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RATIONAL INTEGRATION OF THE LOTKA–VOLTERRA SYSTEM

BY

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ABSTRACT. – The Lotka–Volterra system of autonomous differential equations consists in three homogeneous polynomial equations of degree 2 in three variables.

This system, or the corresponding vector field $V(A, B, C)$, depends on three non-zero parameters and writes $V(A, B, C) = V_x\partial_x + V_y\partial_y + V_z\partial_z$ where

$$V_x = x(Cy + z), \qquad V_y = y(Az + x), \qquad V_z = z(Bx + y).$$

Similar systems of equations have been studied by Volterra in his mathematical approach of the competition of species.

For us, $V(A, B, C)$ is a normal form of a factorisable quadratic system and the study of its first integrals of degree 0 is of great mathematical interest.

A first integral is a non-constant function f which satisfies the identity

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0.$$

As $V(A, B, C)$ is homogeneous, there is a foliation whose leaves are homogeneous surfaces in the three-dimensional space (or curves in the corresponding projective plane), such that the trajectories of the vector field are completely contained in a leaf. A first integral of degree 0 is then a function on the set of all leaves of the previous foliation.

In the present paper, we give all values of the triple (A, B, C) of parameters for which $V(A, B, C)$ has an homogeneous rational first integral of degree 0.

Our proof essentially relies on ideas of algebra and combinatorics, especially in proving that some conditions are necessary. © Elsevier, Paris

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RÉSUMÉ. – Le système d'équations différentielles autonomes de Lotka–Volterra se compose de trois equations polynomiales homogènes de degré 2 et il dépend de trois paramètres non-nuls. On note $V(A, B, C)$ le champ de vecteurs correspondant dont les coordonnées sont

$$V_x = x(Cy + z), \qquad V_y = y(Az + x), \qquad V_z = z(Bx + y).$$

Dans son étude mathématique de la compétition des espèces, Volterra a rencontré et étudié de semblables systèmes d'équations.

Pour nous, $V(A, B, C)$ est une forme normale de système quadratique factorisable et il est intéressant d'étudier ses intégrales premières de degré 0.

On appelle intégrale première d'un champ de vecteurs V une fonction non constante qui vérifie l'identité

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0.$$

Comme le champ $V(A, B, C)$ est homogène, il existe un feuilletage dont les feuilles sont des surfaces homogènes dans l'espace à trois dimensions (ou des courbes dans le plan projectif) tel que chaque trajectoire du champ soit entièrement contenue dans une feuille. Une intégrale première de degré 0 est alors une fonction sur l'espace des feuilles.

Dans ce travail, nous caractérisons toutes les valeurs possibles du triplet (A, B, C) de paramètres pour lesquels $V(A, B, C)$ admet une intégrale première rationnelle homogène de degré 0.

Notre démonstration s'appuie sur des arguments de nature algébrique et combinatoire, particulièrement pour ce qui est d'établir des conditions nécessaires sur les paramètres.

1. Introduction

The search of first integrals is a classical tool in the classification of all trajectories of a dynamical system. Let us simply recall the role of energy in Hamiltonian systems.

We are interested here in some systems consisting in three ordinary autonomous differential equations in three variables: $\dot{x} = V_x,\ \dot{y} = V_y,\ \dot{z} = V_z$.

A *first integral* of this system of equations (or of the corresponding vector field V) is a non-constant function f that satisfies the partial derivative equation

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0.$$

That means that f is constant along all trajectories of the one-parameter local semi-group generated by the vector field V.

The *local* existence of first integrals in a neighborhood of a regular point is a consequence of some classical theorems of differential calculus.

The interesting point for us is the search of *global* solutions; this problem has an algebraic nature if the coordinate functions V_x, V_y and V_z are polynomials in the space variables x, y and z. Related to this problem is the study of singular integrable differential forms, which is of great interest [2–5].

A key point is the specification of the class in which we look for first integrals. In the algebraic case, we follow the classical way of "Integration of differential equations in finite terms" [17–19]. In this frame, it seems reasonable to consider the class of all Liouvillian elements over the differential field $\mathbf{C}(x, y, z)$ of all rational fractions in three variables on the constant field $\mathbf{C}$ of complex numbers. We follow the definition of Liouvillian elements given by Michæl Singer [20].

Despite some specific methods [21], the search of Liouvillian first integrals of polynomial vector fields relies mainly on the study of particular solutions whose use dates back to a memoir by Darboux [6] are we are now used to calling them *Darboux polynomials* of these vector fields [12]. With this vocabulary, polynomial first integrals are Darboux polynomials with the *eigenvalue* 0.

So far as we know, Henri Poincaré [14–16] was the first to notice the difficulty of a decision procedure for the existence of Darboux polynomials.

No procedure is known up to now; Jean-Pierre Jouanolou gives a theorem about this subject but his result is not effective [8].

The Lotka–Volterra vector field $V(A, B, C)$ [7], that we study here, can be considered as a normal form of a factorisable vector field of degree 2 in three variables, at least in what concerns the search of first integrals of degree 0, i.e., first integrals of $V(A, B, C)$ that are also first integrals of the vector field $E = x\partial_x + y\partial_y + z\partial_z$, that we call the Euler field for evident reasons.

Some necessary conditions may be given to allow the existence of a Liouvillian first integral of degree 0 for factorisable vector fields and a categorical result can be obtained: non-integrability is generic [10].

Many *sufficient* conditions can be given that ensure the existence of a Liouvillian first integral of degree 0: it suffices to exhibit a fourth

Darboux polynomial, i.e., a Darboux polynomial with is not divisible by any of the variables x, y and z.

A systematic search of such Darboux polynomials of a given degree can be carried out with the help a computer algebra system [1]. Looking carefully to the lists of parameters produced by such a search shows that the situation remains very intricate.

We turned therefore our interest to the determination of all values of the parameters for which $V(A, B, C)$ has a first integral of degree 0, which is not only Liouvillian, but is a rational fraction. This means that the space of leaves of the foliation generated by the field together with the Euler field, can be well described as an algebraic variety.

In other words, one can consider the 1-form $\omega_0 = i_V i_E \Omega$, where Ω is the 3-form $dx\,dy\,dz$. This 1-form is projective, i.e., $i_E\omega_0 = 0$, and satisfies the Pfaff condition $\omega_0 \wedge d\omega_0 = 0$, which allows the search of an integrating factor ϕ, according to a theorem of Frobenius, and ϕ has to be a rational fraction such that $\phi\omega_0$ has a rational primitive.

This problem, that can be called *rational integration*, turned out to be amenable and its solution is the subject of the present work. In a previous article [13], we were able to determine all cases in which $V(A, B, C)$ has a polynomial first integral. Here too, a combinatorial approach of some systems of linear equations is a key tool to deduce necessary conditions on the parameters.

The present paper is organized as follows. In Section 2, we described useful tools. Some of them are of general interest in the study of the integrability of polynomial vector fields: Darboux curves, Levelt's method, and so on. Others are specific to $V(A, B, C)$: inspection of the linear algebra problem associated to the Darboux property of a polynomial. In Section 3, we prove that some necessary conditions have to be fulfilled by the triple (A, B, C) in order to allow a rational first integral of degree 0 for $V(A, B, C)$. This analysis leads to a classification of candidate triples. In Section 4, we show how to perform the rational integration in all cases of this classification.

So, this paper essentially consists in the proof of the following theorem.

