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POLYNOMIAL ALGEBRA OF CONSTANTS OF THE LOTKA–VOLTERRA SYSTEM

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Abstract. Let k be a field of characteristic zero. We describe the kernel of any quadratic homogeneous derivation $d : k[x,y,z] \to k[x,y,z]$ of the form $d = x(Cy+z)\frac{\partial}{\partial x} + y(Az+x)\frac{\partial}{\partial y} + z(Bx+y)\frac{\partial}{\partial z}$, called the Lotka–Volterra derivation, where $A, B, C \in k$.

1. Introduction. Let $k[x,y,z]$ be the algebra of polynomials in three variables x, y, z over a field k of characteristic zero. By a *derivation* of $k[x,y,z]$ we mean a k-linear mapping $d : k[x,y,z] \to k[x,y,z]$ of the form

$$d = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z},$$

where $f, g, h \in k[x,y,z]$. If the polynomials f, g, h are homogeneous of the same degree s, then we say that d is *homogeneous of degree* s.

For a given derivation d of $k[x,y,z]$ we denote by $k[x,y,z]^d$ the kernel of d, that is,

$$k[x,y,z]^d = \{w \in k[x,y,z] : d(w) = 0\}.$$

The set $k[x,y,z]^d$ is a k-subalgebra of $k[x,y,z]$ containing k, called *the k-algebra of constants of d*. The set $k[x,y,z]^d \setminus k$ coincides with the set of polynomial first integrals of the corresponding system

$$\dot{x} = f(x,y,z), \quad \dot{y} = g(x,y,z), \quad \dot{z} = h(x,y,z),$$

of ordinary differential equations in three variables (see [4], [5] or [6] for details).

It is well known ([7], [5]), that the algebra $k[x,y,z]^d$ is finitely generated over k. This means that either

$$k[x,y,z]^d = k$$

or there exist polynomials $f_1, \ldots, f_r \in k[x,y,z] \setminus k$ (where $r \geq 1$) such that

$$k[x,y,z]^d = k[f_1, \ldots, f_r],$$

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where $k[f_1,\ldots,f_r]$ means the smallest k-subalgebra of $k[x,y,z]$ containing k and $f_1,\ldots,f_r$.

The minimal number of generators of $k[x,y,z]^d$ is not bounded when d runs over the set of all derivations of $k[x,y,z]$, and even if d runs over the set of all homogeneous derivations of degree 1 (see [8]). If $k[x,y,z]^d = k$ then we say that the algebra of constants is *trivial.*

Assume now that A,B,C are elements of k. Following [1], by a *Lotka–Volterra derivation* defined by the triple (A,B,C) we mean the derivation $d : k[x,y,z] \to k[x,y,z]$ given by the formula

$$d = x(Cy+z)\frac{\partial}{\partial x} + y(Az+x)\frac{\partial}{\partial y} + z(Bx+y)\frac{\partial}{\partial z}. \tag{1.1}$$

Note that d is a quadratic homogeneous derivation such that

$$d(x) = x(Cy+z),\qquad d(y) = y(Az+x),\qquad d(z) = z(Bx+y).$$

The autonomous system of differential equations, corresponding to the polynomials $x(Cy+z)$, $y(Az+x)$, $z(Bx+y)$, is called the *Lotka–Volterra system.* This system has been studied for a long time; see for example [1], [2], [3], where many references on this subject can be found. We are interested in an algebraic description of the k-algebra $k[x,y,z]^d$.

In [1; pp. 687–689] a list of polynomials belonging to $k[x,y,z]^d$ is presented. In [2] the first named author characterizes all Lotka–Volterra derivations d such that $k[x,y,z]^d \neq k$, as follows.

THEOREM 1.2 ([3]). *Let $d : k[x,y,z] \to k[x,y,z]$ be the Lotka–Volterra derivation* (1.1) *with respect to* (A,B,C). *The algebra* $k[x,y,z]^d$, *of constants of d, is non-trivial if and only if one of the following cases holds*:

(1) $ABC = -1$.

(2) $C = -1 - 1/A$, $A = -1 - 1/B$ *and* $B = -1 - 1/C$.

