Bull. Sci. math.
1995, 119, p. 195-233.

ON THE NON-EXISTENCE OF CONSTANTS OF DERIVATIONS: THE PROOF OF A THEOREM OF JOUANOLOU AND ITS DEVELOPMENT (*)

BY

JEAN MOULIN OLLAGNIER ([1, 3]),
ANDRZEJ NOWICKI (†) and JEAN-MARIE STRELCYN ([2, 3])

RÉSUMÉ. – En nous inspirant de la démonstration d'un théorème de non-intégrabilité de J.-P. JOUANOLOU, nous décrivons une méthode générale pour prouver l'absence de constantes non-triviales pour certaines dérivations dans des anneaux de polynômes $K[x_1, \ldots, x_n]$, où K est un corps de caractéristique 0.

Le problème plus difficile d'établir l'absence de constantes non-triviales pour des dérivations du corps quotient $K(x_1, \ldots, x_n)$ est également résolu dans certains cas.

Lorsque le corps K est $\mathbf{R}$ ou $\mathbf{C}$, cela revient à démontrer l'absence d'intégrales premières polynomiales ou rationnelles pour des systèmes d'équations différentielles ordinaires polynomiales.

Nous décrivons en détail quelques exemples, parmi lesquels celui de JOUANOLOU.

(*) Manuscript presented by J.-P. FRANÇOISE, received September 1993, revised December 1993.

([1]) GAGE, Centre de Mathématiques, Unité associée CNRS 169, École Polytechnique, 91128 Palaiseau Cedex, France.

(†) Nicolaus Copernicus University, Institute of Mathematics, ul. Chopina 12-18, 87-100 Toruń, Poland.

([2]) Université de Rouen, Département de Mathématiques, Unité associée CNRS 1378, 76821 Mont-Saint-Aignan Cedex, France.

([3]) Laboratoire Analyse, Géométrie et Applications, Unité associée CNRS 742, Institut Gallilée, Université Paris-Nord, avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

BULLETIN DES SCIENCES MATHÉMATIQUES – 0007-4497/95/03/$ 4.00

ABSTRACT. – Inspired by the proof of JOUANOLOU's non-integrability theorem, we describe a method for proving the non-existence of non-trivial constants of some derivations in polynomial rings $K[x_1, \ldots, x_n]$, where K is a field of characteristic 0.

In some cases, even the non-existence of non-trivial constants of derivations in quotient field $K(x_1, \ldots, x_n)$ can be proved.

When $K = \mathbf{R}$ or $\mathbf{C}$, this is equivalent to proving the non-existence of polynomial, respectively rational, first integrals for some systems of polynomial ordinary differential equations. Several examples, among which JOUANOLOU's one, are described in details.

1. Introduction

1.1. THE PROBLEM

The problem studied in this paper is rooted in the classical theory of ordinary differential equations and in the classical mechanics where the search of first integrals is one the main tools of investigation.

Let us consider a system of polynomial ordinary differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n), \qquad 1 \leq i \leq n \tag{1.1}$$

in which all f_i belong to the polynomial ring $K[x_1, \ldots, x_n]$ in n variables where K is either $\mathbf{R}$ or $\mathbf{C}$.

A non-constant element ϕ of $K[x_1, \ldots, x_n]$ is said to be a *first integral* of the system (1.1) if the following identity holds

$$\sum_{i=1}^{n} f_i \frac{\partial \phi}{\partial x_i} = 0 \tag{1.2}$$

It is well-known and easy to be proved that ϕ satisfies (1.2) if, and only if, ϕ is constant on the orbits of system (1.1) of ordinary differential equations.

Given an arbitrary field K, the mapping d from the polynomial ring $K[x_1, \ldots, x_n]$ to itself defined by

$$d(\psi) = \sum_{i=1}^{n} f_i \frac{\partial \psi}{\partial x_i}. \tag{1.3}$$

is not only K-linear but also satisfies Leibnitz's rule

$$\forall\, \alpha, \beta \in K[x_1, \ldots, x_n], \quad d(\alpha\beta) = d(\alpha)\,\beta + \alpha\, d(\beta).$$

In the vocabulary of differential algebra, d is a *K-derivation* or simply a derivation of the polynomial ring $K[x_1, \ldots, x_n]$ and identity (1.2) means that ϕ belongs to the kernel of d, *i.e.* that ϕ is a non-trivial *constant of derivation* d.