THEOREM 1. – *The Lotka–Volterra vector field $V(A, B, C)$ has a rational first integral of degree* 0 *if and only if all three parameters A, B*

and C are rational numbers and moreover, the triple (A, B, C) belongs, up to a natural transformation, to one of the following sets:

- $\mathcal{P}_1 = \{(A, B, C), ABC + 1 = 0\}$,
- $\mathcal{P}_2 = \{(A, B, C), C + 1/A = A + 1/B = B + 1/C = -1\}$,
- $\mathcal{P}_3$, *the set of all triples such that $B = 1$ and $-1/A - C$ is a positive integer while $-C$ and $-1/A$ are not both positive integers,*
- $\mathcal{P}_4$, *the set of all triples such that $B = 2$, $C + 1/A = -1$ and either $A - 1$ or $-1/2 - A$ is a positive integer,*
- *the isolated triple* $\mathcal{P}_5 = \{(-7/3, 3, -4/7)\}$,
- *the isolated triple* $\mathcal{P}_6 = \{(-3/2, 2, -4/3)\}$.

Our redaction will be essentially, but not completely, self-contained. In particular, the reader is asked to refer to [11] for details about Levelt's method.

2. General and specific tools

2.1. Some vocabulary of differential algebra

Given a field k, a k-derivation of an extension field K of k is a k-linear map δ from K in itself that satisfy Leibniz's rule for the derivation of a product. K is then called a *differential field*. The kernel of δ is then a subfield of K and an extension of k. It is known as the *field of constants* of the derivation.

When K is the field $k(x, y, z)$ of rational fractions in three indeterminates, the usual partial derivatives ∂_x, ∂_y and ∂_z are derivations and they commute with one another. Their common field of constants is exactly k. In what follows, k is some finite extension of $\mathbf{Q}$ by parameters and can be thought of as being a subfield of the field $\mathbf{C}$ of complex numbers. A polynomial vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$, where V_x, V_y and V_z are homogeneous polynomials of the same degree in $k[x, y, z]$, defines a k-derivation δ_V of K. When L is an extension of $k(x, y, z)$ such that ∂_x, ∂_y and ∂_z have been extended as commuting k-derivations of L, δ_V is extended to L by the fact.

In particular we will consider the special vector field $E = x\partial_x + y\partial_y + z\partial_z$ and call it the Euler field. Indeed, an element f of $k(x, y, z)$ is homogeneous of degree m if and only if $\delta_E(f) = mf$, according to Euler's relation. In an differential extension field L, this identity may be viewed as a definition of homogeneity.

We will also use freely in the frame of differential algebra the ideas of differential calculus such as exterior derivatives or n-forms.

A 1-form is an element $\omega_x\,dx + \omega_y\,dy + \omega_z\,dz$ in the 3-dimensional vector space K^3 expressed in the "canonical" base $\{dx, dy, dz\}$. The exterior derivative of an element f is the 1-form $\partial_x(f)\,dx + \partial_y(f)\,dy + \partial_z(f)\,dz$.

Given a polynomial vector field V, we call *homogeneous first integral of degree* 0 an element f of some differential extension field L of K that satisfies $\delta_V(f) = \delta_E(f) = 0$ (f is then a constant for these two derivations) without being a constant as we reserve the word *constant* for elements c such that $dc = 0$ (meaning that $\delta_x(c) = \delta_y(c) = \delta_z(c) = 0$).

2.2. Darboux polynomials

Consider a vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$, where V_x, V_y and V_z are homogeneous polynomials in the space variables x, y and z and have the same degree m .

A polynomial f is said to be a *Darboux polynomial* of V if there exists some polynomial Λ such that

$$V_x\frac{\partial f}{\partial x} + V_y\frac{\partial f}{\partial y} + V_z\frac{\partial f}{\partial z} = \Lambda f,$$

where Λ is the corresponding *eigenvalue*.

As V is homogeneous, the consideration of the homogeneous component of highest degree of the above identity shows that Λ has to be an homogeneous polynomial of degree $m - 1$.

Moreover, the homogeneous component f^+ of highest degree of a Darboux polynomial f is a Darboux polynomial. It is therefore sufficient to study homogeneous Darboux polynomials.

Suppose now that some homogeneous Darboux polynomial f for a given homogeneous polynomial vector field V and an eigenvalue Λ factors as the product $f = gh$ of two relatively prime homogeneous polynomials.

As a polynomial ring is a unique factorization domain, Gauß's lemma shows that the factors g and h have to be Darboux polynomials for V with some eigenvalues Λ_1 and Λ_2 such that $\Lambda = \Lambda_1 + \Lambda_2$. Thus, the determination of all Darboux polynomials of a given polynomial vector field V amounts to finding all irreducible Darboux polynomials for V.

According to Euler's identity, homogeneous polynomials are Darboux polynomials for the Euler field; with respect to E, the eigenvalue of a homogeneous polynomial is its degree.

2.3. Levelt's method around Darboux points

As we have previously seen, a homogeneous first integral of degree 0 of some homogeneous polynomial vector field V belongs to the common kernel of V and E, i.e., its derivative is proportional to the 1-form $\omega_0 = i_V i_E \Omega$, where Ω is the 3-form $dx\,dy\,dz$ and we are led to find an integrating factor of the Pfaff form ω_0.

In this framework, the lines (or points of the projective plane) where ω_0 vanishes are of special interest and we call them *Darboux points*.

When the degree of V is 2, there are generically seven Darboux points; in the special case of $V(A, B, C)$, these points can be described more precisely.

Three of them correspond to the axes and we can choose homogeneous coordinates such that $M_1 = [1, 0, 0]$, $M_2 = [0, 1, 0]$, $M_3 = [0, 0, 1]$, three others lie on the "sides" and can be parameterized as $M_4 = [0, A, 1]$, $M_5 = [1, 0, B]$, $M_6 = [C, 1, 0]$, and a seventh one has (in general) all its coordinates different from 0: $M_7 = [AC - A + 1, BA - B + 1, CB - C + 1]$.

In his book [8], Jean-Pierre Jouanolou studies local properties of Darboux polynomials around Darboux points. This allows him to prove the generic absence of Darboux polynomials. A more elementary approach than Jouanolou's one, due to Levelt, is also very powerful. We studied Levelt's method in details and refer the reader to our paper [11], especially to its Section 2.4 entitled "local analysis", for a complete exposition.

To be short, when there are three variables and the degree of V is 2, a linear change of variables can be found such that the Darboux point M under consideration has new coordinates $t = 1, u = 0, v = 0$ while the Darboux property of a polynomial g of degree m writes:

$$\begin{aligned}&\big(\rho u t^2 + t\alpha_2(u, v) + \alpha_3(u, v)\big)g_u + \big(\sigma v t^2 + t\beta_2(u, v) + \beta_3(u, v)\big)g_v\\&\quad = \big(\chi t^2 + t\gamma_1(u, v) + \gamma_2(u, v)\big)g,\end{aligned}$$

where ρ and σ are the eigenvalues of some 2×2 scalar matrix, where χ is a constant and where $\alpha_i, \beta_i, \gamma_i$ are homogeneous two-variable polynomials of degree i (in u and v).

Then g can be ordered as a polynomial $g = \sum t^{m-i} g_i(u, v)$ in the variable t whose coefficients $g_i(u, v)$ are homogeneous polynomials in u and v.

The lowest possible degree μ of such a coefficient is the multiplicity of the "curve" $g = 0$ at M. The corresponding coefficient g_μ satisfies identity $\rho u \partial g_\mu / \partial u + \sigma v \partial g_\mu / \partial v = \chi g_\mu$ and there exist two non-negative integers j and k, whose sum is μ, such that $j\rho + k\sigma = \chi$.

Let us remark that numbers ρ, σ, χ may be freely multiplied by the same factor without any trouble, in such a way that we can make a further choice in the "local coordinates" to ensure $\rho + \sigma = 1$ for instance.