(3) $C = -k_2 - 1/A$, $A = -k_3 - 1/B$, $B = -k_1 - 1/C$, *where*, *up to a permutation*, (k_1,k_2,k_3) *is one of the triples*: $(1,2,2)$, $(1,2,3)$, $(1,2,4)$.

The polynomials $x - Cy + ACz$ *and* $A^2B^2x^2 + y^2 + A^2z^2 - 2ABxy - 2A^2Bxz - 2Ayz$ *belong to* $k[x,y,z]^d$ *in the cases* (1) *and* (2), *respectively. In each of the cases of* (3), *there exists a homogeneous polynomial in* $k[x,y,z]^d$ *of degree* 3, 4 *or* 6 *respectively.* ■

The main result of the present paper is the following theorem, giving a complete description of the algebra $k[x,y,z]^d$ of constants in each of the cases (1), (2), (3) in Theorem 1.2.

THEOREM 1.3. *Let $k[x,y,z]$ be the algebra of polynomials in three variables over a field k of characteristic zero. Let $d : k[x,y,z] \to k[x,y,z]$ be a Lotka–Volterra derivation* (1.1) *such that* $k[x,y,z]^d \neq k$.

(1) *Assume that* $ABC = -1$ *and let* $\mathbb{Q}_- \subseteq k$ *be the set of negative rational numbers.*

(1a) *If* $A, B, C \in \mathbb{Q}_-$ *then there exist positive integers* p, q, r *such that* $\gcd(p, q, r) = 1$, $A = -\frac{p}{q}$, $B = -\frac{q}{r}$, $C = -\frac{r}{p}$, *and* $k[x, y, z]^d = k[t, w]$, *where*

$$\begin{cases} t = pqx + rqy + rpz, \\ w = x^p y^q z^r. \end{cases}$$

(1b) *If some of the scalars* A, B, C *belongs to* $k \setminus \mathbb{Q}_-$ *then* $k[x, y, z]^d = k[x - Cy + ACz]$.

(2) *If* $C = -1 - 1/A$, $A = -1 - 1/B$ *and* $B = -1 - 1/C$, *then* $k[x, y, z]^d = k[g]$, *where* $g = A^2B^2x^2 + y^2 + A^2z^2 - 2ABxy - 2A^2Bxz - 2Ayz$.

(3) *Let* $C = -k_2 - 1/A$, $A = -k_3 - 1/B$, $B = -k_1 - 1/C$, *where, up to a permutation,* (k_1, k_2, k_3) *is one of the triples*: $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$. *In every case there exists a homogeneous irreducible polynomial* g *in* $k[x, y, z]$ (*of degree* 3, 4 *or* 6, *respectively*) *such that* $k[x, y, z]^d = k[g]$.

The proof of Theorem 1.3 is presented in Section 5 and is based on a sequence of preparatory results given in Sections 2–4.

2. Darboux polynomials and strict polynomial constants. Assume that $d : k[x, y, z] \to k[x, y, z]$ is the Lotka–Volterra derivation with respect to (A, B, C).

We say that a nonzero polynomial $f \in k[x, y, z]$ is a *Darboux polynomial* of d if $d(f) = hf$ for some $h \in k[x, y, z]$. In this case the polynomial h is unique and it is called the *eigenvalue* of f.

It is easy to show that the product of Darboux polynomials is a Darboux polynomial. Moreover, if $f \in k[x, y, z]$ is a Darboux polynomial then so is each factor of f. Nonzero polynomials which belong to $k[x, y, z]^d$ are simply Darboux polynomials with the zero eigenvalue.

The variables x, y, z are Darboux polynomials with the eigenvalues $Cy + z$, $Az + x$, $Bx + y$, respectively. Every monomial $x^\alpha y^\beta z^\gamma$ is a Darboux polynomial with the eigenvalue equal to

$$\alpha(Cy + z) + \beta(Az + x) + \gamma(Bx + y).$$

We say that a polynomial $g \in k[x, y, z]$ is *strict* if g is nonzero, homogeneous and not divisible by x, y or z. Every nonzero homogeneous polynomial $f \in k[x, y, z]$ has a unique representation

$$f = x^\alpha y^\beta z^\gamma g,$$

where α, β, γ are nonnegative integers and $g \in k[x, y, z]$ is strict.

