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Around Jouanolou non-integrability theorem*

by Andrzej J. Maciejewski[1], Jean Moulin Ollagnier[2], Andrzej Nowicki[3] and Jean-Marie Strelcyn[4,5]

[1] *Toruń Centre for Astronomy, Nicolas Copernicus University, ul. Gagarina 11, 87-100 Toruń, Poland, e-mail: maciejka@astri.uni.torun.pl*
[2] *Laboratoire Gage, UMS CNRS 658 Medicis, École polytechnique, F 91128 Palaiseau Cedex, France, e-mail: Jean.Moulin-Ollagnier@polytechnique.fr*
[3] *Nicholas Copernicus University, Institute of Mathematics, ul. Chopina 12-18, 87-100 Toruń, Poland, e-mail: anow@mat.uni.torun.pl*
[4] *Université de Rouen, Département de Mathématiques, UPRES-A CNRS 60-85, F 76821 Mont Saint-Aignan Cedex, France, e-mail: Jean-Marie.Strelcyn@univ-rouen.fr*
[5] *Laboratoire Analyse, Géometrie et Applications, UMR CNRS 7539 Institut Galilée, Département de Mathématiques, Université Paris-Nord, Avenue J.-B. Clément, F 93430 Villetaneuse, France, e-mail: strelcyn@math.univ-paris13.fr*

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ABSTRACT

We generalize the well-known Jouanolou non-integrability theorem concerning the system of ordinary differential equations: $x_1' = x_2^s, x_2' = x_3^s, x_3' = x_1^s, s \in \mathbb{N}, s \geq 2$ to an arbitrary prime number $n \geq 3$ of variables and arbitrary integer exponent $s \geq 3$. Our proof is completely elementary.

1. INTRODUCTION

The Jouanolou system of ordinary differential equations is defined in $\mathbb{C}^n$ as follows:

$$(1.1) \qquad \frac{dx_1}{dt} = x_2^s, \ \frac{dx_2}{dt} = x_3^s, \ \cdots \ \frac{dx_{n-1}}{dt} = x_n^s, \ \frac{dx_n}{dt} = x_1^s,$$

where s is a strictly positive integer.

This paper is devoted to the problem of integrability of this system. Namely, we look for Darboux polynomials of this system; indeed, the existence of such polynomials is necessary to have a rational or even a Liouvillian first integral [7].

For $s = 1$ and arbitrary $n \geq 2$, this system has a polynomial first integral, e.g. the following determinant:

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$$F = \det \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_n & x_1 & \cdots & x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & \cdots & x_1 \end{bmatrix}.$$

On the contrary, if $n = 3$ and $s \geq 2$ is an arbitrary integer, Jouanolou [4] proved that the system does not possess any polynomial first integral.

Deciding in which cases the Jouanolou system has a rational first integral remains an open question. A negative answer can be suspected for arbitrary $n \geq 4$ and $s \geq 2$ despite the fact that no proof is known. Trying to solve this problem, we found an elementary proof for a prime number $n \geq 5$ of variables and an exponent $s \geq 3$. This proof also works for $n = 3$ and $s \geq 5$. Taking into account some information provided by [8, 9] enables us to overcome the lacking cases $n = 3, s = 3, 4$ in a simple way. Consequently, we obtain for the first time a completely elementary proof (i.e. without any use of algebraic geometry) of Jouanolou theorem for all prime numbers $n \geq 3$ of variables and all exponents $s \geq 3$.

Unfortunately, our method is not conclusive when $n = 3, s = 2$. Let us repeat that the corresponding integrability problem for $s = 2$ and a prime $n \geq 5$ is still completely open as well as the one for a non-prime $n \geq 4$ and $s \geq 2$.

The main tool is the reduction of our problem for the Jouanolou system (1.1) to the same one for the following factorisable quadratic system

$$(1.2) \qquad \frac{dy_i}{dt} = y_i(-y_i + sy_{i+1}), \quad 1 \leq i \leq n, \quad y_{n+1} \equiv y_1,$$

to which we apply the methods of [10].

For the purposes of the present paper, it may be more convenient to use the notion of a derivation instead of the one of a system of ordinary differential equations. A *derivation* is a $\mathbb{C}$-linear mapping from the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ into itself that satisfies Leibniz rule for the derivation of a product. According to this rule, there is a unique way to define an extension of a derivation from a ring to the field of fractions of this ring:

$$(1.3) \qquad d\left(\frac{P}{Q}\right) = \frac{d(P)Q - Pd(Q)}{Q^2}.$$

Examples of derivations are partial derivatives with respect to the variables, that we denote by ∂_i, $1 \leq i \leq n$, in the general n-variable case and also by $\partial_x, \partial_y\, \partial_z$ in the three-variable case. In fact, all derivations of $\mathbb{C}[x_1, \ldots, x_n]$ are linear combinations of partial derivatives with coefficients in polynomial rings, thus we will call them *polynomial derivations*.

The Jouanolou derivation d_J is thus defined by its values on the variables x_i, which generate the polynomial algebra, as follows:

$$(1.4) \qquad d_J(x_1) = x_2^s,\ d_J(x_2) = x_3^s,\ \cdots,\ d_J(x_{n-1}) = x_n^s,\ d_J(x_n) = x_1^s,$$

or equivalently by $d_J = x_2^s\partial_1 + x_3^s\partial_2 + \cdots + x_n^s\partial_{n-1} + x_1^s\partial_n$.