In the case of $V(A, B, C)$, we will only consider Levelt's approach around the point M_7 to deduce a necessary condition of Diophantine kind for the vector field to have a rational first integral of degree 0.

2.4. Darboux polynomials of the Lotka–Volterra vector field

By the very definition of a factorisable vector field, like $V(A, B, C)$, the space variables x, y and z are Darboux polynomials.

Every homogeneous non-zero polynomial f writes in a unique way

$$f = x^\alpha y^\beta z^\gamma g,$$

where g is not divisible by x, y or z.

If f is a Darboux polynomial, so is g. Such polynomials as g will play an important role in our combinatorial analysis. We will call them *strict* Darboux polynomials.

Let g be a strict Darboux polynomial of degree m for $V(A, B, C)$, which writes

$$x(Cy + z)\partial_x g + y(Az + x)\partial_y g + z(Bx + y)\partial_z g = (\lambda x + \mu y + \nu z)g.$$

As g is supposed not to be divisible by x, y or z, we can consider the three homogeneous non-zero two-variable polynomials of degree m obtained by setting $x = 0$, $y = 0$ and $z = 0$ in g and call them P, Q and R, respectively.

From the previous relation involving g, we deduce some partial differential equations concerning these two-variable polynomials

$$\begin{cases} (\mu y + \nu z)P &= yz(A\partial_y P + \partial_z P),\\ (\nu z + \lambda x)Q &= zx(B\partial_z Q + \partial_x Q),\\ (\lambda x + \mu y)R &= xy(C\partial_x R + \partial_y R).\end{cases}$$

It is not very difficult to prove that there exists six non-negative integers β_1, γ_1, α_2, γ_2, α_3 and β_3 such that P is a non-zero multiple of $y^{\beta_1}z^{\gamma_1}(y - Az)^{m-\beta_1-\gamma_1}$, Q is a non-zero multiple of $z^{\gamma_2}x^{\alpha_2}(z - Bx)^{m-\gamma_2-\alpha_2}$ and R is a non-zero multiple of $x^{\alpha_3}y^{\beta_3}(x - Cy)^{m-\alpha_3-\beta_3}$.

Moreover, these numbers satisfy the following equations and inequalities

$$\begin{cases} \lambda &= \beta_3 = \gamma_2 B,\\ \mu &= \gamma_1 = \alpha_3 C,\\ \nu &= \alpha_2 = \beta_1 A,\\ \beta_1 + \gamma_1 &\leqslant m,\\ \alpha_2 + \gamma_2 &\leqslant m,\\ \alpha_3 + \beta_3 &\leqslant m.\end{cases}$$

In particular, the eigenvalue corresponding to a strict Darboux polynomial of $V(A, B, C)$ is a linear form $\Lambda = \lambda x + \mu y + \nu z$ where λ, μ and ν are non-negative integers.

2.5. Factorisable vector fields

In the present subsection, we try to explain to what extend the Lotka–Volterra vector field $V(A, B, C)$ may be considered as a normal form of a factorisable vector field.

A quadratic vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$, where V_x, V_y and V_z are homogeneous quadratic polynomials in x, y and z, is said to be *factorisable* if x divides V_x, y divides V_y and z divides V_z. In other words, coordinates are Darboux polynomials.

So far as we are interested in integration, a linear change of coordinates preserves qualitative properties of the vector field. A factorisable quadratic vector field is then a vector field with three linearly independent first degree Darboux polynomials. Taking these polynomials as coordinates leads to the factored form: $V_x = x\phi_x$, $V_y = y\phi_y$, $V_z = z\phi_z$.

As we look for homogeneous first integrals of degree 0, V may be freely translated by some multiple of the Euler field E. We then get a first normal form of the factorisable vector fields that we study, by removing "diagonal elements":

$$V_x = x(Cy + C'z), \qquad V_y = y(Az + A'x), \qquad V_z = z(Bx + B'y).$$

Thereafter, it is possible to perform a diagonal change of variables $x' = ax$, $y' = by$, $z' = cz$ or a permutation of the variables.

If A, A', B, B', C, C' are all different from 0, such a diagonal change of variables can be done to put the vector field in Lotka–Volterra normal form, where $A' = B' = C' = 1$ and $ABC \neq 0$. In this case, the field is conjugate with $V(A, B, C)$, which justifies our claim that $V(A, B, C)$ is a normal form.

The only exceptions occur when there are some zeroes among A, A', B, B', C, C'.

In all these exceptional cases, either the vector field V is not irreducible, and we have to study a linear or even a constant vector field [9] instead of a quadratic one, or V is linearly equivalent to $x\partial_y + (Bx + y)\partial_z$ with $B \neq 0$ or to some $V(A, B, C)$ with $ABC = 0$.

We leave the study of these exceptional cases to the reader: it is then possible to prove that there is no rational first integral of degree 0. Slight generalizations of the arguments of Section 2.4 together with the consideration of a system equations similar to the system (2) in the proof of Lemma 3 will be the tools to achieve this task.

2.6. Natural symmetries of the problem

As we have just noticed, some linear changes of variables preserve the factorisable form of the given vector field. Adding thereafter a multiple of the Euler field and making a conjugacy by a diagonal linear change of variables gives a new vector field, which is in Lotka–Volterra normal form for some triple (A', B', C') of parameters.

As we have previously seen, permutations of the coordinates are such transformations. Applying a circular permutation yields a circular permutation of the parameters, in which (A, B, C) becomes (B, C, A) or (C, A, B) while exchanging two coordinates changes (A, B, C) for $(1/B, 1/A, 1/C)$,$(1/C, 1/B, 1/A)$ or $(1/A, 1/C, 1/B)$.

We will refer to such transformations as "natural transformations" of the triples of parameters. When (A', B', C') is the image of (A, B, C) by a natural transformation, it is clear that the vector field $V(A, B, C)$ has a rational (Liouvillian) first integral of degree 0 if and only if $V(A', B', C')$ has such a first integral.

In some cases, the factorisable form can also be preserved by linear changes of variables, different from diagonal and permutations. In particular, when there exists a fourth Darboux polynomial of degree 1, it is possible to take it as a coordinate, together with two among x, y and z and thus to change variables. This happens when $ABC + 1 = 0$ and also when one of the parameters, for instance B, is equal to 1.

In this last case, $y - Az$ is a Darboux polynomial with eigenvalue x. It is then possible to choose new variables $\{X = x, Y = y, Z = y - Az\}$ or $\{X = x, Y = y - Az, Z = z\}$ to get two natural transformations; the first one transforms $(A, 1, C)$ in $(A, 1, 1 - C - 1/A)$ and the second one transforms it in $(A/(A - 1 - AC), 1, C)$.

2.7. Darboux polynomials and linear algebra

Consider some strict Darboux polynomial f for $V(A, B, C)$ of degree m and eigenvalue

$$\Lambda = \lambda x + \mu y + \nu z.$$

Identity

$$V_x f_x + V_y f_y + V_z f_z - \Lambda f = 0$$

appears as a homogeneous linear system of N equations in the n coefficients of a three-variable homogeneous polynomial of degree m as unknowns. This system is thus supposed to have a non-zero solution.

Unknowns are parameterized by triples (i, j, k) of non-negative integers summing to m and there are $n = \binom{m+2}{2}$ of them. Equations are parameterized by triples (i, j, k) of non-negative integers summing to $m + 1$ and there are $N = \binom{m+3}{2}$ of them.

