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Derivations of polynomial algebras without Darboux polynomials

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Abstract

We present several new examples of homogeneous derivations of a polynomial ring $k[X] = k[x_1, \dots, x_n]$ over a field k of characteristic zero without Darboux polynomials. Using a modification of a result of Shamsuddin, we produce these examples by induction on the number n of variables, thus more easily than the previously known example *multidimensional Jouanolou systems* of Żołądek.

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Introduction

Throughout this paper k is a field of characteristic zero, and $k[X] = k[x_1, \dots, x_n]$ is the polynomial ring in n variables over k. A derivation $d = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ of $k[X]$ is said to be *homogeneous of degree s* if all the polynomials $f_1, \dots, f_n$ are homogeneous of the same degree s. If d is a derivation of $k[X]$, then we say that a polynomial $F \in k[X]$ is a *Darboux polynomial* of d if $F \notin k$ and $d(F) = \Lambda F$ for some $\Lambda \in k[X]$. We say that a derivation d of $k[X]$ is *without Darboux polynomials* if d has no Darboux polynomials.

The existence of Darboux polynomials is a necessary condition for a derivation d of $k[X]$ to have a first integral which belongs to $k[X]$, to $k(X)$, or even belongs to a Liouvillian extension of $k(X)$ (see [9]), where $k(X) = k(x_1, \dots, x_n)$ is the field of rational functions in n variables over k.

Denote by $\mathcal{A}(n, s)$ the set (a finite dimensional k-vector space) of all homogeneous derivations of a given degree s, and let $\mathcal{B}(n, s)$ be the subset of $\mathcal{A}(n, s)$ of all derivations which are without Darboux polynomials. It is known [3,6] that $\mathcal{B}(n, s)$ is a countable intersection of Zariski open algebraic sets in $\mathcal{A}(n, s)$. Thus, in the case where $\mathcal{B}(n, s)$ is a nonempty set, the nonexistence of Darboux polynomials is typical, their existence is rare in the Baire category sense if we consider the whole set of all homogeneous derivations of a given degree s. It is easy to prove (see Section 1) that if $n \le 2$ or $s = 1$, then $\mathcal{B}(n, s) = \emptyset$.

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Assume now that $n \geq 3$ and $s \geq 2$. We already know that $\mathcal{B}(n, s)$ is nonempty as there exist examples of derivations without the Darboux polynomial; they are given in the papers [16,7]. However, in these papers, the proofs that the proposed derivations are indeed without Darboux polynomials are very long.

In this paper we present a simple way for a construction of examples of homogeneous derivations of $k[X]$ without Darboux polynomials. In our construction we start from such derivations for $n = 3$. We use, for instance, the fact that the Jouanolou derivation $y^s \frac{\partial}{\partial x} + z^s \frac{\partial}{\partial y} + x^s \frac{\partial}{\partial z}$ is without Darboux polynomials [3]. Other examples for $n = 3$ are given in Section 4.

For $n \geq 4$, we use an induction process to produce examples; then a modification of Shamsuddin's results plays a crucial role in our construction [13,14].

1. Preliminary facts

We are mainly interested in Darboux polynomials of homogeneous derivations of $k[X]$. It is well known from linear algebra that every homogeneous derivation of $k[X]$ of degree 1 has a Darboux polynomial. The same is true for arbitrary degrees and $n \leq 2$. This fact is trivial for $n = 1$. For $n = 2$ we have the following proposition.

Proposition 1.1. *Every homogeneous derivation of $k[x, y]$ has a Darboux polynomial.*

Proof. Let d be a homogeneous derivation of $k[x, y]$ of degree s. Put $f = d(x)$, $g = d(y)$, and let $F = xg - yf$. The Euler equalities $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = sf$ and $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = sg$ imply that $d(F) = \Lambda F$, where $\Lambda = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. So, if $F \neq 0$, then F is a Darboux polynomial of d. If $F = 0$, then $x - y$ is a Darboux polynomial of d. $\square$

If $n \geq 3$ and $s \geq 2$, then the above property does not hold, in general.