A polynomial or a rational fraction F, is said to be *a constant of the polynomial derivation d* if $d(F) = 0$. If this F does not belong to the base field $\mathbb{C}$, we will also call it *a first integral of d*, keeping the simple word *constant* for the elements of the base field, i.e. the common constants of all partial derivatives.

A polynomial $F \in \mathbb{C}[x_1, \ldots, x_n] \setminus \mathbb{C}$ is said to be a *Darboux polynomial of the derivation d* if there exists a polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ such that $d(F) = PF$; this P is then called the *eigenvalue* of F. In particular, polynomial first integrals are Darboux polynomials with the eigenvalue $P = 0$.

Moreover if $F_1, F_2 \in \mathbb{C}[x_1, \ldots, x_n]$ are relatively prime, the fraction F_1/F_2 is a rational constant of d (or a rational first integral) if and only if F_1 and F_2 are Darboux polynomials of d with the same eigenvalue.

On the other hand, according to the well-known Euler identity, a polynomial F of $\mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree m if and only if F is a Darboux polynomial of the *Euler derivation* $E = \sum x_i \partial_i$ with the eigenvalue m.

Refer to [10, 11] for details concerning polynomial derivations, their constants and Darboux polynomials.

In his book [4], J.-P. Jouanolou proved the following theorem.

Theorem 1.1 (Jouanolou, 1979). *For $n = 3$ and any integer $s \geq 2$, the derivation* (1.4) *does not admit any Darboux polynomial.*

The original proof of this theorem is rather lengthy (pp. 160–192 of [4]). The sketch of an alternate proof can also be found on pp. 193–195 of [4]. Written by A.H.M. Levelt, the referee of the book, this proof contains very fruitful ideas but has a gap. Based on Levelt's ideas, complete proofs are given in [10, 11]. Three different proofs can be found in [3, 6, 13]. Nevertheless, all above mentioned proofs use some facts and arguments of algebraic geometry (about plane algebraic curves) and cannot be considered as truly elementary. Moreover, as they use some facts that are specific to the three-dimensional case, these proofs cannot be extended to higher dimensions. Let us also remark that some new examples of homogeneous systems of ordinary differential equations in $\mathbb{C}^3$ without Darboux polynomials can be found in [14].

As explained in Section 4 of [4], the existence of one example of a system of polynomial differential equations

$$(1.5) \qquad \frac{dx_i}{dt} = f_i(x_1, \cdots, x_n), \ 1 \leq i \leq n,$$

where all polynomials f_i are homogeneous with the same degree s, without any Darboux polynomial, implies that the absence of any Darboux polynomial is typical for such homogeneous polynomial systems of differential equations. Thus, typically, such a system is not integrable.

In fact, Jouanolou is interested in the existence of Darboux polynomials for Pfaff forms, but it turns out that this is equivalent to our problem in the three-variable case.

A consequence of the above-mentioned Jouanolou theorem is then that, for

$n = 3$ and an arbitrary given $s \geq 2$, system (1.5) typically does not admit any Darboux polynomial.

On the contrary, let us remark that, for $n = 2$ and arbitrary $s \geq 1$, it is easy to see that systems like (1.5) always have a linear Darboux polynomial.

From our results, the same genericity consequence holds for an arbitrary prime number $n \geq 5$ of variables and an arbitrary exponent $s \geq 3$. This fact is closely related to a problem stated by V.I. Arnold (see Section 10 of [1]) concerning 'the absence of invariant hypersurfaces (or principal ideals)', which is nothing else but the absence of Darboux polynomials. More precisely, V.I. Arnold asked whether the subset of all systems (1.5) without any Darboux polynomial constitutes an open dense subset in the space of all such systems. For $n = 3$ and $s \geq 2$, this fact was already proved by Lins-Neto [6]. Let us stress that the corresponding fact for $n \geq 4$ and $s \geq 2$ remains completely open.

The present paper is written in a self-contained way and thus can be read independently of the given references; it is organized as follows.

In Section 2 we describe the symmetries of Jouanolou system (1.1) while in Section 3 we show how to reduce our problem to the corresponding one for the factorisable quadratic system (1.2). Section 4 is devoted to the proof of the non-existence of non-constant polynomial first integrals for the system (1.2) when $n \geq 5$ is a prime number and $s \geq 3$ an arbitrary integer exponent, as well as in the case where $n = 3, s \geq 5$; this implies the non-existence of Darboux polynomials for the Jouanolou system (1.1) with the same n and s. In Section 5, we restrict our attention to the case $n = 3$ to deal with the values $s = 3$ and $s = 4$.

In Section 6, we present a detailed proof of a generic non-integrability theorem for homogeneous polynomial derivations which is the counterpart of Jouanolou's result for Pfaff forms.

2. SYMMETRIES OF THE JOUANOLOU SYSTEM

We will denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ the subset of strictly positive integers.

Let d be a derivation of $\mathbb{C}[x_1, \dots, x_n]$. An automorphism g of the algebra $\mathbb{C}[x_1, \dots, x_n]$ that commutes with d is called a *symmetry* of d. A derivation d is said to be *G-invariant* if all elements g of a group G of such automorphisms commute with d.

