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Algorithms and methods in differential algebra

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Abstract

Founded by J.F. Ritt, Differential Algebra is a true part of Algebra so that constructive and algorithmic problems and methods appear in this field. In this talk, I do not intend to give an exhaustive survey of algorithmic aspects of Differential Algebra but I only propose some examples to give an insight of the state of knowledge in this domain. Some problems are known to have an effective solution, others have an efficient effective solution which is implemented in recent computer algebra systems, and the decidability of some others is still an open question, which does not prevent computations from leading to interesting results.

Liouville's theory of integration in finite terms and Risch's theorem are examples of problems that computer algebra systems now deal with very efficiently (implementation work by M. Bronstein).

In what concerns linear differential equations of arbitrary order, a basis for the vector space of all liouvillian solutions can "in principle" be computed effectively thanks to a theorem of Singer's [17, 22]; the complexity bound is actually awful and a lot of work is done or in progress, especially by M. Singer and F. Ulmer, to give realistic algorithms [20, 21] for third-order linear differential equations.

Existence of liouvillian first integrals is a way to make precise the notion of integrability of vector fields. Even in the simplest case of three-dimensional polynomial vector fields, no decision procedure is known for this existence.

Nevertheless, explicit computations with computer algebra yield interesting solutions for special examples. In this case, the process of looking for so-called Darboux curves can only be called a method but not an algorithm; for a given degree, this search is a classical algebraic elimination process but no bound is known on the degree of the candidate polynomials.

This paper insists on the search of liouvillian first integrals of polynomial vector fields and a new result is given: the generic absence of such liouvillian first integrals for factorisable polynomial vector fields in three variables.

Resumé

L'algèbre différentielle a été fondée par J.F. Ritt et c'est devenu une partie de l'algèbre où apparaissent des questions et des méthodes de nature algorithmique. Dans cet exposé, je n'ai pas

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la prétention de donner une vue exhaustive des aspects algorithmiques de l'algèbre différentielle. Je propose seulement quelques exemples pour donner une idée de ce qu'est l'état de l'art sur ces questions. On sait que certains problèmes ont une solution algorithmique ; pour d'autres, cette solution peut être considérée comme efficace et elle est implantée dans les systèmes de calcul formel récents ; pour une autre catégorie de problèmes, des questions de décidabilité ne sont pas encore résolues, ce qui n'empêche pas que des calculs effectifs donnent déjà des résultats intéressants.

La théorie de Liouville de l'intégration en termes finis et le théorème de Risch sont des exemples de ce que les systèmes actuels de calculs formel savent mettre en œuvre efficacement (citons en particulier le travail d'implémentation réalisé par Manuel Bronstein).

Pour ce qui est des équations différentielles linéaires d'ordre quelconque, on peut en principe déterminer une base de l'espace vectoriel des solutions liouvilliennes grâce à un théorème de Michael Singer [17, 22]. Mais la complexité rend ce résultat théorique impraticable. Beaucoup de travail a été réalisé ou est en cours pour traiter de façon réaliste le cas des équations du troisième ordre (par Michael Singer et Felix Ulmer en particulier) [20, 21].

L'existence d'intégrales premières liouvilliennes est une manière de préciser ce que peut être l'intégrabilité des champs de vecteurs. Même dans le cas le plus simple des champs de vecteurs polynomiaux à trois dimensions, aucune procédure de décision de cette existence n'est actuellement connue. On ne peut qualifier la recherche de courbes de Darboux que de méthode et non d'algorithme : pour un degré donné, cette recherche est un problème classique d'élimination algébrique, mais on ne sait toujours pas borner a priori le degré d'éventuelles courbes de Darboux pour un champ de vecteurs donné.

Ce travail insiste sur la recherche d'intégrales premières liouvilliennes et présente un résultat nouveau : l'absence générique de ces intégrales premières pour des champs de vecteurs polynomiaux factorisables à trois variables.

1. Three problems in differential algebra

We shall deal with the following definition of a *differential field* K.

