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About a conjecture on quadratic vector fields

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Abstract

The Lotka–Volterra system of autonomous differential equations consists in three homogeneous polynomial equations of degree 2 in three variables. This system, or the corresponding vector field $LV(A,B,C)$, depends on three non-zero (complex) parameters and may be written as $LV(A,B,C) = V_x\partial_x + V_y\partial_y + V_z\partial_z$ where

$$V_x = x(Cy+z), \quad V_y = y(Az+x), \quad V_z = z(Bx+y).$$

As $LV(A,B,C)$ is homogeneous, there is a foliation whose leaves are homogeneous surfaces in the three-dimensional space $\mathbb{C}^3$, or curves in the corresponding projective plane $\mathbb{C}P(2)$, such that the trajectories of the vector field are completely contained in a leaf. An homogeneous first integral of degree 0 is then a non-constant function on the set of all leaves of the previous foliation.

Trying to classify all values of the triple (A,B,C) for which $LV(A,B,C)$ has an homogeneous Liouvillian first integral of degree 0, we discovered the interesting family of Lotka–Volterra systems:

$$SLV_l = LV\left(2, B_l = -\frac{2l+1}{2l-1}, 1/2\right), \quad l \in \mathbb{N}^\star.$$

This family provides a negative answer to the conjecture that there exists a uniform bound M_2 such that, if some homogeneous three-variable vector field of degree 2 has a particular solution (an irreducible Darboux polynomial in our words) of degree at least M_2, then the vector field is rationally integrable (has an homogeneous rational first integral of degree 0). Indeed:

- SLV_l has no homogeneous rational first integral of degree 0.
- SLV_l has an irreducible Darboux polynomial f_l of degree $m = 2l$:

$$x(y/2+z)\frac{\partial f_l}{\partial_x} + y(2z+x)\frac{\partial f_l}{\partial_y} + z(B_l x + y)\frac{\partial f_l}{\partial_z} = ((l-1)y + 2lz)f_l. \qquad (*)$$

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1. Introduction

Algebraic differential equations and integrable 1-forms in three variables are a classical subject of interest for many people; for instance, let us simply quote [1,2] for contemporary work and [10–12] for the great ancestors.

A natural question about polynomial vector fields appeared some years ago. In a recent paper [6], Lins Neto states it in the following way:

> Given $d \geqslant 2$, is there $M(d) \in \mathbb{N}$ such that if a foliation has an algebraic solution of degree greater than or equal to $M(d)$, then it has a rational first integral?

In this note, we give a negative answer to this question in the case $d = 2$.

Using the homogeneous language, we produce a countable family of quadratic homogeneous three-variable vector fields (a family of quadratic foliations of $\mathbb{C}P(2)$) without an homogeneous rational first integral of degree 0 (without a rational first integral) such that each element of the family has an irreducible non-zero homogeneous Darboux polynomial (an algebraic solution) whereas there is no uniform bound on the degrees of these particular solutions for all elements of the family.

We discovered this family in classifying all cases of liouvillian integrability of the Lotka–Volterra system [9]. Our interest in the integration of this particular homogeneous quadratic system of differential equations in three variables dates back to [4].

2. Preliminaries

In this section, we recall some facts and notations from differential algebra.

2.1. Some vocabulary of differential algebra

Given two fields $k \subset K$, a k-derivation of the extension field K is a k-linear map δ from K in itself that satisfies Leibniz rule for the derivation of a product. Endowed with this mapping δ, K becomes a *differential field*. The kernel of δ is then a subfield of K. It is known as the *field of constants* of the derivation and it contains the base field k.

When K is the field $k(x, y, z)$ of rational fractions in three unknowns, the usual partial derivatives ∂_x, ∂_y and ∂_z are derivations and they commute with one another. Their common field of constants is exactly k.

In what follows, k is some finite extension of $\mathbb{Q}$ by parameters and can be thought of as a subfield of the field $\mathbb{C}$ of complex numbers. In particular, the characteristic of all fields will be 0.

A polynomial vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$, where V_x, V_y and V_z are homogeneous polynomials of the same degree in $k[x, y, z]$, defines a k-derivation δ_V of K.

When L is a field extension of $k(x,y,z)$ in which ∂_x, ∂_y and ∂_z have been extended as commuting k-derivations of L, δ_V is extended to L in a natural way.