Let us now assume that d possesses a Darboux polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$, with a corresponding eigenvalue P. If d is G-invariant, for every $g \in G$, $F_g = g(F)$ is also a Darboux polynomial of d and its eigenvalue is $P_g = g(P)$. When the group G of automorphisms is finite, G-invariant Darboux polynomials can be obtained from Darboux polynomials: the product $H = \prod_{g \in G} g(F)$ of all transforms of F is a G-invariant Darboux polynomial of d and its eigenvalue is the sum $Q = \sum_{g \in G} g(P)$ of all transforms of P.

We denote by Γ_n the finite cyclic group of automorphisms of $\mathbb{C}[x_1, \dots, x_n]$ generated by cyclic permutations of the variables. In our considerations, another finite group Δ_n of automorphisms, which is isomorphic to the multiplicative group of all roots of unity of order $s^n - 1$, will play a crucial role. If ϵ is such a root of unity, g_ϵ is the automorphism of the algebra $\mathbb{C}[x_1, \dots, x_n]$ which is completely determined by its values on the set of variables:

$$g_\epsilon(x_1) = \epsilon^{s^{n-1}} x_1,\ g_\epsilon(x_2) = \epsilon^{s^{n-2}} x_2,\ \cdots,\ g_\epsilon(x_n) = \epsilon x_n.$$

These automorphisms arise naturally when we look for 'diagonal' automorphisms g, i.e. such that $g(x_i) = a_i x_i,\ 1 \le i \le n$.

Proving the following lemma is an easy exercise.

Lemma 2.1. *The Jouanolou derivation* (1.4) *is Γ_n-invariant as well as Δ_n-invariant.*

Let us now characterize Δ_n-invariant polynomials. Given a n-tuple $j = (j_1, \cdots, j_n)$ of elements of $\mathbb{N}$, $\sigma(j)$ is the following integer:

$$\sigma(j) = \sum_{i=1}^{n} j_i s^{n-i}. \tag{2.1}$$

It is easy to check that a polynomial is Δ_n-invariant if and only if it is a sum of Δ_n-invariant monomials and that the Δ_n-invariant monomials $x^j = x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ are exactly those whose exponent $j = (j_1, \cdots, j_n)$ is a n-tuple such that $\sigma(j) \equiv 0 \bmod (s^n - 1)$.

Using this characterization, we can prove the following lemma

Lemma 2.2. *If Jouanolou derivation d_J defined in* (1.4) *has a Darboux polynomial, then d_J has also a polynomial first integral i.e. there exists an $F \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$ such that $d_J(F) = 0$.*

Proof. As the polynomials defining the Jouanolou derivation are homogeneous with the same degree, all homogeneous components (with respect to the total degree) of a Darboux polynomial of d are Darboux polynomials. Thus, without loss of generality, we can assume that d has a homogeneous Darboux polynomial F of total degree $m > 0$.

According to the previous lemma, we can also assume that this F is Δ_n-invariant. Its eigenvalue P is then a Δ_n-invariant polynomial. If $P \neq 0$, its degree is $s - 1$, and there exists a n-tuple $j = (j_1, \cdots, j_n)$ such that $\sigma(j) \equiv 0 \bmod (s^n - 1)$, while its total degree $\sum_{i=1}^{n} j_i$ is equal to $s - 1$. These two properties are contradictory: on the one hand, according to (2.1), $\sigma(j)$ has to satisfy $0 < \sigma(j) \le s^n - s$; on the other hand, $\sigma(j)$ has to be a multiple of $s^n - 1$. This contradiction shows that P has to be 0, which means that F is a polynomial constant for d_J. $\square$

The following lemma gives a relation between the two previously described groups of automorphisms, Γ_n and Δ_n, of $\mathbb{C}[x_1, \dots, x_n]$; its proof is easy.

Lemma 2.3. *Let $j = (j_1, \cdots, j_n)$ be a n-tuple of elements of $\mathbb{N}$ such that $\sigma(j)$ is a multiple of $s^n - 1$ (which means that the corresponding monomial is Δ_n-invariant). Then, the n-tuple $\tau(j) = (j_2, j_3, \cdots, j_n, j_1)$ enjoys the same property.*

3. THE CHANGE OF VARIABLES

Let us consider the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ as a subring of the field of rational fractions $\mathbb{C}(x_1, \dots, x_n)$. Let d_J denote the Jouanolou derivation defined in (1.4) for which we suppose $s \geq 2$. Let y_i be the n elements of $\mathbb{C}(x_1, \dots, x_n)$ defined by

$$(3.1) \qquad y_i = \frac{d_J(x_i)}{x_i} = \frac{x_{i+1}^s}{x_i}, \; 1 \leq i \leq n, \; x_{n+1} = x_1$$

It is not difficult to check that the y_i are algebraically independent by writing the Jacobian:

$$\det \frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} = (-1)^n (s^n - 1) \frac{y_1 \dots y_n}{x_1 \dots x_n},$$

which is different from 0 if $s \geq 2$.

Thus, the subring generated by the y_i is a polynomial ring in n variables. By formula (1.3), the derivation d_J of $\mathbb{C}[x_1, \dots, x_n]$ extends in a unique way to a derivation of $\mathbb{C}(x_1, \dots, x_n)$, and the (polynomial) subring $\mathbb{C}[y_1, \dots, y_n]$ is d_J-invariant. Let us call δ the restriction to the polynomial ring $\mathbb{C}[y_1, \dots, y_n]$ of the extension of d_J to $\mathbb{C}(x_1, \dots, x_n)$.