Let K be a field of characteristic zero. A derivation d of K is an additive mapping from K to itself that satisfies Leibnitz rule $d(xy) = d(x)y + xd(y)$ for the derivation of a product. K is then said to be a differential field. The usual rule for the derivation of a quotient can easily be derived from the definition. The subset C of all elements of K whose derivative is 0 is a subfield of K, the field of constants. In certain cases, it can be useful to consider many commuting derivations, in order to mimic the usual partial derivatives with respect to space variables.

I will now say some words on three problems in this domain: elementary integration, liouvillian solutions to linear ODE and liouvillian first integrals of polynomial vector fields in order to emphasize the importance of algebraic and algorithmic approaches in differential algebra. I will only give a brief and very incomplete survey of the first two; I will be more explicit on the third.

2. Elementary Integration

2.1. Elementary extensions

This question dates back to the work of Joseph Liouville who stated the following definition of elementary extensions of a given differential field K of functions.

Let (K,d) be a differential field. A differential extension field (L,d) is said to be *elementary* if there exists a finite tower of intermediate differential fields

$$(K_0,d) = (K,d) \subset (K_1,d) \subset \cdots \subset (K_n,d) = (L,d),$$

such that each K_i is generated over the previous one K_{i-1} by a single element θ_i which is an elementary generator, i.e. ,

either θ_i is algebraic over K_{i-1}, in which case the derivation d on K_i is uniquely defined,

or θ_i is transcendental and an exponential over K_{i-1}, which means that there exists an element η in K_{i-1} such that d is defined on K_i by $d(\theta_i) = \theta_i d(\eta)$,

or θ_i is transcendental and a logarithm over K_{i-1}, which means that there exists an element η in K_{i-1} such that d is defined on K_i by $d(\theta_i) = d(\eta)/\eta$.

2.2. Liouville's principle

Let f be an element of some differential field (K,d). If f admits an elementary integral, i.e. if there exists an elementary differential extension (L,d) of (K,d) and an element F of L such that $d(F) = f$, then, this elementary integral F writes

$$F = v_0 + \sum_{i=1}^{n} c_i \log v_i,$$

where v_0 belongs to K and the other v_i to some extension $\overline{K}$ of K by a finite number of algebraic constants, whereas the c_i are constant elements of $\overline{K}$.

2.3. Algorithmic aspects

Liouville gave an rather analytic proof of his statement. In his "Integration in finite terms", Ritt [16] still used analytic arguments. Rosenlicht [15] was the first to give a purely algebraic proof of this result, which is algebraic in its nature.

This algebraic result turned into an algorithmic one thanks to Risch [14].

Let $(K,d) = (C(x,\theta_1,\ldots,\theta_n),d)$ be a differential field where C is the effective subfield of constants. Moreover, suppose that each θ_i is a transcendental exponential or logarithm over K_{i-1}. Then there exists an algorithm that, given an element f of K, either gives an elementary integral of f or decides that no such integral exists.

Generalizations of this theorem have been given by Davenport [4] (a first algebraic step is allowed in the tower defining K from C) and by Bronstein [1] (real elementary integration) among others.

3. Linear ordinary differential equations

3.1. First-order linear ODE and liouvillian extensions

Let us begin with the first-order affine ODE with coefficients in a given differential field (K, d) in which we also denote the derivation d by a quote: $y' = ay + b$.

A solution to this problem divides classically in two steps, solving the corresponding linear ODE $y' = ay$ and then making the integration constant vary to achieve the solution with an integration.

This splitting method leads to the definition of what is now called a liouvillian extension of a differential field.

