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Simple Darboux points of polynomial planar vector fields

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Abstract

We are interested in some aspects of the integrability of complex polynomial planar vector fields in finite form. Especially, in the case of simple Darboux points, we deduce the famous Baum–Bott formula from a kind of global residue theorem; our elementary proof essentially relies on Hilbert's Nullstellensatz.

As a corollary of our result, we propose formulas relating the various integers involved in the Lagutinskii–Levelt procedure for a Darboux polynomial at the various Darboux points. In particular, from the whole set of our formulas, it is possible to deduce an upper bound on the degree of irreducible Darboux polynomials in classical cases; with respect to such applications, this corollary seems to provide an alternate tool to usual genus formulas.

As many people do in these subjects, we illustrate our corollary by giving a new simple proof of the fact that the polynomial Jouanolou derivation $y^s\partial_x + z^s\partial_y + x^s\partial_z$, with $s \geqslant 2$, has no Darboux polynomial.

MSC: 34C99; 14B99

1. Introduction

We are interested in some aspects of the integrability of complex polynomial planar vector fields in finite form. To put emphasis on algebraic techniques, we will say that the coefficients lie in some fixed algebraic closed field $\mathbb{K}$ of characteristic 0 instead of the field $\mathbb{C}$ of complex numbers.

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There are many possible definitions of polynomial planar vector fields. Let us start by a provisional one. A *polynomial planar vector field of degree* s over $\mathbb{K}$ is a mapping $V = [V_x, V_y, V_z]$ from $\mathbb{K}^3$ to $\mathbb{K}^3$, where V_x, V_y, V_z are homogeneous polynomials in $\mathbb{K}[x, y, z]$ of the same degree s. Such a vector field defines a mapping, may be with singularities, from the projective plane $\mathbb{P}_2(\mathbb{K})$ to itself, whence the name planar.

A homogeneous $\mathbb{K}$-*derivation* δ_V of degree $s-1$ of the ring $\mathbb{K}[x, y, z]$ corresponds to V:

$$\delta_V = V_x\partial_x + V_y\partial_y + V_z\partial_z,$$

where $\partial_x, \partial_y, \partial_z$ are the three partial derivatives. This correspondence is one-to-one.

A specially interesting vector field is the radial or *Euler* vector field $E[x, y, z] = [x, y, z]$ to which corresponds the derivation $\delta_E = x\partial_x + y\partial_y + z\partial_z$.

A *homogeneous first integral of degree* 0 of V is an element f of some differential extension of $\mathbb{K}[x, y, z]$ (for the three partial derivatives) such that

$$\delta_V(f) = 0, \quad \delta_E(f) = 0, \quad \mathrm{d}f = [\partial_x(f), \quad \partial_y(f), \partial_z(f)] \neq [0, 0, 0].$$

In other words, f is a common *constant* for the two derivations δ_V and δ_E without being a constant for the three derivations $\partial_x, \partial_y, \partial_z$. An f of degree 0 may be considered as a function on $\mathbb{P}_2(\mathbb{K})$.

Homogeneous first integrals of degree 0 are constants for all vector fields $V + \Lambda E$, where Λ in any homogeneous polynomial of degree $s-1$ in $\mathbb{K}[x, y, z]$. The problem of finding first integrals of degree 0 thus concerns the equivalence class of V up to the addition of a multiple of E, or, equivalently, the wedge product of the two vector fields, that can be identified with the 1-form ω (dimension is 3):

$$\omega = \omega_x\,\mathrm{d}x + \omega_y\,\mathrm{d}y + \omega_z\,\mathrm{d}z = (V_y z - V_z y)\,\mathrm{d}x + (V_z x - V_x z)\,\mathrm{d}y + (V_x y - V_y x)\,\mathrm{d}z.$$

The 1-form ω is a *Pfaff projective form* [11], which means that ω is orthogonal to its curl, which is a vector or a 2-form and also to the Euler vector field.

We now define a *polynomial planar vector field* as the equivalence class $\mathbf{V}$ of the previous polynomial planar vector field V up to the addition of some ΛE. If necessary, a distinguished representative $\bar{V}$ of the equivalence class $\mathbf{V}$ may be chosen, the one which is divergence-free: $\partial_x(\bar{V}_x) + \partial_y(\bar{V}_y) + \partial_z(\bar{V}_z) = 0$. It turns out that the curl of ω (its exterior derivative) is equal to $(s+2)\bar{V}$.

The singular points of ω are the common zeroes of the three coordinates of ω in $\mathbb{P}_2(\mathbb{K})$. We are used to calling them the *Darboux points* of $\mathbf{V}$. Indeed, at these points, the vectors $V + \Lambda E$ and E are collinear for any $\lambda \in \mathbb{K}_{s-1}[x, y, z]$; this definition is still meaningful in higher dimensions, when this property is no longer equivalent to the vanishing of a 1-form. In any case, being a Darboux point is only related to the equivalence class $\mathbf{V}$, i.e. to the vector field in our new definition.

A natural assumption of irreducibility has to be made on the coordinates of ω: they are relatively prime polynomials (altogether and then pairwise according to the orthogonality relation $x\,\omega_x + y\,\omega_y + z\,\omega_z = 0$). Otherwise, dividing by the greatest common divisor leads to the same problem with a 1-form of smaller degree.

In that case, the number of Darboux points is finite and, if they are properly counted, i.e. with their multiplicities, there are s^2+s+1 of them [8,9,11].

The problem of finding *rational* first integrals of degree 0, that can be called *rational integration*, has been studied by Poincaré [23–25] and Painlevé [17–22] for instance, under the name of *algebraic integration*.

In his articles, Poincaré points out the interest of the singular points of the system and the importance of their nature. As he noticed, a necessary condition for the existence of a homogeneous *rational* first integral of degree 0 of a planar polynomial vector field over $\mathbb{K}$ is that all its Darboux points are either nodes or saddle points.

