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SIMPLE QUADRATIC DERIVATIONS IN TWO VARIABLES

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ABSTRACT

Let $k[x, y]$ be the polynomial ring in two variables over an algebraically closed field k of characteristic zero. We call quadratic derivations the derivations of $k[x, y]$ of the form

$$\frac{\partial}{\partial x} + (y^2 + a(x)y + b(x))\frac{\partial}{\partial y},$$

where $a(x), b(x) \in k[x]$. We are interested in simple derivations of this type; every such derivation is equivalent to $\Delta_p = \partial/\partial x + (y^2 - p(x))\partial/\partial y$ for a suitable p in $k[x]$.

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For some p, we are able to decide the simplicity of Δ_p: if the degree of p is odd, then Δ_p is simple; if p has degree 2, then Δ_p is simple if and only if p fulfills an arithmetic condition.

1 INTRODUCTION

Throughout the paper, k denotes an algebraically closed field of characteristic zero. A derivation d of a commutative k-algebra R is said to be *simple* if R has no other d-invariant ideals than 0 and R. Applications and various properties of simple derivations can be found in many papers (see, for example, [2–4, 7, 10, 11]).

Assume that $R = k[x_1, \dots, x_n]$ is the polynomial ring over k in n variables and consider the derivation d of R given by $d(x_1) = f_1, \dots, d(x_n) = f_n$. It would be of considerable interest to find necessary and sufficient conditions on $f_1, \dots, f_n$ for d to be simple. This question is obvious only for $n = 1$.

If $n = 2$, only some sporadic examples of simple derivations of $R = k[x, y]$ are known. The problem seems to be difficult even with the extra assumption that $d(x) = 1$. The description of all simple derivations d of $k[x, y]$ such that $d(x) = 1$ and $d(y) = a(x)y + b(x)$, where $a(x), b(x)$ are polynomials of $k[x]$, has been given in [10].

In this paper we study simple derivations $d : k[x, y] \to k[x, y]$ such that $d(x) = 1$ and $d(y) = y^2 + a(x)y + b(x)$ for $a(x), b(x) \in k[x]$. It is not difficult to show (see Proposition 7.2) that the problem of simplicity for such derivations reduces to the same problem for the derivations $\Delta_p : k[x, y] \to k[x, y]$ defined by

$$\begin{cases} \Delta_p(x) = 1, \\ \Delta_p(y) = y^2 - p(x), \end{cases}$$

where $p = p(x) \in k[x]$.

The main result of the paper is Theorem 6.1, which states that if Δ_p is not simple, then there exists a Δ_p-invariant principal ideal (F) such that $\deg_y F = 1$. As a consequence of this fact we are able to describe some classes of simple derivations of the form Δ_p. For instance, Δ_p is simple if the degree of $p(x)$ is odd and polynomials p of degree 2 for which Δ_p is simple are characterized by an arithmetic condition (Theorem 8.3).

2 PRELIMINARIES

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over k. A derivation d of R is a k-linear mapping $d : R \to R$ such that $d(FG) = d(F)G + Fd(G)$ for

$F, G \in R$. Let us recall (see for example [1, 10]) that every derivation d of R has a unique decomposition as

$$d = F_1 \frac{\partial}{\partial x_i} + \cdots + F_n \frac{\partial}{\partial x_n},$$

where $F_1, \ldots, F_n \in R$.

Let d be a derivation of R. An ideal A of R is said to be a *d-ideal* if $d(A) \subseteq A$. By definition, d is *simple* if there are no other d-ideals than 0 and R.

As in [9, 10], a polynomial $F \in R$ is said to be a *Darboux polynomial* of d if $F \notin k$ and $d(F) = \Lambda F$ for some $\Lambda \in R$ or equivalently if (F) is a proper d-ideal of R. Using Gauss lemma (and characteristic 0), it is rather simple to prove that factors of Darboux polynomials of d are also Darboux polynomials of d. As usual, R^d stands for the kernel of the k-linear mapping d. Recall that R^d is a subring of R containing k. If $F \in R^d \backslash k$, then F is a Darboux polynomial of R (with the *eigenvalue* $\Lambda = 0$).

Proposition 2.1. *If $d : k[x, y] \to k[x, y]$ is a derivation such that $d(x) = 1$, then d is simple if and only if d has no Darboux polynomial.*

Proof. It is clear that if d is simple then d has no Darboux polynomials. Assume now that d is not simple. Then there exists a proper d-ideal and hence (see, for example, [5]), there exists a prime proper d-ideal P of $k[x, y]$. If P is maximal, then $P = (x - a, y - b)$ for some $a, b \in k$ (since k is algebraically closed) whence a contradiction $1 = d(x - a) \in d(P) \subseteq P$. Therefore, the height of P is equal to 1 and hence, P is a principal proper d-ideal. Let $P = (F)$. Then F is a Darboux polynomial of d. □

Let $k[x]$ and $k(x)$ be the polynomial ring and the field of rational functions, respectively, over k in one variable x. As usual, $\overline{k(x)}$ denotes the algebraic closure of $k(x)$. If $r \in \overline{k(x)}$, r' stands for the derivative of r with respect to the derivation $' : \overline{k(x)} \to \overline{k(x)}$ which is the unique extension of the derivation $\partial/\partial x : k(x) \to k(x)$ to the field $\overline{k(x)}$.

3 DARBOUX POLYNOMIALS AND ALGEBRAIC SOLUTIONS

The aim of this section is to link Darboux polynomials of derivations of $k[x, y]$ with algebraic solutions of some first-order differential equations with coefficients in $k(x)$, which will play an important role in the next sections.

