POLYNOMIAL FIRST INTEGRALS
OF THE LOTKA-VOLTERRA SYSTEM

BY

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ABSTRACT. – The so-called Lotka-Volterra system of autonomous differential equations
consists in three polynomial homogeneous equations in three variables of degree 2.
It depends on three parameters:

$$\dot{x} = V_x = x(Cy + z)$$
$$\dot{y} = V_y = y(Az + x)$$
$$\dot{z} = V_z = z(Bx + y)$$

A first integral of this system is a non-constant function f that satisfies identity:

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0.$$

In the present paper, we give all values of the three parameters A, B and C for which the
previous system has an homogeneous polynomial first integral.
Our proof essentially relies on ideas of algebra and combinatorics.

RÉSUMÉ. – Le système d'équations différentielles autonomes de Lotka-Volterra se compose
de trois équations polynomiales homogènes de degré 2 et il dépend de trois paramètres :

$$\dot{x} = V_x = x(Cy + z)$$
$$\dot{y} = V_y = y(Az + x)$$
$$\dot{z} = V_z = z(Bx + y)$$

On appelle intégrale première d'un tel système une fonction non constante qui vérifie l'identité

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0.$$

Dans ce travail, nous caractérisons toutes les valeurs possibles des trois paramètres A, B et C pour lesquelles le système précédent admet une intégrale première polynomiale. Notre démonstration s'appuie sur des arguments de nature algébrique et combinatoire.

1. Introduction

The search of first integrals is a classical tool in the classification of all trajectories of a dynamical system. Let us simply recall the role of energy in hamiltonian systems.

We are interested here in some systems consisting in three ordinary autonomous differential equations in three variables:

$$\dot{x} = V_x$$
$$\dot{y} = V_y$$
$$\dot{z} = V_z$$

where A, B and C are parameters.

A *first integral* of this system of equations (or of the corresponding vector field V) is a non-constant function f that satisfies the partial derivative equation

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0.$$

That means that f is constant along all trajectories of the one-parameter local semigroup generated by the vector field V.

The *local* existence of first integrals in a neighborhood of a regular point is a consequence of some classical theorems of differential calculus.

The interesting point for us is the search of *global* solutions; this problem has an algebraic nature if the coordinate functions V_x, V_y and V_z are polynomials in the space variables x, y and z.

A key point is the specification of the class in which we look for first integrals. In the algebraic case, it seems reasonable to consider the class of

all Liouvillian elements over the differential field $\mathbf{C}(x, y, z)$ of all rational fractions in three variables on the constant field $\mathbf{C}$ of complex numbers. We follow the definition of Liouvillian elements given by Michael SINGER [8].

The search of Liouvillian first integrals of polynomial vector fields relies mainly on the study of the Darboux polynomials of these fields [1-4].

A polynomial f is said to be a *Darboux polynomial* of the (polynomial) vector field V if there exists some polynomial Λ such that

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = \Lambda f.$$

Later on, we will call Λ an *eigenvalue*. With this vocabulary, polynomial first integrals are simply Darboux polynomials with a 0 eigenvalue.

So far as we know, Henri POINCARÉ ([5], [6], [7]) was the first to notice the difficulty of a decision procedure for the existence of Darboux polynomials.

No procedure is known up to now; Jean-Pierre JOUANOLOU gives a theorem about this subjet but his result is not effective [3].

All that shows the interest of the partial result we present here: the determination of the set of values of the parameters for which a special factorisable system, the Lotka-Volterra one [2], has a polynomial first integral. Necessary conditions on the parameters appear to be a consequence of a combinatorial approach. Let us remark that all these cases appear in [2] and have been found with the help of a computer algebra system.