Following Poincaré, Darboux points of a vector field are defined as *nodes* and *saddle points* according to the local behavior of the vector field that may be described as follows.

Without lost of generality, we can take $[x_0, y_0, 1]$ as a representative of a Darboux point of $\mathbf{V}$.

In order to cancel the coefficient before ∂_z consider the local two variable vector field v that does not depend on the representative of $\mathbf{V}$:

$$v = z\mathbf{V} - V_z E = -\omega_y\partial_x + \omega_x\partial_y.$$

In the local coordinates, $t = x - x_0$, $u = y - y_0$, v vanishes at $[0,0]$. There, its linear part is given by the 2×2-matrix $L\mathbf{V}_M$:

$$L\mathbf{V}_M = \begin{pmatrix} -\partial_x(\omega_y) & -\partial_y(\omega_y) \\ \partial_x(\omega_x) & \partial_y(\omega_x) \end{pmatrix}_{[x_0,y_0,1]}. \tag{1}$$

Let ρ_M and σ_M be the eigenvalues of $L\mathbf{V}_M$. If the ratio of them is a positive rational number, the singular point is said to be a node. If this ratio is a negative rational number, it is said to be a saddle point.

There is no algebraic way to distinguish between the two eigenvalues and moreover, as $L\mathbf{V}_M$ is defined up to conjugacy, we look for a numerical invariant only depending on the ratio of the eigenvalues.

This leads to the definition of β_M at every Darboux point M of $\mathbf{V}$:

$$\beta_M = \rho_M/\sigma_M + \sigma_M/\rho_M + 2 = (\rho_M + \sigma_M)^2/(\rho_M\sigma_M).$$

In the case of a saddle point M, $\beta_M \leqslant 0$, with the limit value $\beta_M = 0$ for a saddle point with opposite eigenvalues; at a node, $\beta_M \geqslant 4$, with the limit value $\beta_M = 4$ for a node with equal eigenvalues.

This invariant β_M is also well defined even if M is neither a node nor a saddle point as the square of the trace of a 2×2 matrix divided by its determinant, provided that $\rho_M\sigma_M \neq 0$, which amounts to saying that the considered Darboux point M is a simple zero of the homogeneous ideal generated by $\omega_x, \omega_y, \omega_z$.

We will denote by $\kappa_V(M)$ the ratio of V and E at a Darboux point M. κ_V is a homogeneous rational fraction of degree $s-1$:

$$\kappa_V(M) = \frac{V_x}{x}(M) = \frac{V_y}{y}(M) = \frac{V_z}{z}(M)$$

and is well defined as one of the denominators (coordinates) at least is not zero. Clearly, $\kappa_V(M)$ depends on the representative V of the vector field; we will use it as an intermediate notation in more canonical definitions and statements.

The homogeneous rational fraction $\Delta_{\mathbf{V}}$ of degree $2s-2$ is well defined at a Darboux point M:

$$\Delta_{\mathbf{V}}(M) = \frac{\partial_x\omega_x\partial_y\omega_y - \partial_x\omega_y\partial_y\omega_x}{z^2} = \frac{\partial_y\omega_y\partial_z\omega_z - \partial_y\omega_z\partial_z\omega_y}{x^2}$$
$$= \frac{\partial_z\omega_z\partial_x\omega_x - \partial_z\omega_x\partial_x\omega_z}{y^2},$$

as one of the denominators at least is not 0 and the product $\rho_M\sigma_M$ is equal to $z^2\Delta_{\mathbf{V}}(M)$ while the sum $\rho_M + \sigma_M$ is $(s+2)\bar{V}_z$.

At every Darboux point M, simple computations show that β_M can be defined from $\Delta_{\mathbf{V}}$, which only depends on the equivalence class $\mathbf{V}$, and from the κ corresponding to the distinguished $\bar{V}$:

$$\beta_M = \left((s+2)^2\,\frac{(\kappa_{\bar{V}})^2}{\Delta_{\mathbf{V}}}\right)(M).$$

So far as we know, β has first been introduced by Baum and Bott [1] and this number is now known as the Baum–Bott index [2,3,12].

Reading Poincaré, it becomes natural to check if there are always some saddle points ($\beta_M \leqslant 0$) and some nodes ($\beta_M \geqslant 4$) when the vector field is rationally integrable. Doing that, one finds a very simple algebraic relation in many cases and it turns out that this relation remains true even if the vector field is not integrable: *the sum of all values of the invariant β_M at all Darboux points of a polynomial vector field* $\mathbf{V}$ *of degree s is equal to* $(s+2)^2$. Thus, there are nodes (otherwise the sum of all β_M would be negative) and there are saddle points (this sum would be greater than $4(s^2+s+1)$).

In fact, this relation is known as Baum–Bott's formula [2,3,12] and it is a deep result whose proof usually involves high-level tools such as Chern classes for instance.

This powerful tool can be used in the study of Poincaré and Painlevé problems that we quoted before; see [13] for a recent work.

In the case we are interested in, Baum–Bott's formula appears as a side-effect of a more general result involving the geometry of the set of all Darboux points when all of them are simple, i.e. when they are s^2+s+1 of them. This more general result may be considered as a special case of a *global residue theorem* where the finite set of points on which we perform the summation is not a (global) complete intersection. The other reason for publishing this work is that our proof is as elementary as possible, i.e. we only use the fundamental theorem of algebraic geometry, Hilbert's Nullstellensatz.

Our main theorem is thus the following.

Main theorem. *Let* $\mathbf{V}$ *be a polynomial planar vector field of degree s whose all Darboux points are distinct in the projective plane* $\mathbb{P}_2(\mathbb{K})$.

Then, for all pairs $[V_1, V_2]$ *of representatives* of $\mathbf{V}$, $\sum_{M\in\mathscr{D}(\mathbf{V})}(\kappa_{V_1}\kappa_{V_2}/\Delta_{\mathbf{V}})(M) = 1$.

Remark that this assertion only depends on the equivalence class $\mathbf{V}$.