Let us note that the derivation d is completely defined by its values on the x_i, that are generators of the K-algebra $K[x_1, \ldots, x_n]$,

$$d(x_i) = f_i(x_1, \ldots, x_n), \qquad 1 \le i \le n \tag{1.4}$$

and by the fact that it is equal to 0 on K.

Although we are first of all interested in differential equations, one of our aims is to consider a more general field K instead of $\mathbf{R}$ or $\mathbf{C}$, or even some commutative rings.

At a first glance, it seems feasible to look for a polynomial solution ϕ of a given degree p of equation (1.2) by the method of "indeterminate coefficients". A polynomial ϕ of degree p in $K[x_1, \ldots, x_n]$ can indeed be written

$$\phi(x_1, \ldots, x_n) = \sum\nolimits_{0 \le i_1 + \cdots + i_n \le p} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}, \tag{1.5}$$

so that the right-hand side of equation (1.5) can be substituted to ϕ in equation (1.2).

All that leads to a linear system $\mathcal{L}(p)$ for the unknowns $\{a_{i_1, \ldots, i_n}\}$. In principle, for a given p, it is possible to write down the system $\mathcal{L}(p)$ and to solve it; but, finding a general rule to get $\mathcal{L}(p)$ for an arbitrary p is much more difficult.

In fact, the general problem of the existence of non-trivial (*i.e.* that do not belong to K) constants of derivations is known to be difficult and so is the corresponding problem of the existence of non-trivial (*i.e.* non-constant) first integrals of systems like (1.1). We are faced with the lack of a general method and have to deal with examples case by case.

Given a ring R endowed with a derivation d, *i.e.* an additive mapping from R to R that satisfies Leibnitz's rule, the subring of its constants is denoted by R^d:

$$R^d = \{a \in R, \ d(a) = 0\}.$$

Recall that when R is without zero divisors, the derivation d can be extended in a unique way to its quotient field by setting

$$d\left(\frac{P}{Q}\right) = \frac{d(P)\,Q - P\,d(Q)}{Q^2}.$$

We shall use the above notations for the ring $K[x_1, \ldots, x_n]$ and its quotient field $K(x_1, \ldots, x_n)$, the field of all rational functions in n indeterminates with coefficients in K:

$$K[x_1, \ldots, x_n]^d = \{P \in K[x_1, \ldots, x_n],\, d(P) = 0\}$$
$$K[x_1, \ldots, x_n]^d = \{f \in K[x_1, \ldots, x_n],\, d(f) = 0\}.$$

Although all results of this paper are formulated and proved for $\mathbf{C}$-derivations, they remain valid if instead of $\mathbf{C}$ one considers an arbitrary field K of characteristic zero or even an arbitrary commutative ring without zero divisors which contains $\mathbf{Z}$ as a subring. This will be proven in Section 7.

1.2. Jouanolou's theorem

In Chapter 4 of his fundamental book [8], J.-P. Jouanolou gives the following beautiful non-integrability result.

Theorem 1.1. – *Let $s \geq 2$ be a natural number and let d be the $\mathbf{C}$-derivation from $\mathbf{C}[x, y, z]$ to itself defined by*

$$(1.6) \qquad d(x) = z^s, \qquad d(y) = x^s, \qquad d(z) = y^s.$$

Then, for every polynomial P in $\mathbf{C}[x, y, z]$, the following equation

$$(1.7) \qquad d(F) = PF$$

does not admit a non-trivial solution F in $\mathbf{C}[x, y, z]$. In particular, the field of constants $\mathbf{C}(x, y, z)^d$ reduces to $\mathbf{C}$, or equivalently, the system of differential equations

$$(1.8) \qquad \frac{dx}{dt} = z^s, \qquad \frac{dy}{dt} = x^s, \qquad \frac{dz}{dt} = y^s$$

does not admit any non-trivial rational first integral.

The theorem would fail for $s = 1$; the subfield of constants does not reduce to $\mathbf{C}$ as $x^3 + y^3 + z^3 - 3\,xyz$ for instance is a constant of d. Moreover, in this case, equation (1.7) has very simple solutions with $P \neq 0$; for example, $P = 1$ and $F = x + y + z$.