The crucial point is in fact to tackle the maximum degree of an *irreducible* Darboux curve. Indeed, if the degree m and the eigenvalue Λ for which we look for a Darboux polynomial of the vector field V are given, the task is a classical algebraic elimination problem. The condition leads to a system of linear equations in which the unknowns are the coefficients of the sought Darboux polynomial. In order to get a non-trivial solution the rank has to lower, and this is equivalent to the nullity of some determinants. These determinants are polynomials in the parameters, and we are *in principle* able to solve a system of polynomial equations.

2. The result

Let us state our main theorem.

THEOREM. – *The Lotka-Volterra system*

$$\dot{x} = V_x = x(Cy + z)$$
$$\dot{y} = V_y = y(Az + x)$$
$$\dot{z} = V_z = z(Bx + y)$$

has a polynomial first integral if and only if one of the following cases holds for the triple (A, B, C) *of parameters*

• *Parameters satisfy relation* $ABC + 1 = 0$.
The field has then an homogeneous polynomial first integral of degree 1.
• *Parameters are related by* $C = -1 - 1/A$, $A = -1 - 1/B$ *and* $B = -1 - 1/C$, *so that one of the parameters can be freely chosen.*
The field has then an homogeneous polynomial first integral of degree 2.
• *Parameters are solutions of a system of equations*

$$C = -k_2 - 1/A$$
$$A = -k_3 - 1/B$$
$$B = -k_1 - 1/C$$

where, up to a permutation (k_1, k_2, k_3) *is one of the following triples:*

$$(1, 2, 2), \quad (1, 2, 3), \quad (1, 2, 4).$$

In these cases, one of the parameters is a root of a rational polynomial of degree 2 and the other two can be deduced from it. In every case, an homogeneous polynomial first integral of degree 3, 4 or 6 respectively can be found.

3. The proof

This section is devoted to the complete proof of the previous result. Intermediate results and arguments are gathered in subsections.

3.1. Homogeneity. – Due to the homogeneity of the three coordinates of the Lotka-Volterra vector field, it is easy to show that the homogeneous components of a polynomial first integral are also first integrals. In the same way, the homogeneous components of a Darboux polynomial are also Darboux polynomials and the eigenvalue has to be an homogeneous polynomial of degree 1 (the common degree 2 of the coordinates of the field minus 1 for the usual partial derivations).

It is therefore sufficient to look for homogeneous first integrals and homogeneous Darboux polynomials.

Let us remark that homogeneous polynomials are Darboux polynomials for the special vector field $E = x\partial_x + y\partial_y + z\partial_z$ according to Euler's identity; with respect to E, the eigenvalue of an homogeneous polynomial is its degree.

3.2. Variables and strict Darboux polynomials. – Suppose now that some Darboux (homogeneous) polynomial f for a given homogeneous polynomial vector field V and an eigenvalue Λ is splitted as the product $f = gh$ of two relatively prime (homogeneous) polynomials.

As a polynomial ring is a unique factorization domain, Gauss lemma shows that the factors g and h have to be Darboux polynomials for V with some eigenvalues Λ_1 and Λ_2 such that $\Lambda = \Lambda_1 + \Lambda_2$.

According to this remark, the determination of all Darboux polynomials of a given polynomial vector field amounts to finding all irreducible Darboux polynomials for V.

In the case of a factorisable vector field that we are interested in, we will not consider a complete factorization in irreducible factors of some first integral or Darboux polynomial.

By the very definition of a factorisable vector field, like the Lotka-Volterra one, the space variables x, y and z are Darboux polynomials.

Every homogeneous polynomial f writes in a unique way

$$f = x^\alpha y^\beta z^\gamma g$$

where g is not divisible by x, y or z.

If f is a Darboux polynomial, so is g. Such polynomials as g will play an important role in our combinatorial analysis. We will call them *strict* polynomials.

3.3. Eigenvalues of strict Darboux polynomials. – Let g be a strict Darboux polynomial of the Lotka-Volterra vector field which writes

$$x(Cy+z)\partial_x g + y(Az+x)\partial_y g + z(Bx+y)\partial_z g = (\lambda x + \mu y + \nu z)g$$

Polynomial g is moreover homogeneous of degree m. As g is supposed not to be divisible by x, y or z, we can consider the three homogeneous non-zero two-variable polynomials of degree m obtained by setting $x=0$, $y=0$ and $z=0$ in g and call them P, Q and R respectively.