Choosing $V_1 = V_2 = \bar{V}$ amounts to summing $\beta_M/(s+2)^2$ and gives the Baum–Bott formula.

As there is a finite number of Darboux points, we will always choose coordinates such that none of them lie on the line $z = 0$ in our computations and proofs.

With such a convention, the previous formula is written in a non-intrinsic but simpler way:

$$\forall \Lambda_1, \Lambda_2 \in \mathbb{K}_{s-1}[x,y,z] \qquad \sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{(V_z + z\Lambda_1)(V_z + z\Lambda_2)}{\partial_x \omega_x \partial_y \omega_y - \partial_x \omega_y \partial_y \omega_x} \right)(M) = 1$$

with an arbitrary choice of the representative V of $\mathbf{V}$.

Developing the numerator leads to

$$\sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{zV_z(\Lambda_1 + \Lambda_2)}{\partial_x \omega_x \partial_y \omega_y - \partial_x \omega_y \partial_y \omega_x} \right)(M) + \sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{z^2 \Lambda_1 \Lambda_2}{\partial_x \omega_x \partial_y \omega_y - \partial_x \omega_y \partial_y \omega_x} \right)(M) = 0.$$

In particular, as products of homogeneous polynomials of degree $s-1$ generate homogeneous polynomials of degree $2(s-1)$,

$$\sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{z^2 H}{\partial_x \omega_x \partial_y \omega_y - \partial_x \omega_y \partial_y \omega_x} \right)(M) = 0 \tag{2}$$

for any $H \in \mathbb{K}_{2(s-1)}[x,y,z]$; this formula, a kind of global residue theorem, will be the first important step of our proof (Proposition 2).

A consequence can be deduced from this theorem. If F is a Darboux polynomial of degree m for the vector field V with cofactor Λ, the so-called Lagutinskii–Levelt procedure (LL for short) [14,16] gives two non-negative integers i_M and j_M for every point $M \in \mathscr{D}(\mathbf{V})$ such that the sum $\rho_M i_M + \sigma_M j_M$ is well known when ρ_M and σ_M are the previously introduced eigenvalues at M. Then, besides the first consequence which is the Baum–Bott theorem, two similar results also hold and we have three formulas:

$$\sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{(\rho_M + \sigma_M)^2}{\rho_M \sigma_M} \right) = (s+2)^2,$$

$$\sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{(\rho_M + \sigma_M)(\rho_M i_M + \sigma_M j_M)}{\rho_M \sigma_M} \right) = (s+2)m,$$

$$\sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{(\rho_M i_M + \sigma_M j_M)^2}{\rho_M \sigma_M} \right) = m^2.$$

In the case where all ratios ρ_M/σ_M are irrational, an upper bound on the degree of an irreducible Darboux polynomial can be given from these formulas.

As an example, we will use this result to produce a new proof of the fact that the Jouanolou derivation $J_s = y^s \partial_x + z^s \partial_y + x^s \partial_z$ has no Darboux polynomial.

2. Proof of the main theorem

We will use several times the following fact.

Proposition 1. *Let $\delta_V = V_x\partial_x + V_y\partial_y + V_z\partial_z$ be a homogeneous $\mathbb{K}$-derivation of degree $s-1$ of the ring $\mathbb{K}[x,y,z]$. Let $\mathscr{I}$ be the homogeneous ideal generated by $\omega_x = V_yz - V_zy$, $\omega_y = V_zx - V_xz$ and $\omega_z = V_xy - V_yx$. Then, if $\mathscr{I}$ has a finite number of zeroes in the projective space $\mathbb{P}_2(\mathbb{K})$ and if all of them are simple, $\mathscr{I}$ is radical.*

Proof. Without loss of generality, we can suppose that there is no zero on the line $z=0$. Simplicity of zeroes means that the ideal $\mathscr{I}$ and its radical $\sqrt{\mathscr{I}}$ agree in large enough degrees.

Let then F be a homogeneous polynomial vanishing at all zeroes of $\mathscr{I}$. Then z^kF belongs to $\mathscr{I}$ for large enough k.

It remains to be proven that $zP \in \mathscr{I}$ implies $P \in \mathscr{I}$ for any homogeneous P. Thus, A,B,C being homogeneous polynomials,

$$zP = A\omega_x + B\omega_y + C\omega_z = A(V_yz - V_zy) + B(V_zx - V_xz) + C(V_xy - V_yx). \quad (3)$$

Given any homogeneous polynomial P of $\mathbb{K}[x,y,z]$, the notation $\bar{P}$ will denote the homogeneous polynomial of $\mathbb{K}[x,y]$ obtained by the evaluation of P at $z=0$. Eq. (3) thus gives

$$0 = -\bar{A}y\overline{V_z} + \bar{B}x\overline{V_z} + \bar{C}\overline{V_xy - V_yx}. \quad (4)$$

As there is no zero of $\mathscr{I}$ on the line $z=0$, $\overline{V_z}$ and $\overline{V_xy - V_yx}$ are coprime and there exists a homogeneous two-variable polynomial K such that $\bar{C} = K\overline{V_z}$ and $\overline{Ay - Bx} = K\overline{V_xy - V_yx}$.

Since x and y are coprime, writing Eq. (4) as $x\overline{KV_y - B} = y\overline{KV_x - A}$ shows that there exists a homogeneous two-variable polynomial L such that $\bar{A} = Lx + K\overline{V_x}$ and $\bar{B} = Ly + K\overline{V_y}$.

Now define $A' = A - Lx - KV_x$, $B' = A - Ly - KV_y$, $C' = C - Lz - KV_z$.

It is easy to check that $(Lx + KV_x)\omega_x + (Ly + KV_y)\omega_y + (Lz + KV_z)\omega_z = 0$.

Thus $zF = A\omega_x + B\omega_y + C\omega_z = A'\omega_x + B'\omega_y + C'\omega_z$.