The corresponding matrix is very sparse; there are at most 3 non-zero coefficients per row or per column and they are given by affine forms of the indices:

$$\text{(1)}\qquad \begin{cases} m_{(i,j,k),(i+1,j,k)} = j + Bk - \lambda, \\ m_{(i,j,k),(i,j+1,k)} = Ci + k - \mu, \\ m_{(i,j,k),(i,j,k+1)} = i + Aj - \nu. \end{cases}$$

As this over-determined system is supposed to have a non-trivial solution, determinants of maximal order are equal to 0 but most of them are not easy to compute.

On the contrary, the extra assumption that f is a strict Darboux polynomial (or only that x does not divide f) means that the linear square subsystem whose unknowns have their index i equal to 0 or 1 and whose equations have their index i equal also to 0 or 1 (except $(0, 0, m+1)$ and $(0, m+1, 0)$) is not Cramer and that its determinant, that we call X, is equal to 0.

It turns out that combinatorial computations yield to a factorization of determinant X. Then saying that $X = 0$ becomes a useful necessary condition on parameters (A, B, C) of the vector field and parameters λ, μ, ν, m of the polynomial.

The same is true for analogous determinants Y and Z defined in a natural way by a "circular permutation". We give this interesting factorization in the following lemma.

LEMMA 1. – *Up to factors that cannot vanish, determinants X, Y and Z factor as follows*:

$$X \equiv [1/A - \beta_1]^{\overline{\beta_1}}\,[C - \gamma_1]^{\overline{\gamma_1}}\,[1/A + C + 1]^{\overline{m-\beta_1-\gamma_1-1}}\,X_4,$$

$$Y \equiv [1/B - \gamma_2]^{\overline{\gamma_2}}\,[A - \alpha_2]^{\overline{\alpha_2}}\,[1/B + A + 1]^{\overline{m-\gamma_2-\alpha_2-1}}\,Y_4,$$

$$Z \equiv [1/C - \alpha_3]^{\overline{\alpha_3}}\,[B - \beta_3]^{\overline{\beta_3}}\,[1/C + B + 1]^{\overline{m-\alpha_3-\beta_3-1}}\,Z_4,$$

with $X_4 = \{m(1 + ABC) - \lambda(1 + AC) - \mu(1 - B) + \nu C(1 - B)\}$ *and corresponding Y_4 and Z_4.*

m is the degree of the supposed strict Darboux polynomial and $\alpha_2 = \nu, \alpha_3, \beta_1, \beta_3 = \lambda, \gamma_1 = \mu, \gamma_2$ *are the previously introduced non-negative*

integers. The notation $[t]^{\overline{k}}$ *for the rising factorial power will be explained in the proof.*

Proof. – With a suitable choice in the order of equations and unknowns, the square matrix $\mathcal{M}$ of our linear system, whose order is $2m+1$, has the following form

$$\mathcal{M} = \begin{pmatrix} \mathcal{O} & \mathcal{A} \\ \mathcal{C} & \mathcal{B} \end{pmatrix},$$

where $\mathcal{O}$ is the zero square matrix of order m, $\mathcal{A}$ is a rectangular matrix with m lines and $m+1$ columns, $\mathcal{C}$ is a rectangular matrix with $m+1$ lines and m columns and $\mathcal{B}$ is a diagonal square matrix of order $m+1$.

These last three matrices look like what follows and they have a lot of zeroes.

$$\mathcal{A} = \begin{pmatrix} m-\mu & A-\nu & 0 & \cdots & \cdots & 0 \\ 0 & m-1-\mu & 2A-\nu & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 2-\mu & (m-1)A-\nu & 0 \\ 0 & \cdots & \cdots & 0 & 1-\mu & mA-\nu \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} 1-\nu & 0 & \cdots & \cdots & 0 \\ C+m-1-\mu & 1+A-\nu & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & C+1-\mu & 1+(m-2)A-\nu & 0 \\ 0 & \cdots & \cdots & C-\mu & 1+(m-1)A-\nu \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} Bm-\lambda & 0 & \cdots & \cdots & 0 \\ 0 & 1+B(m-1)-\lambda & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & (m-1)+B-\lambda & 0 \\ 0 & \cdots & \cdots & 0 & m-\lambda \end{pmatrix}.$$

The development of the determinant X of $\mathcal{M}$ has only $m+1$ non-zero terms, each of them being of course a product of $2m+1$ factors. As all corresponding permutations have the same signature, this leads to the following expression

$$X = \sum_{i=0}^{m} (i + (m-i)B - \lambda) \prod_{j=0}^{i-1} (m-j-\mu)(1+jA-\nu) \times \prod_{j=i+1}^{m} (jA-\nu)(C+m-j-\mu).$$

The reason is the following: in every non-zero term of X, some of the diagonal elements of $\mathcal{B}$ appear as factors. Trying to avoid diagonal elements of $\mathcal{B}$ yields null terms of D_m. In a similar way, if we take two diagonal elements of $\mathcal{B}$, the corresponding cofactor is a $2m-1$ determinant, which is easily transformed by a permutation in a triangular one with a zero element in its diagonal.

The determinant X is thus the sum of products of all diagonal elements of $\mathcal{B}$ by their cofactors; these cofactors are triangular determinants of order $2m$ and thus easy to compute.

Using the notation $[t]^{\overline{k}}$ for the rising factorial power $\prod_{i=0}^{k-1}(t+i)$ and expressing λ, μ and ν in terms of other parameters lead to

$$X = \sum_{i=0}^{m} (i + (m-i)B - \beta_3) \prod_{j=0}^{i-1} (m-j-\gamma_1)(1+jA-\beta_1 A) \times \prod_{j=i+1}^{m} (jA-\beta_1 A)(C+m-j-\alpha_3 C),$$

$$X = A^m \sum_{i=0}^{m} (i + (m-i)B - \beta_3) \prod_{j=0}^{i-1} (m - j - \gamma_1)(1/A + j - \beta_1)$$
$$\times \prod_{j=i+1}^{m} (j - \beta_1)(C + m - j - \alpha_3 C),$$

$$X = A^m \sum_{i=0}^{m} (i + (m-i)B - \beta_3)[m - i + 1 - \gamma_1]^{\overline{i}}[1/A - \beta_1]^{\overline{i}}$$
$$\times [i + 1 - \beta_1]^{\overline{m-i}}[C - \gamma_1]^{\overline{m-i}}.$$

The terms of the sum between β_1 and $m - \gamma_1$ are the only non-zero ones since $[i + 1 - \beta_1]^{\overline{m-i}} = 0$ if $i < \beta_1$ and $[m - i + 1 - \gamma_1]^{\overline{i}} = 0$ if $i > m - \gamma_1$, whence the expression

$$X = A^m \sum_{i=\beta_1}^{m-\gamma_1} (i + (m-i)B - \beta_3)[m - i + 1 - \gamma_1]^{\overline{i}}[1/A - \beta_1]^{\overline{i}}$$
$$\times [i + 1 - \beta_1]^{\overline{m-i}}[C - \gamma_1]^{\overline{m-i}}.$$

When $m = \beta_1 + \gamma_1$, there is only one term in this sum, and some non-zero factors may be ignored to get

$$X \equiv A^m(\beta_1 + \gamma_1 B - \beta_3)[1]^{\overline{\beta_1}}[1/A - \beta_1]^{\overline{\beta_1}}[1]^{\overline{\gamma_1}}[C - \gamma_1]^{\overline{\gamma_1}},$$
$$X \equiv (\beta_1 + \gamma_1 B - \beta_3)[1/A - \beta_1]^{\overline{\beta_1}}[C - \gamma_1]^{\overline{\gamma_1}}.$$

When $n = m - \beta_1 - \gamma_1 - 1 \geqslant 0$, there are many terms in the sum, and the first factor, $(i + (m-i)B - \beta_3)$ can be written as a linear combination $\phi(i - \beta_1) + \psi(m - \gamma_1 - i)$, where ϕ and ψ are constants with respect to the index i.