From the previous relation involving g, we deduce some partial differential equations concerning these two-variable polynomials.

$$\begin{cases} (\mu y + \nu z)P = yz(A\partial_y P + \partial_z P) \\ (\nu z + \lambda x)Q = zx(B\partial_z Q + \partial_x Q) \\ (\lambda x + \mu y)R = xy(C\partial_x R + \partial_y R). \end{cases}$$

It is not very difficult to prove that there exists 6 natural numbers β_1, γ_1, α_2, γ_2, α_3 and β_3 such that P is a nonzero multiple of $y^{\beta_1}z^{\gamma_1}(y-Az)^{m-\beta_1-\gamma_1}$, Q is a nonzero multiple of $z^{\gamma_2}x^{\alpha_2}(z-Bx)^{m-\gamma_2-\alpha_2}$ and R is a nonzero multiple of $x^{\alpha_3}y^{\beta_3}(x-Cy)^{m-\alpha_3-\beta_3}$. Moreover, these numbers satisfy the following equations and inequations

$$\begin{cases} \lambda = \beta_3 = \gamma_2 B, \\ \mu = \gamma_1 = \alpha_3 C, \\ \nu = \alpha_2 = \beta_1 A, \\ \beta_1 + \gamma_1 \le m, \\ \alpha_2 + \gamma_2 \le m, \\ \alpha_3 + \beta_3 \le m. \end{cases}$$

In particular, the eigenvalue corresponding to a strict Darboux polynomial of the Lotka-Volterra vector field is a linear form $\Lambda = \lambda x + \mu y + \nu z$ where λ, μ and ν are non-negative integers.

3.4. The restricted problem. – In this subsection, we discuss some necessary conditions that the parameters A, B and C have to fulfill in order to allow the existence of a polynomial first integral for the Lotka-Volterra vector field.

A polynomial first integral f writes $f = x^\alpha y^\beta z^\gamma g$ where g is a strict Darboux polynomial of some degree m with an eigenvalue $\lambda x + \mu y + \nu z$ and where α, β and γ as well as λ, μ and ν are non-negative integers.

The eigenvalue corresponding to f is 0, which leads to the following system of equations

$$\begin{cases} \lambda + \beta + \gamma B = 0 \\ \mu + \gamma + \alpha C = 0 \\ \nu + \alpha + \beta A = 0 \end{cases}$$

If one of the coefficients λ, μ or ν is different from 0, the parameter appearing in the same equation is a positive rational number according to the results of the previous subsection; this makes the corresponding equation impossible.

Then, the strict Darboux polynomial g has to be a first integral (or a constant) and the system of equations reduces to

$$\begin{cases} \beta + \gamma B = 0 \\ \gamma + \alpha C = 0 \\ \alpha + \beta A = 0 \end{cases}$$

If one or two numbers among α, β and γ are 0, some of the previous equations are impossible.

Two possibilities remain:

$\alpha = \beta = \gamma = 0$ and we have to look for a strict first integral,

$\alpha\beta\gamma \neq 0$ and the parameters A, B and C have to be negative rational numbers with $ABC = -1$.

In the second case, the condition is obviously sufficient.

CONCLUSION 1. – *Moreover, it is easy to show that the condition* $ABC + 1 = 0$ *(without rationality assumption) is sufficient to get a strict first degree homogeneous first integral like, for instance,* $Bx - BCy - z$. *This is the first case of the theorem.*

From now on, we will only consider triples of parameters for which $ABC + 1 \neq 0$ and restrict our search to looking for strict polynomial first integrals.

As the last result of the present subsection, let us prove that in case of a strict polynomial first integral of degree m, the number $\omega = -ABC$ is a m-th root of unity, $\omega^m = 1$.