Theorem 1.2 (*[3]*). *The derivation $y^s \frac{\partial}{\partial x} + z^s \frac{\partial}{\partial y} + x^s \frac{\partial}{\partial z}$, where $s \geq 2$, has no Darboux polynomials.*

We know several different proofs of Theorem 1.2 ([3,5,1,15,10]). The derivation from this theorem is a Jouanolou derivation. By the *Jouanolou derivation* with integer parameters $n \geq 3$ and $s \geq 2$ we mean the homogeneous derivation $d : k[X] \to k[X]$ defined by

$$d(x_1) = x_2^s, \quad d(x_2) = x_3^s, \ldots, d(x_{n-1}) = x_n^s, \quad d(x_n) = x_1^s.$$

We denote this derivation by $J(n, s)$. There exists a proof ([6]) that if $n \geq 3$ is a prime number, then $J(n, s)$ is without Darboux polynomials, for all $s \geq 2$. There are also separate such proofs for $n = 4$ and $s \geq 2$ [16,7]. In 2003 Żołądek [16] proposed an analytical proof that $J(n, s)$ is without Darboux polynomials for all $s \geq 2$ and $n \geq 3$.

2. A modification of Shamsuddin's result

Let R be a commutative ring and $d : R \to R$ be a derivation. We denote by R^d the ring of constants of d, that is, $R^d = \{r \in R;\ d(r) = 0\}$. If $a \in R$, then we say that a is a *Darboux element* of d if $a \neq 0$, a is not invertible in R, and $d(a) = \lambda a$ for some $\lambda \in R$. In other words, a nonzero element a of R is a Darboux element of d if and only if the principal ideal $(a) := \{ra;\ r \in R\}$ is different from R and it is invariant with respect to d, that is $d((a)) \subseteq (a)$. In particular, if a is a nonzero and noninvertible element belonging to R^d, then a is a Darboux element of d. We say that d is *without Darboux elements* if d has no Darboux elements.

Theorem 2.1. *Let R be a commutative domain containing $\mathbb{Q}$ and let $d : R \to R$ be a derivation without Darboux elements. Let $R[t]$ be the polynomial ring in one variable over R, and let $D : R[t] \to R[t]$ be a derivation such that $D(r) = d(r)$ for $r \in R$, and*

$$D(t) = at + b,$$

for some $a, b \in R$. Then the following two conditions are equivalent.

(1) *The derivation D is without Darboux elements.*
(2) *There exist no elements r of R such that $d(r) = ar + b$.*

Proof. (1) $\Rightarrow$ (2). Assume that D is without Darboux elements and suppose that there exists $r \in R$ such that $d(r) = ar + b$. Then $D(t - r) = at + b - (ar + b) = a(t - r)$, so $t - r$ is a Darboux element of D; a contradiction. (2) $\Rightarrow$ (1) Suppose that $0 \neq f \in R[t]$ is a Darboux element of D. Then f is not invertible in $R[t]$ and $D(f) = \lambda f$ for some $\lambda \in R[t]$. Comparing in the equality $D(f) = \lambda f$ the degrees with respect to t, we deduce that $\lambda \in R$. This implies that $f \notin R$, because d is without Darboux elements. Put $n = \deg f$. Note that $n \geq 1$.

Let $f = ct^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$, where $c, c_{n-1}, \ldots, c_1, c_0 \in R$ and $c \neq 0$. Then

$$D(f) = (d(c) + nac)t^n + u_{n-1}t^{n-1} + \cdots u_1 t + u_0,$$

for some $u_0, u_1, \ldots, u_{n-1} \in R$. Comparing in the equality $D(f) = \lambda f$ the coefficients with respect to t^n, we obtain the equality $d(c) + nac = \lambda c$, so

$$d(c) = (\lambda - na)c.$$

Hence, if c is not invertible in R, then c is a Darboux element of d. But d is without Darboux elements, so c is invertible in R. Let $u \in R$ such that $uc = 1$, and consider the polynomial $F = uf$. Then $D(F) = \Lambda F$, where $\Lambda = cd(u) + \lambda$. Hence, $F \in R[t]$ is a monic polynomial of degree n, which is a Darboux element of D.