Let (K, d) be a differential field. A differential extension field (L, d) is said to be *liouvillian* if there exists a finite tower of intermediate differential fields

$$(K_0, d) = (K, d) \subset (K_1, d) \subset \cdots \subset (K_n, d) = (L, d),$$

such that each K_i is generated over the previous one K_{i-1} by a single element θ_i which is an liouvillian generator, i.e.

either θ_i is algebraic over K_{i-1}, in which case the derivation d on K_i is uniquely defined,

or θ_i is transcendental and an exponential-integral over K_{i-1}, which means that there exists an element η in K_{i-1} such that d is defined on K_i by $d(\theta_i) = \theta_i \eta$,

or θ_i is transcendental and a integral over K_{i-1}, which means that there exists an element η in K_{i-1} such that d is defined on K_i by $d(\theta_i) = \eta$.

According to the previous definition, a first-order affine ODE can then be solved in a liouvillian extension (L, d) of the base differential field (K, d).

Moreover, the extension field L can be built with the same field of constants C as K provided that we can decide what are the derivatives and logarithmic derivatives in a differential field.

3.2. Kovačic's theorem

In the classical literature on these subjects, the constant field C is always supposed to be algebraically closed in order to use the so-called Differential Galois Theory of Picard-Vessiot extensions.

This is especially the case in Kovačic's algorithm that deals with the second-order linear differential equation with coefficients in $\mathbb{C}(x)$, where $\mathbb{C}$ is the field of complex numbers.

After standard reductions, the equation to be considered writes $y = ry$, where $r \in \mathbb{C}(x)$.

Kovačic's theorem [8] states that there are precisely four cases that can occur

Case 1. The DE has a solution of the form $\exp(\int \omega)$, where $\omega \in \mathbb{C}(x)$.

Case 2. The DE has a solution of the form $\exp(\int \omega)$, where ω is algebraic over $\mathbb{C}(x)$ of degree 2 and case 1 does not hold.

Case 3. All solutions of the DE are algebraic over $\mathbb{C}(x)$ and cases 1 and 2 do not hold.

Case 4. The DE has no liouvillian solution.

If $\mathbb{C}$ is changed for a effective algebraically closed subfield of it, the proof of the theorem leads to a true decision procedure.

Let us remark that the field of constants does not need to be algebraically closed to perform algorithms [6, 23].

3.3. *Algorithms for linear ODE*

Thanks to an algorithm of Michael Singer [17, 18], it is possible to decide in principle if the nth order ODE with coefficients in a finite algebraic extension of $\mathbb{Q}(x)$, or even with liouvillian coefficients, has a liouvillian solution; but this algorithm is far from being efficient.

Recent algorithmic progresses have been done by Michael Singer and Felix Ulmer to deal with third-order ODE [20, 21].

Another algorithmic question is to find solutions to ODE is the field of coefficients; let us simply quote a work of Manuel Bronstein [2] about that.

4. Liouvillian first integrals

4.1. *First integrals*

In this section, we shall consider 3-dimensional vector fields; the following definitions about liouvillian extensions are also meaningful in higher dimensions while the sole use of the Euler's field as a symmetry is typical of the study of 3-dimensional homogeneous vetor fields.

Let $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$ be a vector field whose coordinates belong to some differential field K (for the three commuting derivations ∂_x, ∂_y and ∂_z).

A *first integral* of V is an element f of a differential extension field L of K that satisfies

$$V_x\partial_x f + V_y\partial_y f + V_z\partial_z f = 0.$$

To be of any interest, f is supposed not to be constant (one of the partial derivatives of f at least is not 0).

This is an algebraic translation of a well-known analytical property: to be constant on the trajectories of the local semi-group generated by the vector field.

A vector field (or the corresponding system of autonomous differential equations) is usually said not to be integrable if there does not exist any first integral of it in a prescribed class. This class strongly depends on the speaker; we are interested in liouvillian first integrals, i.e. in first integrals that belong to liouvillian differential extensions of the base field.

4.2. Liouvillian extensions with several derivations

By analogy with the ordinary differential case, Singer [19] defined the notion of a liouvillian extension of a differential field K when K has several commuting derivations d_i.

This definition still relies on the construction of a tower with finitely many levels.