Theorem 3.1. *Let d be a derivation of $k[x, y]$, let $P(x, y) = d(x)$ and $Q(x, y) = d(y)$. Then the following two conditions are equivalent.*

(1) *There exists a Darboux polynomial* $F \in k[x,y]$ *of* d *such that* $\deg_y F \geq 1$.
(2) *There exists an* $r \in \overline{k(x)}$ *such that* $P(x,r)r' = Q(x,r)$.

Proof. This is a consequence of the two following more precise propositions. □

Proposition 3.2. *Let* d *be a derivation of* $k[x,y]$*, let* $P(x,y) = d(x)$ *and* $Q(x,y) = d(y)$*. If* $F \in k[x,y] = k[x][y]$ *is an irreducible Darboux polynomial of* d *with* $\deg_y F \geq 1$*, then there exists an* $r \in \overline{k(x)}$ *such that* $P(x,r)r' = Q(x,r)$ *and* $F(x,r) = 0$.

Proof. As $\deg_y F \geq 1$, F has a root r in the algebraically closed field $\overline{k(x)}$. Differentiating the identity $F(x,r) = 0$ with respect to x gives

$$\frac{\partial F}{\partial x}(x,r) \cdot 1 + \frac{\partial F}{\partial y}(x,r) \cdot r' = 0. \tag{3.1}$$

The Darboux property $d(F) = \Lambda F$ can be written more precisely as

$$\frac{\partial F}{\partial x}(x,y)P(x,y) + \frac{\partial F}{\partial y}(x,y)Q(x,y) = \Lambda(x,y)F(x,y),$$

and thereafter evaluated at $y = r$ to yield (as $F(x,r) = 0$)

$$\frac{\partial F}{\partial x}(x,r)P(x,r) + \frac{\partial F}{\partial y}(x,r)Q(x,r) = 0.$$

Since F is irreducible, $\partial F/\partial y(x,r) \neq 0$, which gives

$$Q(x,r) = -\frac{\frac{\partial F}{\partial x}(x,r)}{\frac{\partial F}{\partial y}(x,r)}P(x,r) = r'P(x,r).$$

This completes the proof. □

Proposition 3.3. *Let* d *be a derivation of* $k[x,y]$*, let* $P(x,y) = d(x)$ *and* $Q(x,y) = d(y)$*. If* r *is a solution in* $\overline{k(x)}$ *of the differential equation* $P(x,r)r' = Q(x,r)$*, then its primitive minimal polynomial* F *in* $k[x][y]$ *is a Darboux polynomial of* d *with* $\deg_y F \geq 1$.

Proof. By definition, $\deg_y F \geq 1$. Define $H \in k[x,y]$ by

$$H = H(x,y) = d(F(x,y)) = P(x,y)\frac{\partial F}{\partial x}(x,y) + Q(x,y)\frac{\partial F}{\partial y}(x,y).$$

Observe that $H(x,r) = 0$. Indeed:

$$\begin{aligned} H(x,r) &= P(x,r)\frac{\partial F}{\partial x}(x,r) + Q(x,r)\frac{\partial F}{\partial y}(x,r) \\ &= P(x,r)\frac{\partial F}{\partial x}(x,r) + P(x,r)r'\frac{\partial F}{\partial y}(x,r) \\ &= P(x,r)\left(\frac{\partial F}{\partial x}(x,r)\cdot 1 + r'\frac{\partial F}{\partial y}(x,r)\right) \\ &= P(x,r)\frac{\partial}{\partial x}(F(x,r)) \\ &= P(x,r)\frac{\partial}{\partial x}(0) = 0. \end{aligned}$$

Since $F(x,y)$ is a minimal polynomial of r, H has to be a multiple of F in $k(x)[y]$, which means that there exists $\Lambda = \Lambda(x,y) \in k(x)[y]$ such that

$$d(F) = H = \Lambda F.$$

It is easy to see (using Gauss Lemma in $k(x)[y]$) that the content of Λ belongs to $k[x]$, meaning that $\Lambda \in k[x,y]$. Therefore F is a Darboux polynomial of d, which completes the proof. □

Note the following special case of Theorem 3.1.

Corollary 3.4. *Let $d : k[x,y] \to k[x,y]$ be a derivation defined as*

$$\begin{cases} d(x) = 1, \\ d(y) = a(x)y^2 + b(x)y + c(x), \end{cases}$$

where $a(x), b(x), c(x) \in k[x]$, $a(x) \neq 0$. Then d has a Darboux polynomial $F \in \underline{k}[x,y]$ with $\deg_y F \geq 1$ if and only if there exists an algebraic function $r \in \overline{k(x)}$ such that $r' = a(x)r^2 + b(x)r + c(x)$. □

4 DARBOUX POLYNOMIALS OF THE DERIVATION Δ_p

For $p = p(x) \in k[x]$, Δ_p stands for the derivation of $k[x,y]$ defined by

$$\begin{cases} \Delta_p(x) = 1, \\ \Delta_p(y) = y^2 - p(x). \end{cases}$$

In this section we study Darboux polynomials of Δ_p. There are no Darboux polynomials of Δ_p in the subring $k[x]$, thus if F is a Darboux polynomial of

Δ_p in $k[x,y]$, then $\deg_y F \geq 1$ and F is primitive as a polynomial in $k[x][y]$. As a special case of Corollary 3.4 we get

Corollary 4.1. *The derivation Δ_p has a Darboux polynomial $F \in k[x,y]$ with $\deg_y F \geq 1$ if and only if there exists an algebraic function $r \in \overline{k(x)}$ such that $r' = r^2 - p$.* □

In the next two propositions we deal with Darboux polynomials F of Δ_p such that $\deg_y F = 1$.