Let $F = t^n + et^{n-1} + e_{n-2}t^{n-2} + \cdots + e_1 t + e_0$, where $e, e_{n-2}, \ldots, e_1, e_0 \in R$, and consider the polynomial $G := D(F) - naF = (\Lambda - na)F$. Observe that G is of the form $vt^{n-1} + v_{n-2}t^{n-2} + \cdots + v_1 t + v_0$, where $v_{n-2}, \ldots, v_1, v_0 \in R$, and

$$v = nb + d(e) + (n-1)ea - nae = d(e) - ea + nb.$$

So, $\deg G < n = \deg F$ and it is clear that G is divisible by F. Hence, $G = 0$. In particular, $v = 0$, that is, $d(e) = ae - nb$. Put $r := -\frac{1}{n}e$. Then r is an element of R such that $d(r) = ar + b$, and we have a contradiction. $\square$

The above theorem is inspired by results of Ahmad Shamsuddin. In 1982, he proved in [14] a similar theorem for automorphisms leaving no nontrivial proper ideals invariant. In his Ph.D. thesis [13], he also proved a similar theorem for simple derivations. Many consequences of his theorems can be found, for example, in [2,11,8,4].

If $a = 0$, then Theorem 2.1 has the following form.

Theorem 2.2. *Let R be a commutative domain containing $\mathbb{Q}$ and let $d : R \to R$ be a derivation without Darboux elements. Let $R[t]$ be the polynomial ring in one variable over R, and let $D : R[t] \to R[t]$ be a derivation such that $D(r) = d(r)$ for $r \in R$, and $D(t) = b$ for some $b \in R$. Then D is without Darboux elements if and only if $b \notin d(R)$.*

3. Examples for arbitrary number of variables

Starting from the Jouanolou derivation $J(3, s)$ and using Theorems 2.1 and 2.2, we may produce a series of examples of homogeneous derivations of $k[x_1, \ldots, x_n]$ (where $n \geq 4$) without Darboux polynomials.

Example 3.1. If $n \geq 4$, $s \geq 2$, then the derivation

$$x_2^s \frac{\partial}{\partial x_1} + x_3^s \frac{\partial}{\partial x_2} + x_1^s \frac{\partial}{\partial x_3} + x_2 x_3^{s-1} \frac{\partial}{\partial x_4} + \cdots + x_{n-2} x_{n-1}^{s-1} \frac{\partial}{\partial x_n}$$

of $k[x_1, \ldots, x_n]$ is without Darboux polynomials.

Proof. Denote this derivation by δ_n and put $\delta_3 = J(3, s)$. We know, by Theorem 1.2, that δ_3 is without Darboux polynomials. Let $n \geq 4$ and assume that δ_{n-1} is without Darboux polynomials.

Put $R = k[x_1, \ldots, x_{n-1}]$, $d = \delta_{n-1}$, $D = \delta_n$, $b = x_{n-2}x_{n-1}^{s-1}$ and $t = x_n$. Then $R[t] = k[x_1, \ldots, x_n]$, $b \in R$ and D is a derivation of $R[t]$ such that $D(r) = d(r)$ for all $r \in R$, and $D(t) = b$. We will show that $b \notin d(R)$.

Suppose that $b = d(f)$ for some $f \in R$, and denote by g the homogeneous component of f of degree 1. Since d is homogeneous of degree s and b is homogeneous of degree s, we have the equality $b = d(g)$. Let $g = \beta_1 x_1 + \cdots + \beta_{n-1}x_{n-1}$ with $\beta_1, \ldots, \beta_{n-1} \in k$. Then, if $n = 4$ the equality $b = d(g)$ implies the obvious contradiction $x_2 x_3^{s-1} = \beta_1 x_2^s + \beta_2 x_3^s + \beta_3 x_1^s$. If $n \geq 5$, then we have also a contradiction:

$$x_{n-2}x_{n-1}^{s-1} = \beta_1 x_2^s + \beta_2 x_3^s + \beta_3 x_1^s + \beta_4 x_2 x_3^{s-1} + \cdots + \beta_{n-1} x_{n-3} x_{n-2}^{s-1}.$$

Therefore, $b \notin d(R)$. This means, by Theorem 2.2 and an induction, that the derivation $\delta_n = D$ is without Darboux polynomials. $\square$

Using the same proof we obtain the following example for $s = 3$.

