (I_1-I_2)U_1\omega_3\gamma_3 + I_2c_2\gamma_2(-\gamma_2^2+U_2)\right]R\\ &- I_1(I_1-I_2)(-\gamma_2^2+U_2)R^3 - I_2c_1\omega_3\gamma_2^2\gamma_3.\end{aligned}$$

Thus $\Delta(R)=0$ is equivalent to $\widehat{\Delta}(R)=0$.

$\widehat{\Delta}(R)$ depends on $\frac{\partial R}{\partial\gamma_3}$. To determine this derivative we use the same steps as in the case of the first integral of type 1 and obtain

$$\frac{\partial R}{\partial\gamma_3} = -\frac{\frac{\partial A}{\partial\gamma_3}R^2 + \frac{\partial B}{\partial\gamma_3}R + \frac{\partial C}{\partial\gamma_3}}{2AR+B}.$$

We put it in the expression for $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR+B$ appears as a denominator of $\widehat{\Delta}(R)$ and we note

$$\widehat{\Delta}(R) = \frac{\widetilde{\Delta}(R)}{2AR+B}.$$

The identity $\widehat{\Delta}(R)=0$ is equivalent to $\widetilde{\Delta}(R)=0$ but $\widetilde{\Delta}(R)$ depends on the functions A, B and C from (9.5) and their derivatives with respect to γ_3. We put these functions in the expression for $\widetilde{\Delta}(R)$ and obtain that $\widetilde{\Delta}(R)$ has a denominator Γ. We note

$$\widetilde{\Delta}(R) = \frac{\delta(R)}{\Gamma},$$

where $\delta(R)$ is a polynomial of R of degree four.

The identity $\widetilde{\Delta}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_3,\gamma_3)$ of system (9.6) exists, then also $\delta(R)=0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_3,\gamma_2,\gamma_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using Maple we divide δ by Q and obtain a remainder which is a polynomial r of the form

$$r = \frac{r_0x + r_1}{[(I_2-I_1)\gamma_2^2 - I_1\gamma_3^2 + I_1U_2]^3},$$

where

$$r_0 = r_{01}\Gamma + r_{02} \text{ and } r_1 = r_{11}\Gamma + r_{12},$$

Here r_{01}, r_{02}, r_{11} and r_{12} are polynomials of variables ω_3, γ_2, γ_3 and parameters $\mathcal{I}c$ and U_i, $1\le i\le 3$.

According to Propositions 4.2 we have $r_0 = r_1 = 0$. Then by Propositions 4.3 we conclude that $r_{01} = r_{02} = r_{11} = r_{12} = 0$ because $\Gamma \notin \mathbb{C}(\gamma_2,\gamma_3)$. It turns out that for our aims equation $r_{11} = 0$ is sufficient. Equation $r_{11} = 0$ will be identically satisfied if and only if all the coefficients of polynomial r_{11} are zero. The coefficients of r_{11} are 179. We should find all values of the parameters $\mathcal{I}c$ and U_i, $1 \leq i \leq 3$, for which the 179 coefficients are zero. At the three consecutive simplification we obtain the reduced system of seven equations:

$$I_1 - I_2 = 0, \quad c_2c_3 = 0, \quad (I_2 - I_3)c_2 = 0, \quad c_2U_3 = 0,$$
$$c_2U_2 = 0, \quad c_2U_1 = 0, \quad c_1^2 + 2c_2^2 = 0.$$

We solve this system by the MAPLE command `solve` and obtain two solutions. The first of them gives the Lagrange case $I_1 = I_2$, $c_1 = 0$, $c_2 = 0$ and the second - a particular case of the kinetic symmetry case.

Thus the sought partial first integral of type 3, $F(\omega_3, \gamma_2)$ does not exist.

9.3. Elimination of $\gamma_1, \gamma_2, \gamma_3$. Using the MAPLE command `solve` we determine variables γ_1 and γ_2 from equations $H_1 = U_1$ and $H_3 = U_3$ (see (9.1)). Then we put the obtained values of γ_1 and γ_2 in the equation $H_2 = U_2$ from where we determine γ_3. In this way we obtain the following solution:

$$\begin{aligned} \gamma_1 &= \frac{I_2\omega_2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 - U_3) + 2c_2U_1 + 2(I_2c_3\omega_2 - I_3c_2\omega_3)R}{2(I_1c_2\omega_1 - I_2c_1\omega_2)}, \\ \gamma_2 &= -\frac{I_1\omega_1(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 - U_3) + 2c_1U_1 + 2(I_1c_3\omega_1 - I_3c_1\omega_3)R}{2(I_1c_2\omega_1 - I_2c_1\omega_2)}, \\ \gamma_3 &= R, \end{aligned} \tag{9.16}$$

where $R = R(\omega_1, \omega_2, \omega_3)$ is a root of equation

$$Q(x) = Ax^2 + Bx + C = 0,$$

that is

$$Q(R) = AR^2 + BR + C = 0, \tag{9.17}$$

and $A = A(\omega_1, \omega_2, \omega_3)$, $B = B(\omega_1, \omega_2, \omega_3)$ and $C = C(\omega_1, \omega_2, \omega_3)$ are the following polynomials:

$$\begin{aligned} A &= 4I_1^2(c_2^2 + c_3^2)\omega_1^2 - 8I_1I_2c_1c_2\omega_1\omega_2 - 8I_1I_3c_1c_3\omega_1\omega_3 \\ &\quad + 4I_2^2(c_1^2 + c_3^2)\omega_2^2 - 8I_2I_3c_2c_3\omega_2\omega_3 + 4I_3^2(c_1^2 + c_2^2)\omega_3^2, \\ B &= 4I_1^3c_3\omega_1^4 - 4I_1^2I_3c_1\omega_1^3\omega_3 + 4I_1I_2(I_1 + I_2)c_3\omega_1^2\omega_2^2 - 4I_1I_2I_3c_2\omega_1^2\omega_2\omega_3 \\ &\quad + 4I_1^2I_3c_3\omega_1^2\omega_3^2 - 4I_1I_2I_3c_1\omega_1\omega_2^2\omega_3 - 4I_1I_3^2c_1\omega_1\omega_3^3 + 4I_2^3c_3\omega_2^4 \\ &\quad - 4I_2^2I_3c_2\omega_2^3\omega_3 + 4I_2^2I_3c_3\omega_2^2\omega_3^2 - 4I_2I_3^2c_2\omega_2\omega_3^3 - 4I_1^2c_3U_3\omega_1^2 \\ &\quad + 4I_1I_3c_1U_3\omega_1\omega_3 - 4I_2^2c_3U_3\omega_2^2 + 4I_2I_3c_2U_3\omega_2\omega_3 \\ &\quad + 8I_1c_1c_3U_1\omega_1 + 8I_2c_2c_3U_1\omega_2 - 8I_3(c_1^2 + c_2^2)U_1\omega_3, \\ C &= I_1^4\omega_1^6 + I_1^2I_2(2I_1 + I_2)\omega_1^4\omega_2^2 + 2I_1^3I_3\omega_1^4\omega_3^2 + I_1I_2^2(I_1 + 2I_2)\omega_1^2\omega_2^4 \\ &\quad + 2I_1I_2I_3(I_1 + I_2)\omega_1^2\omega_2^2\omega_3^2 + I_1^2I_3^2\omega_1^2\omega_3^4 + I_2^4\omega_2^6 + 2I_2^3I_3\omega_2^4\omega_3^2 + I_3^2I_2^2\omega_2^2\omega_3^4 \\ &\quad - 2I_1^3U_3\omega_1^4 - 2I_1I_2(I_1 + I_2)U_3\omega_1^2\omega_2^2 - 2I_1^2I_3U_3\omega_1^2\omega_3^2 - 2I_2^3U_3\omega_2^4 \\ &\quad - 2I_2^2I_3U_3\omega_2^2\omega_3^2 + 4I_1^2c_1U_1\omega_1^3 + 4I_1I_2c_2U_1\omega_1^2\omega_2 + 4I_1I_2c_1U_1\omega_1\omega_2^2 \end{aligned} \tag{9.18}$$

$$
\begin{aligned}
&+ 4I_1I_3c_1U_1\omega_1\omega_3^2 + 4I_2^2c_2U_1\omega_2^3 + 4I_2I_3c_2U_1\omega_2\omega_3^2 - I_1^2(4c_2^2U_2 - U_3^2)\omega_1^2\\
&+ 8I_1I_2c_1c_2U_2\omega_1\omega_2 - I_2^2(4c_1^2U_2 - U_3^2)\omega_2^2\\
&- 4I_1c_1U_1U_3\omega_1 - 4I_2c_2U_1U_3\omega_2 + 4(c_1^2 + c_2^2)U_1^2.
\end{aligned}
$$

Putting the values of γ_1, γ_2 and γ_3 from (9.16) in the Euler-Poisson equations (1.1) and removing the last three equations we obtain the following system of three equations in unknowns ω_1, ω_2 and ω_3:

$$
\frac{d\omega_i}{dt} = \frac{M_i}{2I_i(I_1c_2\omega_1 - I_2c_1\omega_2)}, \quad 1 \le i \le 3, \tag{9.19}
$$

where M_1, M_2 and M_3 are polynomials of ω_j, γ_j, I_j, c_j, U_j, $1 \le i \le 3$, and of R. The system (9.19) is correctly defined only if

$$
(c_1, c_2) \ne (0, 0). \tag{9.20}
$$

Let us suppose first that the condition (9.20) is satisfied.

As we are going to study the first integrals of system (9.19) we can multiply its right-hand sides by the non-zero factor $2I_1I_2I_3(I_1c_2\omega_1 - I_2c_1\omega_2)$. In this way we come to the following system:

$$
\begin{aligned}
\frac{d\omega_1}{dt} &= -I_2I_3\Big\{2\big[I_1(c_2^2 + c_3^2)\omega_1 - I_2c_1c_2\omega_2 - I_3c_1c_3\omega_3\big]R\\
&\quad + I_1^2c_3\omega_1^3 + I_1I_2c_3\omega_1\omega_2^2 - 2I_1(I_2 - I_3)c_2\omega_1\omega_2\omega_3 + I_1I_3c_3\omega_1\omega_3^2\\
&\quad + 2I_2(I_2 - I_3)c_1\omega_2^2\omega_3 - I_1c_3U_3\omega_1 + 2c_1c_3U_1\Big\},\\
\frac{d\omega_2}{dt} &= I_1I_3\Big\{2\big[I_1c_1c_2\omega_1 - I_2(c_1^2 + c_3^2)\omega_2 + I_3c_2c_3\omega_3\big]R\\
&\quad - I_1I_2c_3\omega_1^2\omega_2 - 2I_1(I_1 - I_3)c_2\omega_1^2\omega_3 + 2I_2(I_1 - I_3)c_1\omega_1\omega_2\omega_3\\
&\quad - I_2^2c_3\omega_2^3 - I_2I_3c_3\omega_2\omega_3^2 + I_2c_3U_3\omega_2 - 2c_2c_3U_1\Big\},\\
\frac{d\omega_3}{dt} &= I_1I_2\Big\{2\big[I_1c_1c_3\omega_1 + I_2c_2c_3\omega_2 - I_3(c_1^2 + c_2^2)\omega_3\big]R\\
&\quad + I_1^2c_1\omega_1^3 + I_1(2I_1 - I_2)c_2\omega_1^2\omega_2 - I_2(I_1 - 2I_2)c_1\omega_1\omega_2^2 + I_1I_3c_1\omega_1\omega_3^2\\
&\quad + I_2^2c_2\omega_2^3 + I_2I_3c_2\omega_2\omega_3^2 - I_1c_1U_3\omega_1 - I_2c_2U_3\omega_2 + 2(c_1^2 + c_2^2)U_1\Big\}.
\end{aligned} \tag{9.21}
$$

We study the existence of a first integral of system (9.21) that depends on at most two variables among the variables $(\omega_1, \omega_2, \omega_3)$. There are three possible types of such a first integral:

1. $F(\omega_1, \omega_2)$, (case(iv))
2. $F(\omega_1, \omega_3)$, (case(iv))
3. $F(\omega_2, \omega_3)$. (case(iv))

As all the three types of first integrals belong to case (iv) it suffices to study only the first type.