Let us note that the passage from non-solvability of equation (1.7) to the triviality of the subfield of constants $\mathbf{C}(x,\ y,\ z)^d$ is very easy. Consider indeed an element P/Q of $\mathbf{C}(x,\ y,\ z)^d$ *i.e.* a constant of derivation d. Without lost of generality, polynomials P and Q can be supposed to be relatively prime and $d\,(P/Q) = 0$ writes $d\,(P)\,Q = d\,(Q)\,P$ so that there exists a polynomial G in $\mathbf{C}(x,\ y,\ z)$ such that $d\,(P) = GP$ and $d\,(Q) = GQ$. By Jouanolou's theorem, P and Q are constants and P/Q belongs to $\mathbf{C}$.

In fact we are here in presence of the completely general statement concerning any $\mathbf{C}$-derivation $d:\ \mathbf{C}[x_1,\ \ldots,\ x_n] \to \mathbf{C}[x_1,\ \ldots,\ x_n]$. If F and G are non-zero relatively prime polynomials from $\mathbf{C}[x_1,\ \ldots,\ x_n]$ then

$$(1.9)\qquad \begin{cases} d\,(F/G) = 0 \\ \text{if and only if} \\ d\,(F) = PF \quad\text{and}\quad d\,(G) = PG, \end{cases}$$

for some $P \in \mathbf{C}[x_1,\ \ldots,\ x_n]$.

Consider now equation (1.7) for $P = 0$, *i.e.* try to find some non-constant polynomial, that will be a first integral of system (1.8). At the present time, we do not know any direct proof of the fact that no such first integral does exist, even for the most simple case $s = 2$. The remarks of the previous section about intrinsic difficulties of the computations of the linear system $\mathcal{L}\,(p)$ of equations are fully confirmed here. In particular, we have to make use of computer algebra to write down $\mathcal{L}\,(10)$ and no general rule for $\mathcal{L}\,(p)$ appears.

In what concerns non-solvability of equation (1.7), the direct proof for second degree polynomials F is already astonishingly long and complicated.

1.3. On the proof of Jouanolou's theorem

In Jouanolou's book, two different proofs of his theorem are given. The first one, described on pages 160-192, is due to Jouanolou and the second one, sketched on pages 193-195, is due to the referee of the book. Both of them essentially use some elementary facts from algebraic geometry in their conclusion.

Trying to understand the second proof, we have gradually realized that the starting point of it relies on some very clever and general ideas, which can be applied to many other derivations, mainly but not exclusively to derivations where polynomials f_i (*see* 1.4) are homogeneous polynomials of the same degree.

This class of derivations is already very large and far from being understood and the same is true in what concerns the corresponding systems (1.1) of ordinary differential equations.

In fact, all arguments used here apply without any significant change to the larger class of derivations where all polynomials f_i are quasi-homogeneous of the same appropriate degree.

The first place where a link between algebraic geometry and the search of first integrals or equivalently with the search of constants of derivations has been made, seems to be the famous memoir [2] by DARBOUX in which non-trivial solutions of equation (1.7) are the main tool of investigation. *See* also the POINCARÉ's papers [15]-[17] related to DARBOUX's ideas. It will not therefore be surprising to find relations with some of DARBOUX's ideas in the proof under consideration.

The second proof of Jouanolou's theorem is unfortunately written in an extremely concise way and there is a gap at the end of it: in fact, the conclusion only holds for a natural integer $s > 1$ that satisfy $s \neq 1$ (mod 3). Nevertheless, the proof is complete in the crucial case $s = 2$.

1.4. ORGANIZATION OF THE PAPER

The aim of the present paper is twofold: first, we give a complete proof of Jouanolou's theorem together with a detailed discussion of all its steps; second, we show on examples how some of the ideas, on which this proof is based, can be used to derive the non-existence of non-trivial constants of derivations.

More precisely, the proof under consideration divides in two parts, the "local analysis", which is fairly general and the "global analysis" which relies on elementary algebraic geometry and is very specific to Jouanolou's example.

This is a remarkable fact that in many non-trivial examples, local analysis is sufficient to yield the non-existence of non-trivial constants of derivations.

The paper is then organized as follows: in Section 2 all general notations and useful facts are presented, Section 3 consists in a detailed description of the proof of Jouanolou's theorem.

In Sections 4-6, we consider three multidimensional examples for which local analysis is a sufficient tool to derive the non-existence of a non-trivial constant of derivation in $\mathbf{C}[x_1, \ldots, x_n]$ or even in $\mathbf{C}(x_1, \ldots, x_n)$.

In the vocabulary of differential equations, this means the non-existence of polynomial, or even rational, first integrals for the corresponding systems of ordinary differential equations.

The short Section 7 contains an extension of the results of this paper from the field $\mathbf{C}$ to the general case of fields of characteristic zero or even to some rings.