Let then g be a strict polynomial first integral of degree m and suppose that one of the three parameters A, B or C at least is equal to 0. Suppose for instance $C = 0$. According to a previous discussion about the eigenvalues of strict Darboux polynomials, polynomial R is a multiple of x^m whereas polynomial Q is a multiple of $(y - Az)^m$, and the coefficients of y^m in g cannot agree.

Thus, all three parameters are different from 0, P is a multiple of $(y - Az)^m$, Q is a multiple of $(z - Bx)^m$ and R is a multiple of $(x - Cy)^m$. The coefficients of x^m, y^m and z^m in g are related in such a way that $\omega = -ABC$ satisfies $\omega^m = 1$.

3.5. Linear algebra and combinatorics. – Consider now some strict polynomial first integral g of the Lotka-Volterra vector field. The degree m of g is greater than 1 and $\omega = -ABC$ satisfies $\omega^m = 1$. Polynomial g writes in the following ways

$$g = P_0(y, z) + x\, P_1(y, z) + x^2 p(x, y, z)$$
$$g = Q_0(x, z) + y\, Q_1(x, z) + y^2 q(x, y, z)$$
$$g = R_0(x, y) + z\, R_1(x, y) + z^2 r(x, y, z)$$

where P_0, Q_0 and R_0 are the two-variable homogeneous non-zero polynomials of degree m that we get by setting $x = 0$, $y = 0$ or $z = 0$ in g (previously denoted by P, Q and R) whereas P_1, Q_1 and R_1 are homogeneous two-variable polynomials of degree $m - 1$ and p, q and r are three-variable homogeneous polynomials of degree $m - 2$.

There are $2m + 1$ coefficients for P_0 and P_1 together. Writing that g is a first integral leads to precisely $2m + 1$ linear equations only involving these coefficients. That the determinant of the corresponding matrix is 0 is a necessary condition to get a non-trivial polynomial g.

The combinatorial computation of this determinant is the key point of our proof. Of course, analogous results for the linear system coming from the consideration of Q_0 and Q_1 as of R_0 and R_1 will be used in the forthcoming conclusion.

With a suitable choice in the order of equations and unknowns, the square matrix $\mathcal{M}$ of our linear system, whose order is $2m + 1$, has the following form

$$\mathcal{M} = \begin{pmatrix} \mathcal{O} & \mathcal{A} \\ \mathcal{C} & \mathcal{B} \end{pmatrix},$$

where $\mathcal{O}$ is the zero square matrix of order m, $\mathcal{A}$ is a rectangular matrix with m rows and $m + 1$ columns, $\mathcal{C}$ is a rectangular matrix with $m + 1$ rows and m columns and $\mathcal{B}$ is a diagonal square matrix of order $m + 1$.

There last three matrices look like what follows and they have a lot of zeroes.

$$\mathcal{A} = \begin{pmatrix} m & A & 0 & \cdots & \cdots & 0 \\ 0 & m-1 & 2A & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 2 & (m-1)A & 0 \\ 0 & \cdots & \cdots & 0 & 1 & mA \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & & 0 \\ C+m-1 & 1+A & 0 & & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & \cdots & C+1 & 1+(m-2)A & & 0 \\ 0 & \cdots & \cdots & C & & 1+(m-1)A \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} Bm & 0 & \cdots & \cdots & 0 \\ 0 & 1+B(m-1) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & (m-1)+B & 0 \\ 0 & \cdots & \cdots & 0 & m \end{pmatrix}$$

The development of the determinant D_m of $\mathcal{M}$ has only $m + 1$ non-zero terms, each of them being of course a product of $2m + 1$ factors.