Let $(K, \{d_i\})$ be a differential field with several commuting derivations $\{d_i\}$.

A differential extension field $(L, \{d_i\})$ is said to be *liouvillian* if there exists a finite tower of intermediate differential fields

$$(K_0, \{d_i\}) = (K, \{d_i\}) \subset (K_1, \{d_i\}) \subset \cdots \subset (K_n, \{d_i\}) = (L, \{d_i\}),$$

such that each K_i is generated over the previous one K_{i-1} by a single element θ_i which is an liouvillian generator, i.e.,

either θ_i is algebraic over K_{i-1}, in which case all derivations d_i on K_i are uniquely defined (and they still commute with one another),

or θ_i is transcendental and an exponential-integral over K_{i-1}, which means that there exists elements η_j in K_{i-1} such that d_j is defined on K_i by $d_j(\theta_i) = \theta_i \eta_j$,

or θ_i is transcendental and a integral over K_{i-1}, which means that there exists elements η_j in K_{i-1} such that d_j is defined on K_i by $d_j(\theta_i) = \eta_j$.

In these last two cases, elements η_j have to satisfy the property of the cross-derivatives, $d_k(\eta_j) = d_j(\eta_k)$, in K_{i-1}.

4.3. 3-dimensional polynomial vector fields

Let $V = V_x \partial_x + V_y \partial_y + V_z \partial_z$ be a polynomial homogeneous vector field, which means that its coordinates are homogeneous polynomials of the same degree in the three space variables x, y, z with coefficients in a given field of constants C.

The base field of the construction is then $K = C(x, y, z)$, the field of rational fractions. This field K is then a differential field for derivations ∂_x, ∂_y and ∂_z.

Due to the homogeneity of the problem, it seems natural to look for liouvillian first integrals f that are moreover homogeneous of degree 0.

According to Euler's formula in the case of polynomials, f will be said to be homogeneous of degree 0 if it is a first integral of Euler's field $E = x\partial_x + y\partial_y + z\partial_z$, i.e. if $x\partial_x f + y\partial_y f + z\partial_z f = 0$.

It can be shown [9] that such a homogeneous polynomial vector field V has a liouvillian first integral of this kind if and only if there exists some 1-form $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ with coordinates in the polynomial ring $C[x, y, z]$ with the following properties:

ω is orthogonal to the given field V, i.e. $V_x\omega_x + V_y\omega_y + V_z\omega_z = 0$,

ω is not projective, i.e. $x\omega_x + y\omega_y + z\omega_z \neq 0$,

ω is integrable, i.e. satisfies Pfaff's condition $\omega \wedge d\omega = 0$, where $d\omega$ is the exterior derivative of ω.

A similar result without 1-forms can be found in [18].

As in Liouville's principle or in Kovačic's theorem, this means that if a solution is in a prescribed class (here the first integral f belongs to a liouvillian extension of K) it has to be of a very special form (here f is built from ω which has very special properties).

4.4. *Darboux curves*

The notion of Darboux functions is closely related to this problem and dates back to a memoir by Darboux [3].

A homogeneous polynomial f is said to be a Darboux polynomial of the homogeneous polynomial vector field V if the polynomial $V_x\partial_x f + V_y\partial_y f + V_z\partial_z f$ is a multiple of f:

$$V_x\partial_x f + V_y\partial_y f + V_z\partial_z f = \Lambda f,$$

where Λ is some homogeneous polynomial in $K[x, y, z]$.

This is an algebraic translation of an analytic fact: the subset where $f = 0$ is invariant under the action of the local semi-group generated by V.

Irreducible factors of a Darboux function are Darboux functions and we will call them Darboux curves because they define plane projective algebraic curves.

Relations of Darboux functions with our integrability result are the following.

If ω is a good 1-form in the sense of the previous result, the inner product $x\omega_x + y\omega_y + z\omega_z$ is a Darboux function.

On the other hand, if there are sufficiently many Darboux curves [3, 18], it is easy to build a good 1-form ω from a linear combination of the logarithmic derivatives of these functions.