Let us finally underline that, as we planned to make this paper self-contained and intended for a wide audience, only a standard mathematical background is required.

Some examples from this paper were presented by the third author in March 1992 at Dynamical Systems Seminar of the Mathematical Department of Warsaw University. During this seminar, the problem of finding a more analytico-geometrical proof of Jouanolou's theorem was formulated.

Recently, one of the participants to this seminar, H. Żołądek, gave in [19] such a proof. As we learnt form this paper two more proofs of Jouanolou's theorem were given in [1] and [12].

2. Preliminaries

2.1. Homogeneity and Darboux polynomials

One of the main tools in our investigations is the well-known Euler's theorem on homogeneous functions ([3], [4]): if Q is an homogeneous polynomial of degree $s \geq 1$ in $\mathbf{C}[x_1, \ldots, x_n]$, then

$$\sum_{i=1}^{n} x_i \frac{\partial Q}{\partial x_i} = s\, Q. \tag{2.1}$$

Let now $V_1, \ldots, V_n$ be n homogeneous polynomials of the same degree s in $\mathbf{C}[x_1, \ldots, x_n]$ and consider the derivation d_V defined by

$$d_V(x_i) = V_i; \qquad 1 \leq i \leq n. \tag{2.2}$$

If polynomial F is a constant of d_V in $\mathbf{C}[x_1, \ldots, x_n]$ *i.e.* satisfies

$$d_V(F) = \sum_{i=1}^{n} V_i \frac{\partial F}{\partial x_i} = 0, \tag{2.3}$$

then this identity also holds for all homogeneous components of F. Thus, when studying the equation (2.3), without any restriction of generality, one can suppose that the unknown non-constant polynomial solution F of equation (2.3) is a homogeneous polynomial of some degree $m \geq 1$.

In fact, we shall be interested by the more general equation

$$d_V(F) = \sum_{i=1}^{n} V_i \frac{\partial F}{\partial x_i} = PF \tag{2.4}$$

in which F is an unknown polynomial of some degree $m \geq 1$, while the "eigenvalue" P is some unknown element of $\mathbf{C}[x_1, \ldots, x_n]$. In fact P is an eigenvalue of the linear differential operator d_V for which F is an eigenvector.

Let us now make precise some notions that date back to DARBOUX's memoir [2]. We are of course responsible for the names given to these notions.

A non-trivial solution F of equation (2.4) will be called a *Darboux polynomial* of derivation d_V and the algebraic hypersurface $\{F = 0\}$ in $\mathbf{C}^n$ a *Darboux manifold.*

When F is non-constant and homogeneous, then instead of $\mathbf{C}^n$ one considers the Darboux manifold $\{F = 0\}$ in the projective space $\mathbf{P}^{n-1}(\mathbf{C})$. In this case, when $n = 3$, Darboux manifolds are called *Darboux curves*. Let us remark that the notion of a Darboux polynomial is still meaningful if the polynomials $V_1, \ldots, V_n$ are not homogeneous.

Darboux polynomials with $P \neq 0$ are well-known in the theory of polynomial differential equations; they coincide with the so-called *partial first integrals:* although F is not a first integral of the vector field

$V = (V_1, \ldots, V_n)$, even if $P \neq 0$, the subset of $\mathbf{C}^n$ where $F = 0$ consists of full orbits of the system (1.1).

The homogeneity of polynomials $V_1, \ldots, V_n$ together with the fact that they are of the same degree, has the following consequence.

LEMMA 2.1. – *Consider the derivation d_V defined by* (2.2) *and its Darboux polynomial F which satisfies* (2.4). *Then P is homogeneous and all homogeneous components of F also satisfy* (2.4).

Proof. – If equation (2.4) is satisfied, then

$$d_V(F^+) = \sum\nolimits_{i=1}^{n} V_i \frac{\partial F^+}{\partial x_i} = P^+ F^+, \tag{2.5}$$

where G^+ denotes the homogeneous component of the highest degree of the polynomial G.

Let us note that if the equation (2.4) is satisfied then also

$$d_V(F^-) = \sum\nolimits_{i=1}^{n} V_i \frac{\partial F^-}{\partial x_i} = P^- F^-, \tag{2.6}$$

where G^- denotes the homogenous component of the lowest degree of G.