These constants are determined by the linear system

$$\phi - \psi = 1 - B,$$
$$-\beta_1\phi + (m - \gamma_1)\psi = mB - \lambda,$$

which can be solved in

$$(m - \beta_1 - \gamma_1)\phi = mB - \beta_3 + (m - \gamma_1)(1 - B),$$
$$(m - \beta_1 - \gamma_1)\psi = mB - \beta_3 + \beta_1(1 - B).$$

Determinant X is then equal to $A^m \frac{(m-\beta_1)!(m-\gamma_1)!}{n!}(\phi X_1 + \psi X_2)$, where

$$X_1 = [1/A - \beta_1]^{\overline{\beta_1+1}}[C - \gamma_1]^{\overline{\gamma_1}} \sum_{i=\beta_1+1}^{m-\gamma_1} \frac{n!}{(i-\beta_1-1)!(m-\gamma_1-i)!} \\ \times [1/A+1]^{\overline{i-\beta_1-1}}[C]^{\overline{m-\gamma_1-i}},$$

$$X_2 = [1/A - \beta_1]^{\overline{\beta_1}}[C - \gamma_1]^{\overline{\gamma_1+1}} \sum_{i=\beta_1}^{m-\gamma_1-1} \frac{n!}{(i-\beta_1)!(m-\gamma_1-i-1)!} \\ \times [1/A]^{\overline{i-\beta_1}}[C+1]^{\overline{m-\gamma_1-i+1}}.$$

Forgetting some factors, that cannot vanish, we get

$$X \equiv \left[\frac{1}{A} - \beta_1\right]^{\bar{\beta}_1} [C - \gamma_1]^{\overline{\gamma_1}}\{T_0 + T_1\},$$

where

$$T_0 = \left[\frac{1}{A} + C\right]^{\overline{m-\beta_1-\gamma_1}} (mB - \beta_3 + \beta_1(1 - B))$$

and

$$T_1 = (m - \beta_1 - \gamma_1)\left(\frac{1-B}{A}\right)\left[\frac{1}{A} + C + 1\right]^{\overline{m-\beta_1-\gamma_1-1}}.$$

Finally, X may be written as announced, which includes the special case $m = \beta_1 + \gamma_1$, the rising factorial power $[1/A + C + 1]^{\overline{-1}}$ corresponding in this case to some non-zero factor.

Similar expressions for Y and Z can be obtained in the same way. □

3. Necessary conditions

In the present section, we present necessary conditions to be fulfilled by the triple (A, B, C) of non-zero parameters in order to make possible the existence of a rational first integral of degree 0 for $V(A, B, C)$.

3.1. Rationality

Our first lemma gives strong limitations to such triples.

LEMMA 2. – *If $V(A, B, C)$ has a rational first integral of degree* 0*, then the three parameters A, B and C are rational numbers.*

Proof. – Let N/D be such a first integral, where N and D are relatively prime homogeneous polynomials of the same degree m. N and D are Darboux polynomials with the same eigenvalue $\Lambda = \lambda x + \mu y + \nu z$ and so is any linear combination $\alpha N + \beta Q$ of them.

It is then possible to find a linear combination which is a strict Darboux polynomial and thus the coefficients λ, μ and ν of the eigenvalue Λ have to be non-negative integers.

The corresponding system (1) of linear equations has then a kernel whose dimension is at least 2 and all determinants of square sub-matrices of size n or $n-1$ have to vanish.

First consider the determinant $\mathcal{X}$ obtained by taking all n unknowns (i, j, k), $i + j + k = m$, and the corresponding n equations $(i + 1, j, k)$, $i + j + k = m$. Due to the sparse structure of this subsystem, there is only one non-zero term in the development of $\mathcal{X}$ which writes

$$\mathcal{X} = \prod_{(i,j,k)} (j + Bk - \lambda).$$

One of the factors $(j + Bk - \lambda)$ has to be 0. Call (i_1, j_1, k_1) the corresponding index.

Now consider another sparse determinant, of size $n-1$, the one you get by erasing from $\mathcal{X}$ the unknown (i_1, j_1, k_1) and the corresponding equation $(i_1 + 1, j_1, k_1)$. There is only one term in the development of this new determinant, which factors in a very similar way, and there is another index (i_2, j_2, k_2) for which the factor $(j + Bk - \lambda)$ vanishes, so that

$$j_1 + Bk_1 - \lambda = j_2 + Bk_2 - \lambda = 0,$$

which proves that B is a rational number.

The same is true for parameters A and C by a circular permutation. □

3.2. Arithmetic conditions

A second remark will help us to use the previously well-factored determinants (Lemma 1) in a more efficient way when a rational first integral exists.

LEMMA 3. – *If $V(A, B, C)$ has a rational first integral of degree* 0, *then either $ABC + 1 = 0$ or the three parameters A, B and C satisfy the following three conditions*:

- *one of the three numbers* $1/A$, C *or* $-C-1/A$ *is a positive integer,*
- *one of the three numbers* $1/B$, A *or* $-A-1/B$ *is a positive integer,*
- *one of the three numbers* $1/C$, B *or* $-B-1/C$ *is a positive integer.*

Proof. – As shown in Section 2.7, the existence of a Darboux polynomial of degree m and eigenvalue $\lambda x+\mu y+\nu z$ which is not divisible by x implies that the corresponding determinant X vanishes.

This determinant can be written as the product of four factors, among which the fourth is the only one depending on λ,μ,ν,m in a linear way:

$$\begin{aligned} &X_4(m,\lambda,\mu,\nu)\\ &\quad = m(1+ABC)-\lambda(1+AC)-\mu(1-B)+\nu C(1-B). \end{aligned}$$

When there exists a rational first integral N/D of degree 0, the 1-form $N\,dD-D\,dN$ is a multiple $\phi\omega_0$ of the Pfaff form ω_0 by some homogeneous non-zero polynomial ϕ whose degree is $2m-4$. This polynomial ϕ is a Darboux polynomial for the vector field V with the eigenvalue $\Lambda'=2\Lambda-\operatorname{div}(V)$, $\operatorname{div}(V)$ being the divergence of the vector field V.

Factoring out powers of x, y and z, ϕ writes $x^\alpha y^\beta z^\gamma\psi$, where ψ is a strict Darboux polynomial with eigenvalue $\Lambda''=\lambda''x+\mu''y+\nu''z$ and degree m'', where λ'', μ'', ν'', α, β and γ are non-negative integers.

All these numbers are then related by the following system of equations:

$$(2)\qquad \begin{cases} m''+\alpha+\beta+\gamma+3 &= 2m,\\ \lambda''+\beta+1+(\gamma+1)B &= 2\lambda,\\ \mu''+\gamma+1+(\alpha+1)C &= 2\mu,\\ \nu''+\alpha+1+(\beta+1)A &= 2\nu. \end{cases}$$

The coefficients of A, B and C in (2) are non-zero integers and the subsystem consisting in the last three equations is a Cramer one with respect to these three unknowns; we then get another proof of the fact that A, B and C are rational numbers.

Applying to ψ what we have applied to a linear combination of N and D, we would obtain $X(m,\lambda,\mu,\nu)=X(m'',\lambda'',\mu'',\nu'')=0$.

It is impossible to get $X_4(m,\lambda,\mu,\nu)=0$ and $X_4(m'',\lambda'',\mu'',\nu'')=0$ unless $ABC+1=0$.

Indeed, these numbers are related by the previous system (2) of equations and the difference $2\,X_4(m,\lambda,\mu,\nu) - X_4(m'',\lambda'',\mu'',\nu'')$ writes $(\alpha+1)(1+ABC)$.