Proposition 4.2. *Let F be a Darboux polynomial of Δ_p such that $\deg_y F = 1$ and denote by $\Lambda \in k[x,y]$ the corresponding eigenvalue: $\Delta_p(F) = \Lambda F$. Write $F = uy + v$ in $k[x][y]$ (with $u \neq 0$ and $\gcd(u,v) = 1$). Then*

(1) $\Lambda = y + s$ *for some* $s \in k[x]$,
(2) $su + v = u'$ *and* $sv = v' - pu$,
(3) $u'' - s'u - 2su' + s^2u - pu = 0$,
(4) $\deg p = 2\deg s$ *(may be both* $-\infty$*)*.

Proof. Developing the equality $\Delta_p(F) = \Lambda F$ yields

$$\Lambda \cdot (uy + v) = \Delta_p(uy + v) = u(y^2 - p) + u'y + v' = uy^2 + u'y + v' - pu,$$

which implies that $\Lambda = y + s$ for some $s \in k[x]$ and that

$$uy^2 + (su + v)y + sv = (y + s)(uy + v) = uy^2 + u'y + (v' - pu).$$

Identifying coefficients in y then gives equalities (2): $su + v = u'$ and $sv = v' - pu$.

According to these equalities, v can be computed from u and thereafter v and v' can be eliminated to produce the second-order differential equation (3).

Since $u \neq 0$, the equalities (2) imply that $p = 0 \Longleftrightarrow s = 0$, that is, $\deg p = -\infty \Longleftrightarrow \deg = -\infty$. Now assume that $p \neq 0$. Then $s \neq 0$ and the following equalities between degrees are a consequence of (2):

$$\deg v = \deg s + \deg u, \quad \deg s + \deg v = \deg p + \deg u.$$

As $u \neq 0$, $\deg(u)$ may be cancelled to give $\deg p = 2\deg s$. □

Proposition 4.3. *Let u and v belong to $k[x]$ with $u \neq 0$ and $\gcd(u,v) = 1$. Let $F = uy + v \in k[x][y]$ and $r = -v/u \in k(x)$. Then the following conditions are equivalent:*

(1) *F is a Darboux polynomial of Δ_p.*
(2) $r' = r^2 - p$.

Proof. (1) $\Rightarrow$ (2). According to Proposition 4.2, $\Delta_p(F) = (y+s)F$ for some $s \in k[x]$, $su + v = u'$ and $sv = v' - pu$. Then, $vu' - v^2 = suv = v'u - pu^2$ and, as $u \neq 0$,

$$r' = \frac{vu' - v'u}{u^2} = \frac{v^2}{u^2} - p = r^2 - p.$$

(2) $\Rightarrow$ (1). Now $r' = r^2 - p$, which can be written as $((vu' - v'u)/u^2 = (v^2/u^2)) - p$ and then multiplied by u^2 to yield

$$u(v' - up) = v(u' - v).$$

Since $\gcd(u, v) = 1$, there exists $s \in k[x]$ such that $v' - up = sv$ and $u' - v = su$. Now, simply compute $\Delta_p(F)$:

$$\begin{aligned}
\Delta_p(F) &= \Delta_p(uy + v) \\
&= u'y + u(y^2 - p) + v' \\
&= uy^2 + (u' - v)y + vy + (v' - up) \\
&= uy^2 + suy + vy + sv \\
&= (y+s)(uy+v) \\
&= (y+s)F.
\end{aligned}$$

So, F is a Darboux polynomial of Δ_p, with the eigenvalue $y + s$. □

5 ON A CLASS OF RICCATI EQUATIONS

Let us consider a differential Riccati equation of the form

$$y' = y^2 - p(x), \tag{5.1}$$

where $p = p(x)$ is a polynomial belonging to $k[x]$. In relation to the simplicity of Δ_p, we are interested in algebraic solutions of this equation, that is, in the elements $y \in \overline{k(x)}$ such that $y' = y^2 - p$. An important simplification occurs: these algebraic functions are in fact rational.

Theorem 5.1. *For $p \in k[x]$, the equation* (5.1) *has no solution y in $\overline{k(x)} \setminus k(x)$. In other words, its algebraic solutions are in fact rational.*

Proof. Suppose that there exists $y \in \overline{k(x)} \setminus k(x)$ which is a solution of (5.1). Let

$$F = F(x, y) = y^n - \sigma_1 y^{n-1} + \sigma_2 y^{n-2} - \cdots + (-1)^n \sigma_n$$

be the minimal monic polynomial for y over $k(x)$. This polynomial belongs to $k(x)[y]$ and its degree n is at least 2.

Let K be the splitting field of F over $k(x)$ and let $y_1, \dots, y_n$ be the n different roots of F in K. By hypothesis, one of them is a solution of the differential equation (5.1).

First observe that every root y_i $(i = 1, \dots, n)$ is a solution of (5.1). Indeed, since the extension $k(x) \subset K$ is algebraic, the derivation $\partial/\partial x$ of $k(x)$ can be extended to a derivation $d : K \to K$ in a unique way. If σ is a $k(x)$-automorphism of K then the mapping $\sigma d \sigma^{-1}$ is a derivation of K and it is an extension of the derivation $\partial/\partial x$ of $k(x)$; hence $\sigma d \sigma^{-1} = d$. This means that the automorphisms of the Galois group of K over $k(x)$ commute with the unique extension of the derivation $\partial/\partial x$ of $k(x)$. Therefore, all elements $y_1, \dots, y_n$ are solutions of (5.1), that is $y_i' = y_i^2 - p$ for $i = 1, \dots, n$.