It is easy to check that δ is entirely defined by the image of y_i:

$$(3.2) \qquad \delta(y_i) = y_i(-y_i + sy_{i+1}), \; 1 \leq i \leq n, \; y_{n+1} = y_1.$$

Despite the fact that an arbitrary polynomial F of $\mathbb{C}[x_1, \dots, x_n]$ cannot be written as a composition $F(x) = G(y(x))$ where G belongs to $\mathbb{C}[y_1, \dots, y_n]$ and $y(x)$ stands for the mapping given by (3.1), some special ones may be written in this way.

Lemma 3.1. *Let $\phi \in \mathbb{C}[x_1, \dots, x_n]$ be a Δ_n-invariant polynomial. Then, there exists a unique polynomial $\Phi \in \mathbb{C}[y_1, \dots, y_n]$ such that $\phi(x) = \Phi(y(x))$.*

Proof. Uniqueness is a consequence of algebraic independence of y_i. To ensure existence, it suffices to prove that a Δ_n-invariant monomial $x^j = x_1^{j_1} \cdots x_n^{j_n}$ can be written as a monomial in y_i. The exponent j is an n-tuple for which $\sigma(j)$ is a multiple of $s^n - 1$, thus we look for elements $k_1, \cdots, k_n$ of $\mathbb{N}$ such that

$$x_1^{j_1}.x_2^{j_2} \cdots x_n^{j_n} = y_1^{k_1}.y_2^{k_2} \cdots y_n^{k_n},$$

which, by (3.1), amounts to solving the following system of equations

$$
(3.3) \qquad \begin{cases} j_1 = sk_n - k_1 \\ j_2 = sk_1 - k_2 \\ \vdots \\ j_n = sk_{n-1} - k_n \end{cases}
$$

The assumption that $\sigma(j)$ is a (non-negative) multiple of $s^n - 1$ allows us to solve the previous system with all $k_i \in \mathbb{N}$. For instance, multiplying the first equation by s^{n-1}, the second by s^{n-2}, etc. and adding them up, yields $(s^n - 1)k_n = \sigma(j)$, which gives a non-negative integer value for k_n. The same is true for the other k_i by circular permutations of system (3.3) and application of Lemma 2.3. □

We can now state as a corollary the conclusion of this reduction process.

Corollary 3.2. *If the derivation δ defined by* (3.2) *has no polynomial first integral (non-trivial constant), then the same is true for the corresponding Jouanolou derivation d_J defined by* (1.4)*. Then, according to Lemma* 2.2*, d_J has no Darboux polynomial.*

Proof. If the Jouanolou derivation (1.4) has a polynomial first integral, then it has also a Δ_n-invariant polynomial first integral ϕ. Then, according to Lemma 3.1, $\phi(x) = \Phi(y(x))$, where Φ is a polynomial first integral of derivation (3.2). □

Finally, let us note that a change of variables like (3.1), which yields the factorisable form (3.2), goes back to Lagutinskii's paper [5]. This is also a special case of Bruno's *power transformations* [2].

4. NON-INTEGRABILITY OF JOUANOLOU SYSTEM FOR A PRIME NUMBER $n \geq 5$ OF VARIABLES AND $s \geq 3$

According to the last corollary of the previous section, to show the non-integrability of the Jouanolou system for a prime number $n \geq 5$ of variables and $s \geq 3$, or $n = 3$ and $s \geq 5$, it is sufficient to prove the following theorem concerning the derivation δ.

Theorem 4.1. *Let $n \geq 3$ be a prime number and let $s \geq 3$ be an integer. If $s > 2\frac{n-1}{n-2}$, then the derivation δ of $\mathbb{C}[x_1, \dots, x_n]$ defined as in* (3.2) *by*

$$
(4.1) \qquad \delta(x_i) = x_i(-x_i + sx_{i+1}), \ 1 \leq i \leq n, \ x_{n+1} = x_1,
$$

has no polynomial first integral.

Proof. Our proof is an example of the so-called Lagutinskii-Levelt procedure (LL for short) described in [10, 11]. The various steps of this procedure will appear along the proof.

Suppose that F is a first integral of the derivation δ. As the polynomial coefficients defining δ from the partial derivatives are homogeneous of the same degree, the homogeneous component of F of highest total degree is also a first integral of δ.

Thus, without loss of generality, we can suppose this F to be homogeneous of some degree $m \geq 1$, i.e. a Darboux polynomial of the Euler derivation E with eigenvalue m.

> The first step of the LL procedure consists in choosing some 'Darboux points' of δ around which we write the facts that F is a Darboux polynomial for δ and E in local coordinates.

For a homogeneous polynomial derivation $d = \sum V_i \partial_i$, a *Darboux point* is a non-zero vector $M = (X_1, \cdots, X_n)$ of $\mathbb{C}^n$ where d and E are collinear, which means that there exists a k in $\mathbb{C}$ such that $V_i(M) = kX_i$, $1 \leq i \leq n$.

Of course, this property only depends on the one-dimensional vector space generated by M, i.e. this is a property of points in the projective space. In the present proof, we will only need the Darboux point $M_0 = (1, \cdots, 1)$.

Fix the last coordinate $x_n = 1$ and choose $n-1$ local coordinates around M_0: $x_i = 1 + y_i$, $1 \leq i \leq n-1$. Homogeneous polynomials in n variables (denoted by capital letters) become non-homogeneous polynomials in $n-1$ variables (denoted by the corresponding small letters).

