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Constants and Darboux polynomials for tensor products of polynomial algebras with derivations

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Abstract

Let $d_1 : k[X] \to k[X]$ and $d_2 : k[Y] \to k[Y]$ be k-derivations, where $k[X] := k[x_1, \dots, x_n]$, $k[Y] := k[y_1, \dots, y_m]$ are polynomial algebras over a field k of characteristic zero. Denote by $d_1 \oplus d_2$ the unique k-derivation of $k[X, Y]$ such that $d|_{k[X]} = d_1$ and $d|_{k[Y]} = d_2$. We prove that if d_1 and d_2 are *positively homogeneous* and if d_1 has no nontrivial Darboux polynomials, then every Darboux polynomial of $d_1 \oplus d_2$ belongs to $k[Y]$ and is a Darboux polynomial of d_2. We prove a similar fact for the algebra of constants of $d_1 \oplus d_2$ and present several applications of our results.

1 Introduction

Throughout this paper, k is a field of characteristic zero. If A is a commutative k-algebra and $d : A \to A$ is a k-derivation, then the pair (A, d) is called a differential k-algebra; in this case, we denote by A^d the k-subalgebra of constants of d, that is, $A^d = \{a \in A;\ d(a) = 0\}$.

Let (A, d_1) and (B, d_2) be differential k-algebras and consider the pair $(A \otimes_k B, d)$ in which the k-algebra $A \otimes_k B$ is the tensor product of A and B, and d is the k-derivation of $A \otimes_k B$ defined as $d = d_1 \otimes 1 + 1 \otimes d_2$. This pair is the coproduct of (A, d_1) and (B, d_2) in the category of differential k-algebras. Assume that we know the algebras of constants A^{d_1} and B^{d_2}. How to describe the structure of the algebra $(A \otimes_k B)^d$ in terms of A^{d_1} and B^{d_2} ? In general, this is a very difficult question and there are examples where such description is impossible. Every element of $A^{d_1} \otimes_k B^{d_2}$ is a constant for d but, in general, it is not true that the algebra $(A \otimes_k B)^d$ is isomorphic to $A^{d_1} \otimes_k B^{d_2}$.

In this paper we study this problem when A and B are polynomial algebras over k. Given positive integers n and m, $A = k[X] := k[x_1, \dots, x_n]$, $B = k[Y] := k[y_1, \dots, y_m]$. Then $A \otimes_k B$ is the polynomial algebra $k[X, Y] := k[x_1, \dots, x_n, y_1, \dots, y_m]$. If $d_1 : k[X] \to k[X]$ and $d_2 : k[Y] \to k[Y]$ are k-derivations, then we denote by $d_1 \oplus d_2$ the derivation $d_1 \otimes 1 + 1 \otimes d_2$, which means that $d_1 \oplus d_2$ is the derivation d of $k[X, Y]$ such that $d|_{k[X]} = d_1$ and $d|_{k[Y]} = d_2$.

We prove (Theorem **??**) that if d_1 and d_2 are *positively homogeneous* (in some sense to be defined) and if $k[X]^{d_1} = k$, then $k[X,Y]^{d_1 \oplus d_2} = k[Y]^{d_2}$.

The main result of this paper is Theorem **??**, which, in some cases, describes the set of all Darboux polynomials of $d_1 \oplus d_2$.

2 Preliminaries

Let δ be a k-derivation of $k[X]$. Then $k[X]^\delta$ stands for the algebra of constants of δ: $k[X]^\delta = \{F \in k[X];\ \delta(F) = 0\}$. We still denote by δ the unique extension of δ to the field of quotients $k(X) := k(x_1, \dots, x_n)$ of $k[X]$ and $k(X)^\delta$ then denotes the field of constants of δ.

We say that a polynomial $F \in k[X]$ is a *Darboux polynomial* of δ (see [**?**, **?**]) if there exists a polynomial $\Lambda \in k[X]$ such that $\delta(F) = \Lambda F$. In this case, we say that Λ is a *cofactor* of F (if $F \neq 0$, then this cofactor is uniquely determined). Moreover, we say that a Darboux polynomial F is *trivial* if $F \in k$.

By a *direction* in $\mathbb{Z}^n$ we mean a nonzero sequence $\gamma = (\gamma_1, \dots, \gamma_n)$ of integers.

Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a direction. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a sequence of nonnegative integers, then $|\alpha|_\gamma$ denotes the sum $\gamma_1\alpha_1 + \cdots + \gamma_n\alpha_n$ whereas X^α is the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $s \in \mathbb{Z}$; a nonzero polynomial $F \in k[X]$ is said to be a *γ-homogeneous polynomial of degree s* if F is of the form:

$$F = \sum_{|\alpha|_\gamma = s} a_\alpha X^\alpha,$$

where each a_α belongs to k. We assume that the zero polynomial is γ-homogeneous of any degree. It is standard to check (Cf. [**?**] 19) that:

Proposition 2.1. *If F is a nonzero polynomial in $k[X]$, then the following conditions are equivalent:*

(1) *F is γ-homogeneous of degree s;*

(2) $F(t^{\gamma_1}x_1, \dots, t^{\gamma_n}x_n) = t^s F(x_1, \dots, x_n)$ *(in the ring $k(t)[X]$);*

(3) $\gamma_1 x_1 \frac{\partial F}{\partial x_1} + \cdots + \gamma_n x_n \frac{\partial F}{\partial x_n} = sF$. $\square$

In other words, a γ-homogeneous F of degree s is a Darboux polynomial with cofactor s for the linear derivation $\epsilon_\gamma = \sum \gamma_i\, x_i\, \frac{\partial}{\partial x_i}$.

Denote by A_s^γ the group of all γ-homogeneous polynomials from $k[X]$ of degree s. Each A_s^γ is a k-subspace of $k[X]$ and $k[X] = \bigoplus_{s\in\mathbb{Z}} A_s^\gamma$. Moreover, $A_s^\gamma A_t^\gamma \subseteq A_{s+t}^\gamma$ for all $s, t \in \mathbb{Z}$ and $k[X]$ is thus a graded ring. Such a gradation on $k[X]$ is said to be a *γ-gradation*. Every polynomial $F \in k[X]$ has *the γ-decomposition* $F = \sum F_s$ into *γ-components f_s of degree s*. If $F \neq 0$, then γ-deg(F) denotes *the γ-degree of F*, that is, the maximal s such that $F_s \neq 0$. We assume also that γ-deg$(0) = -\infty$.

The derivation δ is called *γ-homogeneous of degree s* if $\delta(A_p^\gamma) \subseteq A_{s+p}^\gamma$, for any $p \in \mathbb{Z}$ (equivalently, if $\delta(x_i) \in A_{s+\gamma_i}^\gamma$ for $i = 1, \dots, n$). Note that each partial derivative $\partial/\partial x_i$, for $i = 1, \dots, n$, is γ-homogeneous of degree $-\gamma_i$. We say that the derivation δ is *linear* if $\delta(x_i) = \sum_{j=1}^n a_{ij}x_j$, for all $i = 1, \dots, n$, where each a_{ij} belongs to k. Every linear derivation is $(1, \ldots, 1)$-homogeneous of degree zero. We say that the derivation δ is *γ-homogeneous* if there exists an integer s such that δ is γ-homogeneous of degree s.

Assume now that δ is γ-homogeneous and let $F \in k[X]$. If $F \in k[X]^\delta$, then each γ-homogeneous component of F belongs also to $k[X]^\delta$. Thus, in this case the algebra $k[X]^\delta$ is generated over k by γ-homogeneous polynomials. Note also:

Proposition 2.2 ([?, ?]). *Let $\delta : k[X] \to k[X]$ be γ-homogeneous of degree s, and let $F \in k[X]$ be a nonzero Darboux polynomial of δ with a cofactor Λ. Then Λ is γ-homogeneous of degree s and all γ-components of F are also Darboux polynomials of δ with a common cofactor equal to Λ.* $\square$

A direction $\gamma = (\gamma_1, \ldots, \gamma_n)$ is said to be *positive* if all integers $\gamma_1, \ldots, \gamma_n$ are positive. We say that a derivation δ of $k[X]$ is *positively homogeneous* if there exists a positive direction γ in $\mathbb{N}^n$ such that δ is γ-homogeneous of a positive degree.

Proposition 2.3. *Assume that $d_1 : k[X] \to k[X]$ and $d_2 : k[Y] \to k[Y]$ are positively homogeneous derivations. Then there exist positive directions η and ε in $\mathbb{N}^n$ and $\mathbb{N}^m$, respectively, such that:*

(*a*) *d_1 is η-homogeneous of a positive degree s,*
(*b*) *d_2 is ε-homogeneous of the same degree s and*
(*c*) *$d_1 \oplus d_2$ is γ-homogeneous of degree s, where $\gamma = (\eta, \varepsilon)$.*

In particular, if $d_1 : k[X] \to k[X]$ and $d_2 : k[Y] \to k[Y]$ are positively homogeneous derivations, then the derivation $d_1 \oplus d_2 : k[X, Y] \to k[X, Y]$ is also positively homogeneous.