Consider now the discriminant of F:

$$\Delta = (-1)^{n(n-1)/2} \prod_{i \neq j} (y_i - y_j).$$

The logarithmic derivative of Δ is easy to compute:

$$\frac{\Delta'}{\Delta} = \sum_{i \neq j} \frac{y_i' - y_j'}{y_i - y_j} = \sum_{i \neq j} (y_i + y_j) = 2(n-1)\sigma_1.$$

From the above equality, it follows that σ_1 is, up to a factor $(1/(2(n-1)))$, the logarithmic derivative of the discriminant Δ, which belongs to $k(x)$. Thus the partial fraction decomposition of σ_1 has the form of a finite sum

$$\sigma_1 = \sum_{\alpha} \frac{\lambda_\alpha}{x - \alpha},$$

where each α belongs to k and each λ_α is a rational number.

According to Proposition 3.3, the fact that the roots of F are solutions of (5.1) can be expressed by the following Darboux property of F:

$$\frac{\partial F}{\partial x} + (y^2 - p)\frac{\partial F}{\partial y} = (ny + a)F. \tag{5.2}$$

The eigenvalue $(ny + a)$ is easily seen to be a polynomial of degree 1 in y. The leading term n in this eigenvalue comes from a simple consideration concerning the coefficients of degree $n + 1$ (in y) in (5.2). The "constant term" a has now to be studied in detail.

In (5.2), consider all other degrees in y from n to 1 and then degree 0. In the corresponding system (Σ) of $n + 1$ equations, all coefficients of F are inductively defined from a and, after substitutions, the last equation becomes a differential equation for a:

$$(\Sigma):\begin{cases} \sigma_1 = a\sigma_0 - \sigma_0' & = a \quad (\sigma_0 = 1), \\ 2\sigma_2 = a\sigma_1 - \sigma_1' & -np\sigma_0, \\ 3\sigma_3 = a\sigma_2 - \sigma_2' & -(n-1)p\sigma_1, \\ \quad\vdots & \\ i\sigma_i = a\sigma_{i-1} - \sigma_{i-1}' & -(n+2-i)p\sigma_{i-2}, \\ \quad\vdots & \\ n\sigma_n = a\sigma_{n-1} - \sigma_{n-1}' & -2p\sigma_{n-2}, \\ 0 = a\sigma_n - \sigma_n' & -p\sigma_{n-1} \quad . \end{cases} \tag{5.3}$$

The system (Σ) may also be thought of as the inductive definition of a sequence of rational fractions $(\sigma_i), i \in \mathbb{N}$, by its two initial values $\sigma_0 = 1$, $\sigma_1 = a$ and the induction rule $i\sigma_i = a\sigma_{i-1} - \sigma_{i-1}' - (n+2-i)p\sigma_{i-2}$. We then demand $\sigma_{n+1} = \sigma_{n+2} = \cdots = 0$.

For every pole α of $\sigma_1 = a, p$ is not involved in this polar part of the partial fraction decomposition of each equation of (Σ); it follows that α is a pole of σ_i with an order at most i. Let $\overline{\sigma_i}$ stand for the polar part of σ_i of order i at the pole α; $\overline{\sigma_i}$ can be computed by induction:

$$i! \cdot \overline{\sigma_i} = \frac{\lambda_\alpha(\lambda_\alpha + 1)\cdots(\lambda_\alpha + i - 1)}{(x-\alpha)^i}.$$

The last equation of (Σ) then gives an equation for λ_α:

$$\lambda_\alpha(\lambda_\alpha + 1)\cdots(\lambda_\alpha + n) = 0.$$

Thus, all λ_α are negative integers in the range $[-n, -1]$.

Assume now that $p \neq 0$. We can perform "at infinity" the previous analysis of the system (Σ) that we did around every pole, which means that we consider the degrees and the leading coefficients of $\sigma_1, \ldots, \sigma_n$.

In this projection, p is now strongly involved. As $\deg(p) \geq 0$, it is easy to prove by induction that $\deg(\sigma_{2i}) = i\deg(p)$ and $\deg(\sigma_{2i+1}) \leq i\deg(p) - 1$. Equality holds for even indices, for instance $\deg(\sigma_0) = 0$. For odd indices, there may be a gap, especially if $a = \sigma_1 = 0$, in which case $\deg(\sigma_1) = -\infty < -1$.

Denote by $\bar{p}$ the leading coefficient of p and by $\overline{\sigma_i}$ the coefficient of σ_i corresponding to its nominal highest degree (in x): $\overline{\sigma_{2i}}$ is the coefficient of σ_{2i} of degree $i\deg(p)$ while $\overline{\sigma_{2i+1}}$ is the coefficient of σ_{2i+1} of degree $i\deg(p) - 1$. $\Lambda = -\sum \lambda_\alpha$ and $\delta = \deg(p)$ are non-negative integers. In (5.3), the following relations hold between nominal leading coefficients:

$$\begin{cases} \overline{\sigma_1} = \bar{a} = -\Lambda, \\ 2\overline{\sigma_2} = -n\bar{p}, \\ 3\overline{\sigma_3} = \overline{a\sigma_2} - \overline{\sigma_2}' - (n-1)\overline{p\sigma_1}, \\ \vdots \\ (2i)\overline{\sigma_{2i}} = -(n+2-2i)\overline{p\sigma_{2i-2}}, \\ (2i+1)\overline{\sigma_{2i+1}} = \overline{a\sigma_{2i}} - \overline{\sigma_{2i}}' - (n-2i+1)\overline{p\sigma_{2i-2}}. \end{cases}$$

It turns out that $\overline{\sigma_{2s}} = (-1)^s \overline{p}^s M_{2s}$ for even indices and $\overline{\sigma_{2s+1}} = -(-1)^s \overline{p}^s M_{2s+1}$ for odd indices, where M_i are non-negative rational factors given by the rules

$$\begin{cases} M_0 = 1, \\ M_1 = \Lambda, \\ M_{2s} = \frac{n+2-2s}{2s} M_{2s-}, \\ M_{2s+1} = (\Lambda + s\delta) M_{2s} + (n+1-2s) M_{2s-1}. \end{cases}$$

If $n = 2s+1$ is odd, $M_{n+1} = M_{2s+2}$ has to be 0, which is impossible. Thus, $n = 2s$ is even and $M_{n+1} = M_{2s+1} = 0$, which implies $\Lambda = \delta = 0$ ($M_{2s} > 0 \Rightarrow \Lambda + s\delta = 0 \Rightarrow \Lambda = 0$ and (as $s > 0$) $\delta = 0$).