Type 1. We consider a first integral F of system (9.21) of type 1, i.e. $F(\omega_1, \omega_2)$. We have

$$\frac{dF}{dt} = \frac{d\omega_1}{dt}\frac{\partial F}{\partial \omega_1} + \frac{d\omega_2}{dt}\frac{\partial F}{\partial \omega_2} = Y_1(F) = 0, \tag{9.22}$$

where $\frac{d\omega_1}{dt}$ and $\frac{d\omega_2}{dt}$ are taken from (9.21) and Y_1 is the corresponding vector field defined on $\mathbb{C}^3(\omega_1, \omega_2, \omega_3)$.

This equation should be identically equal to zero with respect to variables ω_1, ω_2 and ω_3. As function F does not depend on ω_3 then its partial derivatives will not depend on ω_3 too. Thus if we differentiate identity (9.22) with respect to ω_3 we obtain again a linear partial differential equation for F

$$\frac{\partial Y_1(F)}{\partial \omega_3} = \frac{\partial}{\partial \omega_3}\left(\frac{d\omega_1}{dt}\right)\frac{\partial F}{\partial \omega_1} + \frac{\partial}{\partial \omega_3}\left(\frac{d\omega_2}{dt}\right)\frac{\partial F}{\partial \omega_2} = Y_2(F) = 0, \tag{9.23}$$

where Y_2 is the corresponding vector field defined on $\mathbb{C}^3(\omega_1, \omega_2, \omega_3)$.

Equations (9.22) and (9.23) can be considered as a system of two homogeneous linear algebraic equations with unknowns $\operatorname{grad} F = \left(\frac{\partial F}{\partial \omega_1}, \frac{\partial F}{\partial \omega_2}\right)$. This linear system admits a non-zero solution if and only if its determinant $\Delta(R)$ vanishes identically with respect to variables ω_1, ω_2 and ω_3 provided that R is a root of equation (9.17).

We compute $\Delta(R)$, remove its non-zero factor $2I_1I_2I_3^2(I_1c_2\omega_1 - I_2c_1\omega_2)$ and obtain

$$\begin{aligned}
\widehat{\Delta}(R) = \frac{\Delta(R)}{2I_1I_2I_3^2(I_1c_2\omega_1 - I_2c_1\omega_2)} &= -2I_3c_3(c_1^2 + c_2^2 + c_3^2)R^2 \\
&- \Big[I_1(2I_3c_2^2 - 2I_1c_2^2 - 2I_1c_3^2 + 3I_3c_3^2)\omega_1^2 + 2(2I_1I_2 - I_1I_3 - I_2I_3)c_1c_2\omega_1\omega_2 \\
&- 2I_1I_3c_1c_3\omega_1\omega_3 + I_2(2I_3c_1^2 - 2I_2c_1^2 - 2I_2c_3^2 + 3I_3c_3^2)\omega_2^2 \\
&- 2I_2I_3c_2c_3\omega_2\omega_3 - I_3^2c_3^2\omega_3^2 - I_3c_3^2U_3\Big]R \\
&- \Big[I_1^2c_1c_3\omega_1^3 + I_1I_2c_2c_3\omega_1^2\omega_2 + I_1(2I_1c_2^2 - 2I_3c_2^2 + 2I_1c_3^2 - I_3c_3^2)\omega_1^2\omega_3 \\
&+ I_1I_2c_1c_3\omega_1\omega_2^2 - 2(2I_1I_2 - I_1I_3 - I_2I_3)c_1c_2\omega_1\omega_2\omega_3 \\
&- I_3(I_1 - 2I_3)c_1c_3\omega_1\omega_3^2 + I_2^2c_2c_3\omega_2^3 \\
&+ I_2(2I_2c_1^2 - 2I_3c_1^2 + 2I_2c_3^2 - I_3c_3^2)\omega_2^2\omega_3 \\
&+ I_3(2I_3 - I_2)c_2c_3\omega_2\omega_3^2 + I_3^2c_3^2\omega_3^3 - I_1c_1c_3U_3\omega_1 - I_2c_2c_3U_3\omega_2 \\
&- I_3c_3^2U_3\omega_3 + 2c_3(c_1^2 + c_2^2 + c_3^2)U_1\Big]\frac{\partial R}{\partial \omega_3} \\
&+ c_3\Big[I_1^2(I_1 - I_3)\omega_1^4 + I_1I_2(I_1 + I_2 - 2I_3)\omega_1^2\omega_2^2 - I_1I_3(I_1 - I_3)\omega_1^2\omega_3^2 \\
&+ I_2^2(I_2 - I_3)\omega_2^4 - I_2I_3(I_2 - I_3)\omega_2^2\omega_3^2 - I_1(I_1 - I_3)U_3\omega_1^2 \\
&- I_2(I_2 - I_3)U_3\omega_2^2 + 2(I_1 - I_3)c_1U_1\omega_1 + 2(I_2 - I_3)c_2U_1\omega_2 - 2I_3c_3U_1\omega_3\Big].
\end{aligned} \tag{9.24}$$

Like in Sec. 9.2 we should obtain $\widehat{\Delta}(R)$ as a polynomial of R that is we should determine $\frac{\partial R}{\partial \omega_3}$ as a function of R. For the purpose we use equation (9.17) where polynomials $A(\omega_1, \omega_2, \omega_3)$, $B(\omega_1, \omega_2, \omega_3)$ and $C(\omega_1, \omega_2, \omega_3)$ are taken from (9.18). We differentiate

(9.17) with respect to ω_3 and obtain

$$\frac{\partial Q(R)}{\partial \omega_3} = \frac{\partial A}{\partial \omega_3}R^2 + \frac{\partial B}{\partial \omega_3}R + \frac{\partial C}{\partial \omega_3} + \frac{dQ}{dR}\frac{\partial R}{\partial \omega_3} = 0. \tag{9.25}$$

The determination of $\frac{\partial R}{\partial \omega_3}$ from (9.25) is possible if and only if $\frac{dQ}{dR} = 2AR + B$ is not zero when R is a root of polynomial Q. Then we obtain

$$\frac{\partial R}{\partial \omega_3} = -\frac{\frac{\partial A}{\partial \omega_3}R^2 + \frac{\partial B}{\partial \omega_3}R + \frac{\partial C}{\partial \omega_3}}{2AR + B}.$$