Proof. Let $\gamma' = (\gamma'_1, \ldots, \gamma'_n)$ and $\gamma'' = (\gamma''_1, \ldots, \gamma''_m)$ be positive directions in $\mathbb{N}^n$ and $\mathbb{N}^m$, respectively, such that d_1 is γ'-homogeneous of a positive degree s' and d_2 is γ''-homogeneous of a positive degree s''. Put $s := s's''$, $\eta := s''\gamma' = (s''\gamma'_1, \ldots, s''\gamma'_n)$ and $\varepsilon := s'\gamma'' = (s'\gamma''_1, \ldots, s'\gamma''_n)$. Using Proposition **??**, we see that d_1 is η-homogeneous of degree s and d_2 is ε-homogeneous of the same degree s. Let $\gamma = (\eta, \varepsilon) = (s''\gamma'_1, \ldots, s''\gamma'_n, s'\gamma''_1, \ldots, s'\gamma''_m)$. Then γ is a positive direction in $\mathbb{N}^{n+m}$ and the derivation $d_1 \oplus d_2$ is γ-homogeneous of the positive degree s. $\square$

3 Darboux polynomials

Theorem 3.1. *Let $d_1 : k[X] \to k[X]$ and $d_2 : k[Y] \to k[Y]$ be positively homogeneous derivations and let $d = d_1 \oplus d_2$. Assume that d_1 has no Darboux polynomials belonging to $k[X] \smallsetminus k$. Then every Darboux polynomial of d belongs to $k[Y]$ and it is a Darboux polynomial of d_2.*

Proof. Let s, η, ε and γ be as in Proposition **??**, and let $F \in k[X, Y] \smallsetminus k$ be a Darboux polynomial of d with a cofactor $\Lambda \in k[X, Y]$. Then, by Proposition **??**, Λ is γ-homogeneous of degree s and we may assume that F is γ-homogeneous of degree $p \geqslant 1$. The polynomials F and Λ have the following forms:

$$F = \sum_{|\alpha|_\varepsilon \leqslant p} F_\alpha Y^\alpha, \quad \Lambda = \sum_{|\beta|_\varepsilon \leqslant s} \Lambda_\beta Y^\beta,$$

where F_α, Λ_β (for any α and β) are γ-homogeneous polynomials from $k[X]$ of degrees $p-|\alpha|_\varepsilon$ and $s-|\beta|_\varepsilon$, respectively. The polynomials $d(F)$ and ΛF may then be developed as

$$d(F) = \sum_{|\alpha|_\varepsilon \leqslant p} \left(d_1(F_\alpha)Y^\alpha + F_\alpha d_2(Y^\alpha) \right), \quad \Lambda F = \sum_{|\beta|_\varepsilon \leqslant s} \sum_{|\alpha|_\varepsilon \leqslant p} \Lambda_\beta F_\alpha Y^{\alpha+\beta},$$

Since $d(F) = \Lambda F$, we have:

$$0 = \sum_{|\alpha|_\varepsilon \leqslant p} \left(d_1(F_\alpha)Y^\alpha + F_\alpha d_2(Y^\alpha) \right) - \sum_{|\beta|_\varepsilon \leqslant s} \sum_{|\alpha|_\varepsilon \leqslant p} \Lambda_\beta F_\alpha Y^{\alpha+\beta}.$$

Denote by H the polynomial on the right-hand side of this equality and consider the ε-gradation on the polynomial ring $k[X, Y]$ over $k[X]$.
With respect to this gradation, the polynomial H has the form $H = H_0 + H_1 + \cdots + H_{p+s}$, where $H_i \in \bigoplus_{|\alpha|_\varepsilon = i} k[X]Y^\alpha$ for all $i = 0, 1, \ldots, p+s$. Since $H = 0$, we have $H_0 = H_1 = \ldots = H_{p+s} = 0$.