We can now conclude; if there existed a strictly algebraic solution to the differential equation (5.1), p would be a constant and the coefficient $a = \sigma_1$ would be 0. In this case, F would have an even degree $n = 2s$ and an easy computation from the above system on $\overline{\sigma_i}$ then showed that $\sigma_{2i} = (-1)^i \binom{s}{i}$. Then F would be $(y^2 - p)^s$. Since F is supposed to be irreducible, F would equal $y^2 - p$ with a constant p. But the field k is algebraically closed, so we have a contradiction.

Therefore, we proved that the differential equation (5.1), with a nonzero polynomial p, has no strictly algebraic solution. In the case $p = 0$, the local analysis at infinity also leads to $\Lambda = 0$, meaning $\sigma_1 = a = 0$. Then, all σ_i are 0, whence again a contradiction. This completes the proof. □

6 SIMPLICITY OF THE DERIVATION Δ_p

Let us recall that, for $p = p(x)$ in $k[x]$, Δ_p stands for the derivation of $k[x, y]$ defined by

$$\begin{cases} \Delta_p(x) = 1, \\ \Delta_p(y) = y^2 - p(x). \end{cases}$$

The following theorem is one of the main results of our paper.

Theorem 6.1. *If the derivation Δ_p is not simple, then there exists a Darboux polynomial F of Δ_p such that* $\deg_y F = 1$.

Proof. If $p = 0$ then Δ_p is not simple and $F = y$ is a Darboux polynomial of Δ_p. So we may assume that $p \neq 0$.

Since Δ_p is not simple, there exists (by Proposition 2.1) a Darboux polynomial F in $k[x, y]$ of Δ_p. As we have previously seen, $\deg_y F \geq 1$. Hence, by Corollary 4.1, there exists an algebraic function $r \in \overline{k(x)}$ such that $r' = r^2 - p$. This means that the differential equation $y'' = y^2 - p$ has an algebraic solution. According to Theorem 5.1, there is no solution in $\overline{k(x)} \setminus k(x)$. Therefore the equation $y' = y^2 - p(x)$ has a solution y in $k(x)$. Since $p \neq 0$, $y \neq 0$ and, by Proposition 4.3, the derivation Δ_p has a Darboux polynomial $F \in k[x, y]$ such that $\deg_y F = 1$. □

The following results are now consequences of Theorem 6.1 and Proposition 4.2.

Theorem 6.2. *If p is a nonzero polynomial of odd degree, then Δ_p is simple.* □

Proposition 6.3. *Every derivation $d : k[x, y] \to k[x, y]$ of the following form is simple:*

$$\begin{cases} d(x) = 1, \\ d(y) = y^2 \pm x^n, \quad 0 \neq n \in \mathbb{N}. \end{cases}$$

Proof. If n is odd then d is simple, by Theorem 6.2. Let us assume that $n = 2m$, $0 \neq m \in \mathbb{N}$, and suppose that d is not simple. Then (Theorem 6.1) there exists a Darboux polynomial F of d such that $F = uy + v$ for $0 \neq u, v \in k[x]$, and then, by Proposition 4.2 (3),

$$u'' - s'u - 2su' + s^2u - pu = 0, \tag{6.1}$$

for $p = \pm x^{2m}$ and some $0 \neq s \in k[x]$ with $\deg s = m$. Put

$$s = s_m x^m + \cdots + s_1 x + s_0, \quad s_0, \ldots, s_m \in k, \quad s_m \neq 0.$$

Comparing in the equality (6.1) the leading coefficients of the powers of x, we see that $s_m^2 = \mp 1$ and we deduce successively that $s_{m-1} = s_{m-2} = \cdots = s_0 = 0$. Therefore, $s = s_m x^m$, $0 \neq s_m \in k$, and so

$$u'' - m s_m x^{m-1} u - 2 s_m x^m u' = 0.$$

Comparing again the leading coefficients we obtain the equality $m + 2\deg u = 0$ which is a contradiction (because $m > 0$). This completes the proof. □

Example 6.4. Every derivation $d : k[x, y] \to k[x, y]$ of the form

$$\begin{cases} d(x) = 1, \\ d(y) = y^2 - x^{2m} + mx^{m-1}, \quad 0 \neq m \in \mathbb{N}, \end{cases}$$

is not simple (since $y - x^m$ is a Darboux polynomial of d). □

7 SIMPLICITY AND EQUIVALENT DERIVATIONS

Two derivations d and δ of $k[x, y]$ are said to be *equivalent* if there exists a k-algebra automorphism σ of $k[x, y]$ such that $\delta = \sigma d \sigma^{-1}$. Clearly, if d and δ are equivalent derivations, then d is simple if and only if δ is simple.

