Complements to *Generic polynomial vector fields are not integrable*

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Abstract

In the paper *Generic polynomial vector fields are not integrable* [1], we study some generic aspects of polynomial vector fields or polynomial derivations with respect to their integration.

Using direct sums of derivations together with our previous results we showed that, for all $n \geq 3$ and $s \geq 2$, the absence of polynomial first integrals, or even of Darboux polynomials, is generic for homogeneous polynomial vector fields of degree s in n variables.

To achieve this task, we need an example of such vector fields of degree $s \geq 2$ for any prime number $n \geq 3$ of variables and also for $n = 4$.

The purpose of this note is to correct a gap in our paper for $n = 4$ by completing the corresponding proof.

1 Introduction

We are interested in some generic aspects of polynomial vector fields or polynomial derivations with respect to integration. Precisely, we want to show that the absence of polynomial first integrals, or even of Darboux polynomials, is generic (in the Baire category sense) for homogeneous polynomial vector fields of degree s in n variables for all $n \geq 3$ and $s \geq 2$.

Using direct sums of derivations together with previous results of us [2], this fact can be settled provided that there is an example of such vector fields of degree $s \geq 2$ for any prime number $n \geq 3$ of variables and also for $n = 4$.

There is a gap in our proof when we exhibit the Jouanolou derivation $J_{4,s}$ as the sought example for $n = 4$; the weak point lies at the very end of the paper [1]. To use our Lemma 1, we only need an exponent α

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- which belongs to the exposed face $\mathcal{E}$,
- such that α_2 is 0 or 1,
- with a total degree $|\alpha| \leq m$.

The discussion is subsections 3.9 and 3.10 does prove, at least in the situations we are interested in, that, for a square-free Darboux polynomial at a Darboux point, there exists, for every local coordinate u_i, an exponent α in $\mathcal{E}$ for which the component α_i is either 0 or 1.

But, there is no *general* reason for the exposed face $\mathcal{E}$ to consist only of exponents of total degree less than or equal to the degree m of the polynomial.

What remains true of the local analysis around a Darboux point that we did in our paper is the following.

Remark 1 *If a square-free Darboux polynomial F of some derivation d of $\mathbb{K}[x_1, \cdots, x_n]$ vanishes at a Darboux point M of d, the corresponding powers series ϕ in the local variables is square-free in the unique factorization domain $\mathbb{K}[[u_1, \cdots, u_{n-1}]]$ of power series.*

If moreover the linear part of the local derivation at M can be put in diagonal form $\sum_{i=1}^{n-1} \lambda_i u_i \partial_i$ without non-trivial tuple α in $\mathbb{Z}^{n-1}$ such that $\sum_{i=1}^{n-1} \lambda_i \alpha_i = 0$ and $\sum_{i=1}^{n-1} \alpha_i = 0$, then the exposed face $\mathcal{E}$ of F at M either consists of only one exponent or is a line and the 0–1 constraint holds for the square-free F at M: for every i, there exists an $\alpha \in \mathcal{E}$ such that α_i is 0 or 1.

This is exactly what happens for $FJ_{4,s}$ at $[1, 1, 1, 1]$ and for $J_{n,2}$ (n is an odd prime) at $[1, \cdots, 1]$.

Using this remark, our correction will thus consist of three points :

- for $s \geq 3$, the factored derivation $FJ_{4,s}$ has no strict Darboux polynomial,
- for any prime number $n \geq 3$, the Jouanolou derivation $J_{n,2}$ has no Darboux polynomial,
- the Jouanolou derivation $J_{4,2}$ has no Darboux polynomial.

2 $FJ_{4,s}$ has no strict Darboux polynomial

Suppose that F is a strict irreducible Darboux polynomial of degree m for $FJ_{4,s}$.
At $U = [1, 1, 1, 1]$, the exposed face for F would consist of all $[\alpha_1, \alpha_2, \alpha_3]$ in $\mathbb{N}^3$ such that

$$\alpha_1 = \alpha_3, \quad \alpha_1 + (1 + s)\alpha_2 + \alpha_3 = (s - 1)m + L,$$

for some integer $L \geq 2$.
$\alpha_1 = \alpha_3$ takes its minimal value ϵ, (which is 0 or 1 according to Remark 1) for the exponent of minimal total degree $\mu \leq m$.
Let $\overline{\alpha_2}$ be the corresponding value of α_2 ($\mu = \overline{\alpha_2} + 2\epsilon$).
From degree μ to degree m, there is a *propagation of non-support* : the *minimal* degree in u_2 of all monomials in the support of F cannot decrease too fast, by 1 if there is no exponent in the exposed face $\mathcal{E}$ of degree $\mu + k$ or have a larger jump if there is an exponent in $\mathcal{E}$ of degree $\mu + k$.
Precisely, define a finite sequence $(d_k), 0 \leq k \leq m - \mu$ of nonnegative integers in the following way