Let us prove that $\frac{dQ}{dR}$ is not zero. We use Proposition 4.1. Let R be a root of equation $Q(R) = 0$. We consider the resultant ρ of Q and $\frac{dQ}{dR}$ and prove that it can never be identically zero with respect to ω_1, ω_2 and ω_3. We have

$$\rho = A(4AC - B^2)$$

and as A never vanishes identically we do not consider ρ but $\widehat{\rho} = 4AC - B^2$ instead. Putting in $\widehat{\rho}$ the expressions for A, B and C from (9.18) we obtain

$$\widehat{\rho} = 16(I_1c_2\omega_1 - I_2c_1\omega_2)^2\widetilde{\rho}.$$

As we consider the case (9.20), then the first factor never vanishes identically. The second one, i.e. $\widetilde{\rho}$ is a long polynomial of ω_1, ω_2 and ω_3 that has 35 monomials. Among them is the monomial $I_1^4\omega_1^6$ and therefore $\widetilde{\rho}$ never vanishes identically. Consequently ρ never vanishes identically either.

We put the value of $\frac{\partial R}{\partial \omega_3}$ obtained from equation (9.25) in (9.24) and find $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR + B$ appears as a denominator of $\widehat{\Delta}(R)$ and we note

$$\widehat{\Delta}(R) = \frac{\delta(R)}{2AR + B},$$

where $\delta(R)$ is a polynomial of R of degree three.

It is clear that $\widehat{\Delta}(R) = 0$ is equivalent to $\delta(R) = 0$. We know that if $Q(R) = 0$, then if in addition some supplementary first integral $F(\omega_1, \omega_2)$ of system (9.21) exists, then also $\delta(R) = 0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K} = \mathbb{A}\mathrm{lg}(\omega_1, \omega_2, \omega_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

Using the MAPLE command `rem` we compute the remainder of the division of polynomial δ by Q and obtain a remainder r of the form:

$$r(x) = \frac{4(I_1c_2\omega_1 - I_2c_1\omega_2)}{(I_1c_2\omega_1 - I_2c_1\omega_2)^2 + (I_1c_3\omega_1 - I_3c_1\omega_3)^2 + (I_2c_3\omega_2 - I_3c_2\omega_3)^2}(r_0x + r_1),$$

where r_0 and r_1 are polynomials of ω_1, ω_2 and ω_3.

It is easily seen that when $(c_1, c_2) \neq (0, 0)$ the fraction in the above equality is non-zero on open dense subset of $\mathbb{C}^3(\omega_1, \omega_2, \omega_3)$.

Thus $r_0 = r_1 = 0$ identically with respect to ω_1, ω_2 and ω_3. Below we consider only $r_0 = 0$ which turns out sufficient for our needs.

As r_0 has a non-zero factor I_3 we remove it. The obtained polynomial has 74 coefficients. To find all values of the parameters $\mathcal{I}c$ and U_i, $1 \leq i \leq 3$, for which these coefficients are zero we apply simplification and after four consecutive simplifications we obtain the reduced system of five equations:

$$c_2c_3 = 0, \quad c_1c_3 = 0, \quad (I_1 - I_2)c_3 = 0, \quad (I_1 - I_3)c_2 = 0, \quad (I_2 - I_3)c_1 = 0.$$

Solving it by the MAPLE command `solve` we obtain the following five solutions:

$$\begin{aligned}
&c_1 = 0,\ c_2 = 0,\ c_3 = 0 \text{ with arbitrary } I_1,\ I_2,\ I_3,\ U_1,\ U_2,\ U_3,\\
&I_1 = I_2,\ c_1 = 0,\ c_2 = 0 \text{ with arbitrary } I_2,\ I_3,\ c_3,\ U_1,\ U_2,\ U_3,\\
&I_2 = I_3,\ c_2 = 0,\ c_3 = 0 \text{ with arbitrary } I_1,\ I_3,\ c_1,\ U_1,\ U_2,\ U_3,\\
&I_1 = I_3,\ c_1 = 0,\ c_3 = 0 \text{ with arbitrary } I_2,\ I_3,\ c_2,\ U_1,\ U_2,\ U_3,\\
&I_1 = I_3,\ I_2 = I_3,\ c_3 = 0 \text{ with arbitrary } I_3,\ c_1,\ c_2,\ U_1,\ U_2,\ U_3.
\end{aligned}$$

As $(c_1, c_2) \neq (0, 0)$ we remove the first and second solutions. Third and fourth solutions give the Lagrange case and fifth one – the kinetic symmetry case. Thus the sought partial first integral of type 1, i.e. $F(\omega_1, \omega_2)$ does not exist when $(c_1, c_2) \neq (0, 0)$.

Let us suppose now that (9.20) is not fulfilled, i.e. $(c_1, c_2) = (0, 0)$. To avoid the Euler case we suppose that $c_3 \neq 0$. Solving equations (9.1) with respect to γ_1, γ_2 and γ_3 by the MAPLE command `solve` we obtain

$$\begin{aligned}
\gamma_1 &= \frac{I_1I_3\omega_1^2\omega_3 + I_2I_3\omega_2^2\omega_3 + I_3^2\omega_2^3 - I_3U_3\omega_3 + 2c_3U_1 - 2I_2\omega_2R}{2I_1c_3\omega_1},\\
\gamma_2 &= \frac{R}{c_3}, \qquad \gamma_3 = -\frac{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 - U_3}{2c_3},
\end{aligned} \tag{9.26}$$

where $R = R(\omega_1, \omega_2, \omega_3)$ is a root of equation

$$Q(x) = Ax^2 + Bx + C = 0,$$

that is

$$Q(R) = AR^2 + BR + C = 0. \tag{9.27}$$

Here $A = A(\omega_1, \omega_2)$, $B = B(\omega_1, \omega_2, \omega_3)$ and $C = C(\omega_1, \omega_2, \omega_3)$ are the following polynomials:

$$\begin{aligned}
A &= 4(I_1^2\omega_1^2 + I_2^2\omega_2^2),\\
B &= -4I_2\omega_2(I_1I_3\omega_1^2\omega_3 + I_2I_3\omega_2^2\omega_3 + I_3^3\omega_3^3 - I_3U_3\omega_3 + 2c_3U_1),\\
C &= I_1^4\omega_1^6 + 2I_1^3I_2\omega_1^4\omega_2^2 + I_1^2I_3(2I_1 + I_3)\omega_1^4\omega_3^2 + I_1^2I_2^2\omega_1^2\omega_2^4\\
&\quad + 2I_1I_2I_3(I_1 + I_3)\omega_1^2\omega_2^2\omega_3^2 + I_1I_3^2(I_1 + 2I_3)\omega_1^2\omega_3^4 + I_2^2I_3^2\omega_2^4\omega_3^2\\
&\quad + 2I_2I_3^3\omega_2^2\omega_3^4 + I_3^4\omega_3^6 - 2I_1^3U_3\omega_1^4 - 2I_1^2I_2U_3\omega_1^2\omega_2^2 - 2I_1I_3(I_1 + I_3)U_3\omega_1^2\omega_3^2\\
&\quad - 2I_2I_3^2U_3\omega_2^2\omega_3^2 - 2I_3^3U_3\omega_3^4 + 4I_1I_3c_3U_1\omega_1^2\omega_3 + 4I_2I_3c_3U_1\omega_1^2\omega_3\\
&\quad + 4I_3^2c_3U_1\omega_3^3 - I_1^2(4c_3^2U_2 - U_3^2)\omega_1^2 + I_3^2U_3^2\omega_3^2 - 4I_3c_3U_1U_3\omega_3 + 4U_1^2c_3^2.
\end{aligned} \tag{9.28}$$

After substitution of γ_1, γ_2 and γ_3 from (9.26) in the first three Euler-Poisson equations (1.1) we obtain the following system for $\omega_1,\ \omega_2$ and ω_3:

$$\begin{aligned}
\frac{d\omega_1}{dt} &= \frac{R+(I_2-I_3)\omega_2\omega_3}{I_1},\\
\frac{d\omega_2}{dt} &= \frac{2I_2\omega_2R-I_1(2I_1-I_3)\omega_1^2\omega_3-I_2I_3\omega_2^2\omega_3-I_3^2\omega_3^3+I_3U_3\omega_3-2c_3U_1}{2I_1I_2\omega_1},\\
\frac{d\omega_3}{dt} &= \frac{(I_1-I_2)\omega_1\omega_2}{I_3}.
\end{aligned} \tag{9.29}$$

Like in case (9.20) we examine only the type 1 of first integrals of system (9.29).

Type 1. As in the case when $(c_1,c_2)\neq(0,0)$ we define the vector fields Y_1 and Y_2 by $Y_1(F)=\frac{dF}{dt}$ (see (9.22)) and $Y_2(F)=\frac{\partial Y_1(F)}{\partial\omega_3}$ (see (9.23)) but now $\frac{d\omega_1}{dt}$ and $\frac{d\omega_2}{dt}$ are taken from (9.29).

Determinant $\Delta(R)$ of linear system (9.22) and (9.23) should vanish identically with respect to variables ω_1, ω_2 and ω_3 provided that R is a root of equation (9.27).

We compute $\Delta(R)$. It has a non-zero denominator $2I_1^2I_2\omega_1$. We note

$$\begin{aligned}
\widehat{\Delta}(R)=2I_1^2I_2\omega_1\Delta(R) &= -\Big[I_1(2I_1-I_3)\omega_1^2+I_2(2I_2-I_3)\omega_2^2+3I_3^2\omega_3^2-I_3U_3\Big]R\\
&\quad+\Big[I_1(2I_1-I_3)\omega_1^2\omega_3+I_2(2I_2-I_3)\omega_2^2\omega_3+I_3^2\omega_3^3-I_3U_3\omega_3+2U_1c_3\Big]\frac{\partial R}{\partial\omega_3}\\
&\quad-2(I_2-I_3)(I_3^2\omega_3^3-U_1c_3)\omega_2.
\end{aligned} \tag{9.30}$$

In order to obtain $\widehat{\Delta}(R)$ as a polynomial of R we determine $\frac{\partial R}{\partial\omega_3}$ using equation (9.27) where polynomials $A(\omega_1,\omega_2)$, $B(\omega_1,\omega_2,\omega_3)$ and $C(\omega_1,\omega_2,\omega_3)$ are taken from (9.28). After differentiating (9.27) with respect to ω_3 we obtain

$$\frac{\partial Q}{\partial\omega_3}=\frac{\partial B}{\partial\omega_3}R+\frac{\partial C}{\partial\omega_3}+\frac{dQ}{dR}\frac{\partial R}{\partial\omega_3}=0. \tag{9.31}$$

In the same way as in the case $(c_1,c_2)\neq(0,0)$ we prove by Proposition 4.1 that $\frac{dQ}{dR}$ is not zero and determine $\frac{\partial R}{\partial\omega_3}$ from (9.31). Then we put it in (9.30) and find $\widehat{\Delta}(R)$. After this substitution the non-zero expression $2AR+B$ appears as a denominator of $\widehat{\Delta}(R)$ and we note

$$\widehat{\Delta}(R)=\frac{\delta(R)}{2AR+B},$$

where $\delta(R)$ is a polynomial of R of degree two.

It is clear that $\widehat{\Delta}(R)=0$ is equivalent to $\delta(R)=0$. We know that if $Q(R)=0$, then if in addition some supplementary first integral $F(\omega_1,\omega_2)$ of system (9.29) exists, then also $\delta(R)=0$. Thus all assumptions of Proposition 4.2 are fulfilled. Consequently in polynomial ring $\mathbb{K}[x]$, where $\mathbb{K}=\mathbb{A}\mathrm{lg}(\omega_1,\omega_2,\omega_3)$, the polynomial $Q(x)$ divides the polynomial $\delta(x)$.