Example 3.2. If $n \geq 4$, then the derivation

$$x_2^3 \frac{\partial}{\partial x_1} + x_3^3 \frac{\partial}{\partial x_2} + x_1^3 \frac{\partial}{\partial x_3} + x_1 x_2 x_3 \frac{\partial}{\partial x_4} + \cdots + x_{n-3} x_{n-2} x_{n-1} \frac{\partial}{\partial x_n}$$

of $k[x_1, \ldots, x_n]$ is without Darboux polynomials.

In the same way we obtain the following generalization of the above example.

Example 3.3. If $n \geq 4$, $n \geq 3$, then the derivation

$$x_2^s \frac{\partial}{\partial x_1} + x_3^s \frac{\partial}{\partial x_2} + x_1^s \frac{\partial}{\partial x_3} + x_1 x_2 x_3^{s-2} \frac{\partial}{\partial x_4} + \cdots + x_{n-3} x_{n-2} x_{n-1}^{s-2} \frac{\partial}{\partial x_n}$$

of $k[x_1, \ldots, x_n]$ is without Darboux polynomials.

Note also an example for four variables.

Example 3.4. The derivation $y^2 \frac{\partial}{\partial x} + z^2 \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z} + (xt + yz) \frac{\partial}{\partial t}$ of the polynomial ring $k[x, y, z, t]$ is without Darboux polynomials.

Proof. Denote this derivation by D and put $R = k[x, y, z]$, $d = J(3, 2)$, $a = x$ and $b = yz$. Then $R[t] = k[x, y, z, t]$, $a, b \in R$ and D is a derivation of $R[t]$ such that $D(r) = d(r)$ for all $r \in R$, and $D(t) = at + b$. Note that d is homogeneous of degree 2 and, by Theorem 1.2, d is without Darboux polynomials. We will show that there exist no elements f of R such that $d(f) = af + b$.

Suppose that $d(f) = af + b$ for some $f \in R$. It is clear that $f \notin k$. Denote by g the initial homogeneous component of f. If $\deg g \geq 2$ then, by the homogeneity, $d(g) = xg$ which contradicts the fact that d is without Darboux polynomials. Hence g is of the form $\alpha x + \beta y + \gamma z$, for some $\alpha, \beta, \gamma \in k$. Comparing the initial homogeneous components in the equality $d(f) = af + b$, we obtain the equality

$$\alpha y^2 + \beta z^2 + \gamma x^2 = \alpha x^2 + \beta xy + \gamma xz + yz.$$

But this equality is an obvious contradiction. Thus, there exist no elements f of R such that $d(f) = af + b$. This implies, by Theorem 2.1, that the derivation D is without Darboux polynomials. □

Using the same proof and an induction, we have the following generalization of Example 3.4.

Example 3.5. If $n \geq 4$, then the derivation

$$x_2^2 \frac{\partial}{\partial x_1} + x_3^2 \frac{\partial}{\partial x_2} + x_1^2 \frac{\partial}{\partial x_3} + (x_1 x_4 + x_2 x_3) \frac{\partial}{\partial x_4} + \cdots + (x_{n-3} x_n + x_{n-2} x_{n-1}) \frac{\partial}{\partial x_n}$$

of $k[x_1, \ldots, x_n]$ is without Darboux polynomials.