- $d_0 = \overline{\alpha_2}$
- if there is no exponent of degree $\mu + k$ in $\mathcal{E}$, then $d_k = \sup(0, d_{k-1} - 1)$
- if there is an exponent $[., \alpha_2, .]$ of degree $\mu + k$ in $\mathcal{E}$, then $d_k = \sup(0, \inf(d_{k-1} - 1), \alpha_2)$.

Then, every exponent α in the support of F with a total degree $\mu + k$ has a second coordinate greater than or equal to d_k.
Exposed exponents are such that $\mu + k + s\alpha_2 = \mu + \overline{\alpha_2}$, i. e. $s(\overline{\alpha_2} - \alpha_2) = k$ and thus, in fact, they play no role in the definition of the previous sequence : $d_k = \sup(0, d_0 - k)$.

Now, as u_2 is not a Darboux polynomial, it does not divide F and there is an exponent *in the support of* F with a 0 second coordinate.
Then d_k has to vanish from some k; then $d_{m-\mu} = 0$ which means that $d_0 - (m - \mu) \leq 0$, whence

$$d_0 \leq m - (d_0 + 2\epsilon), \text{hence } d_0 \leq (m/2) - \epsilon$$

and the inequality :

$$(s-1)m + L = 2\epsilon + (1+s)d_0 \leq 2\epsilon + \frac{1+s}{2}(m - 2\epsilon)$$

which yields an upper bound on L

$$L \leq \frac{3-s}{2}m - (s-1)\epsilon.$$

Even if we do not take into consideration the fact that ϵ is 0 or 1, this gives $L \leq 0$ when $s \geq 3$, which is contradictory with $L \geq 2$ $\square$.

3 $J_{n,2}$ has no Darboux polynomial for an odd prime n

Consider the two following automorphisms of the polynomial ring $\mathbb{K}[x_1, \cdots, x_n]$.
Given ϵ, a primitive S-root of unity where $S = 2^n - 1$, the *multiplication* M is defined on the variables by $M(x_i) = \epsilon^{2^{n-i}} x_i$.
The *rotation* R is defined by $R(x_i) = x_{i+1}$ for $1 \leq i < n$ and $R(x_n) = x_1$.
The Jouanolou derivation $J_{n,2}$ commute with the two automorphisms M and R.
M generates a finite cyclic group of order S whereas R generates a finite cyclic group of order n.
Moreover, M and R are related by $R\,M\,R^{-1} = M^2$. Thus, M and R together generate a finite solvable group G of automorphisms of $\mathbb{K}[x_1, \cdots, x_n]$, whose order is $n\,S$; these automorphisms commute with the Jouanolou derivation $J_{n,2}$.
Suppose now that $J_{n,2}$ has an irreducible Darboux polynomial f and consider the subgroup G_f of G consisting of all g that leave f *projectively invariant*, which means that its transform $g(f)$ is a scalar multiple of f. Take the product ϕ of all $g(f)$ for all g in the right-quotient of G by G_f.
This polynomial ϕ is square-free, it is a Darboux polynomial for $J_{n,2}$, its cofactor is G-invariant and thus is 0, which means that $J_{n,2}(\phi) = 0$; moreover, ϕ is projectively invariant under G. Let m be its degree.

The support of ϕ is thus contained in the subset Γ_m of $\mathbb{N}^n$ consisting of all n-tuples α of total mass $|\alpha| = m$ and such that $\sigma(\alpha) = 0$, where $\sigma(\alpha)$ is defined as

$$\sigma(\alpha) = \sum_{i=1}^{n} 2^{n-i}\,\alpha_i \pmod{2^n - 1}.$$

Looking now locally at the Darboux point $[1, \cdots, 1]$, as we did in [2], Theorem 4.1, we see that m and n are related by $m - (n-1)\nu = 2\nu$, where ν is the common value of the exponents of the local variables in the form of minimal degree (n is an odd prime).
According to Remark 1, it can be deduced from the fact that ϕ is square-free that the integer ν is either 0 or 1. Supposing that $\nu = 0$ leads to $m = 0$. Thus ν must be 1 and $m = n + 1$.