Thus, either $ABC+1=0$ or the product of the first three factors of $X(m,\lambda,\mu,\nu)$ or of $X(m'',\lambda'',\mu'',\nu'')$ has to be 0. Our assertion on numbers A, C and $-C-1/A$ is then an easy consequence of one of these factors being equal to 0.

The same is true by a circular permutation. □

3.3. Levelt's method around M_7

Let us denote by Δ the discriminant of the second degree equation satisfied by local eigenvalues ρ and σ at the Darboux point M_7 of $V(A,B,C)$ where $ABC+1\neq 0$ and the convention that $\rho+\sigma=1$. This number writes as follows:

$$\Delta = \Delta(A,B,C) = 1 + 4\frac{\omega(p+1)(q+1)(r+1)}{(\omega+1)^2},$$

where $p=-A-1/B$, $q=-B-1/C$, $r=-C-1/A$ and $\omega=ABC$.

The following necessary condition will be very useful.

LEMMA 4. – *If $V(A,B,C)$, where $ABC+1\neq 0$, has a rational first integral of degree* 0*, then $\Delta(A,B,C)$ is the square of a rational number.*

Proof. – Let us use Levelt's method around the point M_7. If there exists a rational first integral $F=N/D$ of degree 0, then the non-proportional polynomials N and D are Darboux polynomials with the same degree and the same eigenvalue. The corresponding linear algebra problem has then a kernel whose dimension is at least 2. Translating this property in the "local" coordinates, we get the fact that eigenvalues ρ and σ at M_7 have to satisfy two different relations $j\rho+k\sigma=\chi$ and $j'\rho+k'\sigma=\chi$, which implies that ρ and σ are rationally dependent. Due to the normalization $\rho+\sigma=1$, they are rational numbers and their discriminant Δ is the square of a rational number. □

3.4. Finding candidates

Using the arguments of the previous lemmas, we can now discuss the various cases where a rational first integral of degree 0 can occur.

The necessary condition we give in the next proposition appears as a disjunction of possibilities.

Later on, an additional restriction (Lemma 6) will be given which has been inspired by the actual possibility to find a first integral (next section: Sufficient conditions).

PROPOSITION 1. – *If $V(A, B, C)$ has a rational first integral of degree* 0, *then, up to a natural transformation, the triple* (A, B, C) *of rational parameters belongs to one of the following sets*:

- $\mathcal{P}_1 = \{(A, B, C), ABC + 1 = 0\}$,
- $\mathcal{P}_2 = \{(A, B, C), C + 1/A = A + 1/B = B + 1/C = -1\}$,
- $\mathcal{P}_3$, *the set of all triples such that* $B = 1$ *and* $r = -1/A - C$ *is a positive integer,*
- $\mathcal{P}_4$, *the set of all triples such that* $B = 2$, $C + 1/A = -1$ *and either* $A - 1$ *or* $-1/2 - A$ *is a positive integer,*
- *the isolated triple* $\mathcal{P}_5 = \{(-7/3, 3, -4/7)\}$,
- *the isolated triple* $\mathcal{P}_6 = \{(-3/2, 2, -4/3)\}$.

Proof. – That $ABC + 1 = 0$ cannot be excluded and this gives the first set $\mathcal{P}_1$ of parameters.

Otherwise, if $ABC + 1 \neq 0$, the triple (A, B, C) of rational parameters satisfies the following conditions of Lemma 3 which can then be written

- $1/A$ or C or r is a positive integer,
- $1/B$ or A or p is a positive integer,
- $1/C$ or B or q is a positive integer,

where $p = -A - 1/B,\ q = -B - 1/C,\ r = -C - 1/A$.

Discussing how this conjunction of disjunctions can be fulfilled gives a classification of the remaining possible triples (sets $\mathcal{P}_2$ to $\mathcal{P}_6$). We will use natural transformations to reduce possibilities and Lemma 4 to derive subsequent reductions.

Among numbers p, q and r, some have to be positive integers; we take into account how many of them are positive integers, and, due to symmetries coming from natural transformations, we can choose which ones.

The first choice is that none of p, q and r is a positive integer. Then either one among A, B, and C is equal to 1 or all three are positive integers or inverse of positive integers. These two last options are equivalent by a natural transformation and they are to be avoided since $\Delta(A, B, C)$ is strictly negative for $A \geqslant 2$, $B \geqslant 2$, $C \geqslant 2$ and thus cannot be the square a a rational number. So, up to a natural transformation, this first choice leads to $B = 1$ and $1/A \in \mathbf{N}^\star$, which can be transformed in $B = 1, r = -1/A - C \in \mathbf{N}$ according to the discussion developed in Section 2.6. This is the set $\mathcal{P}_3$ of parameters.

The second choice is that one among p, q and r is a positive integer and we can choose r to be this one. Then B has to be a positive integer or the inverse of a positive integer. We choose B to be a positive integer. Then, either $B = 1$ or A is a positive integer. If $B = 1$, the triple $(A, 1, C)$, where $r = -C - 1/A$ is a positive integer, belongs to $\mathcal{P}_3$. If $A = 1$, the triple $(1, B, -r - 1)$ can be transformed to belong to $\mathcal{P}_3$.

Thus, within our second choice, it remains to consider triples (A, B, C) of rational numbers where $B \in \mathbf{N}$, $B \geqslant 2$, $A \in \mathbf{N}$, $A \geqslant 2$ and $r = -C - 1/A$ is a positive integer. $\Delta(A, B, -r - 1/A)$ is strictly negative and thus cannot be the square a a rational number except for $A \geqslant 2, B = 2, C = -1 - 1/A$. This constitutes one of the possibilities in the set $\mathcal{P}_4$ of parameters.

The third choice is that there are two positive integers among p, q and r and we can choose p and r to be these ones. Then either B or $1/C$ is a positive integer. We choose B to be a positive integer.

If $B = 1$, the triple $(-p - 1, 1, -r + 1/(p + 1))$ can be transformed to belong to $\mathcal{P}_3$.

Thus, within our third choice, it remains to consider triples (A, B, C) where $B \in \mathbf{N}$, $B \geqslant 2$ and p and r are positive integers.

$\Delta(-p - 1/B, B, -r + B/(pB + 1))$ is then strictly negative and thus cannot be the square of a rational number except

- for $B = 2, C = -1 - 1/A, A = -1/2 - p, p \in \mathbf{N}^\star$, which constitutes the other possibility in the set $\mathcal{P}_4$ of parameters,
- and for two other exceptional triples $(-7/3, 3, -4/7)$ and $(-3/2, 2, -4/3)$ (subsets $\mathcal{P}_5$ and $\mathcal{P}_6$).

The last and fourth choice consists in the assumption that p, q and r are positive integers.

In this case, C writes as a rational function of A, B as a rational function of C and A as rational function of B.

Combining these relations gives an equation of degree two to be satisfied by the rational number A; its discriminant is equal to $(pqr - p - q - r + 2)(pqr - p - q - r - 2)$.

Thus, the integer $D = (pqr - p - q - r)^2 - 4$ has to be the square of an integer. But $D + 4$ is already a square and the only possibility is $D = 0$.

Thus, we have to solve the equation $(pqr - p - q - r)^2 = 4$ in positive integers.

Within a permutation of (p, q, r), corresponding to a natural transformation of (A, B, C), the only possibilities are $(1, 1, r)$, $(1, 2, 5)$, $(1, 3, 3)$ and $(2, 2, 2)$. The first one can be solved in (A, B, C) if and only if $r = 1$ while the other three lead to triples (A, B, C) such that $ABC + 1 = 0$.