The reason is the following: in every non-zero term of D_m, some of the diagonal elements of $\mathcal{B}$ appear as factors. Trying to avoid diagonal elements of $\mathcal{B}$ yields null terms of D_m. In a similar way, if we take two diagonal elements of $\mathcal{B}$, the corresponding cofactor is a $2m-1$ determinant, which is easily transformed. by a permutation in a triangular one with a zero element in its diagonal.

The determinant D_m is thus the sum of products of all diagonal elements of $\mathcal{B}$ by their cofactors; these cofactors are triangular determinants of order $2m$ and thus easy to compute.

All that amounts to the following expression of D_m

$$D_m = \sum_{i=0}^{m} (i + (m-i)B) \prod_{j=0}^{i-1} (m-j)(1+jA) \times$$
$$\times \prod_{j=i+1}^{m} (jA)(C+m-j).$$

Introducing the notation $t^{\overline{k}}$ for the ascending power $\prod_{i=0}^{k-1} (t+i)$, binomial-like expansions yield to a factorization of D_m :

$$D_m = A^{m-1}m!\left(\frac{1}{A} + C + 1\right)^{\overline{m-1}} m(ABC+1).$$

As D_m is equal to 0, there exists some integer k_2, $1 \le k_2 \le m$, such that $1/A + C + k_2 = 0$ (recall that $ABC + 1 \ne 0$).

Similar results are true if we take into consideration coefficients of Q_0 and Q_1 or those of R_0 and R_1.

Whence a necessary condition to be fulfilled by the parameters in order to get a strict polynomial first integral of degree m:

there exist three integers k_1, k_2 and k_3 all three strictly positive and less than or equal to m such that

$$B = -k_1 - 1/C$$
$$C = -k_2 - 1/A$$
$$A = -k_3 - 1/B$$

and the product $\omega = -ABC$ satisfies $\omega^m = 1$.

3.6. Cyclotomic polynomials. – Eliminating B and C in the previous system of equations yields

$$(1 - k_1 k_2)A^2 + (k_2 + k_3 - k_1 - k_1 k_2 k_3)A + 1 - k_1 k_3.$$

If k_1 and k_2 are equal to 1, this equation has degree 0 so that k_3 has to be equal to 1.

CONCLUSION 2. – *This is in fact a possibility. The parameters are related by the relations*

$$B = -1 - 1/C$$
$$C = -1 - 1/A$$
$$A = -1 - 1/B,$$

one of them can be freely chosen, and the two others are computed from it and the product ABC is equal to 1 (not to -1!).

In such a case, a simple computation yields a strict polynomial first integral of degree 2.

$$A^2 B^2 x^2 + y^2 + A^2 z^2 - 2ABxy - 2A^2 Bxz - 2Ayz.$$

This is the second case of the theorem.

Otherwise, A is an algebraic number as a solution of a second degree equation with rational coefficients.

It is not difficult to compute ω from A

$$\omega = -ABC = -A\left(\frac{-k_1 C - 1}{C}\right)C =$$

$$= -A\left(k_1 \frac{-k_2 A - 1}{A} - 1\right) = k_1 k_2 A + k_1$$

and to deduce a second degree equation that ω satisfies:

$$\Psi(\omega) = \omega^2 + (k_1 + k_2 + k_3 - k_1 k_2 k_3)\omega + 1 = 0$$

where k_1, k_2 and k_3 are positive integers and not all equal to 1.

Then, ω is an algebraic integer of degree 2 but also a m-th root of unity. The above polynomial Ψ of degree 2 is thus a multiple of the cyclotomic polynomial Φ_p where p is a divisor of m such that ω is a primitive p-th root of unity.

Cyclotomic polynomials of degree 1 or 2 only correspond to values 1, 2, 3, 4 and 6 of p.

If $p = 1$, $\omega = 1$, which has been excluded (first case of the theorem).

If $p = 2$, $\Psi(-1) = 2 - k_1 - k_2 - k_3 + k_1 k_2 k_3 = 0$, which leads to

$$\frac{1}{k_1 k_2} + \frac{1}{k_1 k_3} + \frac{1}{k_2 k_3} = 1 + \frac{2}{k_1 k_2 k_3}$$

with three positive integers.