As z divides A', B', C', F itself can be written as a linear combination of $\omega_x, \omega_y, \omega_z$. □

The next lemma is the first example of techniques that we will use repeatedly in the present proof.

Lemma 1. *Let* **V** *be a planar polynomial vector field of degree s with simple Darboux points. If a homogeneous polynomial H of degree $2s-2$ vanishes at all points of $\mathscr{D}(\mathbf{V})$ but one, then it also vanishes at this last one.*

Proof. By a linear change of coordinates, we can suppose that all points of $\mathscr{D}(\mathbf{V})$ have a non-zero third coordinate (no Darboux point on the "line at infinity") and that the Darboux point at which H is not known to vanish is $[0,0,1]$.

Then the polynomials xH and yH vanish at all points of $\mathscr{D}(\mathbf{V})$. As Darboux points are simple, the ideal $(\omega_x, \omega_y, \omega_z)$ is radical according to Proposition 1 and xH and yH belong to it. Taking into consideration the orthogonality relation $x\omega_x + y\omega_y + z\omega_z$, we can get rid of ω_z and find four homogeneous polynomials A,B,C,D of degree $s-1$

such that

$$xzH = A\omega_x + B\omega_y,$$

$$yzH = C\omega_x + D\omega_y,$$

from which we deduce two developments of $xyzH$:

$$xyzH = Ay\omega_x + By\omega_y = Cx\omega_x + Dx\omega_y$$

and the relation equating them

$$(Ay - Cx)\omega_x = -(By - Dx)\omega_y.$$

As ω_x and ω_y are coprime, $Ay - Cx$ is divisible by ω_y but its degree s is smaller than the degree $s+1$ of ω_y; thus $Ay - Cx = 0$ and similarly $By - Cx = 0$.

Then there exists homogeneous polynomials E and F of degree $s-2$ such that

$$A = xE, \quad B = xF, \quad C = yE, \quad D = yF$$

from which we deduce $zH = E\omega_x + F\omega_y$. Then H vanishes at $[0, 0, 1]$. $\square$

Lemma 2. *Let* $\mathbf{V}$ *be a planar polynomial vector field of degree s with simple Darboux points. Then the vector space* $\mathbb{K}_{2s-2}[x, y, z]$ *is generated by elements vanishing at all points of* $\mathscr{D}(\mathbf{V})$ *but three at most.*

Proof. By a linear change of coordinates, we can suppose without lost of generality that all points of $\mathscr{D}(\mathbf{V})$ have a non-zero third coordinate.

According to Darboux lemma [8,9,11], there are $S = s^2 + s + 1$ elements in $\mathscr{D}(\mathbf{V})$. Fixing a representative for each of them by $z = 1$ gives a finite set $\bar{\mathscr{E}}$ of S elements of $\mathbb{K}^3$.

For every non-zero $M \in \mathbb{K}^3$, ε_M is the evaluation map on $\mathbb{K}[x, y, z]$: $\varepsilon_M(P) = P(M)$. Restricted to the finite-dimensional $\mathbb{K}$-vector space $\mathbb{K}_{s-1}[x, y, z]$, ε_M is a linear form. A finite set of elements of $\mathbb{K}^3$ is said to be a *testing set* for homogeneous polynomials of degree $s-1$ if the only polynomial of degree $s-1$ vanishing at all these points is the 0 polynomial; in other words, the corresponding set of evaluation maps generates the dual vector space of $\mathbb{K}_{s-1}[x, y, z]$.

For degree reasons, a non-zero element H of $\mathbb{K}_{s-1}[x, y, z]$ cannot evaluate to 0 for all $M \in \bar{\mathscr{E}}$. H would indeed belong to the radical ideal $(\omega_x, \omega_y, \omega_z)$ generated by homogeneous polynomials of degree $s+1$. This means that $\bar{\mathscr{E}}$ is a testing set for $\mathbb{K}_{s-1}[x, y, z]$.

The dimension of this vector space is $\bar{s} = s(s+1)/2$ and there exists a subset $\bar{\mathscr{E}}_1 = \{M_1, \ldots, M_{\bar{s}}\}$ of $\bar{\mathscr{E}}$ with $\bar{s}$ elements which is a (minimal) testing set for $\mathbb{K}_{s-1}[x, y, z]$.

The corresponding evaluation maps constitute a basis of the dual space of $\mathbb{K}_{s-1}[x, y, z]$.

Thus, there exists a dual basis of this basis, i.e. $\bar{s}$ polynomials H_i such that

$$H_i(M_i) = 1 \text{ and } \forall j \neq i, \quad H_i(M_j) = 0.$$

Consider now the finite set $\bar{\mathscr{E}} \setminus \bar{\mathscr{E}}_1$ of $\mathbb{K}^3$. We claim that $\bar{\mathscr{E}} \setminus \bar{\mathscr{E}}_1$ is also a testing set for $\mathbb{K}_{s-1}[x,y,z]$. Indeed, if some $H \in \mathbb{K}_{s-1}[x,y,z]$ vanishes for all elements of $\bar{\mathscr{E}} \setminus \bar{\mathscr{E}}_1$, then, for every $M_i \in \bar{\mathscr{E}}_1$, HH_i vanishes at all Darboux points but possibly M_i; the degree of this product is $2s-2$ and, according to Lemma 1, it also vanishes at M_i. Then H would be zero on all points of $\mathscr{D}(\mathbf{V})$, whereas its degree is $s-1$, a contradiction.

Then, there exists a subset $\bar{\mathscr{E}}_2 = \{\bar{M}_1, \ldots, \bar{M}_{\bar{s}}\}$ of $\bar{\mathscr{E}} \setminus \bar{\mathscr{E}}_1$ with $\bar{s}$ elements which is a (minimal) testing set for $\mathbb{K}_{s-1}[x,y,z]$. There then exists a dual basis of this basis, i.e. $\bar{s}$ polynomials $\bar{H}_i$ such that

$$\bar{H}_i(\bar{M}_i) = 1, \quad \forall j \neq i, \quad \bar{H}_i(\bar{M}_j) = 0.$$

Let remark that $S = \bar{s} + \bar{s} + 1$, so that $\bar{\mathscr{E}} \setminus (\bar{\mathscr{E}}_1 \cup \bar{\mathscr{E}}_2)$ has only one element, that we will call *the extra point*.