The remainder $r(x)$ of the division of polynomial $\delta(x)$ by polynomial $Q(x)$ is a polynomial of x of degree one

$$r(x)=r_0x+r_1,$$

where r_0 and r_1 are polynomials of ω_1, ω_2 and ω_3 which, by Proposition 4.2, should be identically equal to zero. We consider only the leading coefficient of $r(x)$

$$r_0 = -8I_1I_3(I_1 - I_2)\omega_1^2\omega_2(I_3^2\omega_3^3 - c_3U_1) = 0.$$

It is easily seen that r_0 vanish identically if and only if $I_1 = I_2$ which together with restriction $c_1 = c_2 = 0$ considered now leads to the Lagrange case. Thus the sought partial first integral of type 1, $F(\omega_1, \omega_2)$ does not exist.

This concludes our study.

References

[1] *Algebraic Functions, Encyclopedia of Mathematics*, http://encyclopediaofmath.org/index.php?title=Algebraic_function&oldid=45136 In book format: M.Hazewinkel (Ed.), *Encyclopedia of Mathematics*, 1(A-B), Kluwer, 1987.

[2] T. Archibald, *Differential equations and algebraic transcendents: French efforts at the creation of a Galois theory of differential equations 1880–1910*, Rev. Histoire Math. 17 (2011), no. 2, 373–401

[3] Yu. A. Arkhangelskii, *Analytical dynamics of the rigid body*, Nauka, Moscow, 1977 (in Russian).

[4] Yu. A. Arkhangelskii, *On one new property of Euler-Poisson equations*, Doklady Akad. Nauk USSR 258 (1981), 810-811 (in Russian).

[5] V. I. Arnold, *Mathematical Methods of Classical Mechanics, Graduate Texts in Math.* 60, 2nd edition, Springer-Verlag, Berlin, 1989.

[6] M. Audin, *Spinning Tops. A Course on Integrable Systems*, Cambridge Studies in Advanced Mathematics 51, Cambridge University Press, 1996.

[7] M. Audin, *Les systèmes hamiltoniens et leur intégrabilité, Cours spécialisé*, vol. 8, Société Mathématique de France, EDP Sciences (2001) (in French); English translation: *Hamiltonian Systems and Their Integrability*, SMF/AMS Texts and Monographs 15, American Mathematical Society (2008).

[8] V. N. Bogaevskii, *On general integrals of the equations of motion of a rigid body about a fixed point*, Vestnik Moskov. Univ. Ser. I Mat. Meh., no. 5 (1965), 92–95 (in Russian).

[9] O. I. Bogoyavlensky, *Integrable Euler Equations on Six-Dimensional Lie Algebras*, Doklady Akad. Nauk USSR 268 (1983), 11-15 (in Russian). English transl. Soviet Math. Doklady 27, N1 (1983), 1-5.

[10] A. V. Borisov and I. S. Mamaev, *Poisson structures and Lie algebras in Hamiltonian mechanics*, Library “R&C Dynamics”, Edited by Izdatel’skiĭ Dom “Udmurtskiĭ Universitet”, Izhevsk, (1999) (in Russian). http://ics.org.ru/publications/index.php?cat=103&author=23

[11] A. V. Borisov and I. S. Mamaev, *Modern Methods of the Theory of Integrable Systems*, Moscow-Izhevsk: Institute of Computer Science (2003) (in Russian). http://ics.org.ru/publications/index.php?cat=103&author=23

[12] A. V. Borisov and I. S. Mamaev, *Rigid Body Dynamics. Hamiltonian Methods, Integrability, Chaos*, Moscow-Izhevsk: Institute of Computer Science, (2005) (in Russian). http://ics.org.ru/publications/index.php?cat=103&author=23 English translation: *Rigid Body Dynamics*, De Gruyter Studies in Mathematical Physics 52, De Gruyter, Higher Education Press, 2019.

[13] N. Bourbaki, *Éléments d’histoire des mathématiques*, Springer-Verlag, 2007.

[14] S. A. Chaplygin, *A new case of rotation of a heavy rigid body supported at one point*, Trudy Otdel. Fiz. Nauk Obshchestva Lyubit. Estestvozn. 10 (2) (1901), 31-34 (in Russsian). Reproduced in:

a) Complete works of S. A. Chaplygin, Vol. 1. Publish. USSR Academy of Sciences, Leningrad, 1933, 241-245 (in Russian).

b) S. A. Chaplygin, Collected Works, Vol. 1, Moscow, Leningrad, Publish. Gostekhizdat, 1948, 118-124. (in Russian).

c) S. A. Chaplygin, Selected works on mechanics and mathematics. Publish. Gostekhizdat, Moscow, 1954, 472-476. (in Russian).

https://1lib.fr/book/5234667/03fd98

[15] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Springer, 4th ed., 2015 (preceding editions, 1992, 1997, 2012).

[16] A. I. Dokshevich, *On the fourth integral in the problem of the rotation of a heavy rigid body around a fixed point*, Integration of some differential equations of mathematical physics, Edition "Nauka" of Uzbek Soviet Republic, Tashkent, 1964, 256 pp. (in Russian).

[17] A. I. Dokshevich, *Solutions in finite terms of the Euler-Poisson equations*, Naukova Dumka, Kiev, 1992 (in Russian).

[18] D. S. Dummit and R. M. Foote, *Abstract Algebra*, Third Edition, John Wiley and Sons, 2004.

[19] A. T. Fomenko, *Integrability and nonintegrability in geometry and mechanics*, Dodrecht: Kluwer Academic Publishers, 1988.

[20] I. N. Ganshenko, G. V. Gorr and A. M. Kovalev, *Classical Problems of Rigid Body Dynamics*, Kiev, Naukova Dumka, 2012 (in Russian).

[21] L. Gavrilov, *Non-integrability of the equations of heavy gyrostat*, Compositio Mathematica 82 (3) (1992), 275-291.

[22] V. V. Golubev, *Lectures on integration of the equations of motion of a rigid body about a fixed point*, Moscow, Gostekhizdat, 1953 (in Russian); English translation: Israel program for scientific translations, Haifa, 1960.

[23] A. Goriely, *A brief history of Kovalevskaya exponents and modern developments*, Regul. Chaotic Dyn. 5 (2000), no. 1, 3–15.