Up to an evident symmetry, one can assume $1 \leq k_1 \leq k_2 \leq k_3$. Then k_1 is equal to 1 else the left-hand side would be bounded above 3/4 and the right-hand side bounded below by 1. We then look for integers k_2 and $k_3 (1 \leq k_2 \leq k_3)$ such that

$$\frac{1}{k_2} + \frac{1}{k_3} = 1 + \frac{1}{k_2 k_3}$$

and the only possibility is $k_2 = k_3 = 1$.

This situation has already be treated (second case of the theorem).

If $p = 3$, Ψ is the cyclotomic polynomial Φ_3 and $k_1 + k_2 + k_3 - k_1 k_2 k_3 = 1$ which leads to

$$\frac{1}{k_1 k_2} + \frac{1}{k_1 k_3} + \frac{1}{k_2 k_3} = 1 + \frac{1}{k_1 k_2 k_3}$$

with three positive integers.

Still assuming the order $1 \leq k_1 \leq k_2 \leq k_3$, we get $k_1 = 1$ then $k_2 = k_3 = 2$.

Corresponding to this triple $(1,2,2)$ of positive integers, there are two triples of parameters (A, B, C) as A is solution of an algebraic equation of degree 2 and B and C are then computed from A.

A computer-aided verification then shows the existence of a strict polynomial first integral of degree 3 of the Lotka-Volterra vector field for these triples of parameters.

If $p = 4$, Ψ is the cyclotomic polynomial Φ_4 and $k_1 + k_2 + k_3 - k_1 k_2 k_3 = 0$ which leads to

$$\frac{1}{k_1 k_2} + \frac{1}{k_1 k_3} + \frac{1}{k_2 k_3} = 1$$

with three positive integers.

Still assuming the order $1 \leq k_1 \leq k_2 \leq k_3$, we get $k_1 = 1$ and the equation

$$\frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_2 k_3} = 1.$$

We have to choose $k_2 = 2$ whence $k_3 = 3$.

Corresponding to this triple $(1,2,3)$ of positive integers, there are two triples of parameters (A, B, C) as A is solution of an algebraic equation of degree 2 and B and C are then computed from A.

A computer-aided verification then shows the existence of a strict polynomial first integral of degree 4 of the Lotka-Volterra vector field for these triples of parameters.

If $p = 6$, Ψ is the cyclotomic polynomial Φ_6 and $k_1 + k_2 + k_3 - k_1 k_2 k_3 = -1$ which leads to

$$\frac{1}{k_1 k_2} + \frac{1}{k_1 k_3} + \frac{1}{k_2 k_3} = 1 - \frac{1}{k_1 k_2 k_3}$$

with three positive integers.

Still assuming the order $1 \leq k_1 \leq k_2 \leq k_3$, we get $k_1 = 1$ and the equation

$$\frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_2 k_3} = 1.$$

We have to choose $k_2 = 2$ whence $k_3 = 4$.

Corresponding to this triple (1,2,4) of positive integers, there are two triples of parameters (A, B, C) as A is solution of an algebraic equation of degree 2 and B and C are then computed from A.

A computer-aided verification then shows the existence of a strict polynomial first integral of degree 6 of the Lotka-Volterra vector field for these triples of parameters.

CONCLUSION 3. – *We have thus found the third case of our theorem.*

It is worth noting the relationship of these last arithmetic computations with the so-called Painlevé analysis of the Lotka-Volterra system (*see* for instance [2] page 770).

3.7. Conclusion. – The proof by case inspection of our theorem is now complete. We hope that such combinatorial techniques, bypassing the decision problem about the maximum degree of an irreducible Darboux polynomial, will allow to go further towards a description of all triples of parameters for which a strict Darboux polynomial exists, which would implies the Liouvillian integrability of the corresponding system [4].

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