Thus, within our fourth choice, the only novelty is exactly the set $\mathcal{P}_2$ of parameters. $\square$

4. Sufficient conditions

This section is devoted to proving that the previous necessary condition on the triple (A, B, C) of rational non-zero parameters is in fact sufficient for the existence of a rational first integral of $V(A, B, C)$.

In the case of $\mathcal{P}_3$, it turns out that an additional assumption is to be made in order to perform the rational integration: that $-1/A$ and $-C$ are not simultaneously positive integers. A final complementary lemma will then show that this restriction is in fact necessary, and our main theorem will be completely proved.

Our proof relies on various computations to deal with the six cases of Proposition 1. A general well-known idea will also be useful in that context and we state it as a lemma.

LEMMA 5. – *If $V(A, B, C)$ has an algebraic first integral of degree* 0*, it also has a rational one.*

Proof. – Our assumption means the following: there exists an algebraic field extension L of $k(x, y, z)$, the derivations δ_x, δ_y and δ_z can be extended to L in a unique way, which gives an extension of derivations δ_V and δ_E. The supposed algebraic first integral of degree 0 is then an element F of L such that $\delta_V(F) = \delta_E(F) = 0$ while the three partial derivatives $\delta_x(F)$, $\delta_y(F)$ and $\delta_z(F)$ are not all equal to 0.

Consider now the minimal unitary polynomial of F with coefficients in the field $k(x, y, z)$ of rational fractions:

$$P(t) = t^n + p_{n-1}t^{n-1} + \cdots + p_0.$$

Applying a derivation δ to identity $P(F)=0$ leads to

$$\overline{P}(F)+P'(F)\delta(F)=0,$$

where P' is the usual derivative of P in the polynomial ring $k(x,y,z)[t]$ while $\overline{P}$ is the polynomial of $k(x,y,z)[t]$ obtained by applying component-wise derivation δ to the coefficients of P. This $\overline{P}$ has a degree at most $n-1$.

In the case of δ_V and δ_E, $\delta(F)=0$ and thus $\overline{P}(F)=0$. As P is supposed to be the minimal polynomial of F, $\overline{P}$ has to be the 0 polynomial in $k(x,y,z)[t]$, which means that the coefficients of P are in the common kernel of δ_V and δ_E.

With respect to δ_x, δ_y and δ_z, one of the coefficients p_i of P at least is not a constant, otherwise F would be a constant and p_i is then a rational first integral of degree 0. □

We give now the headlines of the computations: the use of a computer algebra system has been very useful to carry out some of them.

COMPUTATION 1. – *If the rational parameters A, B and C are such that $ABC+1=0$, then $V(A,B,C)$ has a rational first integral of degree* 0.

Proof. – In the present situation, there exists a polynomial first integral $l=ax+by+cz$ of degree 1, where the non-zero triple of integers (a,b,c) satisfies $b=-aC,\ c=-bA,\ a=-cB$, due to the rank condition on this system.

Polynomials x, y and z are Darboux polynomials with respective eigenvalues $Cy+z$, $Az+x$ and $Bx+y$ and, due to a similar rank condition, a non-trivial triple (α,β,γ) of integers can be found such that $x^\alpha y^\beta z^\gamma$ is a rational first integral of $V(A,B,C)$. Its degree is $\delta=\alpha+\beta+\gamma$.

Then, the rational fraction $x^\alpha y^\beta z^\gamma l^{-\delta}$ is the sought first integral of degree 0. □

COMPUTATION 2. – *If the triple of (A,B,C) of rational parameters belongs to $\mathcal{P}_2$, then $V(A,B,C)$ has a rational first integral of degree* 0.

Proof. – In the present situation, one among A, B and C is positive. Using symmetries, we can choose B to be positive and less than or equal to 1 without restricting the proof. Thus, there exist two relatively prime positive integers p and q such that

$$0 < p \leqslant q, \qquad B = \frac{p}{q}, \qquad A = -\frac{p+q}{p}, \qquad C = -\frac{q}{p+q}.$$

Then a Darboux polynomial ϕ of degree 2 and eigenvalue 0 can be found:

$$\phi = X^2 - 2\,XY - 2\,XZ + Y^2 - 2\,ZY + Z^2,$$

where the linear change of variables is given by $X = -p(p+q)x$, $Y = pqy$, $Z = -q(p+q)z$.

It is thereafter possible to build an algebraic field extension $L = \mathbf{Q}(u, v, w)$ of $\mathbf{Q}(x, y, z) = \mathbf{Q}(X, Y, Z)$ by setting

$$\begin{aligned} X &= (u+v-w)(-u+v-w), \\ Y &= 2v(-u+v+w), \\ Z &= 2w(-u+v+w). \end{aligned}$$

Derivations δ_X, δ_Y and δ_Z can be extended in a unique way to L and so are δ_V and δ_E.

The factors appearing in X, Y and Z are homogeneous of degree $1/2$ and they are Darboux elements; it turns out that the following element F of L is a first integral of degree 0, i.e., it satisfies $\delta_V(F) = \delta_E(F) = 0$:

$$F = \left(\frac{(u+v-w)(-u+v-w)}{v(u-v+w)}\right)^p \left(\frac{w(u+v-w)}{(-u+v+w)(u-v+w)}\right)^q,$$

which also writes (up to a constant factor)

$$F = \left(\frac{Z(4WY - Z^2)}{WY(Z-4W)}\right)^p \left(\frac{W(4WY - Z^2)}{Z^2(Z-4W)}\right)^q,$$

where $W = w^2$ is a root of the second degree polynomial

$$16YW^2 + (4XZ - 4YZ - 4Z^2)W + Z^3 = 0.$$

F is thus algebraic of degree 2 over $\mathbf{Q}(X, Y, Z)$, F is not a constant and it satisfies $\delta_V(F) = \delta_E(F) = 0$. This achieves the proof. To be more precise, the product of F by its conjugate is a constant, while the sum of

F and its conjugate is not a constant and thus provides the sought rational first integral of degree 0. □

COMPUTATION 3. – *If the triple of* (A, B, C) *of rational parameters belongs to* $\mathcal{P}_3$*, i.e., if* $B = 1$ *and* $r = -1/A - C$ *is a positive integer and if moreover* $-C$ *and* $-1/A$ *are not both positive integers, then* $V(A, B, C)$ *has a rational first integral of degree* 0.

Proof. – When $B = 1$ and $A \neq 0$, there exists a fourth Darboux polynomial of degree 1, $\phi = y - Az$. We then get an integrating factor ψ for the Pfaff form ω_0:

$$\psi = x^{-2} y^{\frac{1}{A}-1} z^{C-1} \phi^{-C-\frac{1}{A}}$$

which is algebraic when A and C are rational.

The closed form $\psi\omega_0$ then writes

$$\psi\omega_0 = d\left(\frac{F}{x}\right) + \theta\left(\frac{y}{z}\right) d\left(\frac{y}{z}\right),$$

where F is a homogeneous algebraic function of degree 1 of y and z solely while θ is the one-variable algebraic function

$$\theta(t) = -\frac{(t-A)^r t^{1/A}}{t}.$$

If the primitives of θ are algebraic functions of one variable, the proof is achieved.

To check that last point, denote $1/A$ by $-p/q$ with relatively prime integers p, q and positive q.