4. Homogeneous monomial derivations of $k[x, y, z]$

If d is a derivation from the examples of the previous section, then $d([x_1, x_2, x_3]) \subseteq k[x_1, x_2, x_3]$ and the restriction of d to $k[x_1, x_2, x_3]$ is the Jouanolou derivation $J(3, s)$. We used $J(3, s)$, because we know (by Theorem 1.2) that $J(3, s)$ is a homogeneous derivation of $k[x, y, z] = k[x_1, x_2, x_3]$ without Darboux polynomials. We will show that there exist other examples of homogeneous derivations of $k[x, y, z]$ without Darboux polynomials.

Let $s \geq 2$ be a fixed integer. In this section we consider derivations of the form $d : k[x, y, z] \to k[x, y, z]$, where $d(x), d(y), d(z)$ are monic monomials of the same degree s. We denote such a derivation by $[d(x), d(y), d(z)]$, and we say that d is *irreducible* if $\gcd(d(x), d(y), d(z)) = 1$. Moreover, we say that such a monomial derivation is *strict* if $d(x)$ is not divisible by x, $d(y)$ is not divisible by y and $d(z)$ is not divisible by z. It is clear that if d is strict then d is irreducible, and if d is without Darboux polynomials then d is strict. Note also (see for example [12]) that if d is without Darboux polynomials, then d is without nontrivial rational constants.

In [12] (see Section 10) there exists a list of all monomial derivations of $k[x, y, z]$ of degree 2 without nontrivial rational constants. This list contains 40 derivations divided into 8 parts. The derivations in each part are the same, up to permutations of variables. We see here only two derivations, up to permutations of variables, which are strict. One of them is the Jouanolou derivation $J(2,3) = [y^2, z^2, x^2]$. The second derivation is of the form $[y^2, z^2, xy]$.

Theorem 4.1. *The derivation* $[y^2, z^2, xy]$ *is without Darboux polynomials.*

Proof. Put $d = [y^2, z^2, xy]$, and suppose that there exists a polynomial $F \in k[x, y, z] \setminus k$ such that $d(F) = \Lambda F$ for some $\Lambda \in k[x, y, z]$.

Let $\bar{k}$ be the algebraic closure of the field k and let $\bar{d}$ mean the derivation $[y^2, z^2, xy]$ of the polynomial ring $\bar{k}[x, y, z]$. Then we have the equality $\bar{d}(F) = \Lambda F$, where $\Lambda, F \in \bar{k}[x, y, z]$ and $F \notin \bar{k}$. Thus, we may assume that the field k is algebraically closed. Moreover, since d is homogeneous, we may assume that the polynomials F and Λ are also homogeneous.

Let $\sigma : k[x, y, z] \to k[x, y, z]$ be a k-algebra automorphism defined by

$$\sigma(x) = \varepsilon^4 x, \quad \sigma(y) = \varepsilon^2 y, \quad \sigma(z) = \varepsilon z,$$

where ε is a primitive root of unity of degree 5. Denote by G the subgroup of the group of all k-algebra automorphisms of $k[x, y, z]$, generated by σ. Then we have $G = \{\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$, where σ^5 is the identity. We say that a polynomial $H \in k[x, y, z]$ is G-invariant, if $\tau(H) = H$ for every $\tau \in G$. It is clear that, in our case, a nonzero polynomial H is G-invariant if and only if each monomial of H is G-invariant. Moreover, a monomial $ux^p y^q z^r$, where $0 \neq u \in k$, is G-invariant if and only if the integer $4p + 2q + r$ is divisible by 5. In particular, there are no nonzero homogeneous G-invariant polynomials of degree 1.