Isolating the contribution of the last partial derivative, the Darboux property of F for δ ($\delta(F) = 0$ here) and the homogeneity of F ($E(F) = mF$) can be written as

$$(4.2) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n-1} x_i(-x_i + sx_{i+1})\partial_i(F) + x_n(-x_n + sx_1)\partial_n(F) = 0, \\ \displaystyle\sum_{i=1}^{n-1} x_i \partial_i(F) + x_n \partial_n(F) = mF. \end{cases}$$

A linear combination of these two equations makes the last partial derivative disappear:

$$(4.3) \qquad \sum_{i=1}^{n-1} x_i[(-x_i + sx_{i+1}) - sx_1 + x_n]\partial_i(F) = m(-sx_1 + x_n)F.$$

This can be expressed in local coordinates around M_0:

$$(4.4) \qquad \sum_{i=1}^{n-1}(1 + y_i)[-y_i + sy_{i+1} - sy_1]\frac{\partial f}{\partial y_i} = m[-s(1 + y_1) + 1]f,$$

where the last term of the left-hand side sum deserves a special treatment as $y_n = 0$:

$$(4.5)\qquad \sum_{i=1}^{n-2}(1+y_i)[-y_i+sy_{i+1}-sy_1]\frac{\partial f}{\partial y_i}+(1+y_{n-1})[-y_{n-1}-sy_1]\frac{\partial f}{\partial y_{n-1}} = -m[s(1+y_1)-1]f.$$

The second step of the LL procedure consists in the consideration of the homogeneous component h of lowest degree $l \leq m$ of f. Let us note that l is nothing else but the multiplicity of F at the chosen Darboux point M.

Looking at terms of lowest degree in (4.5) we get an equation for h:

$$(4.6)\qquad \sum_{i=1}^{n-2}[-y_i+sy_{i+1}-sy_1]\frac{\partial h}{\partial y_i}+[-y_{n-1}-sy_1]\frac{\partial h}{\partial y_{n-1}} = -m(s-1)h.$$

To simplify matters, we add the Euler identity for the homogeneous polynomial h of degree l to (4.6) and then divide by s:

$$(4.7)\qquad \sum_{i=1}^{n-2}[y_{i+1}-y_1]\frac{\partial h}{\partial y_i}-y_1\frac{\partial h}{\partial y_{n-1}} = \frac{-m(s-1)+l}{s}h.$$

The above relation (4.7) is a linear equation relating h and its $n-1$ partial derivatives with respect to the local coordinates y_i, whose coefficients are homogeneous polynomials of degree 1 with respect to y_i.

Let T be the $(n-1)\times(n-1)$ square matrix whose entries in the i-th row are the coefficients of the linear form near $\partial h/\partial y_i$ in (4.7). In our case, the matrix T is

$$(4.8)\qquad T = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The conclusion of the second step of the LL procedure is that there exist non-negative integers $i_1, \cdots, i_{n-1}$ such that

$$(4.9)\qquad \sum_{k=1}^{n-1}\lambda_k i_k = \frac{-m(s-1)+l}{s}, \quad \sum_{k=1}^{n-1} i_k = l,$$

where λ_i are the eigenvalues of T.

In the general case, the right-hand side of the first equation in (4.9) has to be replaced by the numerical coefficient before h in (4.7), which in turn comes from the right-hand side of the first equation of (4.2), this value being 0 in the present case.

Refer to [10] for a proof.

In our case, the characteristic polynomial of T is $Q(\lambda) = (-1)^n \sum_{r=0}^{n-1}\lambda^r$ and its roots are all n-th roots of unity except 1 itself. Choosing some primitive n-th root ϵ of unity, we deduce from (4.9) that

$$(4.10)\quad \sum_{k=1}^{n-1} i_k \epsilon^k + \frac{m(s-1)-l}{s} = 0.$$

As n is a prime number, Q is the minimal polynomial of ϵ and the polynomial defined by the above relation (4.10) has to be a multiple of Q. As their degrees are the same, these two polynomials are proportional so that

$$i_1 = i_2 = \cdots = i_{n-1} = \frac{m(s-1)-l}{s}.$$

Summing from 1 to $n-1$, (4.9) gives

$$(4.11)\quad (n-1)\,\frac{m(s-1)-l}{s} = l.$$

As $l \leq m$, we get $(n-1)\frac{s-2}{s} \leq 1$ and finally $s \leq 2\frac{n-1}{n-2}$.

Thus if $s > 2\frac{n-1}{n-2}$, the derivation δ has no first integral. □

Now, using Corollary 3.2, we derive two consequences of the previous theorem.

Corollary 4.2. *Let $n \geq 3$ be a prime number and let $s \in \mathbb{N}$, $s \geq 3$ be such that $s > 2\frac{n-1}{n-2}$. Then the Jouanolou derivation* (1.4) *has no Darboux polynomial.*

Corollary 4.3. *The conclusions of the previous Corollary are valid for a prime number $n \geq 5$ as soon as $s \geq 3$ and for the prime number* 3 *when $s \geq 5$.*

Thus, the three-variable case needs special arguments when $s = 3$ or $s = 4$; this is the purpose of the next section. Remark that these arguments will be valid for any $s \geq 3$.