It is clear that the $\bar{s}^2$ products $H_i\bar{H}_j$ generate the vector space $\mathbb{K}_{2s-2}[x,y,z]$. Moreover each such product $H_i\bar{H}_j$ vanishes at all points of $\mathscr{D}(\mathbf{V})$ except M_i, $\bar{M}_j$ and maybe the extra point i.e. at all Darboux points but at most three. □

Proposition 2. *Let* $\mathbf{V}$ *be a planar polynomial vector field of degree* s *with simple Darboux points. Then, for every* $H \in \mathbb{K}_{2s-2}[x,y,z]$

$$\sum_{M \in \mathscr{D}(\mathbf{V})} \left(\frac{H}{\Delta_{\mathbf{V}}}\right)(M) = 0.$$

Proof. As the vector space $\mathbb{K}_{2s-2}[x,y,z]$ is generated by elements vanishing at all points of $\mathscr{D}(\mathbf{V})$ but three at most and because the involved sum is $\mathbb{K}$-linear, it suffices to prove the announced result for polynomials of this kind.

Let then H be a homogeneous polynomial of degree $2s-2$ in $\mathbb{K}[x,y,z]$ vanishing at all points of $\mathscr{D}(\mathbf{V})$ but three at most.

Without loss of generality, the coordinates may be chosen in such a way that all Darboux points lie outside the line $z=0$, and we can moreover suppose that the three particular Darboux points are $[0,0,1]$, $[0,1,1]$ and $[1,0,1]$.

Given a homogeneous polynomial P of $\mathbb{K}[x,y,z]$, the notation $\bar{P}$ will denote the homogeneous polynomial of $\mathbb{K}[x,y]$ obtained by the evaluation of P at $z=0$.

As there is no Darboux point at infinity, the polynomials $\overline{V_z}$ and $\overline{yV_x - xV_y}$ are coprime; in particular, $\overline{V_z}$ is not zero. Choose a $\lambda \in \mathbb{K}^\star$ such that $y - \lambda x$ does not divide $\overline{V_z}$. Then define two homogeneous polynomials of $\mathbb{K}_2[x,y,z]$: $\phi = x(y - \lambda x + \lambda z)$ and $\psi = y(y - \lambda x - z)$. They vanish at the three special Darboux points $[0,0,1]$, $[0,1,1]$ and $[1,0,1]$.

Then the polynomials ϕH and ψH vanish at all points of $\mathscr{D}(\mathbf{V})$. As Darboux points are simple, the ideal $(\omega_x, \omega_y, \omega_z)$ is radical according to Proposition 1 and ϕH and ψH belong to it. Taking into consideration the orthogonality relation $x\omega_x + y\omega_y + z\omega_z$, we can get rid of ω_z and find four homogeneous polynomials A, B, C, D of degree s such that

$$\phi z H = A\omega_x + B\omega_y,$$

$$\psi zH = C\omega_x + D\omega_y, \tag{5}$$

Whence $\phi\psi zH = \psi(A\omega_x + B\omega_y) = \phi(C\omega_x + D\omega_y)$.

As ω_x and ω_y are coprime, there exists a linear polynomial L such that

$$L\omega_x = D\phi - B\psi,$$

$$L\omega_y = -C\phi + A\psi. \tag{6}$$

Multiplying matrices, we get $LzH = AD - BC$ as $[\phi, \psi] \neq [0, 0]$.

Taking partial derivatives with respect to x and y in linear system (5) and then evaluating at the three special Darboux points, where ω_x, ω_y, ϕ and ψ vanish, we get

$$zH \begin{bmatrix} \partial_x(\phi) & \partial_y(\phi) \\ \partial_x(\psi) & \partial_y(\psi) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \partial_x(\omega_x) & \partial_y(\omega_x) \\ \partial_x(\omega_y) & \partial_y(\omega_y) \end{bmatrix}, \tag{7}$$

from which can be deduced a relation between determinants which is valid at these three points:

$$z^2H^2. \begin{vmatrix} \partial_x(\phi) & \partial_y(\phi) \\ \partial_x(\psi) & \partial_y(\psi) \end{vmatrix} = (AD - BC). \begin{vmatrix} \partial_x(\omega_x) & \partial_y(\omega_x) \\ \partial_x(\omega_y) & \partial_y(\omega_y) \end{vmatrix}. \tag{8}$$

The ratio $H/\Delta_{\mathbf{V}}$, which is homogeneous of degree 0, vanishes at all Darboux points but $[0, 0, 1]$, $[0, 1, 1]$ and $[1, 0, 1]$ where it is given by

$$\frac{H}{\Delta_{\mathbf{V}}} = \frac{zL}{\begin{vmatrix} \partial_x(\phi) & \partial_y(\phi) \\ \partial_x(\psi) & \partial_y(\psi) \end{vmatrix}}. \tag{9}$$

The sum of the right-hand side ratio at the three points is easily shown to be 0 by elementary computations when $L = z$ or $L = y - \lambda x$. Our proof is then complete once we have shown that $y - \lambda x$ divides $\bar{L}$.