[24] G. V. Gorr, L. V. Kudriashova and L. A. Stepanova, *Classical Problems in Dynamics of rigid body. Development and contemporary state*, Kiev Editions "Naukova Dumka", 1978 (in Russian).

[25] D. N. Goryachev, *On the motion of heavy rigid body about a fixed point in the case* $A = B = 4C$, Mat. Sbornik 21(3) (1900), 431-438 (in Russian).

[26] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, J.-M. Strelcyn and S. Wojciechowski, *Integrals of quadratic ordinary differential equations in* $\mathbb{R}^3$*: the Lotka-Volterra system*, Phys. A 163 (1990), no. 5, 683–722.

[27] A. T. Grigorian, B. N. Fradlin, *History of Mechanics of Rigid Body*, Nauka, Moscow, 1982 (in Russian).

[28] F. Haas, *Frobenius method and invariants for one-dimensional time-dependent Hamiltonian systems*, J. Phys. A, 34 (2001), no. 5, 1005–1017.

[29] T. Hawkins, *Emergence of the theory of Lie groups, An essay in the history of mathematics 1869–1926. Sources and Studies in the History of Mathematics and Physical Sciences*, Springer-Verlag, 2000.

[30] D. D. Holm, *Geometric Mechanics. Part I: Dynamics and Symmetry. Part II: Rotating,*

Translating and Rolling., Imerial College Press. 1st edit., 2008, 2nd edit., 2011.

[31] D. D. Holm, T. Schmah and C. Stoica, *Geometric Mechanics and Symmetry. From Finite to Infinite Dimensions*, Cambridge University Press, 2009.

[32] E. Husson, Recherches des intégrales algébriques dans le mouvement d'un corps pesant autour d'un point fixe, Ann. Fac. Sci. Toulouse 8 (1906), 73-152.

[33] A. A. Iliukhin, *Spacial Problems of Nonlinear Theory of Elastic Rods*, Kiev, Naukova Dumka, 1979 (in Russian).

[34] Yu. Ilyashenko and S. Yakovenko, *Lectures on analytical differential equations*, Graduate Studies in Math. 86, AMS, 2007.

[35] A. V. Kavinov, *On Calculation of the First Integrals for Three-dimensional ODE Systems.* Mathematics and Mathematical Modeling. 2018, no. 6, 11-21. (In Russian).

[36] V. V. Kozlov, *Symmetries, Topology and Resonances in Hamiltonian Mechanics*, Springer-Verlag, 1996.

[37] E. Leimanis, *The General Problem of the Motion of Coupled Rigid Bodies about a fixed Point*, Springer-Verlag, 1965.

[38] T. Levi Civita and U. Amaldi, *Lezioni di meccanica razionale*, 2, Nicola Zanichelli (Ed.), Bologna. XIV + 807 pp, 1927.

[39] Y. Z. Liu and Y. Xue, *Formulation of Kirchhoff Rod Based on Quasi-coordinates*, Technische Mechanik, 24(3-4) (2004), 206-210.

[40] J. Lützen, *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics*, Springer-Verlag, New York, 1990, xx+884 pp.

[41] A. J. Maciejewski and S. I. Popov, *Invariants of homogeneous ordinary differential equations*, Reports on Mathematical Physics 41(3) (1998), 287-310.

[42] A. J. Maciejewski, S. I. Popov and J.-M. Strelcyn, *The Euler Equations on Lie algebra so(4); an elementary approach to integrability condition*, Journal of Mathematical Physics 42(6) (2001), 2701-2717.

[43] A. J. Maciejewski and M. Przybylska, *Differential Galois approach to the non-integrability of the heavy top problem*, Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6(14) (2005), 123-160.

[44] K. Magnus, *Kreisel, Theorie und Anwendungen*, Springer Verlag, 1971.

[45] J. E. Marsden, *Lectures on Mechanics*, London Math. Soc. Lecture Notes Series 174, Cambridge University Press, 1992.

[46] J. Montaldi and T. Ratiu (Eds), *Geometric Mechanics and Symmetry: The Peyresq Lectures*, London Math. Soc. Lecture Note Series 306, Cambridge University Press, 2005.

[47] J. J. Morales-Ruiz, *Kovalevskaya, Liapounov, Painlevé, Ziglin and the differential Galois theory*, Regul. Chaotic Dyn. 5 (2000), no. 3, 251–272.

[48] J. J. Morales-Ruiz, *Picard-Vessiot theory and integrability*, J. Geom. Phys. 87 (2015), 314343.

[49] J. J. Morales-Ruiz and C. Simó, *Picard-Vessiot theory and Ziglin's theorem*, J. Differential Equations 107 (1994), no. 1, 140–162.

[50] J. Moulin-Ollagnier and J.-M. Strelcyn, *On first integrals of linear systems, Frobenius integrability theorem and linear representations of Lie algebras*, Bifurcations of planar vector fields (Luminy, 1989), 243–271, Lecture Notes in Math., 1455, Springer, Berlin, (1990).

[51] R. Narasimhan, *Analysis on Real and Complex Manifolds*, third printing, North-Holland, Amsterdam-NewYork-Oxford, 1985.

[52] P. J. Olver, *Applications of Lie groups to differential equations, Graduate Texts in Math.*

107, Springer-Verlag, Berlin, 1986.

[53] A. M. Perelomov, *Integrable systems of classical mechanics and Lie algebras.* Vol.I, Birkhauser Verlag, Basel, 1990.

[54] P. Ya. Polubarinova-Kochina, *On single-valued solutions and algebraic integrals of the problem about rotation of the heavy rigid body problem around a fixed point* in "Motion of the Rigid Body around a fixed Point. S.V.Kovalevskaya Memorial Collection" Edition of Academy of Sciences of USSR, Moscow-Leningrad (1940), 157-186 (in Russian).

[55] S. I. Popov, *On the existence of depending on* p, q, r, γ *first integral of the Euler-Poisson system*, Theoretical and Applied Mechanics (Bulgarian Academy of Sciences) 2 (1981), 28-33 (in Bulgarian).

[56] S. I. Popov, *On the nonexistence of a new first integral* $F(p,q,r,\gamma,\gamma') = const$ *of the problem of a heavy rigid body motion about a fixed point*, Comptes rendus de l'Académie bulgare des Sciences, 38(5) (1985), 583-586 (in Russian).