Using an induction with respect to j, we will now show, for all $j \in \{0, 1, \ldots, p-1\}$, that

$$(*) \qquad F_\alpha = 0, \quad \text{for any } \alpha \text{ with } |\alpha|_\varepsilon = j.$$

First consider the case $j = 0$. We know that $0 = H_0 = d_1(F_0)Y^0 - \Lambda_0 F_0 Y^0$, and $d_1(F_0) = \Lambda_0 F_0$. But d_1 has only trivial Darboux polynomials, so $F_0 \in k$. Moreover, F_0 is γ-homogeneous and γ-$\deg(F_0) = p - |0|_\varepsilon = p \geqslant 1$, so $F_0 = 0$. Thus, the condition $(*)$ holds for $j = 0$.

Assume now that $j > 0$ (and of course $j \in \{0, 1, \ldots, p-1\}$) and that $(*)$ is true for all numbers smaller than j. Consider the polynomial H_j.

Case 1. Let $j < s+1$. Then we have:

$$H_j = \sum_{|\alpha|_\varepsilon = j} d_1(F_\alpha)Y^\alpha - \sum_{|\alpha|_\varepsilon + |\beta|_\varepsilon = j} \Lambda_\beta F_\alpha Y^{\alpha+\beta}.$$

By induction, $F_\alpha = 0$ for $|\alpha|_\varepsilon < j$, so

$$0 = H_j = \sum_{|\alpha|_\varepsilon = j} d_1(F_\alpha)Y^\alpha - \sum_{|\alpha|_\varepsilon = j} \Lambda_0 F_\alpha Y^\alpha = \sum_{|\alpha|_\varepsilon = j} \left(d_1(F_\alpha) - \Lambda_0 F_\alpha \right) Y^\alpha,$$

and this implies that $d_1(F_\alpha) = \Lambda_0 F_\alpha$ for $|\alpha|_\varepsilon = j$. But d_1 has only trivial Darboux polynomials, so $F_\alpha \in k$ for $|\alpha|_\varepsilon = j$. Moreover, each such F_α is γ-homogeneous and γ-$\deg(F_\alpha) = p - |\alpha|_\varepsilon = p - j \geqslant 1$, so $F_\alpha = 0$.

Case 2. Let $j \geq s+1$. In this case

$$H_j = \sum_{|\alpha|_\varepsilon = j} d_1(F_\alpha)Y^\alpha + \sum_{|\alpha'|_\varepsilon = j-s} F_{\alpha'} d_2(Y^{\alpha'}) - \sum_{|\alpha|_\varepsilon + |\beta|_\varepsilon = j} \Lambda_\beta F_\alpha Y^{\alpha+\beta}.$$

By induction, each $F_{\alpha'}$ is equal to 0 and $F_\alpha = 0$ for $|\alpha|_\varepsilon < j$, so

$$0 = H_j = \sum_{|\alpha|_\varepsilon = j} d_1(F_\alpha)Y^\alpha - \sum_{|\alpha|_\varepsilon = j} \Lambda_0 F_\alpha Y^\alpha = \sum_{|\alpha|_\varepsilon = j} \left(d_1(F_\alpha) - \Lambda_0 F_\alpha \right) Y^\alpha.$$

Hence, $d_1(F_\alpha) = \Lambda_0 F_\alpha$ and $F_\alpha \in k$ for $|\alpha|_\varepsilon = j$. Each such F_α is γ-homogeneous and γ-$\deg(F_\alpha) = p - |\alpha|_\varepsilon = p - j \geqslant 1$, so $F_\alpha = 0$ for $|\alpha|_\varepsilon = j$. Therefore, we proved the condition $(*)$.

We can now conclude that F reduces to $\sum_{|\alpha|_\varepsilon = p} F_\alpha Y^\alpha$, where the γ-degree of each F_α is equal to $p - |\alpha|_\varepsilon = p - p = 0$. Since the direction γ is positive, each F_α belongs to k. Thus, $F \in k[Y]$. Moreover, $\Lambda F = d(F) = d_2(F) \in k[Y]$, that is, $\Lambda \in k[Y]$. Therefore, F is a Darboux polynomial of d_2. This completes the proof. □

As a consequence of this theorem we obtain:

Corollary 3.2. *Let $d_1 : k[X] \to k[X]$ and $d_2 : k[Y] \to k[Y]$ be positively homogeneous derivations and assume that d_1 and d_2 have only trivial Darboux polynomials. Then every Darboux polynomial of $d_1 \oplus d_2$ is trivial.* □

Every γ-homogeneous derivation of the polynomial ring over k in two variables has a nontrivial Darboux polynomial [?]. There exist homogeneous derivations in three variables without nontrivial Darboux polynomials. It is well known [?, ?, ?, ?] that if $s \geqslant 2$, then the derivation δ of $k[x, y, z]$ defined by $\delta(x) = y^s$, $\delta(y) = z^s$ and $\delta(z) = x^s$ has only trivial Darboux polynomials. Using this fact and Corollary **??** we get the following example in the case $n = m = 3$.