Now remark that, on the one hand, $[1, \cdots, 1] \ \in \Gamma_n$ whereas, on the other hand, the n-tuple $[-1, 2, 0, \cdots, 0]$ and its transforms by rotations ρ^i generate the $\mathbb{Z}$-module of all $\alpha \in \mathbb{Z}$ such that $|\alpha| = 1$ and $\sigma(\alpha) = 0$.
Γ_{n+1} thus consists of all $\alpha \in \mathbb{N}^n$ that can be written

$$\alpha \ = \ [1, \cdots, 1] + \sum_{i=1}^{i=n} k_i \rho^i([-1, 2, 0, \cdots, 0]),$$

where the k_i are integers (in $\mathbb{Z}$) and their sum is 1. This is only possible if some k_i is 1 while all others are 0 and Γ_{n+1} consists of only one orbit under rotations, the orbit of the exponent $[0, 3, 1, \cdots, 1]$. Then ϕ could be written

$$\phi = \sum_{i=1}^{n} a_i\, x_{i+1}^2 \prod_{j \neq i} x_j.$$

The candidate support of $J_{n,2}(\phi)$ consists of all n-tuples of nonnegative integers

$$[1, \cdots, 1] \ + \ \rho^i([-1, 2, 0, \cdots, 0]) \ + \ \rho^j([-1, 2, 0, \cdots, 0]), \quad i \neq j.$$

and $J_{n,2}(\phi) = 0$ gives $\frac{n(n-1)}{2}$ linear equations involving the a_i:

$$3\,a_i \ + \ a_j \ = \ 0, \ (\text{when } j = i+1), \quad a_i \ + \ a_j \ = \ 0, \ (\text{when } |j - i| > 1).$$

To conclude that ϕ has to be 0, the sought contradiction, we have to distinguish two cases.
When $n \geq 5$, equations $a_i + a_{i+2} = 0$ and $a_i + a_{i+3} = 0$ give $a_{i+2} = a_{i+3}$ for any i; all a_i are equal and then equal to 0.
In the case $n = 3$ we get a square linear system whose determinant is not 0.

4 $J_{4,2}$ has no Darboux polynomial

Let $J_{4,2}$ be the Jouanolou derivation for $s = 2$ and $n = 4$. It is convenient to introduce new variables to study this derivation. Let u_0, u_1, u_2, u_3 be these *Fourier* coordinates (i is a square root of -1, i. e. a primitive 4-root of unity):

$$\left\{ \begin{array}{rcl} u_0 & = & x_1 + x_2 + x_3 + x_4 \\ u_1 & = & i\,x_1 - x_2 - i\,x_3 + x_4 \\ u_2 & = & -x_1 + x_2 - x_3 + x_4 \\ u_3 & = & -i\,x_1 - x_2 + i\,x_3 + x_4 \end{array} \right.$$

Let d be the derivation $4J_{4,2}$:

$$\begin{cases} d(u_0) &= u_0^2 + u_2^2 + 2u_1u_3 \\ d(u_1) &= -i(2u_0u_1 + 2u_2u_3) \\ d(u_2) &= -2u_0u_2 - u_1^2 - u_3^2 \\ d(u_3) &= i(2u_0u_3 + 2u_2u_1) \end{cases}$$

Let f be a non-trivial irreducible Darboux polynomial of $J_{4,2}$. Using automorphisms like in the previous section, we can build some product ϕ of transforms of f which is a square-free constant of $J_{4,2}$.
Now remark that d increases the partial degree in u_1 and u_3. Write $\phi = \phi_0 + \cdots \phi_m$, where ϕ_k is homogeneous of degree k in u_1 and u_3 and homogeneous of degree $m-k$ in u_0 and u_2.

Let k_0 be the smallest k for which $\phi_k \neq 0$.