5. THE CASE $n = 3$ AND $s = 3, 4$

Theorem 5.1. *In the three-variable case, the derivation δ defined in* (4.1) *has no polynomial constant, when $s \geq 3$.*

Proof. In this three-variable case, the factorisable quadratic derivation δ turns out to be a linear combination

$$\delta = \mathrm{LV}(s+1, s+1, s+1) - (x+y+z)E,$$

where E stands for the three-variable Euler derivation $E = x\partial_x + y\partial_y + z\partial_z$, and where LV is the so-called Lotka-Volterra derivation, which can be considered as a normal form of a factorisable quadratic derivation in three variables and depends on three parameters:

$$\mathrm{LV}(A, B, C) = x(Cy+z)\partial_x + y(Az+x)\partial_y + z(Bx+y)\partial_z.$$

We will simply denote $LV(s+1, s+1, s+1)$ by LV_S.

A homogeneous polynomial F of degree m is then a constant of δ if and only if F is a Darboux polynomial for LV_S with an eigenvalue $m(x+y+z)$. Due to the cyclic symmetry of the problem, without loss of generality, F can be supposed to be Γ_3-invariant. In this case, F factors as

$$(5.1) \qquad F = (xyz)^\alpha G,$$

where α is a non-negative integer and G is a *strict* Darboux polynomial of LV_S, i.e. G is no longer divisible by x, y or z.

The eigenvalue of G as a Darboux polynomial for LV_S is $[m-\alpha(s+2)](x+y+z)$ and its degree is equal to $m-3\alpha$; moreover, G is also Γ_3-invariant.

Let us recall an elementary discussion, which can be found in [8, 9], about a strict Darboux polynomial of the Lotka-Volterra derivation.

Let G be a strict Darboux polynomial of degree $\tilde{m}$ for LV(A,B,C):

$$(5.2) \qquad x(Cy+z)\partial_x G + y(Az+x)\partial_y G + z(Bx+y)\partial_z G = (\lambda x + \mu y + \nu z)G.$$

As G is supposed to be divisible neither by x, nor by y nor by z, we can consider the three homogeneous non-zero two-variable polynomials of degree $\tilde{m}$ obtained by setting $x=0$ (resp. $y=0$, $z=0$) in G and call them P (resp. Q, R).

From relation (5.2) involving G, we deduce some partial differential equations concerning these three two-variable polynomials:

$$\begin{cases} (\mu y + \nu z)P &= yz(A\partial_y P + \partial_z P), \\ (\nu z + \lambda x)Q &= zx(B\partial_z Q + \partial_x Q), \\ (\lambda x + \mu y)R &= xy(C\partial_x R + \partial_y R). \end{cases}$$

Using a partial fraction decomposition (we deal with homogeneous two-variable polynomials), we have no difficulty to prove that there exist six non-negative integers β_1, γ_1, α_2, γ_2, α_3 and β_3 such that P is a non-zero multiple of $y^{\beta_1} z^{\gamma_1}(y-Az)^{\tilde{m}-\beta_1-\gamma_1}$, Q is a non-zero multiple of $z^{\gamma_2}x^{\alpha_2}(z-Bx)^{\tilde{m}-\gamma_2-\alpha_2}$ and R is a non-zero multiple of $x^{\alpha_3}y^{\beta_3}(x-Cy)^{\tilde{m}-\alpha_3-\beta_3}$.

Moreover, these integers satisfy the following relations

$$\begin{cases} \lambda &= \beta_3 &= \gamma_2 B, \\ \mu &= \gamma_1 &= \alpha_3 C, \\ \nu &= \alpha_2 &= \beta_1 A, \\ \beta_1 &+ \gamma_1 &\leq \tilde{m}, \\ \alpha_2 &+ \gamma_2 &\leq \tilde{m}, \\ \alpha_3 &+ \beta_3 &\leq \tilde{m}. \end{cases}$$

In particular, the eigenvalue corresponding to a strict Darboux polynomial of LV is a linear form $\lambda x + \mu y + \nu z$ where λ, μ and ν are non-negative integers.

In our case, $A = B = C = s+1$ and, as G is Γ_3-invariant, $\lambda = \mu = \nu$. Thus, there exist non-negative integers λ and ρ such that $\lambda = (s+1)\rho$, $\lambda + \rho \leq \deg(G) = m - 3\alpha$.

On the other hand, equality (4.11), which we got near the end of the proof of

Theorem 4.1, becomes here $2m(s-1) = l(s+2)$, where l is the multiplicity of F at the Darboux point $M_0 = (1,1,1)$. As x, y and z do not vanish at M_0, l is also the multiplicity of G, in such a way that $l \leq \deg(G) = m - 3\alpha$.

We can now achieve the proof; if there existed a polynomial first integral F of δ, then the previously defined non-negative integers would satisfy the following relations

$$(5.3) \qquad \begin{cases} l(s+2) & = & 2m(s-1), \\ \alpha(s+2) & = & m-\lambda, \\ l & \leq & m-3\alpha, \\ \lambda(1+1/(s+1)) & \leq & m-3\alpha, \end{cases}$$

where the last inequality is nothing else but $\lambda + \rho \leq m - 3\alpha$.

This system (5.3) of equations and inequalities has no solution. Indeed, its first equation would yield

$$l = 2m\frac{s-1}{s+2},$$

and its first inequality would then give an upper bound for α:

$$(5.4) \qquad 3\alpha \leq m\frac{4-s}{s+2}.$$

We then recover what we know from Section 4: $s \geq 5$ is impossible.