[57] S. I. Popov, *On the nonexistence of a new first integral* $F(p,q,r,\gamma,\gamma') = const$ *of the problem of a heavy rigid body motion about a fixed point*, Theoretical and Applied Mechanics (Bulgarian Academy of Sciences) 4 (1988), 17-23 (in Russian).

[58] S. I. Popov and J.-M. Strelcyn, *An elementary approach to integrability condition for the Euler equations on Lie algebra so(4)*, Hamiltonian mechanics (Toruń, 1993), NATO Adv. Sci. Inst. Ser. B: Phys., 331, Plenum, New York (1994), 371–376.

[59] S. I. Popov and J.-M. Strelcyn, *The Euler-Poisson equations: an elementary approach to integrability conditions*, Journal of Geometric Mechanics 10(3) (2018), 293-329.

[60] V. V. Prasolov, *Polynomials*, Springer-Verlag, 2004.

[61] S. Spodzieja, *The field of Nash functions and factorization of polynomials*, Ann. Polon. Math. 65 (1996), no. 1, 81–94.

[62] L. N. Sretenskii, *On some cases of integration of the gyrostat motion equations*, Dokl. of USSR Acad. of Sciences 149(2) (1963), 292-294 (in Russian).

[63] L. N. Sretenskii, *On certain cases of motion of a heavy rigid body with a gyroscope*, (Russian) Vestnik Moskov. Univ. Ser. I Mat. Meh. (3) (1963), 60–71.

[64] J.-M. Strelcyn and S. Wojciechowski, *A method of finding integrals for three-dimensional dynamical systems*, Phys. Lett. A 133 (1988), no. 4-5, 207–212.

[65] V. V. Trofimov, *Introduction to the Geometry of Manifolds with Symmetry*, Mathematics and its Applications 270, Kluwer Academic Publishers Group, Dodrecht, 1994.

[66] E. Vessiot, *Gewöhnliche Differentialgleichungen; elementare Integrationsmethoden*, in Burkhardt (H.), Wirtinger (W.) & Fricke (Robert), eds., Encyklopädie der mathematischen Wissenschaften, Band II, 1. Teil, 1. Hälfte, Teubner, Leipzig, 1900, pp. 232-293.

[67] E. Vessiot, *Méthodes d'intégration élémentaires. Étude des équations différentielles ordinaires au point de vue formel*, in Molk (Jules), éd., Encyclopédie des sciences mathématiques pures et appliquées, Tome II (Vol. 3), Équations différentielles ordinaires, 1910, pp. 65-179. Gauthier-Villars, Paris et Teubner, Leipzig. Reprint: Éditions Jacques Gabay, Sceaux, 1992. vi+86 pp.

[68] G. D. Villa Salvador, *Topics in the theory of algebraic function fields*, Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2006. xviii+652 pp.

[69] G. N. Watson, *A treatise on the theory of Bessel functions*, Reprint of the second (1944) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. viii+804 pp.

[70] E. von Weber, *Partielle Differentialgleichungen. Allgemaine Eigenschaften der Differentialsysteme, Die linearen partiellen Differentialgleichungen erster Ordnung mit einer Um-*

bekanten, in Burkhardt (H.), Wirtinger (W.) & Fricke (Robert), eds., Encyklopädie der mathematischen Wissenschaften, Band II, 1. Teil, 1. Hälfte, Teubner, Leipzig, 1900, pp. 294-322

[71] E. von Weber, *Propriétés générales des systèmes d'équations aux dérivées partielles. Equations linéaires du premier ordre*, in Molk (Jules), éd., Encyclopédie des sciences mathématiques pures et appliquées, Tome II (Vol. 4), Équations aux dérivées partielles, 1913, pp. 1-55. Gauthier-Villars, Paris et Teubner, Lepzig. Reprint: Éditions Jacques Gabay, Sceaux, 1991. 127 pp.

[72] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*, B.C.Teubner, 1972.

[73] S. Wojciechowski, *Construction of integrable Riccati systems by the use of low-dimensional Lie algebras*, in Structure, coherence and chaos in dynamical systems (Lyngby, 1986), 523–527, Proceedings of the MIDIT Workshop held at the Technical University of Denmark, Lyngby, August 12–16, 1986. Edited by Peter L. Christiansen and Robert D. Parmentier. Proceedings in Nonlinear Science. Manchester University Press, Manchester, 1989, xx+663 pp.

[74] S. Wojciechowski, Communication in conference Euromech 216, Integrable systems in nonlinear analytical mechanics , 22-25 September 1986 Leeds, England.

[75] S. Wojciechowski, *A new test for integrability of Riccati systems of equations*, in Finite-dimensional integrable nonlinear dynamical systems, 46–59. Proceedings of the workshop held in Johannesburg, January 11-15, 1988. Edited by P. G. L. Leach and W.-H. Steeb. World Scientific Publishing Co., Singapore, 1988, x+346 pp.

[76] S. Wojciechowski, *A method of studying integrals of dynamical systems based on Frobenius' integrability theorem*, in Symmetries in science, III, 493-503. Papers from the symposium held in Vorarlberg, July 25–28, 1988. Edited by Bruno Gruber and Francesco Iachello. Plenum Press, New York, 1989, x+613 pp.

[77] S. Wojciechowski, *Method of Compatible Vector Field in Studying Integrability of 3-Dimensional Dynamical Systems*, Proceedings of a NATO Advanced Research Workshop on Complex Dynamics and Biological Evolution, held August 6-10, 1990, in Hindsgavl, Denmark, 383-388, in Complexity, Chaos, and Biological Evolution, Edited by E. Mosekilde and L. Mosekilde, Plenum Press, New York. 1991. xi+431 pp.

[78] H. Yehia, *New integrable case in the dynamics of rigid bodies*, Mech. Res. Commun. 13 (1986), 169-172.

[79] S. L. Ziglin, *Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics.* II, Funktsional. Anal. i Prilozhen. 17(1) (1983), 8-23 (in Russian); English translation: part II, Functional Anal. Appl. 17(1) (1983), 8-23.