In particular, we consider the vector field $E = x\partial_x + y\partial_y + z\partial_z$ and call it the *Euler field*. Indeed, an element f of $k(x,y,z)$ is homogeneous of degree m if and only if $\delta_E(f) = mf$, according to Euler relation. In a differential extension field L, this identity may be taken as a definition of homogeneity.

We will also freely use the ideas of differential calculus such as exterior derivatives or n-forms in the frame of differential algebra.

A 1-form is an element $\omega = \omega_x\,\mathrm{d}x + \omega_y\,\mathrm{d}y + \omega_z\,\mathrm{d}z$ in the 3-dimensional vector space K^3 written in the "canonical" base $[\mathrm{d}x, \mathrm{d}y, \mathrm{d}z]$. The exterior derivative of an element f of K is the 1-form $\mathrm{d}f = \partial_x(f)\,\mathrm{d}x + \partial_y(f)\,\mathrm{d}y + \partial_z(f)\,\mathrm{d}z$.

Given a polynomial vector field V, an element f of some differential extension field L of K that satisfies $\delta_V(f) = \delta_E(f) = 0$ is called an *homogeneous first integral of degree 0*. The element f of L is then a constant of both derivations without being a "constant". It is better to keep the word *constant* for the elements c such that $\mathrm{d}c = 0$, which means $\partial_x(c) = \partial_y(c) = \partial_z(c) = 0$.

2.2. Darboux polynomials

Let $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$ be a vector field, where V_x, V_y and V_z are homogeneous polynomials in the space variables x, y and z and have the same degree m.

A non-zero polynomial f is said to be a *Darboux polynomial* of V if there exists some polynomial Λ such that

$$V_x\frac{\partial f}{\partial x} + V_y\frac{\partial f}{\partial y} + V_z\frac{\partial f}{\partial z} = \Lambda f. \tag{1}$$

This polynomial Λ is then called the *eigenvalue* or the *cofactor* corresponding to the Darboux polynomial f.

The homogeneous components of highest degree of the above identity (1) agree, which shows that Λ has to be an homogeneous polynomial of degree $m-1$.

Moreover, the homogeneous component f^+ of highest degree of a Darboux polynomial f is a Darboux polynomial. It is therefore sufficient to study homogeneous Darboux polynomials (*Darboux curves* in $\mathbb{C}P(2)$).

Suppose now that some homogeneous Darboux polynomial f of a given homogeneous polynomial vector field V with an eigenvalue Λ factors as the product $f = gh$ of two relatively prime homogeneous polynomials. The polynomial ring $k[x,y,z]$ is a unique factorization domain and the factors g and h have to be Darboux polynomials of V with respective eigenvalues Λ_1 and Λ_2 such that $\Lambda = \Lambda_1 + \Lambda_2$, according to Gauss lemma.

Let us recall that we have supposed the characteristic of the base field k to be 0. Thus, if some positive power f^n of an irreducible homogeneous polynomial f is a Darboux polynomial of V, f itself is a Darboux polynomial of V.

Thus, the determination of all Darboux polynomials of a given homogeneous polynomial vector field V amounts to finding all irreducible homogeneous Darboux polynomials of V.

According to Euler identity, homogeneous polynomials are Darboux polynomials of the Euler field; with respect to E, the eigenvalue of an homogeneous polynomial is its degree.

An homogeneous irreducible Darboux polynomial corresponds to a *particular algebraic solution* of the system of ODE.

2.3. Darboux points

As we have previously seen, a homogeneous first integral f of degree 0 of some homogeneous polynomial vector field V belongs to the common kernel of δ_V and δ_E:

$$V_x\frac{\partial f}{\partial x}+V_y\frac{\partial f}{\partial y}+V_z\frac{\partial f}{\partial z}=0,$$
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=0.$$

Its derivative $\mathrm{d}f$ is then proportional to the 1-form

$$\omega_0=(yV_z-zV_y)\,\mathrm{d}x+(zV_x-xV_z)\,\mathrm{d}y+(xV_y-yV_x)\,\mathrm{d}z.$$

In this framework, the lines in $\mathbb{C}^3$ (or points of the projective plane $\mathbb{C}P(2)$) where ω_0 vanishes (i.e. where V and E are collinear) are of special interest. We call these singular points of ω_0 the *Darboux points* of V.

When the degree of V is 2 and ω_0 is irreducible (*its three coordinates are relatively prime*), there are generically seven Darboux points according to a result by Darboux [3,5].