Example 3.3. *Let $R = k[x_1, \dots, x_6]$ and let d be the derivation of R defined by*

$$d(x_1) = x_2^p,\ d(x_2) = x_3^p,\ d(x_3) = x_1^p,\ d(x_4) = x_5^q,\ d(x_5) = x_6^q,\ d(x_6) = x_4^q,$$

where p and q are integers greater than 1. This derivation has only trivial Darboux polynomials. □

Similar examples we can produce for any number of variables divided by 3. Using Theorem **??** and the above mentioned fact we get the following next example.

Example 3.4. *Let $R = k[x_1, x_2, x_3, x_4]$ and let d be the derivation of $k[X]$ defined by*

$$d(x_1) = x_2^p,\ d(x_2) = x_3^p,\ d(x_3) = x_1^p,\ d(x_4) = x_4^q,$$

where p and q are integers greater than 1. Then every nontrivial Darboux polynomial of d is of the form ax_4^s with $0 \neq a \in k$ and $s \geqslant 1$. □

This example is valid also for $q = 1$, as the following more general theorem shows.

Theorem 3.5. *Let $d_1 : k[X] \to k[X]$ be a positively homogeneous derivation without nontrivial Darboux polynomials. If $d_2 : k[Y] \to k[Y]$ is a linear derivation, then every Darboux polynomial of $d_1 \oplus d_2$ belongs to $k[Y]$ and is a Darboux polynomial of d_2.*

Proof. Let γ be a positive direction in $k[X]$ such that d_1 is γ-homogeneous of a positive degree s. Let $F \in k[X,Y]$ be a Darboux polynomial of $d := d_1 \oplus d_2$ with a cofactor $\Lambda \in k[X,Y]$.

Case 1. Assume that the field k is algebraically closed. In this case, using a change of coordinates, we may assume that the matrix of d_2 is triangular. Let

$$d_2(y_i) = a_{i,i}y_i + a_{i,i+1}y_{i+1} + \cdots + a_{i,m}y_m$$

for $i = 1, \dots, m$, where each a_{ij} belongs to k.

Let $F = \sum_\alpha F_\alpha Y^\alpha$, with $F_\alpha \in k[X]$. Then $d(F) = \sum_\alpha d_1(F_\alpha)Y^\alpha + F_\alpha d_2(Y^\alpha)$. Comparing in the equality $d(F) = \Lambda F$ the degrees with respect to Y, we observe that $\Lambda \in k[X]$.

We will show that each F_α belongs to k. Suppose that this is not true. Let β be the maximal element, with respect to lexicographic order on $k[Y]$, such that $F_\beta \notin k$. Comparing in the equality $0 = \sum_\alpha (d_1(F_\alpha)Y^\alpha + F_\alpha d_2(Y^\alpha) - \Lambda F_\alpha Y^\alpha)$ the coefficients of Y^β, we obtain (thanks to our assumptions) that

$$d_1(F_\beta) = aF_\beta + \Lambda F_\beta + b = (a + \Lambda)F_\beta + b,$$

for some $a, b \in k$. Let $H \in k[X]$ be the initial γ-component of F_β and let $p := \gamma\text{-}\deg H$. Note that $d_1(H) \in \cap A^\gamma_{p+s}$.

Suppose that $\Lambda + a = 0$. Then $d_1(F_\beta) = b \in k$, so $d_1(H) \in k$ (because d_1 is γ-homogeneous of a positive degree). Hence, $d_1(H) \in A^\gamma_0 \cap A^\gamma_{p+s} = 0$, that is, $H \in k[X]^{d_1}$ and, in particular, H is a Darboux polynomial of d_1. Since d_1 has only trivial Darboux polynomials, $H \in k$ and consequently, $F_\beta \in k$. But it is a contradiction.