Let us recall the following result.

PROPOSITION 2.1 ([2], [3]). *If g is a strict Darboux polynomial of d then its eigenvalue is a linear form*

$$\lambda x + \mu y + \nu z,$$

where λ, μ, ν are nonnegative integers. ∎

Using Proposition 2.1 we get an important consequence.

PROPOSITION 2.2. *Let $g \in k[x,y,z]$ be a strict polynomial and let $g = g_1 g_2$, for some $g_1, g_2 \in k[x,y,z]$. If $d(g) = 0$ then $d(g_1) = d(g_2) = 0$.*

Proof. Let $d(g)=0$. Then g_1, g_2 are strict Darboux polynomials of d, and hence (by Proposition 2.1) $d(g_1) = h_1 g_1$, $d(g_2) = h_2 g_2$, where $h_1 = \lambda_1 x + \mu_1 y + \nu_1 z$, $h_2 = \lambda_2 x + \mu_2 y + \nu_2 z$, for some nonnegative integers λ_1, μ_1, ν_1, λ_2, μ_2 and ν_2. The equalities $0 = d(g) = d(g_1 g_2) = (h_1 + h_2) g$ imply that $h_1 + h_2 = 0$, and hence $\lambda_1 + \lambda_2 = 0$, $\mu_1 + \mu_2 = 0$ and $\nu_1 + \nu_2 = 0$, that is, $\lambda_1 = \mu_1 = \nu_1 z = \lambda_2 = \mu_2 = \nu_2 = 0$. Therefore $d(g_1) = 0 g_1 = 0$, $d(g_2) = 0 g_2 = 0$. ∎

COROLLARY 2.3. *If the set $k[x,y,z]^d \setminus k$ contains a strict polynomial then it contains a strict irreducible polynomial.* ∎

Now we recall some facts from [2].

PROPOSITION 2.4 ([2]). *If $k[x,y,z]^d \neq k$, then $A \neq 0$, $B \neq 0$ and $C \neq 0$.* ∎

PROPOSITION 2.5 ([2]). *If g is a strict polynomial of degree m, belonging to $k[x,y,z]^d$, then*

$$g(0,y,z) = a(y - Az)^m, \quad g(x,0,z) = b(z - Bx)^m, \quad g(x,y,0) = c(x - Cy)^m,$$

for some nonzero elements $a, b, c \in k$. Moreover, $a = c(-C)^m$, $b = a(-A)^m$ and $c = b(-B)^m$. ∎

PROPOSITION 2.6 ([2]). *The ring $k[x,y,z]^d$ contains a nonzero homogeneous polynomial of degree 1 if and only if $ABC = -1$.* ∎

3. Monomial constants. In this section we characterize all the Lotka–Volterra derivations d such that the algebra $k[x,y,z]^d$ contains a nontrivial monomial.

Assume again that $d : k[x,y,z] \to k[x,y,z]$ is the Lotka–Volterra derivation with respect to (A, B, C).

PROPOSITION 3.1. *The following two conditions are equivalent*:

(1) *The set $k[x,y,z]^d \setminus k$ contains a monomial.*

(2) *The parameters A, B, C are negative rational numbers and $ABC = -1$.*

P r o o f. (1)⇒(2). Let $d(x^\alpha y^\beta z^\gamma) = 0$, where α, β, γ are nonnegative integers with $\alpha + \beta + \gamma > 0$. Then $\alpha(Cy + z) + \beta(Az + x) + \gamma(Bx + y) = 0$ and so

$$\alpha C = -\gamma, \quad \beta A = -\alpha, \quad \gamma B = -\beta.$$

If $\alpha = 0$, then $\gamma = -0C = 0$, $\beta = -0B = 0$, and we have a contradiction because $\alpha + \beta + \gamma > 0$. Hence $\alpha > 0$ and analogously $\beta > 0$, $\gamma > 0$. This implies that $A = -\alpha/\beta$, $B = -\beta/\gamma$, $C = -\gamma/\alpha$ are negative rational numbers and $ABC = (-\alpha/\beta)(-\beta/\gamma)(-\gamma/\alpha) = -1$.