This last argument comes from the line at infinity. Evaluating linear systems (5) and (6) at $z = 0$ indeed yields

$$0 = \bar{A}\overline{\omega_x} + \bar{B}\overline{\omega_y} = -\bar{A}y\overline{V_z} + \bar{B}x\overline{V_z},$$

$$0 = \bar{C}\overline{\omega_x} + \bar{D}\overline{\omega_y} = -\bar{C}y\overline{V_z} + \bar{D}x\overline{V_z}, \tag{10}$$

$$-\bar{L}y\overline{V_z} = \bar{D}x(y - \lambda x) - \bar{B}y(y - \lambda x),$$

$$\bar{L}x\overline{V_z} = -\bar{C}x(y - \lambda x) + \bar{A}y(y - \lambda x). \tag{11}$$

From the first system (10), we deduce $\bar{A}y = \bar{B}x$ and $\bar{C}y = \bar{D}x$. Substituting these results in (11) and canceling factors shows that $y - \lambda x$ divides $\overline{LV_z}$. As we chose $y - \lambda x$ so that it is coprime with $\overline{V_z}$, it divides $\bar{L}$. □

Proposition 3. *Let* $\mathbf{V}$ *be a planar polynomial vector field of degree* $s \geqslant 2$ *with simple Darboux points. Then, for every* $H \in \mathbb{K}_{s-1}[x, y, z]$ *and every representative* V

of $\mathbf{V}$,

$$\sum_{M\in\mathscr{D}(\mathbf{V})}\left(\frac{\kappa_V H}{\Delta_{\mathbf{V}}}\right)=0.$$

Proof. As usual, without lost of generality, the coordinates are chosen in such a way that all Darboux points lie outside the line $z=0$.

It suffices to prove, under the additional assumption that the degree s is greater than 1, that there exists a $H_1\in\mathbb{K}_{2s-2}[x,y,z]$ such that

$$V_zH-zH_1\in(\omega_x,\omega_y)$$

in order to apply Proposition 2 to H_1 to get the conclusion.

The degree $s-1$ of H is at least 1 and $H=zH_2+H_3$ where $H_3\in\mathbb{K}_{s-2}[x,y]$. Thus H_3 is a sum of monomials of positive degrees in x and y, whereas $\omega_x=-yV_z+zV_y$ and $\omega_y=xV_z-zV_x$. Each product of V_z by such a monomial is then equal to a multiple of z modulo the ideal (ω_x,ω_y). □

Proposition 4. *Let* $\mathbf{V}$ *be a planar polynomial vector field of degree* s *with simple Darboux points. There exist two representatives* V_1 *and* V_2 *of* $\mathbf{V}$ *such that the product* $\kappa_{V_1}\kappa_{V_2}$ *vanishes at all points of* $\mathscr{D}(\mathbf{V})$ *but one*, M_0, *and*

$$\sum_{M\in\mathscr{D}(\mathbf{V})}\left(\frac{\kappa_{V_1}\kappa_{V_2}}{\Delta_{\mathbf{V}}}\right)(M)=\left(\frac{\kappa_{V_1}\kappa_{V_2}}{\Delta_{\mathbf{V}}}\right)(M_0)=1.$$

Proof. As usual, without loss of generality, the coordinates are chosen in such a way that all Darboux points lie outside the line $z=0$.

Start with any representative V of $\mathbf{V}$. Following the proof of Lemma 2, there exists a partition of $\mathscr{D}(\mathbf{V})$ in two testing sets for $\mathbb{K}_{s-1}[x,y,z]$ and a singleton. Thus, there exits a $H_1\in\mathbb{K}_{s-1}[x,y,z]$ such that $V_z-zH_1=V_{1z}$ vanishes on the first testing set and a $H_2\in\mathbb{K}_{s-1}[x,y,z]$ such that $V_z-zH_2=V_{2z}$ vanishes on the second testing set.

Then the product $H=(V_z-zH_1)(V_z-zH_2)$ vanishes at all Darboux points of $\mathbf{V}$ but one and so does $\kappa_{V_1}\kappa_{V_2}$.

An affine change of variables for x and y put this extra point at $[0,0,1]$. Moreover, as $\overline{V_z}$ has only finitely many irreducible factors of degree 1, the last change of variables can be performed in such a way that x and y do not divide $\overline{V_z}$.

The involved sum reduces to one term and we have to prove that this term is equal to 1:

$$\frac{(V_z-zH_1)(V_z-zH_2)}{\begin{vmatrix}\partial_x(\omega_x) & \partial_y(\omega_x)\\ \partial_x(\omega_y) & \partial_y(\omega_y)\end{vmatrix}}([0,0,1])=1.$$

The polynomials xH and yH vanish at all points of $\mathscr{D}(\mathbf{V})$. As Darboux points are simple, the ideal $(\omega_x,\omega_y,\omega_z)$ is radical according to Proposition 1 and xH and yH belong to it. Taking into consideration the orthogonality relation $x\omega_x+y\omega_y+z\omega_z$, we can get rid of ω_z and find four homogeneous polynomials A, B, C, D of degree $s+1$ such that

$$xzH = A\omega_x + B\omega_y,$$

$$yzH = C\omega_x + D\omega_y. \tag{12}$$

These various polynomials may be written $A = \bar{A} + z\tilde{A}$, $B = \bar{B} + z\tilde{B}$, $C = \bar{C} + z\tilde{C}$, $D = \bar{D} + z\tilde{D}$, where the $\bar{\square}$ are homogeneous polynomials in $\mathbb{K}_{s+1}[x, y]$ and the $\tilde{\square}$ are homogeneous polynomials in $\mathbb{K}_s[x, y, z]$.

Making $z = 0$ in the above linear system (12) leads to

$$0 = -\bar{A}y\overline{V_z} + Bx\overline{V_z},$$

$$0 = -\bar{C}y\overline{V_z} + Dx\overline{V_z}, \tag{13}$$

whence the existence of $\bar{E}$ and $\bar{F}$ in $\mathbb{K}_s[x, y]$ such that $\bar{A} = x\bar{E}$, $\bar{B} = y\bar{E}$, $\bar{C} = x\bar{F}$, $\bar{D} = y\bar{F}$. Consider now the homogeneous components of degree 1 in z in linear system (12):

$$x\overline{V_z}^2 = -\tilde{A}y\overline{V_z} + \tilde{B}x\overline{V_z} + (\overline{xV_y - yV_x})\bar{E},$$

$$y\overline{V_z}^2 = -\tilde{C}y\overline{V_z} + \tilde{D}x\overline{V_z} + (\overline{xV_y - yV_x})\bar{F}. \tag{14}$$

As $\overline{V_z}$ and $(\overline{xV_y - yV_x})$ are coprime (no Darboux point on $z = 0$), $\overline{V_z}$ divides $\bar{E}$ and $\bar{F}$: $\bar{E} = K_1\overline{V_z}$, $\bar{F} = K_2\overline{V_z}$, where $K_1, K_2 \in \mathbb{K}$.