Using the new variable $u = t^{1/q}$, which is algebraic over t, the following algebraic function χ of u remains to be integrated

$$\chi(u) = u^{-p-1}(u^q - A)^r.$$

Primitives of χ are algebraic provided that the exponent -1 does not appear in the development of χ as a finite linear combination of rational powers of u. This is equivalent to saying that there is no integer l in the range $[0, r]$ such that $-p - 1 + lq = -1$, i.e., that p/q is not an integer in the range $[0, r]$; this is precisely what is excluded by the extra assumption that $-1/A$ and $-C$ are not simultaneously positive integers. □

COMPUTATION 4. – *If the triple of* (A, B, C) *of rational parameters belongs to* $\mathcal{P}_4$*, i.e., if* $B = 2, C = -1 - 1/A$ *and* A *is either a positive integer* ($A \geqslant 2$) *or equal to* $-k - 1/2$*, where* k *is a positive integer, then* $V(A, 2, -1 - 1/A)$ *has a rational first integral of degree* 0.

Proof. – When $B = 2$, $C = -1 - 1/A$ and $A \neq 0$, there exists a fourth Darboux polynomial of degree 2, $\phi = (y - Az)^2 - 2A^2xz$. We then get an integrating factor ψ for the Pfaff form ω_0:

$$\psi = x^{A-1}y^{-2}z^A\phi^{-A-\frac{1}{2}}$$

which is algebraic when A and C are rational.

But, this is not the easiest way to achieve the proof. It is more convenient to parameterize the projective surface $\phi = t^2$ like in our second computation. We can therefore build an algebraic field extension $L = \mathbf{Q}(u, v, w)$ of $\mathbf{Q}(x, y, z)$ by setting

$$x = (v - Aw)^2 - u^2, \quad y = 2A^2vw, \quad z = 2A^2w^2.$$

In the new variables, V becomes an homogeneous vector field $\widetilde{V}$ of degree 3, but w is a common factor of the components of $\widetilde{V}$ and it suffices to find a rational (with respect to u, v, w) first integral of degree 0 for the vector field $W = \widetilde{V}/w$.

It turns out that this homogeneous vector field W has 4 Darboux polynomials of degree 1 (in u, v, w): u, v, $v - Aw + u$ and $v - Aw - u$. W is therefore factorisable and can be put in Lotka–Volterra normal form to become $V(1/A, 1, 1 - 2A)$ so that we have reduced our study to a previous one.

It is indeed sufficient to find a rational first integral of degree 0 for $V(1/A, 1, 1 - 2A)$.

If A is an integer greater than 1, this is true according to Computation 3. If $-1/2 - A$ is a positive integer, the triple $(1/A, 1, 1 - 2A)$ is equivalent through a natural transformation to $(1/A, 1, A)$ and Computation 3 can be applied to this last triple. $\square$

COMPUTATION 5. – *The Lotka–Volterra vector field* $V(-7/3, 3, -4/7)$ *has a rational first integral of degree* 0.

Proof. – A solution is $x^7y^3z^4fg^{-3}$, where f and g are irreducible homogeneous polynomials of degrees 4 and 6:

$$\begin{aligned}f = {} & -259308\,x^3z - 185220\,x^2yz + 259308\,x^2z^2 + 567\,xy^3\\ & - 13230\,xy^2z - 71001\,xyz^2 - 86436\,xz^3 + 324\,y^4 + 3024\,y^3z\\ & + 10584\,y^2z^2 + 16464\,yz^3 + 9604\,z^4,\end{aligned}$$

$$\begin{aligned}g = {} & 9529569\,x^4z^2 + 2722734\,x^3yz^2 - 12706092\,x^3z^3\\ & - 333396\,x^2y^3z - 388962\,x^2y^2z^2 + 3630312\,x^2yz^3\\ & + 6353046\,x^2z^4 - 47628\,xy^4z - 444528\,xy^3z^2 - 1555848\,xy^2z^3\\ & - 2420208\,xyz^4 - 1411788\,xz^5 + 729\,y^6 + 10206\,y^5z + 59535\,y^4z^2\\ & + 185220\,y^3z^3 + 324135\,y^2z^4 + 302526\,yz^5 + 117649\,z^6.\end{aligned}$$

□

COMPUTATION 6. – *The Lotka–Volterra vector field* $V(-3/2, 2, -4/3)$ *has a rational first integral of degree* 0.

Proof. – A solution is $x^3y^2z^4fg^{-3}$, where f and g are irreducible homogeneous polynomials of degrees 3 and 4:

$$\begin{aligned}g = {} & 324\,x^2z^2 + 288\,xy^2z + 216\,xyz^2 - 324\,xz^3 + 16\,y^4\\ & + 96\,y^3z + 216\,y^2z^2 + 216\,yz^3 + 81\,z^4,\end{aligned}$$

$$\begin{aligned}f = {} & 108\,x^2z + 6\,xy^2 + 180\,xyz - 108\,xz^2 + 8\,y^3 + 36\,y^2z\\ & + 54\,yz^2 + 27\,z^3.\end{aligned}$$

□

4.1. A special final trick

The purpose of the following lemma is to prove that the sub-cases of $\mathcal{P}_3$ in which we were unable to perform the rational integral are in fact to be excluded.

LEMMA 6. – *When* $B = 1$ *while* $a = -1/A$ *and* $c = -C$ *are both positive integers,* $V(A, B, C)$ *does not have a rational first integral of degree* 0.

Proof. – The consideration of some special determinants of small order of the linear system studied in Subsection 2.7 is the tool to get this further restriction on the possible candidate triples. We use freely the notations and ideas of this subsection.

Suppose that there exists some strict Darboux polynomial f of degree m for $V(A,B,C)$ and consider the square subsystem (with a non-trivial kernel) of the corresponding linear one consisting in the unknowns $(0,j,m-j)$, $0 \leqslant j \leqslant a$, and $(1,j,m-1-j)$, $0 \leqslant j \leqslant a-1$, on the one hand, and of the equations $(0,j,m+1-j)$, $1 \leqslant j \leqslant a$, and $(1,j,m-j)$, $0 \leqslant j \leqslant a-1$ on the other hand.

According to computations similar to previous ones, the determinant of this subsystem finally writes:

$$[1/A + C + 1]^{\overline{a-1}}\,(m-\lambda),$$

up to some non-zero factors.

$[1/A + C + 1]^{\overline{a-1}}$ cannot vanish and λ has to be equal to m. Then, trying to solve level by level to find the x-homogeneous components of f shows that the Darboux polynomial has to be a multiple of $(y-Az)^m$.

Thus, there is only one fourth Darboux irreducible polynomial, $y-Az$.

But it it impossible to find a non-constant rational first integral of degree 0, that would write $x^\alpha y^\beta z^\gamma (y-Az)^\delta$ with $\alpha, \beta, \gamma, \delta$ integers. Such a non-trivial quadruple $[\alpha, \beta, \gamma, \delta]$ of integers would indeed satisfy equations $\alpha + \beta + \gamma + \delta = 0$, $\beta + \gamma + \delta = 0$, $C\alpha + \gamma = 0$ and $\alpha + A\beta = 0$. $\square$

5. Conclusion and remarks

We have thus proved that the rational integration of factorisable quadratic homogeneous three-variable vector fields is a decidable question by giving a complete characterization of all triples (A,B,C) for which $V(A,B,C)$ has a rational first integral of degree 0.

Lists of parameters (A,B,C) can be written down, for which there exists a fourth Darboux polynomial of $V(A,B,C)$ with the help of which the Liouvillian integration of the field can be achieved, whereas the rational integration of it is impossible. Unfortunately, the problem of the Liouvillian integration of $V(A,B,C)$ cannot be considered as solved up to now, because no effective bound is known on the degree m of possible Darboux polynomials for a given triple (A,B,C) of parameters.

On the contrary, it can be thought that the specific linear algebra tools, that we have developed in the present paper to find necessary conditions of rational integrability, will find a wider use, for instance in the study of the rational integrability of general quadratic homogeneous three-variable vector fields.

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