(2)⇒(1). If A, B, C are negative rational numbers and $ABC = -1$, then there exist integers $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ such that $A = -\alpha/\beta$, $B = -\beta/\gamma$, $C = -\gamma/\alpha$. Then $d(x^\alpha y^\beta z^\gamma) = 0$. ∎

Let us note the following corollary from the above proof.

COROLLARY 3.2. *Let α, β, γ be nonnegative integers with $\alpha + \beta + \gamma > 0$. If $d(x^\alpha y^\beta z^\gamma) = 0$, then $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $A = -\alpha/\beta$, $B = -\beta/\gamma$, $C = -\gamma/\alpha$.* ∎

We say that a monomial $x^p y^q z^r$ is *primitive* if $p > 0$, $q > 0$, $r > 0$ and $\gcd(p, q, r) = 1$. As a consequence of the above facts we obtain

COROLLARY 3.3. *Assume that the set $k[x, y, z]^d \setminus k$ contains a monomial. Then there exists a unique primitive monomial w belonging to $k[x, y, z]^d$. Every monomial belonging to $k[x, y, z]^d$ is, up to a nonzero coefficient, a power of w.* ∎

Let us also note a fact from [2].

PROPOSITION 3.4 ([2]). *Let $f = x^\alpha y^\beta z^\gamma g$, where α, β, γ are nonnegative integers and $g \in k[x, y, z]$ is strict. If $d(f) = 0$, then $d(x^\alpha y^\beta z^\gamma) = 0$ and $d(g) = 0$.* ∎

4. The algebra of constants. The following theorem describes the algebra $k[x, y, z]^d$ in the case when a monomial belongs to $k[x, y, z]^d \setminus k$. This proves the statement (1a) of Theorem 1.3.

THEOREM 4.1. *Let d be a Lotka–Volterra derivation with respect to $(-p/q, -q/r, -r/p)$, where p, q, r are positive integers and* $\gcd(p, q, r) = 1$. *Then $k[x, y, z]^d = k[t, w]$, where*

$$t = pqx + rqy + rpz, \quad w = x^p y^q z^r.$$

P r o o f. It is clear that $k[t, w] \subseteq k[x, y, z]^d$. Since d is homogeneous, it is sufficient to prove that if $f \in k[x, y, z]$ is a homogeneous polynomial such that $d(f) = 0$, then $f \in k[t, w]$. Assume therefore that $0 \neq f \in k[x, y, z]^d$ and f is homogeneous.

Let $f = x^\alpha y^\beta z^\gamma g$, where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $g \in k[x, y, z]$ is strict. Then $d(x^\alpha y^\beta z^\gamma) = 0$ and $d(g) = 0$ (see Proposition 3.4).

The equality $d(x^\alpha y^\beta z^\gamma) = 0$ implies (by Corollary 3.3) that $x^\alpha y^\beta z^\gamma$ is, up to a nonzero coefficient, a power of w (because w is a unique primitive monomial belonging to $k[x, y, z]^d$). This means that $x^\alpha y^\beta z^\gamma$ belongs to $k[t, w]$.

Therefore it is sufficient to prove that if g is a strict polynomial belonging to $k[x, y, z]^d$, then $g \in k[t, w]$. We will prove it by induction on the degree of g. If $\deg g = 1$ then it is obvious. Assume now that $\deg g = m > 1$.

Since g is strict, there exists (by Proposition 2.5) a nonzero element $c \in k$ such that

$$g(x, y, 0) = c\left(x + \frac{r}{p}y\right)^m.$$

Consider now the polynomial

$$h = g - \frac{c}{p^m q^m}t^m.$$

It is a homogeneous polynomial belonging to $k[x, y, z]^d$. Observe that

$$h(x, y, 0) = c\left(x + \frac{r}{p}y\right)^m - \frac{c}{p^m q^m}(pqx + rqy)^m = 0.$$

This implies that h is divisible by z.