These relations can then be used in two different ways. In one direction, finding sufficiently many Darboux curves leads to an integrability proof. In the other direction, proving the nonexistence of Darboux curves excludes integrability.

4.5. *A result of Jouanolou and an effectiveness problem*

Jouanolou considers the general homogeneous polynomial 3-dimensional vector field of a given degree m [7]. He proves that such a field is generically not integrable (here the constant field is the field $\mathbb{C}$ of complex numbers and genericity refers to Baire category classification).

To get this result, as we have just said, it suffices to prove that a generic vector field has no Darboux curve.

Moreover, due to simple algebraic remarks about Zariski closed sets, it is even sufficient to prove that there exists some vector field without any Darboux curve.

For a given $m \geqslant 2$, Jouanolou chooses the special vector field

$$V_m = z^m\partial_x + x^m\partial_y + y^m\partial_z$$

and proves that this vector field V_m has no Darboux curve.

His proof is very clever and uses subtle arithmetic properties.

In a recent work with Andrzej Nowicki and Jean-Marie Strelcyn [10], we gave some development of these ideas in more than 3 dimensions in particular.

A difficult algorithmic question, that dates back to some papers of Poincaré [11–13], remains open: effectively decide if a given vector field in integrable or not, in the previous sense, or at least, decide if there are sufficiently many Darboux curves. Given the degree, it is not too difficult, in principle and with the help of computer algebra systems, to decide if there is a Darboux function of a given vector field with this prescribed degree: it is a matter of algebraic elimination. But there is no known reason in general to give a bound to the degree of the candidate functions.

In order to get some insight in this decision problem, we studied a special family of 3-dimensional quadratic vector fields that are called factorisable. The last section of this paper is devoted to this study.

5. Integrability of factorisable vector fields

5.1. Factorisable vector fields

Consider 3-dimensional quadratic vector fields of the following form:

$$V = x\phi_x\partial_x + y\phi_y\partial_y + z\phi_z\partial_z,$$

where ϕ_x, ϕ_y and ϕ_z are linear forms i.e. first degree homogeneous polynomials in x, y and z (in [10] we only asked that the ϕ_i are homogeneous polynomials of the same degree, not necessarily 1).

Such vector fields are called factorisable. Sonia Kovalewska was apparently the first to consider these vector fields and to ask when they are integrable.

A factorisable vector field is characterized by 9 parameters, the coefficients of the three linear forms.

It is possible to do some reductions in order to restrict the problem to a 3-parameter one by only considering vector fields $V_{A,B,C}$, where

$$V_x = x(Cy + z), \quad V_y = y(Az + x), \quad V_z = z(Bx + y).$$

In this form, we give the vector field the name of Lotka-Volterra, because the corresponding system of autonomous differential equations appears in the study of a predator-prey system by Lotka and Volterra.

The first argument for the previous reduction consists in the following remark. We are interested in finding liouvillian first integrals homogeneous of degree 0 for a factorisable vector field. The problem is then invariant by addition of a multiple of Euler's field to our field, in such a way that we can choose canonical elements of the corresponding equivalence classes: factorisable vector fields for which the "diagonal" coefficients $\phi_{xx}, \phi_{yy}, \phi_{zz}$ are 0. We thus restrict ourselves to a six-parameter problem.

A second argument can be used to reduce the space of parameters: our problem is invariant under the action of the (3-dimensional) diagonal group of linear changes of variables. By only excluding some rare cases it is then possible to choose the previous Lotka-Volterra normal form with only three parameters.

The problem is then to study for which triples of parameters (A,B,C), the vector field $V_{A,B,C}$ has a good 1-form ω.