The derivation d is the sum $d = d_0 + d_2$ of two derivations, where d_0 is homogeneous of degree 0 and d_2 is homogeneous of degree 2 for the partial degree in u_1 and u_3:

$$\begin{cases} d_0(u_0) &= u_0^2 + u_2^2 \\ d_0(u_1) &= -i(2u_0u_1 + 2u_2u_3) \\ d_0(u_2) &= -2u_0u_2 \\ d_0(u_3) &= i(2u_0u_3 + 2u_2u_1) \end{cases} \qquad \begin{cases} d_2(u_0) &= 2u_1u_3 \\ d_2(u_1) &= 0 \\ d_2(u_2) &= -u_1^2 - u_3^2 \\ d_2(u_3) &= 0 \end{cases}$$

Once again, Remark 1 shows that the minimal value of $\alpha_1 = \alpha_3$ is either 0 or 1, which means that the smallest possible k_0 is either 0 or 2. It remains to show that the two possibilities lead to contradictions.

This will be the purpose of the rest of this note for which we give appropriate new notations in subsection 4.1.

If $k_0 = 2$, then $\phi_2 = A\,u_1^2 + B\,u_1\,u_3 + C\,u_3^2$ would be a nonzero constant of the derivation d_0, where A, B, C are homogeneous polynomials of degree $m-2$ in u_0 and u_2; in subsection 4.2, we show this is impossible.

If $k_0 = 0$, then the equation $d_0(\phi_0)$ quickly gives $m = 3\mu$, $\phi_0 = (u_2\,(3u_0^2 + u_2^2))^\mu$ (up to a nonzero constant), and the second term ϕ_2 would satisfy $d_0(\phi_2) = -d_2(\phi_0)$; in subsection 4.3, we show this is impossible.

4.1 Notations

We will use the following new notations.

$$x := u_0,\ y := u_2,\ d := d_0,\ U := 3x^2 + y^2,\ V := Uy = (3x^2 + y^2)y.$$

So, $d : \mathbb{Q}[x, y] \to \mathbb{Q}[x, y]$ is a derivation defined by

$$d(x) = x^2 + y^2,\ d(y) = -2xy.$$

The polynomial V is a constant of d, and U, y are Darboux polynomials; $d(U) = 2xU$, $d(y) = (-2x)y$.

Consider the factor ring $\mathbb{A} = \mathbb{Q}[x, y]/(U)$, and let $\phi : \mathbb{Q}[x, y] \to \mathbb{A}$ be the natural homomorphism of rings (if $w \in \mathbb{Q}[x, y]$, then $\phi(w)$ is the remainder of w with respect to U). Every polynomial w from $\mathbb{Q}[x, y]$ has a unique presentation

$$w = (3x^2 + y^2)w_0 + a(x)y + b(x),$$

where $w_0 \in Q[x,y]$, $a(x), b(x) \in \mathbb{Q}[x]$. Thus, the ring $\mathbb{A}$, as a set, is the set of all pairs (a,b), where $a, b \in \mathbb{Q}[x]$, and

$$\begin{array}{rcl} (a_1,b_1)+(a_2,b_2) & = & (a_1+a_2, b_1+b_2) \\ (a_1,b_1)\cdot(a_2,b_2) & = & (a_1b_2+a_2b_1, b_1b_2-3x^2a_1a_2). \end{array}$$

We will write the elements of $\mathbb{A}$ in the form $a(x)y+b(x)$, where $y^2=-3x^2$. If $w \in \mathbb{Q}[x,y]$ with $w=(3x^2+y^2)w_0+a(x)y+b(x)$, then $\phi(w)=a(x)y+b(x)$.

Consider the derivation $\Delta : \mathbb{Q}[x,y] \to \mathbb{Q}[x,y]$ defined by $\Delta(x)=x^2$, $\Delta(y)=xy$. Observe that $\Delta(U)=2xU$. In fact, $\Delta(U)=\Delta(3x^2+y^2)=6x\cdot x^2+2y\cdot xy=2x(3x^2+y^2)=2xU$. So, the ideal (U) is Δ-invariant. This derivation induces the derivation $\delta : \mathbb{A} \to \mathbb{A}$ such that

$$\delta(x)=x^2,\ \ \delta(y)=xy.$$

Let $\bar{d} : \mathbb{Q}[x,y] \to \mathbb{Q}[x,y]$ be the derivation defined by:

$$\bar{d} := -\frac{1}{2}d.$$

Then we have:

Lemma 1 $\delta\phi=\phi\bar{d}$.