Therefore $\Lambda + a \neq 0$. Let $\lambda \in k[X]$ be the initial γ-component of $\Lambda + a$. Then $\lambda \neq 0$. Comparing the initial γ-components in the equality $d_1(F_\beta) = (a+\Lambda)F_\beta + b$, we obtain that $d_1(H) = \lambda H$, that is, H is a Darboux polynomial of d_1. But d_1 has only trivial Darboux polynomial, so $H \in k$. This implies that $F_\beta \in k$, but it is again a contradiction.

Thus, each F_α belongs to k, that is, $F \in k[Y]$. Since, $\Lambda F = d(F) = d_2(F) \in k[Y]$, we have $\Lambda \in k[Y]$, and this means that F is a Darboux polynomial of d_2.

Case 2. Now we do not assume that k is algebraically closed. Let $\overline{k}$ be an algebraic closure of k, and let $\overline{d}_1 : \overline{k}[X] \to \overline{k}[X]$, $\overline{d}_2 : \overline{k}[Y] \to \overline{k}[Y]$, be the derivations such that $\overline{d}_1(x_i) = d_1(x_i)$ for $i = 1, \dots, n$, and $\overline{d}_2(y_j) = d_2(y_j)$ for $j = 1, \dots, m$. Then F (as an element from $\overline{k}[X,Y]$) is a Darboux polynomial of $\overline{d}_1 \oplus \overline{d}_2$ and hence, by Case 1, $F \in \overline{k}[Y] \cap k[X,Y] = k[Y]$. So, $F \in k[Y]$ and it is clear that F is a Darboux polynomial of d_2. $\square$

4 Constants

Theorem 4.1. *Let $d_1 : k[X] \to k[X]$ and $d_2 : k[Y] \to k[Y]$ be positively homogeneous derivations and let $d = d_1 \oplus d_2$. Assume that $k[X]^{d_1} = k$. Then $k[X,Y]^d = k[Y]^{d_2}$.*

Proof. Let s, η, ε and γ be as in Proposition **??**, and let $F \in k[X,Y]$ be a γ-homogeneous polynomial of degree $p \geqslant 1$ belonging to $k[X,Y]^d \smallsetminus k$. Put

$$F = \sum_{|\alpha|_\varepsilon \leqslant p} F_\alpha Y^\alpha,$$

where each F_α is a γ-homogeneous polynomial from $k[X]$ of degree $p-|\alpha|_\varepsilon$. Since $d(F)=0$, we have
$$0=\sum_{|\alpha|_\varepsilon\leqslant p}\left(d_1(F_\alpha)Y^\alpha+F_\alpha d_2(Y^\alpha)\right).$$
Repeating the proof of Theorem **??** for the Darboux polynomial F with cofactor $\Lambda=0$ we deduce that if $j\in\{0,1,\ldots,p-1\}$, then $F_\alpha=0$ for any α with $|\alpha|_\varepsilon=j$. Thus, we have:
$$F=\sum_{|\lambda|_\varepsilon=p}F_\alpha Y^\alpha,$$
where γ-degree of each F_α is equal to $p-|\alpha|_\varepsilon=p-p=0$. Since the direction γ is positive, each F_α belongs to k. This means that $F\in k[Y]$. Moreover, $d_2(F)=d(F)=0$. Therefore, $F\in k[Y]^{d_2}$. This completes the proof. □

As a consequence of this theorem we obtain:

Corollary 4.2. *Let $d_1:k[X]\to k[X]$ and $d_2:k[Y]\to k[Y]$ be positively homogeneous derivations. If $k[X]^{d_1}=k$ and $k[Y]^{d_2}=k$, then $k[X,Y]^{d_1\oplus d_2}=k$.* □

The following example shows that a similar corollary is not true if instead of rings of constants we consider fields of constants.

Example 4.3. *Let $n=m=1$ and let $d_1:k[X]\to k[X]$, $d_2:k[Y]\to k[Y]$ be the derivations such that $d_1(x_1)=x_1^2$ and $d_2(y_1)=y_1^2$. Then d_1 and d_2 are positively homogeneous, $k(X)^{d_1}=k$, $k(Y)^{d_2}=k$, and $k(X,Y)^{d_1\oplus d_2}\neq k$. The rational function $\frac{x_1-y_1}{x_1y_1}$ belongs to $k(X,Y)^{d_1\oplus d_2}\smallsetminus k$.* □

The next example shows that if homogeneous derivations d_1 and d_2 are not positively homogeneous, then Corollary **??** is not true, in general.