5.2. *Integrability results for Lotka-Volterra vector fields*

Let us call $K = \mathbb{Q}(A,B,C)$ the field of constants. A good 1-form ω exists if and only if we can find a closed 1-form $\overline{\omega}$ with coefficients in $K(x,y,z)$ which is orthogonal to V. Indeed, we can choose $\overline{\omega} = \omega/P$, where polynomial $P = x\omega_x + y\omega_y + z\omega_z$. In the other direction, we get ω from $\overline{\omega}$ by a multiplication and, as $\overline{\omega}$ is closed, its multiples satisfy Pfaff's integrability condition.

By the very definition of factorisable vector fields, curves x, y and z are Darboux curves. We will say that a factorisable vector field has a fourth Darboux curve if there exists a Darboux function f which is not a multiple of x, y or z. In such a case, we get four Darboux functions x, y, z and f with the eigenvalues $Cy + z$, $Az + x$, $Bx + y$ and $\Lambda = \lambda_x x + \lambda_y y + \lambda_z z$.

The four linear forms cannot be linearly independent and some non-trivial linear combination with coefficients in K, $(\alpha(Cy+z)+\beta(Az+x)+\gamma(Bx+y)+\delta\Lambda)$ is equal to 0.

Then $(\alpha(dx/x) + \beta(dy/y) + \gamma(dz/z) + \delta(df/f))$ is the sought closed 1-form $\overline{\omega}$ and the vector field is integrable.

Careful computations with a specially designed computer algebra program allowed us to find all values of (A,B,C) for which such a fourth Darboux curve exists up to degree 6. There are general families of such triples and also exceptional ones that seem to be of a special arithmetic type.

We were able to state a genericity result for which we give some details below: generically, $V_{A,B,C}$ has no fourth Darboux curve.

As we have seen above, a fourth Darboux curve is a sufficient reason to have a liouvillian first integral but this reason is not necessary.

Another way to build such a liouvillian first integral is to find some nonconstant rational fraction $F = N/D$ where the denominator D is a product of powers of the coordinates, where the numerator and denominator are homogeneous of the same degree and where $V_x\partial_x F + V_y\partial_y F + V_z\partial_z F$ is a first degree homogeneous polynomial.

A good 1-form can then be built as a linear combination of the exterior derivative dF of F with the logarithmic derivatives of the coordinate functions.

There are examples of triples (A,B,C) of parameters (for instance $(A,B,C) = (-1, 1/2, 0)$) for which no fourth Darboux curve exists while a substitute fraction can be used to prove integrability [9]. This show that the original method of Darboux is not the only one to get liouvillian first integrals.

It is also possible to state a genericity result in this case: generically, $V_{A,B,C}$ has no substitute fraction. We give the details below.

It is not difficult to show that a fourth Darboux curve or a substitute fraction are the only two possibilities to build a good 1-form. Thus, generically, a factorisable vector field has no homogeneous liouvillian first integral of degree 0.

5.3. Generic absence of a fourth Darboux curve

The announced result of the generic absence of a fourth Darboux curve for factorisable 3-dimensional polynomial vector fields refers to the Baire category classification of "small" and "big" subsets of $\mathbb{R}^k$ or $\mathbb{C}^k$.

This result relies on the two following propositions.

Proposition 5.1. *Let $V = V_{A,B,C}$ be a factorisable vector field in Lotka-Volterra normal form. Then, if A, B and C are not rational numbers, a fourth Darboux function f for V has to be a polynomial first integral, i.e. the eigenvalue Λ has to be 0.*

Proof. Let f be some Darboux function for V of degree m and suppose that f in not a multiple of x, y or z. This means that the three 2-variable homogeneous polynomials of degree m, $P(y,z)$, $Q(x,z)$ and $R(x,y)$ obtained by setting x, y or z to 0 in f, are different from 0.