3. Solving the conjecture

3.1. The solution

Theorem 1. *Consider the family of homogeneous three-variable quadratic vector fields*

$$SLV_l=x(y/2+z)\partial_x+y(2z+x)\partial_y+z\left(y-\frac{2l+1}{2l-1}x\right)\partial_z,$$

where l is a positive integer. This family gives a negative answer to the conjecture:

- *On the one hand, for every $l\in\mathbb{N}^\star$, the field SLV_l has no homogeneous rational first integral of degree* 0.
- *On the other hand, SLV_l has an irreducible Darboux curve (a particular solution) of degree $2l$ and thus there is no uniform bound on these degrees.*

Proof. We deal with these two points in the next two subsections. □

3.2. These systems are not rationally integrable

The vector fields SLV_l, $l \in \mathbb{N}^\star$ are special cases of the three-parameter Lotka–Volterra vector field

$$LV(A,B,C) = V_x\partial_x + V_y\partial_y + V_z\partial_z$$
$$\text{where } V_x = x(Cy+z),\ V_y = y(Az+x),\ V_z = z(Bx+y).$$

In particular, SLV_l corresponds to the triple $[A,B,C] = [2, -(2l+1)/(2l-1), 1/2]$.

Now recall the main result of [8], in which *natural transformation* essentially means circular permutation of the parameters.

Theorem 2. *The Lotka–Volterra vector field $LV(A,B,C)$ has an homogeneous rational first integral of degree 0 if and only if the triple (A,B,C) of parameters consists of rational numbers and belongs, up to a natural transformation, to one of the following sets*:

- $\mathscr{P}_1 = \{(A,B,C) \in \mathbb{Q}^3, ABC+1 = 0\}$,
- $\mathscr{P}_2 = \{(A,B,C) \in \mathbb{Q}^3, C+1/A = A+1/B = B+1/C = -1\}$,
- $\mathscr{P}_3 = \{(A,B,C) \in \mathbb{Q}^3, B = 1, -1/A - C \in \mathbb{N}^\star, -C \notin \mathbb{N}^\star \text{ or} -1/A \notin \mathbb{N}^\star\}$,
- $\mathscr{P}_4 = \{(A,B,C) \in \mathbb{Q}^3, B = 2, C+1/A = -1, A-1 \in \mathbb{N}^\star \text{ or} -1/2 - A \in \mathbb{N}^\star\}$,
- $\mathscr{P}_5 = \{(-7/3, 3, -4/7)\}$,
- $\mathscr{P}_6 = \{(-3/2, 2, -4/3)\}$.

Obviously, no SLV_l can be found in the previous list, whence the first announced result.

Remark 1. Proving that SLV_l has no rational first integral of degree 0 was not evident. Indeed, at every Darboux point, the eigenvalues of the vector field have a rational ratio, which is a usual necessary condition for rational integrability.

3.3. Irreducible Darboux curves of unbounded degree

Now fix $l \in \mathbb{N}^\star$. We claim that SLV_l has an irreducible Darboux polynomial f of degree $2l$ with the eigenvalue $(l-1)y + 2lz$.

First, we have to prove that there exists a non-zero homogeneous polynomial f of degree $2l$ that satisfies Eq. $(*)$ and thereafter, we will have to check that f is irreducible.

To be precise, Eq. $(*)$ can be written as

$$x(y/2+z)\partial f/\partial_x + y(2z+\mathrm{d}x)\partial f/\partial_y + z\left(y - \frac{2l+1}{2l-1}x\right)\partial f/\partial_z$$
$$-((l-1)y+2lz)f = 0. \tag{2}$$

Eq. (2) means that some homogeneous polynomial F of degree $2l+1$ is 0.

All coefficients before the monomials in F are 0: we get an homogeneous linear system of equations; there is one equation for every monomial in F and the unknowns are the coefficients of the three-variable homogeneous polynomial f of degree $2l$.

The unknowns are then parameterized by the triples (i,j,k) of nonnegative integers summing to $2l$ and there are $n=\binom{2l+2}{2}$ of them. The equations are parameterized by the triples (i,j,k) of nonnegative integers summing to $2l+1$ and there are $N=\binom{2l+3}{2}$ of them.

The corresponding matrix M is very sparse; there are at most three non-zero coefficients per row or per column and they are given by affine forms of the indices:

$$\begin{aligned}
M_{(i+1,j,k),(i,j,k)} &= j+Bk-\lambda,\\
M_{(i,j+1,k),(i,j,k)} &= Ci+k-\mu,\\
M_{(i,j,k+1),(i,j,k)} &= i+Aj-\nu,
\end{aligned} \tag{3}$$

where $[A,B,C]=[2,-(2l+1)/(2l-1),1/2]$ and $\lambda=0,\ \mu=l-1,\ \nu=2l$.