We can then change A for $A - K_1\omega_y$, B for $B + K_1\omega_x$, C for $C - K_2\omega_y$ and D for $D + K_2\omega_x$ in linear system (12) to get coefficients with $\bar{\square} = 0$; thereafter, we can divide the equations by z to get the simpler system where a, b, c and d have now the degree s:

$$xH = a\omega_x + b\omega_y,$$

$$yH = c\omega_x + d\omega_y. \tag{15}$$

As ω_x and ω_y are coprime, there exists a $K \in \mathbb{K}$ such that

$$K\omega_x = dx + -by,$$

$$K\omega_y = -cx + ay. \tag{16}$$

Composing the two linear systems easily gives (x and y are not 0)

$$KH = ad - bc. \tag{17}$$

Now, taking the partial derivatives with respect to x and y and then evaluating at the point $[0, 0, 1]$ gives the sought quotient

$$\frac{H}{\begin{vmatrix} \partial_x(\omega_x) & \partial_y(\omega_x) \\ \partial_x(\omega_y) & \partial_y(\omega_y) \end{vmatrix}}([0, 0, 1]) = \frac{ad - bc}{H} = K \tag{18}$$

and it remains to be proven that the constant K is 1.

Cancellation by $\overline{V_z}$ in the evaluation of systems (15) and (16) at $z=0$ shows that $\bar{a}$ is divisible by x, $\bar{a}=x\overline{a'}$ and that $\bar{d}$ is divisible by y, $\bar{d}=y\overline{d'}$.

The following linear system can then be deduced:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{b} \\ \bar{c} \\ x\overline{a'} \\ y\overline{d'} \end{bmatrix} = \overline{V_z} \begin{bmatrix} K+1 \\ K-1 \\ K \\ K \end{bmatrix}. \tag{19}$$

The 4×4 matrix is invertible, $x\overline{a'}$ and $y\overline{d'}$ are then multiples of $\overline{V_z}$ and have the same degree. As x and y do not divide $\overline{V_z}$, a' and d' have to be 0 and the second line gives the sought value: $K=1$. □

We can now present the main theorem and achieve its proof.

Theorem 1. *Let* $\mathbf{V}$ *be a polynomial planar vector field of degree s whose all Darboux points are distinct in the projective plane* $\mathbb{P}_2(\mathbb{K})$.

Then, for all pairs $[V_1, V_2]$ *of representatives of* $\mathbf{V}$,

$$\sum_{M\in\mathscr{D}(\mathbf{V})} \left(\frac{\kappa_{V_1}\kappa_{V_2}}{\Delta_{\mathbf{V}}}\right)(M) = 1.$$

Proof. After having chosen new coordinates such that all Darboux points are outside $z=0$, the announced result is equivalent to:

$$\sum_{M\in\mathscr{D}(\mathbf{V})} \left(\frac{(V_z+z\Lambda_1)(V_z+z\Lambda_2)}{\begin{vmatrix} \partial_x(\omega_x) & \partial_y(\omega_x) \\ \partial_x(\omega_y) & \partial_y(\omega_y) \end{vmatrix}}\right)(M) = 1.$$

When the degree s is greater than 1, the proof is an easy consequence of the above three Propositions 2–4.

When $s=1$, Proposition 3 no longer holds and an elementary computational proof leads to the same result; we leave it to the reader. □

3. Consequences and applications

3.1. Formulas for Darboux polynomials

A *Darboux polynomial* of a homogeneous derivation V is a homogeneous polynomial F of degree m such that $V_x\partial_x(F)+V_y\partial_y(F)+V_z\partial_z(F)=\Lambda_V F$ for some $\Lambda_V\in\mathbb{K}_{s-1}[x,y,z]$, called the *cofactor*.

Being a Darboux polynomial only depends on $\mathbf{V}$ but the cofactor depends on the chosen representative V.

Clearly, $\Lambda_{V+\Lambda_1 E} = \Lambda_V + m\Lambda_1$ and the *vector cofactor* $W = V - (\Lambda_V/m)E$ of F is well defined and this representative satisfies

$$W_x\partial_x(F) + W_y\partial_y(F) + W_z\partial_z(F) = 0.$$

The so-called Lagutinskii–Levelt procedure (LL for short) [14,16] gives two non-negative integers i_M and j_M for every point $M \in \mathscr{D}(\mathbf{V})$ such that $\rho_M i_M + \sigma_M j_M = m\kappa_W$, where ρ_M and σ_M are the previously introduced eigenvalues at M normalized in such a way such that $\rho_M + \sigma_M = (s+2)\kappa_{\bar{V}}(M)$.

Then, applying Theorem 1 to the pairs $[\bar{V}, \bar{V}], [\bar{V}, W], [W, W]$ of representatives of $\mathbf{V}$ gives the following three formulas after some simplifications:

$$\sum_{M\in\mathscr{D}(\mathbf{V})} \left(\frac{(\rho_M + \sigma_M)^2}{\rho_M\sigma_M}\right) = (s+2)^2,$$

$$\sum_{M\in\mathscr{D}(\mathbf{V})} \left(\frac{(\rho_M + \sigma_M)(\rho_M i_M + \sigma_M j_M)}{\rho_M\sigma_M}\right) = (s+2)m,$$

$$\sum_{M\in\mathscr{D}(\mathbf{V})} \left(\frac{(\rho_M i_M + \sigma_M j_M)^2}{\rho_M\sigma_M}\right) = m^2. \tag{20}$$

Remarks. The Baum–Bott theorem corresponds to the first of the three above formulas (20).