Proposition 7.1. *Let $a, b, \varphi \in k[x]$ and let d, δ be derivations of $k[x, y]$ defined by:*

$$\begin{cases} d(x) = 1, \\ d(y) = y^2 + ay + b, \end{cases} \qquad \begin{cases} \delta(x) = 1, \\ \delta(y) = y^2 + (2\varphi + a)y + b + \varphi^2 + a\varphi - \varphi'. \end{cases}$$

These derivations are equivalent.

Proof. Consider the automorphism $\sigma : k[x, y] \to k[x, y]$ such that $\sigma(x) = x$ and $\sigma(y) = y - \varphi$. Then $\delta = \sigma d \sigma^{-1}$. □

Proposition 7.2. *Let $d : k[x, y] \to k[x, y]$ be a derivation such that*

$$\begin{cases} d(x) = 1, \\ d(y) = y^2 + ay + b, \end{cases}$$

where $a, b \in k[x]$. This derivation is equivalent to the derivation Δ_p where

$$p = \tfrac{1}{4}(a^2 - 4b) - \tfrac{1}{2}a'.$$

Proof. Use Proposition 7.1 for $\varphi = -a/2$. □

Theorem 7.3. *Let $d : k[x, y] \to k[x, y]$ be a derivation such that*

$$\begin{cases} d(x) = 1, \\ d(y) = y^2 + ay + b, \end{cases}$$

where $a, b \in k[x]$. The derivation d is not simple if and only if there exists a Darboux polynomial F of d such that $\deg_y F = 1$.

Proof. Let $0 \neq F \in k[x, y]$. The derivation d is not simple if and only if the derivation Δ_p is not simple, where p is such as in Proposition 7.2. We know (see the proofs of Propositions 7.1 and 7.2) that $\Delta_p = \sigma d \sigma^{-1}$, where $\sigma(x) = x$ and $\sigma(y) = y + a/2$. This implies that F is a Darboux polynomial of d if and only if $\sigma(F)$ is a Darboux polynomial of Δ_p. Moreover, $\deg_y F = 1$ if and only if $\deg_y \sigma(F) = 1$. Hence, this theorem follows from Theorem 6.1. □

Theorem 7.4. *Let $d : k[x, y] \to k[x, y]$ be a derivation such that*

$$\begin{cases} d(x) = 1, \\ d(y) = y^2 + ay + b, \end{cases}$$

where $a, b \in k[x]$. If $\deg b$ is odd and $\deg b > 2 \deg a$, then d is simple.

Proof. It follows from Theorem 6.2 because d is equivalent (by Proposition 7.2) to Δ_p, where p is a polynomial of odd degree in $k[x]$. □

8 THE CASE DEG $p(x) = 2$

In this section we study the derivations Δ_p for $p = Ax^2 + Bx + C$, where $A, B, C \in k$ and $A \neq 0$. First we shall show that the problem of simplicity for such derivations reduces to the same problem for derivations of the form δ_e, for $e \in k$, where $\delta_e = \Delta_{x^2 - e}$, that is, δ_e is the derivation of $k[x, y]$ defined by

$$\begin{cases} \delta_e(x) = 1, \\ \delta_e(y) = y^2 - x^2 + e. \end{cases}$$

Lemma 8.1. *Let $p = p(x) \in k[x]$, $\alpha \in k$ and let $q(x) = p(x + \alpha)$. Then the derivations Δ_p and Δ_q are equivalent.*

Proof. $\Delta_q = \sigma \Delta_p \sigma^{-1}$, where σ is given by $\sigma(x) = x + \alpha$ and $\sigma(y) = y$. □

Lemma 8.2. *Let $p = p(x) \in k[x]$, $0 \neq \beta \in k$ and let $r(x) = \beta^2 p(\beta x)$. Then Δ_p is simple if and only if Δ_r is simple.*

Proof. Consider the automorphism τ of $k[x, y]$ defined by $\tau(x) = \beta x$, $\tau(y) = \beta^{-1} y$. The conclusion follows from the equality $\tau \Delta_p \tau^{-1} = \beta^{-1} \Delta_r$.

Assume now that $p = Ax^2 + Bx + C$, $A, B, C \in k$, $0 \neq A$, and consider the derivation $d = \Delta_p$. Since the field k is algebraically closed, there exists $\beta \in k \setminus \{0\}$ such that $A\beta^4 = 1$. Let $r = r(x) = \beta^2 p(\beta x)$. Then we have

$$r(x) = \beta^2(A(\beta x)^2 + B\beta x + C) = x^2 + \beta^3 Bx + \beta^2 C.$$

This means, by Lemma 8.2, that if we study the problem of simplicity of the above derivation d, then we may assume that $A = 1$. Moreover, by Lemma 8.1, we may also assume that d is of the form δ_e for $e \in k$.

The next theorem is the second main result of the present paper.

Theorem 8.3. *Let $e \in k$. The derivation δ_e is not simple if and only if e is an odd integer.*

The proof of this theorem consists of three lemmas.

Lemma 8.4. *If F and Λ are polynomials in $k[x, y]$ such that $\delta_e(F) = \Lambda F$, $F \neq 0$ and $\deg_y F = 1$, then $\Lambda = y \pm x$.*

Proof. Let $F = uy + v$, where $u, v \in k[x]$, $u \neq 0$. According to Proposition 4.2, $\Lambda = y + s$ for some $s \in k[x]$ of degree 1. Thus $s = ax + b$, where $a, b \in k$ with $a \neq 0$ and

$$\begin{cases} u' = v + (ax + b)u, \\ v' + (-x^2 + e)u = (ax + b)v. \end{cases} \tag{8.1}$$

Let u^* and v^* be the leading forms of the polynomials u and v, respectively. Then $u^* \neq 0$, $v^* \neq 0$ and by the above equalities,

$$v^* = -axu^* \quad \text{and} \quad axv^* = -x^2 u^*,$$

and hence $a^2 = 1$, that is, $a = \pm 1$.