Example 4.4. *Let $n=m=1$ and let $d_1:k[X]\to k[X]$, $d_2:k[Y]\to k[Y]$ be the derivations such that $d_1(x_1)=-x_1$ and $d_2(y_1)=y_1$. Then d_1 and d_2 are homogeneous with respect to the ordinary directions and they are not positively homogeneous; both derivations are homogeneous of degree zero. In this case $k[X]^{d_1}=k$ and $k[Y]^{d_2}=k$, but $k[X,Y]^{d_1\oplus d_2}\neq k$. The polynomial x_1y_1 belongs to $k[X,Y]^{d_1\oplus d_2}\smallsetminus k$.* □

In this example both derivations are homogeneous but not positively homogeneous. There is no similar examples in which d_2 is as above and d_1 positively homogeneous. This follows from the following, more general, theorem.

Theorem 4.5. *Let γ be a direction in $k[X]$ (not necessary positive) and let $d_1:K[X]\to k[X]$ be a γ-homogeneous derivation of degree $s\neq 0$. If $d_2:k[Y]\to k[Y]$ is a linear derivation with $k[Y]^{d_2}=k$, then $k[X,Y]^{d_1\oplus d_2}=k[X]^{d_1}$.*

Proof. As in the proof of Theorem **??** we may assume that the field k is algebraically closed, and that $d_2(y_i) = a_{i,i}y_i + a_{i,i+1}y_{i+1} + \cdots + a_{i,m}y_m$ for $i = 1, \ldots, m$, where each a_{ij} belongs to k. Since $k[X]^{d_2} = k$, the elements $\lambda_1 := a_{1,1}, \ldots, \lambda_m := a_{m,m}$ are linearly independent over $\mathbb{N}$ (see for example [**?**]).

Let $F = \sum_\alpha F_\alpha Y^\alpha$, with $F_\alpha \in k[X]$, be a nonzero polynomial from $k[X,Y]$ belonging to $k[X,Y]^d$, where $d := d_1 \oplus d_2$. Let $\beta = (\beta_1, \ldots, \beta_m)$ be the maximal element, with respect to lexicographic order on $k[Y]$, such that $F_\beta \neq 0$. Since $d(F) = 0$, we have the equality $0 = \sum_\alpha (d_1(F_\alpha)Y^\alpha + F_\alpha d_2(Y^\alpha))$. Comparing in this equality the coefficients of Y^β we obtain (thanks to our assumptions) that

$$d_1\left(F_\beta\right) = \mu F_\beta,$$

where $\mu = -(\beta_1\lambda_1 + \cdots + \beta_m\lambda_m)$. Let $H \in k[X]$ be the initial γ-component of F_β and let $p := \gamma\text{-}\deg H$. Since F_β is a Darboux polynomial of d_1 and d_1 is γ-homogeneous, Proposition **??** implies that $d_1(H) = \mu H$.

If $\mu \neq 0$, then we have a contradiction: $0 \neq \mu H = d_1(H) \in A_p^\gamma \cap A_{p+s}^\gamma = 0$. Thus, $\beta_1\lambda_1 + \cdots + \beta_m\lambda_m = -\mu = 0$ and this implies (since $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{N}$) that $\beta = (0, \ldots, 0)$. Hence, $F = F_0 \in k[X]^{d_1}$. $\square$

In Example **??** the derivations d_1 and d_2 are linear. There is no similar example in which d_2 is linear and d_1 is positively homogeneous. This follows from the following theorem.

Theorem 4.6. *Let d_1 be a positively homogeneous derivation of $k[X]$ with $k[X]^{d_1} = k$. If d_2 is a linear derivation of $k[Y]$, then $k[X,Y]^{d_1 \oplus d_2} = k[Y]^{d_2}$.*

Proof. As in the proof of Theorem **??** we may assume that the field k is algebraically closed, and that $d_2(y_i) = a_{i,i}y_i + a_{i,i+1}y_{i+1} + \cdots + a_{i,m}y_m$ for $i = 1, \ldots, m$, where each a_{ij} belongs to k. Assume moreover, that γ is such a positive direction in $k[X]$ that d_1 is γ-homogeneous of a positive degree s.

Let $F = \sum_\alpha F_\alpha Y^\alpha$, with $F_\alpha \in k[X]$, be an element from $k[X,Y]$ belonging to $k[X,Y]^d$, where $d := d_1 \oplus d_2$. We will show that each F_α belongs to k.