Identity

$$x(Cy+z)f_x + y(Az+x)f_y + z(Bx+y)f_z = (\lambda x + \mu y + \nu z)f$$

yields

$$\begin{aligned} AyzP_y + yzP_z &= (\mu y + \nu z)P, \\ xzQ_x + BxzQ_z &= (\lambda x + \nu z)Q, \\ CxyR_x + xyR_y &= (\lambda x + \mu y)R. \end{aligned}$$

Consider now the terms of highest degree in the variable x in the last two equations. The second equation shows that λ is an integral multiple of B while the third shows that λ is an integer. As B is not a rational number, λ has to be 0. The same is true in what concerns μ and ν, which achieves the proof of the proposition. $\square$

Proposition 5.2. *Let $V = V_{A,B,C}$ be a factorisable vector field in Lotka-Volterra normal form. If f is a polynomial first integral of V which is not a multiple of x, y or z, then $(-ABC)^m = 1$, where m is the degree of f.*

Proof. With the notations of the above proposition, we get

$$\begin{aligned} AyzP_y + yzP_z &= 0, \\ xzQ_x + BxzQ_z &= 0, \\ CxyR_x + xyR_y &= 0. \end{aligned}$$

It can easily be shown that P has then to be some multiple of $(y - Az)^m$; in the same way, Q is a multiple of $(z - Bx)^m$ and R a multiple of $(x - Cy)^m$.

Comparing now the coefficients of x^m, y^m and z^m in f yields $(-ABC)^m = 1$. ☐

Now, if the triple (A, B, C) of complex parameters belongs to the countable intersection of the Zariski open dense subsets of $\mathbb{C}^3$ given by conditions

$$A \notin \mathbb{Q}, \quad B \notin \mathbb{Q}, \quad C \notin \mathbb{Q}, \quad \forall m \in N(-ABC)^m \neq 1,$$

the corresponding factorisable vector field $V_{A,B,C}$ does not admit any fourth Darboux curve.

5.4. *Generic absence of a substitute fraction*

The purpose of this section is to prove the following result: the set of all triples (A, B, C) of parameters for which there is no substitute fraction contains the intersection of a countable family of Zariski dense open subsets of $\mathbb{C}^3$.

This substitute fraction would write

$$F = \frac{f}{x^\alpha y^\beta z^\gamma},$$

where α, β and γ are natural numbers and f is an homogeneous polynomial of degree $\alpha+\beta+\gamma$ such that $V_x F_x + V_y F_y + V_z F_z = \Lambda$, where Λ is some linear form $\lambda x + \mu y + \nu z$. Moreover, f is not proportional to the denominator $x^\alpha y^\beta z^\gamma$.

This leads to the fact that the polynomial

$$x(Cy + z)f_x + y(Az + x)f_y + z(Bx + y)f_z$$

$$-f(\alpha(Cy + z) + \beta(Az + x) + \gamma(Bx + y))$$

is equal to some multiple $\Lambda x^\alpha y^\beta z^\gamma$ of $x^\alpha y^\beta z^\gamma$.

For every triple of natural numbers (α, β, γ), the existence of such an f is a linear algebra problem in the coefficients of the unknown polynomial f of degree $\alpha + \beta + \gamma$ (setting to 0 some linear combinations).

Classical elimination by determinants shows that this problem has a non-trivial solution if and only if (A, B, C) belongs to some Zariski closed subset of $\mathbb{C}^3$.

Either a Zariski closed set is the whole space or its interior is empty. To conclude the proof, it remains to show that none of these sets (depending on (α, β, γ)) is the whole space $\mathbb{C}^3$.

At that point, our proof is similar to Jouanolou's proof: it suffices to find one triple of parameters for which no substitute fraction exists.

Choose $(A, B, C) = (0, 0, 0)$, i.e. $V_x = xz, V_y = yx, V_z = zy$. A substitute fraction F would be of the form $F = f/x^\alpha y^\beta z^\gamma$, with for instance, $\alpha \neq 0$ so that f would not be divisible by x.

Now call P the homogeneous 2-variable non-zero polynomial obtained by setting $x = 0$ in f. This polynomial P would satisfy identity

$$zyP_z - P(\alpha z + \gamma y) = 0.$$

In order to balance terms of maximal degree in z, α has to be 0, which is a contradiction that achieves the proof.

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