The second formula is a version of Brunella's formula [3,4] for an invariant curve C:

$$C^2 + Z(\mathscr{F}, C) = N_{\mathscr{F}}C.$$

The third one is a version of Suwa's generalization of Camacho–Sad theorem [4,5,26]:

$$CS(\mathscr{F}, C) = C^2.$$

In both cases, our version gives a more explicit first member and it could be thought that this could be generalized for foliations in more general surfaces.

3.2. *Bounding the degree when Darboux points are irrational*

Let us now suppose that all Darboux points are *irrational*, which means that they are neither nodes not saddle points: the ratio of the eigenvalues of $L\mathbf{V}_M$ is not a rational number.

Let F be an irreducible homogeneous Darboux polynomial of degree m with vector cofactor W. At any Darboux point M of $\mathbf{V}$, the LL indices are then either 0 or 1 according to [15], Proposition 14.

At every point M we will denote by i_m the index corresponding to ρ_M and by j_m the index corresponding to σ_M. The set $\mathscr{D}(\mathbf{V})$ of all Darboux points may be partitioned into four subsets, $\mathscr{D}(\mathbf{V})_0$ where $i_M = j_M = 0$, $\mathscr{D}(\mathbf{V})_1$ where $i_M = 1,\ j_M = 0$, $\mathscr{D}(\mathbf{V})_2$ where $i_M = 0,\ j_M = 1$, and $\mathscr{D}(\mathbf{V})_3$ where $i_M = j_M = 1$.

General system (20) then becomes

$$\sum_{M\in\mathscr{D}(\mathbf{V})}\frac{(\rho_M+\sigma_M)^2}{\rho_M\sigma_M}=(s+2)^2,$$
$$\sum_{M\in\mathscr{D}(\mathbf{V})_1}1+\frac{\rho_M}{\sigma_M}+\sum_{M\in\mathscr{D}(\mathbf{V})_2}1+\frac{\sigma_M}{\rho_M}+\sum_{M\in\mathscr{D}(\mathbf{V})_3}\frac{(\rho_M+\sigma_M)^2}{\rho_M\sigma_M}=(s+2)m,$$
$$\sum_{M\in\mathscr{D}(\mathbf{V})_1}\frac{\rho_M}{\sigma_M}+\sum_{M\in\mathscr{D}(\mathbf{V})_2}\frac{\sigma_M}{\rho_M}+\sum_{M\in\mathscr{D}(\mathbf{V})_3}\frac{(\rho_M+\sigma_M)^2}{\rho_M\sigma_M}=m^2. \tag{21}$$

Subtracting the third equation to the second in (21) gives

$$\sharp(\mathscr{D}(\mathbf{V})_1)+\sharp(\mathscr{D}(\mathbf{V})_2)=m(s+2-m)\geqslant 0, \tag{22}$$

whence the upper bound $s+2$ for the degree of irreducible Darboux polynomials when all Darboux points are irrational.

Remark that similar results usually come from some genus formula [6,7].

3.3. $J_{3,s}$ has no Darboux polynomial

As an example, we will use this result to produce a new elementary proof of the fact that the Jouanolou derivation $J_{3,s}=y^s\partial_x+z^s\partial_y+x^s\partial_z$ has no Darboux polynomial.

Our proof will be simpler than the one given in [12]: thanks to the complete result (our three formulas), we only need Bézout's theorem on the intersection index of two algebraic plane projective curve without considerations about the genus or about Milnor numbers.

First computations are well known [14,16]. There are $N=s^2+s+1$ simple Darboux points and the eigenvalues do not depend on the chosen point: $\rho=\left(s+2+\mathrm{i}s\sqrt{3}\right)/2(s+2)$, $\sigma=\left(s+2-\mathrm{i}s\sqrt{3}\right)/2(s+2)$ if we normalize by $\rho+\sigma=1$. The ratio ρ/σ is irrational.

Let now F be an irreducible Darboux polynomial of degree m with cofactor Λ for the given derivation. According to the previous subsection, the degree of F is bounded by $s+2$. Moreover, the three formulas (21) may then be written

$$n_0+n_1+n_2+n_3=s^2+s+1,$$
$$(n_1+n_3)(1+\rho/\sigma)+(n_2+n_3)(1+\sigma/\rho)=(s+2)m,$$
$$(n_1+n_3)(\rho/\sigma)+(n_2+n_3)(\sigma/\rho)+2n_3=m^2, \tag{23}$$

if we denote by n_i the cardinal number of $\mathscr{D}(\mathbf{V})_i$, for $i=0,1,2,3$.

From the irrationality of the ratio ρ/σ, we receive $n_2=n_1$ and after some simplifications, the system of three equations in the non-negative integer unknowns n_0,n_1,n_3:

$$n_0+2n_1+n_3=s^2+s+1,$$
$$(n_1+n_3)(s+2)=(s^2+s+1)m,$$
$$2n_1=m(s+2-m). \tag{24}$$

As in the discussion we had in [16], the greatest common divisor of s^2+s+1 and $s+2$ is 3 when $s \equiv 1 \pmod 3$ and 1 otherwise.

In the case $s \equiv 1 \pmod 3$ there exists an integer κ, $1 \leqslant \kappa \leqslant 3$ such that $m = \kappa(s+2)/3$, $n_1+n_3 = \kappa(s^2+s+1)/3$. Otherwise, $m = s+2$, $n_1+n_3 = s^2+s+1$.

n_3 is the total intersection index of all three partial derivatives of F.

From a corollary of Bézout's theorem, (see [10, Chap. 5, Theorem 2]), $n_3 \leqslant (m-1)(m-2)$.

When κ is 1 or 2, $n_3 = (s-1)^2/9$, which is contradictory. Thus, in any case, we have to take $m = s+2$, $n_3 = s^2+s+1 > (m-1)(m-2) = s(s+1)$, a contradiction.

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