Suppose that this is not true. Let β be the maximal element, with respect to lexicographic order on $k[Y]$, such that $F_\beta \notin k$. Since $d(F) = 0$, we have the equality $0 = \sum_\alpha (d_1(F_\alpha)Y^\alpha + F_\alpha d_2(Y^\alpha))$. Comparing in this equality the coefficients of Y^β we obtain (thanks to our assumptions) that

$$d_1\left(F_\beta\right) = aF_\beta + b,$$

for some $a, b \in k$. Let $H \in k[X]$ be the initial γ-component of F_β and let $p := \gamma\text{-}\deg H$.

Suppose that $a = 0$. Then $d_1(F_\beta) = b \in k$, so $d_1(H) \in k$ (because d_1 is γ-homogeneous). Hence, $d_1(H) \in A_0^\gamma \cap A_{p+s}^\gamma = 0$, that is, $H \in k[X]^{d_1}$. But $k[X]^{d_1} = k$, so $H \in k$ and we have a contradiction.

Therefore $a \neq 0$. Comparing the initial γ-components is the equality $d_1\left(F_\beta\right) = aF_\beta + b$, we obtain that $d_1(H) = aH$. Now we have: $aH = d_1(H) \in A_p^\gamma \cap A_{p+s}^\gamma = 0$, that is $H = 0$, but it is again a contradiction.

Thus, each F_α belongs to k, and this implies that $F \in k[Y]^{d_2}$. $\square$

As a consequence of this theorem and the classical result of Weitzenböck (see for example [**?**]) we get

Corollary 4.7. *Let d_1 be a positively homogeneous derivation of $k[X]$ with $k[X]^{d_1} = k$. If d_2 is a linear derivation of $k[Y]$, then the algebra $k[X,Y]^{d_1 \oplus d_2}$ is finitely generated over k.* $\square$

Let δ_5, δ_6 and δ_7 be the derivations of polynomial rings over k in 5, 6 and 7 variables, respectively, defined as follows:

$$\begin{aligned}
\delta_5 &= y_1^2\frac{\partial}{\partial y_3} + (y_1y_3 + y_2)\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_5},\\
\delta_6 &= y_1^3\frac{\partial}{\partial y_3} + y_2^3y_3\frac{\partial}{\partial y_4} + y_2^3y_4\frac{\partial}{\partial y_5} + y_1^2y_2^2\frac{\partial}{\partial y_6},\\
\delta_7 &= y_1^3\frac{\partial}{\partial y_4} + y_2^3\frac{\partial}{\partial y_5} + y_3^3\frac{\partial}{\partial y_6} + y_1^2y_2^2y_3^2\frac{\partial}{\partial y_7}.
\end{aligned}$$

It is known that the rings of constants of these derivations are not finitely generated over k. The proofs of this fact we have in [**?**] and [**?**] (for δ_7), [**?**] (for δ_6), [**?**] and [**?**] (for δ_5). It is easy to check that these derivations are positively homogeneous. The derivation δ_7 is γ-homogeneous of degree 1 for $\gamma = (1,1,1,2,2,2,5)$. The derivation δ_6 is γ-homogeneous of degree 1 for $\gamma = (1,1,2,4,6,3)$, and δ_5 is γ-homogeneous of degree 1 for $\gamma = (2,5,2,4,3)$. Using the derivations δ_5, δ_6, δ_7 and Theorem **??** we may construct new examples of polynomial derivations with non-finitely generated ring of constants. Let us note:

Proposition 4.8. *Let $d_1 : k[X] \to k[X]$ be a positively homogeneous derivation such that $k[X]^{d_1} = k$, and let $d_2 : k[Y] \to k[Y]$ be a derivation belonging to the set $\{\delta_5, \delta_6, \delta_7\}$. Then the ring of constants of the derivation $d_1 \oplus d_2 : k[X,Y] \to k[X,Y]$ is not finitely generated over k.* $\square$

If the ring of constants of one of the derivations d_1 and d_2 is not finitely generated over k, then it is not true, in general, that the ring of constants of $d_1 \oplus d_2$ is also not finitely generated.

Example 4.9. *Let $d_2 = \delta_5$ and d_1 be the derivation of $k[X] := k[x_1]$ such that $d_1(x_1) = 1$. Then $k[Y]^{d_2}$ is not finitely generated over k and $k[X,Y]^{d_1 \oplus d_2}$ is finitely generated over k.*

Proof. It is obvious, because the derivation $d_1 \oplus d_2$ is locally nilpotent with a slice (see for example, [**?**] or [**?**]). $\square$

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