It remains to prove that $b = 0$. Denoting by $g = axu + v$, we write (8.1) as

$$\begin{cases} u' = ub + g, \\ (e - a)u + g' = (2ax + b)g. \end{cases}$$

If b were different from 0, then, by the first of the two above equalities, $\deg(g) = \deg(u)$, a contradiction with the second equality. Thus, $b = 0$ and $\Lambda = y \pm x$. □

Lemma 8.5. *Let $e \in k$. The derivation δ_e has a Darboux polynomial F such that $\deg_y F = 1$ if and only if there exists a nonzero polynomial $u \in k[x]$ such that*

$$u'' - 2xu' + (e - 1)u = 0 \quad \text{or} \quad u'' + 2xu' + (e + 1)u = 0. \tag{8.2}$$

Proof. Let $F = uy + v$, where $u, v \in k[x]$, $u \neq 0$ be a Darboux polynomial of δ_e. By Lemma 8.4 and its proof, $v = u' - axu$ and $v' = axv + (x^2 - e)u$ (see (8.1)), where $a = \pm 1$. Hence, if $a = -1$ then $u'' + 2xu' + (e+1)u = 0$, and if $a = 1$ then $u'' - 2xu' + (e-1)u = 0$.

Assume now that there exists a nonzero polynomial $u \in k[x]$ satisfying (8.2). If $u'' - 2xu' + (e-1)u = 0$, then $\delta_e(F) = (y+x)F$, where $F = uy + (u' - xu)$. If $u'' + 2xu' + (e+1)u = 0$, then $\delta_e(F) = (y-x)F$, where $F = uy + (u' + xu)$.

Lemma 8.6. *Let $e \in k$. The derivation δ_e has a Darboux polynomial F such that $\deg_y F = 1$ if and only if e is an odd integer.*

Proof. Assume that δ_e has a Darboux polynomial F with $\deg_y F = 1$. Then it follows from Lemma 8.5 that there exists a nonzero polynomial $u \in k[x]$ satisfying (8.2). Without loss of generality, u may be assumed monic:

$$u = x^s + a_{s-1}x^{s-1} + \cdots + a_1 x + a_0,$$

where $s \geq 1$ and $a_0, \ldots, a_{s-1} \in k$. If $u'' - 2xu' + (e-1)u = 0$, then comparing the coefficients of x^s in this equality we get

$$-2s + (e-1) = 0,$$

that is, $e = 2s + 1$. In a similar way, if $u'' + 2xu' + (e+1)u = 0$, then $e = -2s - 1$.

Now assume $e = 2s + 1$, where $s \geq 0$. A monic polynomial u_s of degree s is defined in $k[x]$ as follows:

$u_0 = 1$, $u_1 = x$ and for $s \geq 2$, $u_s = a_s x^s + a_{s-1}x^{s-1} + \cdots + a_1 x + a_0$, where

$$\begin{cases} a_s &= 1, \\ a_{s-1} &= 0, \\ a_i &= \frac{(i+1)(i+2)}{2(i-s)} a_{i+2}, \quad \text{for } i = 0, 1, \ldots, s-2. \end{cases}$$

It is easy to check that u_s satisfies the differential equation $u_s'' - 2xu_s' + (e-1)u_s = 0$.

Consider now a negative odd integer $e = -2s - 1$, $s \geq 0$. A similar definition can be given. A monic polynomial v_s of degree s is defined in $k[x]$ as follows: $v_0 = 1$, $v_1 = x$ and for $s \geq 2$, $v_s = a_s x^s + a_{s-1}x^{s-1} + \cdots + a_1 x + a_0$, where

$$\begin{cases} a_s &= 1, \\ a_{s-1} &= 0, \\ a_i &= -\frac{(i+1)(i+2)}{2(i-s)} a_{i+2}, \quad \text{for } i = 0, 1, \ldots, s-2. \end{cases}$$

Here, as it is easily seen, v_s satisfies the differential equation $v_s'' + 2xv_s' + (e+1)v_s = 0$.

Hence, if e is an odd integer then, by Lemma 8.5, there exists a Darboux polynomial F of δ_e such that $\deg_y F = 1$.

Combining Theorem 6.1 and Lemma 8.6 gives the proof of Theorem 8.3. □

Example 8.7. If $e = 1, 3, 5, 7$ or 9 then $\delta_e(F) = (y+x)F$, where F is given by the following table.

e	F
1	$y - x$
3	$xy - x^2 + 1$
5	$(2x^2 - 1)y - 2x^3 + 5x$
7	$(2x^3 - 3x)y - 2x^4 + 9x^2 - 3$
9	$(4x^4 - 12x^3 + 3)y - 4x^5 + 28x^3 - 27x$

This follows from the proofs of Lemmas 8.4–8.6. □

9 EXAMPLES IN n VARIABLES

Let us recall the following result of Shamsuddin [12] (see also [3, 10]).

Theorem 9.1 [12]. *Let R be a ring containing $\mathbb{Q}$ and let d be a simple derivation of R. Extend the derivation d to a derivation $\tilde{d}$ of the polynomial ring $R[t]$ by setting $\tilde{d}(t) = at + b$ where $a, b \in R$. Then the following two conditions are equivalent.*

(1) The derivation $\tilde{d}$ is simple.
(2) There exist no elements r of R such that $d(r) = ar + b$. □

The next two propositions are consequences of the above theorem.

Proposition 9.2. *Let $d : k[x, y, z] \to k[x, y, z]$ be a derivation such that*

$$\begin{cases} d(x) = 1, \\ d(y) = f(x, y), \\ d(z) = y, \end{cases}$$

where $f(x, y) \in k[x, y]$, $\deg_y f(x, y) \geq 2$. Let $\delta : k[x, y] \to k[x, y]$ be the restriction of d to $k[x, y]$. If δ is simple, then d is simple.