Proof. It is enough to check that $\delta\phi(x)=\phi\bar{d}(x)$ and $\delta\phi(y)=\phi\bar{d}(y)$. Let us check:

$$\begin{array}{rcl} \delta\phi(x) & = & \delta(x)=x^2, \\ \phi\bar{d}(x) & = & -\frac{1}{2}\phi(x^2+y^2)=\frac{1}{2}\phi((3x^2+y^2)-2x^2)=-\frac{1}{2}(-2x^2)=x^2, \\ \delta\phi(y) & = & d(y)=xy, \\ \phi\bar{d}(y) & = & -\frac{1}{2}\phi d(y)=-\frac{1}{2}\phi(-2xy)=-\frac{1}{2}(-2)xy=xy. \end{array}$$

So, $\delta\phi=\phi\bar{d}$. □

4.2 $k_0=2$ is impossible

Suppose that $\phi_2 = A\,u_1^2 + B\,u_1\,u_3 + C\,u_3^2$ with $d_0(\phi_2)=0$ and $[A,B,C]\neq[0,0,0]$. This can be detailed as

$$\left\{\begin{array}{rcl} d_0(A) - 4\,i\,u_0\,A + 2\,i\,u_2\,B & = & 0 \\ d_0(C) + 4\,i\,u_0\,C - 2\,i\,u_2\,B & = & 0 \\ d_0(B) - 4\,i\,u_2\,A + 4\,i\,u_2\,C & = & 0. \end{array}\right.$$

Then $d_0(B^2-4AC)=0$ and $B^2-4AC = (u_2\,(3u_0^2+u_2^2))^{\overline{\mu}}$ up to a nonzero constant.
Thus $3\overline{\mu}=2m-4=2(m-2)$ and 3 divides $m-2$: $m=3\mu+2$.
It is now convenient to write $A=D+iE$ and $C=D-iE$. The polynomials B, D, E would then satisfy the following system

$$\left\{\begin{array}{rcl} d_0(D) + 4\,u_0\,E & = & 0 \\ d_0(E) - 4\,u_0\,D + 2\,u_2\,B & = & 0 \\ d_0(B) + 8\,u_2\,E & = & 0. \end{array}\right.$$

Using the notations of subsection 4.1, the following proposition then gives the conclusion.

Proposition 1 *Let D, E, B be homogeneous polynomials from $\mathbb{Q}[x,y]$ of the same degree 3μ, where $\mu \geqslant 1$. Assume that*

$$(1) \qquad \begin{cases} d(D) + 4xE & = & 0 \\ d(B) + 8yE & = & 0 \\ d(E) - 4xD + 2yB & = & 0. \end{cases}$$

Then $D = E = B = 0$.

Suppose that there is a nonzero polynomial among D, E, B. Then it is easy to see that all of them are nonzero.

Lemma 2 *If the polynomials D, E, B are as above, then all of them are divisible by y^μ.*

Proof. Let n be the maximal degree in x for B, D, E together. Let then $[d_n, e_n, b_n] \neq [0,0,0]$ be the vector of the corresponding coefficients. We have

$$\begin{bmatrix} n - 2(3\mu - n) & 4 & 0 \\ -4 & n - 2(3\mu - n) & 0 \\ 0 & 0 & n - 2(3\mu - n) \end{bmatrix} \begin{bmatrix} d_n \\ e_n \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the matrix has to be 0, which is only possible if $n = 2\mu$.
Thus the *minimal* degree in y for all monomials of D, E, B is $3\mu - 2\mu = \mu$. $\square$

Lemma 3 *If the polynomials D, E, B are as above, then all of them are divisible by U^μ.*

Proof. Let $p \geqslant 0$ be such integer that

$$D = U^p D_0, \;\; E = U^p E_0, \;\; B = U^p B_0,$$

where $D_0, E_0, B_0 \in \mathbb{Q}[x,y]$ and among D_0, E_0, B_0 there exists a polynomial which is not divisible by U. The polynomials D_0, E_0, B_0 are homogeneous of the same degree $3\mu - 2p$. We will show that $p = \mu$.