This linear system can be partially put in "triangular form", which leads to a partial solution of it: all coefficients $\{f_{i,j,k}, i+j+k=2l, k>j\}$ and all coefficients $\{f_{i,j,k}, i+j+k=2l, j>k+2\}$ are 0. Moreover, $M_{(i,j+1,k),(i,j,k)}=Ci+k-\mu=0$ when $j=k+2$ and $M_{(i,j,k+1),(i,j,k)}=i+Aj-\nu=0$ when $j=k$.

We are thus left with a linear sub-system: the remaining unknowns correspond to the triples (i,j,k), $i+j+k=2l$ where $k\leqslant j\leqslant k+2$ and the remaining equations to the triples (i,j,k), $i+j+k=2l+1$ where $k\leqslant j\leqslant k+2$.

There are as many equations as unknowns in this subsystem: $3l+1$. To prove that this square linear sub-system has a one-dimensional kernel, do as follows:

- choose an arbitrary non-zero value for $f_{2l,0,0}$,
- determine the other coefficients in a unique way by a suitable induction, using all equations but the last one indexed by $(0,l,l+1)$,
- the given value of B_l is exactly the good one to solve this last equation.

In other words, the corresponding determinant has many factors and it has in fact only two non-zero terms; it is easy to compute and the value of B_l is exactly the condition for this determinant to vanish.

It remains to be proven that f is irreducible.

First remark that f is a *strict Darboux polynomial* of a Lotka–Volterra vector field, which means that f is divisible neither by x, nor by y nor by z.

If f were factored as $f_1 f_2$, with non-constant factors, f_1 and f_2 would also be homogeneous strict Darboux polynomials of the same vector field.

Let us now recall the technical result of [7] about the strict Darboux polynomials of the Lotka–Volterra vector field.

Proposition 1 (Eigenvalues of strict Darboux polynomials.). *Let g be a strict Darboux polynomial of the Lotka–Volterra vector field with the eigenvalue $\Lambda=\lambda x+\mu y+\nu z$. There exists 6 natural numbers β_1, γ_1, α_2, γ_2, α_3 and β_3 such that the* *following equations and inequalities hold*:

$$\lambda = \beta_3 = \gamma_2 B,$$
$$\mu = \gamma_1 = \alpha_3 C,$$
$$v = \alpha_2 = \beta_1 A,$$
$$\beta_1 + \gamma_1 \leqslant m,$$
$$\alpha_2 + \gamma_2 \leqslant m,$$
$$\alpha_3 + \beta_3 \leqslant m.$$

In particular, the coefficients λ, μ and v of the eigenvalue are non-negative integers.

According to Proposition 1, the degree of f_1 would be $2l_1$ and its eigenvalue $(l_1-\varepsilon_1)y + 2l_1 z$ whereas the degree of f_2 would be $2l_2$ and its eigenvalue $(l_2-\varepsilon_2)y + 2l_2 z$; l_1 and l_2 are positive integers. Moreover, one of the ε_i would take the value 0 and the other the value 1.

Solving the linear system corresponding to f_1 or f_2 would lead (by arguments similar to those used before) to another value of B, a contradiction.

Thus, the family SLV_l, $l \in \mathbb{N}^\star$ of vector fields provides a negative answer to the conjecture that there exists a uniform bound M_2 such that, if some homogeneous 3-variable vector field of degree 2 has a particular solution of degree at least M_2, then the vector field is rationally integrable.

Remark 2. The vector fields of this family are very special with respect to their Darboux points. To make notations precise, let us choose homogeneous coordinates for the Darboux points M_i, $1 \leqslant i \leqslant 7$, of a Lotka–Volterra system in the following way

$$M_1 = [1,0,0], \quad M_2 = [0,1,0], \quad M_3 = [0,0,1],$$
$$M_4 = [0,A,1], \quad M_5 = [1,0,B], \quad M_6 = [C,1,0],$$
$$M_7 = [1 + CA - A, 1 + AB - B, 1 + BC - C].$$

Then, in the case of SLV_l,

- Among these Darboux points, two of them collapse: $M_7 = M_4$.
- At every M_i, the eigenvalues have a rational ratio.
- Three Darboux points are nodes: M_2, M_3, M_6 (the previous ratio is positive).

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