Observe that the derivation d is G-invariant, that is, $\tau d \tau^{-1} = d$ for any $\tau \in G$. Hence, the equality $d(F) = \Lambda F$ implies the equality $d(F') = \Lambda' F'$, where

$$F' = F \cdot \sigma(F) \cdot \sigma^2(F) \cdot \sigma^3(F) \cdot \sigma^4(F), \quad \Lambda' = \Lambda + \sigma(\Lambda) + \sigma^2(\Lambda) + \sigma^3(\Lambda) + \sigma^4(\Lambda).$$

The polynomials F', Λ' belong to $k[x, y, z]$ and $F' \notin k$. They are homogeneous and G-invariant. Suppose that $\Lambda' \neq 0$. Then, since d is homogeneous of degree 2, the degree of Λ' is equal to 1. We already know that there are no nonzero homogeneous G-invariant polynomials of degree 1. This means that $\Lambda' = 0$. Now we have: $d(F') = 0$ and $F' \notin k$. But this is a contradiction, because – by [12] – the derivation d is without nontrivial rational constants. □

As a consequence of the above theorem, Theorem 1.2 and [12] Proposition 10.1 we obtain the following proposition.

Proposition 4.2. *Let* $d : k[x, y, z] \to k[x, y, z]$ *be an irreducible derivation such that* $d(x), d(y), d(z)$ *are monic monomials of the same degree* 2. *Then d is without Darboux polynomials if and only if d can be found in the following list of* 8 *derivations divided into* 2 *parts. The derivations in each part are the same, up to permutations of variables.*

(1) $[y^2, z^2, xy]$, $[z^2, xz, y^2]$, $[z^2, x^2, xy]$, $[yz, z^2, x^2]$, $[y^2, xz, x^2]$, $[yz, x^2, y^2]$;
(2) $[y^2, z^2, x^2]$, $[z^2, x^2, y^2]$.

Using the results of [12] and various modifications of the proof of Theorem 4.1 we may describe all monomial homogeneous derivations of $k[x, y, z]$ without Darboux polynomials. Now we present such descriptions for $s = 3$ and 4.

Proposition 4.3. *Let* $d : k[x, y, z] \to k[x, y, z]$ *be an irreducible derivation such that* $d(x), d(y), d(z)$ *are monic monomials of the same degree 3. Then d is without Darboux polynomials if and only if d can be found in the following list of 28 derivations divided into 6 parts. The derivations in each part are the same, up to permutations of variables.*

(1) $[z^3, xz^2, y^3]$, $[y^3, x^2z, x^3]$, $[yz^2, z^3, x^3]$, $[z^3, x^3, x^2y]$, $[y^2z, x^3, y^3]$, $[y^3, z^3, xy^2]$;
(2) $[y^3, x^2z, xy^2]$, $[y^2z, xz^2, y^3]$, $[yz^2, x^2z, x^3]$, $[yz^2, z^3, xy^2]$, $[y^2z, x^3, x^2y]$, $[z^3, xz^2, x^2y]$;
(3) $[y^2z, z^3, x^3]$, $[y^3, z^3, x^2y]$, $[z^3, x^3, xy^2]$, $[y^3, xz^2, x^3]$, $[yz^2, x^3, y^3]$, $[z^3, x^2z, y^3]$;
(4) $[yz^2, x^3, xy^2]$, $[y^2z, z^3, x^2y]$, $[y^3, xz^2, x^2y]$, $[y^2z, xz^2, x^3]$, $[yz^2, x^2z, y^3]$, $[z^3, x^2z, xy^2]$;
(5) $[z^3, x^3, y^3]$, $[y^3, z^3, x^3]$;
(6) $[yz^2, x^2z, xy^2]$, $[y^2z, xz^2, x^2y]$.

Proposition 4.4. *Let $d : k[x, y, z] \to k[x, y, z]$ be an irreducible derivation such that $d(x), d(y), d(z)$ are monic monomials of the same degree 4. Then d is without Darboux polynomials if and only if, up to permutations of variables, d can be found in the following list of 11 derivations.*

$$[z^4, x^2z^2, y^4],\ [z^4, xz^3, y^4],\ [z^4, x^3z, y^4],\ [z^4, x^2z^2, xy^3],\ [z^4, x^3z, xy^3],\ [z^4, xz^3, x^2y^2],$$
$$[z^4, x^3z, x^2y^2],\ [z^4, xz^3, x^3y],\ [yz^3, x^2z^2, xy^3],\ [z^4, x^2z^2, x^3y],\ [z^4, x^4, y^4].$$

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