Put the above to (1) and remove the factor U^p. Then we have the following three equalities.

$$(2) \qquad \begin{cases} d(D_0) + 2pxD_0 + 4xE_0 & = & 0 \\ d(B_0) + 2pxB_0 + 8yE_0 & = & 0 \\ d(E_0) + 2pxE_0 - 4xD_0 + 2yB_0 & = & 0. \end{cases}$$

Multiplying the above equalities by $-\frac{1}{2}$ we obtain:

$$(3) \qquad \begin{cases} \bar{d}(D_0) - pxD_0 - 2xE_0 & = & 0 \\ \bar{d}(B_0) - pxB_0 - 4yE_0 & = & 0 \\ \bar{d}(E_0) - pxE_0 + 2xD_0 - yB_0 & = & 0. \end{cases}$$

Now, using the homomorphism ϕ and Lemma 1, we have the following equalities in the ring $\mathbb{A}$.

$$(4) \qquad \begin{cases} \delta(\bar{D}_0) - px\bar{D}_0 - 2x\bar{E}_0 & = & 0 \\ \delta(\bar{B}_0) - px\bar{B}_0 - 4y\bar{E}_0 & = & 0 \\ \delta(\bar{E}_0) - px\bar{E}_0 + 2x\bar{D}_0 - y\bar{B}_0 & = & 0, \end{cases}$$

where $\bar{D}_0 = \phi(D_0)$, $\bar{E}_0 = \phi(E_0)$, $\bar{B}_0 = \phi(B_0)$.

Recall that all the polynomials D_0, E_0, B_0 are homogeneous of the same degree $3\mu - 2p$. Moreover, at least one of them is not divisible by U. Put

$$s = 3\mu - 2p.$$

Then we have

$$(5) \qquad \begin{cases} \bar{D}_0 &= a_1 x^{s-1} y + a_0 x^s, \\ \bar{B}_0 &= b_1 x^{s-1} y + b_0 x^s, \\ \bar{E}_0 &= c_1 x^{s-1} y + c_0 x^s, \end{cases}$$

where $a_0, b_0, c_0, a_1, b_1, c_1 \in \mathbb{Q}$. Observe that

$$\delta(\bar{D}_0) - px\bar{D}_0 = 3(\mu - p)a_1 x^s y + 3(\mu - p)a_0 x^{s+1}.$$

Let us check:

$$\begin{aligned} \delta(\bar{D}_0) - px\bar{D}_0 &= \delta(a_1 x^{s-1} y + a_0 x^s) - px(a_1 x^{s-1} y + a_0 x^s) \\ &= (s-1)a_1 x^{s-2} y x^2 + a_1 x^{s-1} x y + s a_0 x^{s-1} x^2 - p a_1 x^s y - p a_0 x^{s+1} \\ &= (s - 1 + 1 - p)a_1 x^s y + (s - p)a_0 x^{s+1} \\ &= (s-p)a_1 x^s y + (s-p)a_0 x^{s+1} \\ &= 3(\mu - p)a_1 x^s y + 3(\mu - p)a_0 x^{s+1}. \end{aligned}$$

We have also similar two equalities for $\bar{B}_0$ and $\bar{E}_0$. Putting this to (4) we obtain the following three equalities in the ring $\mathbb{A}$.

$$\begin{aligned} \gamma a_1 x^s y + \gamma a_0 x^{s+1} - 2c_1 x^s y - 2c_0 x^{s+1} &= 0 \\ \gamma b_1 x^s y + \gamma b_0 x^{s+1} - 4c_0 x^s y + 12 c_1 x^{s+1} &= 0 \\ \gamma c_1 x^s y + \gamma c_0 x^{s+1} + 2a_1 x^s y + 2a_0 x^{s+1} - b_0 x^s y + 3b_1 x^{s+1} &= 0, \end{aligned}$$

where $\gamma = 3(\mu - p)$.

Comparing the coefficients we get:

$$\begin{array}{lll} (1a) & \gamma a_1 - 2c_1 & = 0 \\ (2a) & \gamma a_0 - 2c_0 & = 0 \\ (3a) & \gamma b_1 - 4c_0 & = 0 \\ (4a) & \gamma b_0 + 12c_1 & = 0 \\ (5a) & \gamma c_1 + 2a_1 - b_0 & = 0 \\ (6a) & \gamma c_0 + 2a_0 + 3b_1 & = 0. \end{array}$$