Proof. Suppose that there exists $r \in k[x, y]$ such that $\delta(r) = y$. Then $\deg_y r \geq 1$. Let $r = r_n y^n + \cdots + r_1 y + r_0$, where $n \geq 1$, $r_0, \ldots, r_n \in k[x]$,

$r_n \neq 0$, and let $f(x,y) = f_m y^m + \cdots + f_1 y + f_0$, where $m \geq 2$, $f_0, \ldots, f_m \in k[x]$, $f_m \neq 0$.
Then

$$y = d(r) = r'_n y^n + \cdots + r'_1 y + r'_0 \\ + (nr_n y^{n-1} + \cdots + r_1)(f_m y^m + \cdots + f_1 y + f_0).$$

Comparing the coefficients of y^{m+n-1} we get a contradiction: $0 = nr_n f_m \neq 0$.

Hence, there is no polynomial $r \in k[x,y]$ such that $\delta(r) = y$ and hence, by Theorem 9.1, d is simple. □

Proposition 9.3. *Let $d : k[x,y,z] \to k[x,y,z]$ be a derivation such that*

$$\begin{cases} d(x) = 1, \\ d(y) = g(x,y), \\ d(z) = a(x)z + b(x)y + c(x), \end{cases}$$

where $g(x,y) \in k[x,y]$, $\deg_y g(x,y) = 2$, $a(x), b(x), c(x) \in k[x]$ and $b(x) \neq 0$. Let $\delta : k[x,y] \to k[x,y]$ be the restriction of d to $k[x,y]$. If δ is simple, then d is simple.

Proof. Suppose that there exists $r \in k[x,y]$ such that $\delta(r) = a(x)r + b(x)y + c(x)$. Then it is clear that $\deg_y r \geq 1$. Comparing the leading coefficients in y in the above equality, we get a contradiction. Therefore, this proposition follows from Theorem 9.1. □

Repeating the same argument as in the proofs of Propositions 9.2 and 9.3, and using facts from previous sections, we get the following example.

Example 9.4. Let d_1 and d_2 be derivations of $k[x,y,z,t_1,\ldots,t_n]$ defined as follows

$$\begin{cases} d_1(x) = 1 \\ d_1(y) = y^2 + x \\ d_1(z) = y \\ d_1(t_1) = zt_1 + 1 \\ d_1(t_2) = t_1 t_2 + 1 \\ d_1(t_3) = t_2 t_3 + 1 \\ \quad\vdots \\ d_1(t_n) = t_{n-1} t_n + 1, \end{cases} \qquad \begin{cases} d_2(x) = 1 \\ d_2(y) = y^2 + x^5 + 2x \\ d_2(z) = x^2 z + xy \\ d_2(t_1) = z^2 t_1 + z \\ d_2(t_2) = t_1^2 t_2 + t_1 \\ d_2(t_3) = t_2^2 t_3 + t_2 \\ \quad\vdots \\ d_2(t_n) = t_{n-1}^2 t_n + t_{n-1}, \end{cases}$$

The derivations d_1 and d_2 are simple. □

10 FINAL REMARKS

10.1 About the Present Work – Acknowledgments

We started this work while one of us (J.M.O.) visited N. Copernicus University in Toruń. He would like to express his gratitude for the excellent conditions of that visit.

As a matter of fact, the starting point of our considerations was the following:

- Kovačic's algorithm [6] is a very powerful tool to decide whether a linear differential equation (L) of order 2 with coefficients in $\mathbb{C}(\cap)$ has a nontrivial *liouvillian* solution. A liouvillian solution belongs to a differential extension of $\mathbb{C}(\cap)$ of a special type, called liouvillian. This algorithm can be used to prove some non-existence theorems in a rather general setting.
- The existence of a liouvillian solution for (L) is in turn related to the existence of an algebraic solution for a Riccati type (nonlinear) first order differential equation (R); this is a folklore result in Differential Algebra which is well described in Kaplansky's book [5]. According to Differential Galois Theory, if (L) has a liouvillian solution, either (L) has an algebraic solution or (R) has an algebraic solution whose minimal polynomial over $\mathbb{C}$ has a degree 1 or 2.

Let us remark that Kaplansky studies "simple *linear* derivations" (in our vocabulary) as an application of the previous result.

Refer also to the work of Magid [8] for Differential Galois Theory.

- Then algebraic solutions for (R) correspond to Darboux polynomials for some derivation that we are interested in; this last fact is still present and important in our paper.

But all these inspiring remarks and powerful tools (Kovačic's algorithm, Liouvillian extensions, Differential Galois Theory) had to disappear from the body of the paper: our Theorem 5.1 now excludes algebraic non-rational solutions to some differential equations of Riccati type and this is enough to go further and prove our main results.

Nevertheless, we hope that the reader will forgive us having said some words about these interesting facts.

10.2 A Technical Remark

Although all results of this paper are formulated and proved for polynomial rings over an algebraically closed field k, they remain valid for an arbitrary field K of characteristic zero instead of k. It is a consequence of the following proposition given in [10] (see Propositions 13.1.1 and 5.1.4 in [10]).

Proposition 10.1. *Let $K \subset K'$ be an extension of fields (of characteristic zero) and let d be a derivation of $K[x_1, \ldots, x_n]$. Consider the derivation d' of $K'[x_1, \ldots, x_n]$ such that $d'(x_i) = d(x_i)$ for $i = 1, \ldots, n$. Then d is simple if and only if d' is simple.* □

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