From the previous upper bound (5.4) for α and the second equation of (5.3) a lower bound for λ follows:

$$(5.5) \qquad \lambda \geq m\frac{s-1}{3}.$$

On the other hand, as α is known from λ, s and m from the second equation of (5.3), an upper bound for λ can be deduced from the second inequality of (5.3):

$$(5.6) \qquad \lambda \leq m\frac{s^2-1}{s^2+s+1}.$$

As m and $s-1$ are nonnegative, bounds (5.5) and (5.6) for λ are contradictory as soon as $s^2 - 2s - 2 > 0$, which is true for $s \geq 3$. □

6. GENERIC NON-INTEGRABILITY OF HOMOGENEOUS POLYNOMIAL DERIVATIONS

The existence of Darboux polynomials is a necessary condition for a polynomial derivation to have a first integral, which is a polynomial, a rational fraction, or even belongs to a Liouvillian extension of $\mathbb{C}(x_1, \ldots, x_n)$ [7]. Therefore, the non-existence of Darboux polynomials is a reasonable definition of non-integrability (among many equally reasonable ones).

In his book [4], J.-P. Jouanolou proves the following result, that describes the fact that, for three variables, the absence of Darboux polynomials is typical while their existence is rare.

> Theorem 1.1 of [4] p. 158 : In the projective space $X(3, m)$ of all complex algebraic Pfaff equations of a given degree m in three variables, the subset of equations without algebraic solutions is a countable intersection of non-empty Zariski open sets. It is then a dense G_δ for the ordinary topology of the space $X(3, m)$.

In this last section, we give a counterpart of Jouanolou's result in the case of homogeneous polynomial derivations. We will give two versions of our theorem.

The first one, which we will call *natural*, deals with homogeneous polynomial derivations or vector fields.

In the second version, which we will refer to as the *quotient version*, we take into account the fact that a homogeneous Darboux polynomial f of a homogeneous polynomial derivation δ of degree s in n variables is also a Darboux polynomial of the sum $\delta + \phi E$, where E is the n-variable Euler derivation and where ϕ is an arbitrary homogeneous polynomial of degree $s - 1$. Of course, the eigenvalue changes whereas the fact that f is Darboux remains. Thus, instead of considering the finite-dimensional vector space of all homogeneous polynomial derivations of degree s in n variables, we will consider the quotient space of this vector space by the subspace of all ϕE. We can then choose derivations with a 'zero divergence' as representatives of elements of the quotient space.

The second point of view agrees exactly with Jouanolou's theorem in the case of three variables. Indeed a Pfaff 1-form can be built from the wedge product of an homogeneous vector field V and the Euler field E. Conversely, the exterior derivative of a Pfaff 1-form is a closed 2-form that can be identified with a vector field with a zero divergence in the case of three variables. In more variables, polynomial derivations and algebraic Pfaff equations cannot be reduced to one another.

A preliminary remark has to be made in order to show the meaning of both versions of our theorem.

We are interested in the finite dimensional vector space of all homogeneous polynomial derivations of some degree s in n variables (or, for the quotient version, in its quotient by the subspace generated by E). If a non-zero homogeneous polynomial derivation (with a zero divergence in the quotient version) has a Darboux polynomial, the same is true for a non-zero multiple of it in such a way that the natural spaces to consider are the projective spaces corresponding to the previous vector spaces. So, by generic non-integrability, we mean the following results:

> In the two previous projective spaces, the absence of Darboux polynomials is a generic property in the Baire categorical sense.

We give now some precise notation and state the main algebraic propositions.

Let us fix the integers n and s, $n \geq 2$ and $s \geq 1$.

Let $\mathcal{H}_{n,s}$ denote the finite-dimensional $\mathbb{C}$-vector space of all homogeneous polynomials of degree s in n variables. Its dimension is $\binom{n+s-1}{s}$ and a natural basis is given by products of powers of the variables; in the sequel, coordinates will be taken with respect to this basis, which means that we will use the usual coefficients of the polynomials of $\mathcal{H}_{n,s}$ as coordinates.

Let $\mathcal{V}_{n,s}$ denote the vector space of all homogeneous polynomial derivations of degree s in n variables. An element of $\mathcal{V}_{n,s}$ is given by its n coordinates, which belong to $\mathcal{H}_{n,s}$ and the dimension of $\mathcal{V}_{n,s}$ is then $n\binom{n+s-1}{s}$.

In order to state the quotient version of the theorem, let $\mathcal{W}_{n,s}$ denote the quotient space of $\mathcal{V}_{n,s}$ by the subspace $\mathcal{H}_{n,s-1}\,E$ of all multiples of E by homogeneous polynomials of degree $s-1$, whose dimension is $\binom{n+s-2}{s-1}$.

A natural supplementary subspace of $\mathcal{H}_{n,s-1}\,E$ in $\mathcal{V}_{n,s}$ consists of all homogeneous polynomial derivations of degree s whose divergence is 0. Let us recall that the *divergence* of $\delta = \sum V_i \partial_i$ is the sum $\operatorname{div}(\delta) = \sum \partial_i(V_i)$.

We will naturally identify the quotient space $\mathcal{W}_{n,s}$ with the subspace of $\mathcal{V}_{n,s}$ of all derivations whose divergence is 0.