Now suppose that $\gamma \neq 0$. Then, from $(2a)$ and $(3a)$, $a_0 = \frac{2}{\gamma}c_0$ and $b_1 = \frac{4}{\gamma}c_0$. So, by $(6a)$, $\gamma c_0 + \frac{4}{\gamma}c_0 + \frac{12}{\gamma}c_0 = 0$, so $(\gamma^2 + 16)c_0 = 0$ and so, $c_0 = 0$. Hence $a_0 = b_1 = c_0 = 0$. Moreover, using $(1a)$, $(4a)$ and $(5a)$, we have $a_1 = b_0 = c_1 = 0$. Hence, if $\gamma \neq 0$, then $\phi(D_0) = \phi(E_0) = \phi(B_0) = 0$ and this means that all the polynomials D_0, E_0, B_0 are divisible by U. But it is a contradiction. Therefore, $\gamma = 0$, that is, $3(\mu - p) = 0$, so $p = \mu$. □

Now we are ready to complete the proof of Proposition 1.

Proof of Proposition 1. Suppose that $[D, E, B] \neq [0, 0, 0]$. Then all of the polynomials D, E, B are homogeneous of the same degree 3μ and, by Lemmas 2 and 3, they are of the form

$$D = aV^\mu, \;\; B = bV^\mu, \;\; E = cV^\mu,$$

where $V = Uy = (3x^2 + y^2)y$. But $d(V) = 0$, so the equalities (1) imply that: $4xE = 0$ so $E = 0$; $2xD = yB$ so $B = 0$ (because if $B \neq 0$, then $x \nmid B$), and so $D = 0$.

Hence, $D = E = B = 0$, a contradiction □.

4.3 $k_0 = 0$ is impossible

If $k_0 = 0$ then $\phi_0 = (u_2\,(3u_0^2 + u_2^2))^\mu$ up to a nonzero multiplicative constant.
The second term $\phi_2 = A\,u_1^2 + B\,u_1\,u_3 + C\,u_3^2$ satisfies $d_0(\phi_2) = -d_2(\phi_0)$.
A, B, C are homogeneous polynomials is u_0 and u_2 of the same degree $3\mu - 2$.
Writing again $A = D + iE$ and $C = D - iE$, we receive the sought contradiction from the following proposition.

Proposition 2 *Let D, E, B be homogeneous polynomials from $\mathbb{Q}[x, y]$ of the same degree $3\mu - 2$, where $\mu \geqslant 1$. The following system has no solution.*

$$(6) \qquad \begin{cases} d(D) + 4xE & = & -3\mu V^{\mu-1}(x^2 + y^2) \\ d(B) + 8yE & = & 12\mu V^{\mu-1}(xy) \\ d(E) - 4xD + 2yB & = & 0. \end{cases}$$

Lemma 4 *If the polynomials D, E, B are as above, then all of them are divisible by $y^{\mu-1}$.*

Proof. The proof is the same as the proof of Lemma 2. □

Lemma 5 *If the polynomials D, E, B are as above, then all of them are divisible by $U^{\mu-1}$, where $U = 3x^2 + y^2$.*

Proof. The proof is the same as the proof of Lemma 3. □

Now we are ready to complete the proof of Proposition 2.

Proof of Proposition 2. Since D, E, B are homogeneous of the same degree $3\mu - 2$, Lemmas 4 and 5 imply that they are of the following form:

$$D = (d_1x + d_2y)V^{\mu-1},\ E = (e_1x + e_2y)V^{\mu-1},\ B = (b_1x + b_2y)V^{\mu-1},$$

where $d_1, d_2, b_1, b_2, e_1, e_2 \in \mathbb{Q}$. Put this to (6). Recall that $d(V) = 0$. We can remove the factor $V^{\mu-1}$. Then we have:

$$\begin{array}{lcl} d_1x^2 + d_1y^2 - 2d_2xy + 4e_1x^2 + 4e_2xy & = & -3\mu x^2 - 3\mu y^2 \\ b_1x^2 + b_1y^2 - 2b_2xy + 8e_1xy + 8e_2y^2 & = & 12\mu xy \\ e_1x^2 + e_1y^2 - 2e_2xy - 4d_1x^2 - 4d_2xy + 2b_1xy + 2b_2y^2 & = & 0. \end{array}$$

Comparing the coefficients we obtain, the following equalities (among others):

$$d_1 + 4e_1 + 3\mu = 0,\ d_1 + 3\mu = 0,\ e_1 - 4d_1 = 0,$$

Hence, $e_1 = 0, d_1 = 0$ and so, $\mu = 0$, a contradiction □.

References

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