If $h = 0$, then

$$g = \frac{c}{p^m q^m}t^m \in k[t, w].$$

Suppose now that $h \neq 0$. Then $h = x^\alpha y^\beta z^\gamma g_1$, where $g_1 \in k[x, y, z]$ is strict and $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha + \beta + \gamma \geq 1$. The equality $d(h) = 0$ implies (by Proposition 3.4) that $d\left(x^\alpha y^\beta z^\gamma\right) = 0$ and $d(g_1) = 0$. But $\deg g_1 < \deg g$ so, by induction, $g_1 \in k[t, w]$. Moreover, the monomial $x^\alpha y^\beta z^\gamma$ also belongs to $k[t, w]$, because it is (by Corollary 3.3), up to a nonzero coefficient, a power of w. Therefore $g \in k[t, w]$. ■

EXAMPLE 4.2. Let d be the derivation of $k[x, y, z]$ such that

$$d(x) = x(z - y), \quad d(y) = y(x - z), \quad d(z) = z(y - x).$$

Then (by Theorem 4.1) $k[x, y, z]^d = k[x + y + z, xyz]$. It is easy to check that d coincides with the jacobian derivation $\mathrm{Jac}(xyz, x + y + z, \_)$. ■

The next theorem decribes the algebra of constants in the case when the set $k[x, y, z]^d \setminus k$ has no monomials.

THEOREM 4.3. *Let d be a Lotka–Volterra derivation. Assume that $k[x, y, z]^d \neq k$ and the set $k[x, y, z]^d \setminus k$ has no monomials. Then there exists an irreducible homogeneous polynomial $g \in k[x, y, z]$ such that $k[x, y, z]^d = k[g]$.*

Proof. The idea of the proof is similar to that in Theorem 4.1. It follows from the assumptions and Proposition 3.4 that there exists a strict polynomial g belonging to $k[x,y,z]^d$. We may assume (by Corollary 2.3) that g is irreducible. Let $m = \deg g$.

It is sufficient to prove that every nonzero homogeneous polynomial belonging to $k[x,y,z]^d$ is, up to a nonzero coefficient, a power of g.

Assume that f is a nonzero homogeneous polynomial, of degree $n \geq 1$, belonging to $k[x,y,z]^d$. Then f is strict (by the assumptions and Proposition 3.4) and hence, by Proposition 2.5,

$$f(0,y,z) = p(y - Az)^n,$$

for some $0 \neq p \in k$. Moreover, also by Proposition 2.5,

$$g(0,y,z) = a(y - Az)^m,$$

for some $0 \neq a \in k$. Consider now the polynomial

$$h = a^n f^m - p^m g^n.$$

It is a homogeneous polynomial belonging to $k[x,y,z]^d$. Observe that

$$h(0,y,z) = a^n p^m (y - Az)^{nm} - a^n p^m (y - Az)^{nm} = 0.$$

This implies that h is divisible by x.

Suppose that $h \neq 0$. Then $h = x^\alpha y^\beta z^\gamma h_1$, where $h_1 \in k[x,y,z]$ is strict and $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha + \beta + \gamma \geq 1$. Since $d(h) = 0$, Proposition 3.4 implies that $d(x^\alpha y^\beta z^\gamma) = 0$, which is a contradiction with our assumptions.

Therefore $h = 0$, that is, $a^n f^m = p^m g^n$ and we see that f is, up to a nonzero coefficient, a power of g (since g is irreducible). ∎

5. Conclusion. Now it is easy to prove our main result.

Proof of Theorem 1.3. The statement (1a) is a consequence of Theorem 4.1.

Let $ABC = -1$ and assume that some of the scalars A, B, C belong to $k \setminus \mathbb{Q}_-$. Then $k[x,y,z]^d \neq k$ (by Theorem 1.2) and the set $k[x,y,z]^d \setminus k$ has no monomials (Proposition 3.1). Hence, by Theorem 4.3, there exists an irreducible homogeneous polynomial $g \in k[x,y,z]$ such that $k[x,y,z]^d = k[g]$. Since $ABC = -1$, Proposition 2.6 implies that $\deg g = 1$. It is easy to check that $g = x - Cy + ACz$. This completes the proof of (1b).

The statements (2) and (3) are simple consequences of Theorems 1.2, 4.3 and Proposition 3.1. ∎

COROLLARY 5.1. *Let d be a Lotka–Volterra derivation. If the ring of constants of d is nontrivial, then it is a polynomial ring in one or two variables.*

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