For any vector space $\mathcal{V}$ over $\mathbb{C}$, $\mathbb{P}(\mathcal{V})$ stands for the corresponding projective space and π denotes the canonical mapping from $\mathcal{V} \setminus \{0\}$ onto $\mathbb{P}(\mathcal{V})$.

Proposition 6.1. *Let $\mathcal{D}_{n,s,m}$ be the subset of all non-zero elements of $\mathcal{V}_{n,s}$ for which there exists a Darboux polynomial of degree m. For every positive integer m, the subset $\pi(\mathcal{D}_{n,s,m})$ of $\mathbb{P}(\mathcal{V}_{n,s})$ is closed in the Zariski topology.*

Proof. We study the subset of all non-zero δ in $\mathcal{V}_{n,s}$ for which there exists an element Λ of $\mathcal{H}_{n,s-1}$ such that there exists a non-zero Darboux polynomial f of degree m (an element of $\mathcal{H}_{n,m}$) for δ with the eigenvalue Λ, i.e. $\delta(f) = \Lambda f$.

The fact that $\mathcal{D}_{n,s,m}$ is a closed Zariski set is the result of a two-step elimination process:

1. the elimination of f will show that the set of all non-zero couples (δ, Λ), for which there exists a non-zero Darboux polynomial f for δ with the eigenvalue Λ, is Zariski closed,
2. the further elimination of the eigenvalue Λ gives the conclusion.

The first elimination amounts to saying that a linear system has a non-zero solution. The unknowns of this system are the coefficients of the unknown Darboux polynomial f of degree m and the equations correspond to coefficients of all monomials in the polynomial $\delta(f) - \Lambda f$. Coordinates of δ and Λ in their vector spaces, i.e. coefficients of polynomials $\delta(x_1), \cdots, \delta(x_n)$ and Λ, appear as parameters with a degree 1 in the previous linear equations.

The existence of a non-trivial solution of this linear system is then equivalent to the fact that many determinants are 0.

These determinants are homogeneous polynomials in the coordinates of δ and Λ, and thus they vanish on a closed Zariski subset $\mathcal{Z}_{n,s,m}$ of the projective space $\mathbb{P}(\mathcal{V}_{n,s} \times \mathcal{H}_{n,s-1})$.

As the only possible eigenvalue of the derivation 0 is 0, the first projection p_1 of the product vector space $\mathcal{V}_{n,s} \times \mathcal{H}_{n,s-1}$ defines a regular mapping from $\mathcal{Z}_{n,s,m}$ to $\mathbb{P}(\mathcal{V}_{n,s})$ whose image is exactly $\pi(\mathcal{D}_{n,s,m})$.

Then $\pi(\mathcal{D}_{n,s,m})$ is a closed Zariski subset of $\mathbb{P}(\mathcal{V}_{n,s})$ as the image of a projective closed set by a regular mapping. This classical result is stated as Theorem 3 in the 5th section of the first chapter of [12]. □

The quotient version of this proposition can be proved in the same manner.

Proposition 6.2. *Let $\mathcal{P}_{n,s,m}$ be the subset of all non-zero elements of $\mathcal{W}_{n,s}$ for which there exists a Darboux polynomial of degree m. For every positive integer m, the subset $\pi(\mathcal{P}_{n,s,m})$ of $\mathbb{P}(\mathcal{W}_{n,s})$ is closed in the Zariski topology.*

We can now derive the announced genericity conclusions from these propositions.

Theorem 6.3. *Let $\mathcal{G}_{n,s}$ be the subset of all elements of $\mathcal{V}_{n,s}$ without a Darboux polynomial. If the number $n \geq 3$ of variables is a prime number and if the degree s is such that $s \geq 3$, then $\mathcal{G}_{n,s}$ is a dense G_δ for the ordinary topology of $\mathcal{V}_{n,s}$.*

The same is true for the subset of all elements of $\mathcal{W}_{n,s}$ without a Darboux polynomial.

Proof. For every integer m, the closed Zariski set $\mathcal{D}_{n,s,m}$ is also closed for the ordinary topology. Moreover, either it is the whole space or it has an empty interior.

Then, the complement $\mathcal{G}_{n,s}$ of the union of all $\mathcal{D}_{n,s,m}$ in $\mathbb{P}_{n,s}$ is a G_δ for the ordinary topology and, by Baire category theorem, $\mathcal{G}_{n,s}$ is either dense or empty.

To conclude that it is dense, meaning that the absence of Darboux polynomials is generic, it then suffices to show that none of the $\mathcal{D}_{n,s,m}$ is the whole space.

Jouanolou's argument consists in proving that a given derivation (the one we called Jouanolou derivation) has no Darboux polynomial of any degree.

In our case, that is exactly what we did in Sections 3 and 4 for the announced values of n and s.

All that goes for the quotient version of the theorem as well as it goes for the natural version of it. □

It is worth noting that Proposition 6.2 allows the following dichotomy result which is true for all $n \geq 3$ and $s \geq 2$.

Proposition 6.4. *Fix $n \geq 3$ and $s \geq 2$. Either $\mathcal{G}_{n,s}$ is a dense G_δ for the ordinary topology of $\mathcal{V}_{n,s}$ or $\mathcal{G}_{n,s}$ is empty.*

Finally, let us note that the same genericity results also hold in the real case.

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