Let us write now

$$P = \sum\nolimits_{i=\nu}^{\nu+k} P_i, \quad k \geq 0, \qquad \text{and} \qquad F = \sum\nolimits_{i=\mu}^{\mu+l} F_i, \quad l \geq 0,$$

the homogeneous decomposition of P and F respectively. We suppose that $P_\nu \neq 0$, $P_{\nu+k} \neq 0$, $F_\mu \neq 0$ and $F_{\mu+l} \neq 0$.

Comparing the degrees of both sides of the equalities (2.5) and (2.6) one obtains that

$$s - 1 + \mu + l = (\nu + k) + (\mu + l) \quad \text{and that } s - 1 + \mu = \nu + \mu$$

respectively. Consequently $k = 0$ and thus P is homogeneous.

Now our assertion is evident. ■

Then when proving the non-existence of the non-trivial solution F of the equation (2.4), without any restriction of generality one can suppose F and P to be homogeneous.

Let us note that even for the derivation d_V defined by (2.2), its Darboux polynomials are not necessarily homogeneous. Indeed, let $n = 2$ and let $d(x_1) = x_1$, $d(x_2) = 2\,x_2$. Then $d(F) = 2\,F$ for $F = x_1^2 + x_2$.

Let us also note that for $n = 2$ and for any derivation d_V defined by (2.2) with V_1, V_2 homogeneous polynomials of the same degree, a Darboux polynomial always exists. Indeed, when $F = x_1\,V_2 - x_2\,V_1 \neq 0$ than $d_V(F) = PF$, with

$$P = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2}.$$

When $F = 0$, then it is easy to see that $d_V(x_1 - x_2) = (x_1 - x_2)\,g$ for some $g \in \mathbf{C}[x_1,\ x_2]$.

Thus in future, when studying the non-existence of Darboux polynomials, we will only consider the case $n \geq 3$.

Independently of homogeneity conditions, Darboux polynomials enjoy the following stability property.

LEMMA 2.2. – *Let F be a Darboux polynomial of the derivation d defined by* (1.3). *Then all factors of F are also Darboux polynomials of d.*

Proof. – First, let G be an irreducible factor of polynomial F, which writes $F = G^\alpha\,H$, where G and H are relatively prime in $\mathbf{C}[x_1,\ \ldots,\ x_n]$ and α is a strictly positive integer. Let P be the eigenvalue corresponding to F:

$$d(F) = \alpha\,G^{\alpha-1}\,H\,d(G) + G^\alpha\,d(H) = PF = PG^\alpha\,H.$$

As G and H are relatively prime, G must divide $d(G)$, *i.e.* G is a Darboux polynomial of d.

Now, as a product of Darboux polynomials is also a Darboux polynomial, every factor of F is a Darboux polynomial of d. ■

Thus, looking for non-trivial Darboux polynomials of a given derivation d reduces to looking for irreducible ones. Moreover, if a rational function which a first integral of derivation d is written as a quotient of two relatively prime polynomials, then all irreducible factors of its numerator and denominator are Darboux polynomials of d.

2.2. DEGREE AND MULTIPLICITIES OF PLANE ALGEBRAIC CURVES

In the two-dimensional projective case, irreducible homogeneous polynomials (in three variables) define plane algebraic curves. The multiplicities of a curve at various points of $\mathbf{P}^2(\mathbf{C})$ and the degree of the curve are related by an inequality which plays a crucial role in the second part of the proof of Jouanolou's theorem.

Let us now make the definition precise and state this inequality.

Let F be a homogeneous irreducible polynomial of degree m in $\mathbf{C}[x, y, z]$.

Take some point M of the projective plane $\mathbf{P}^2(\mathbf{C})$ and let (a, b, c) be a representation of M in homogeneous coordinates. In order to define the multiplicity of F at M, we have to choose local affine coordinates; without lost of generality, we can assume that $c \neq 0$ and that it can be set to 1.

Denote then by f the (non-homogeneous) two-variable irreducible polynomial defined by $f(x, y) = F(x, y, 1)$. Polynomial f is not 0 and its degree is at most m, the degree of F.

Consider now the Taylor's development of f around point (a, b):

$$f = \sum_{i=0}^{m} h_i(x-a, y-b),$$

where each h_i is an homogeneous two-variable polynomial of degree i.

Let μ be the lowest degree i for which h_i is not 0; this natural number does not depend on the choice of local affine coordinates, but only on polynomial F and point M. Thus, it can be written $\mu_M(F)$ and defined as the *multiplicity* of F at M.

The multiplicity is strictly positive ($\mu_M(F) > 0$) iff $F(M) = 0$ *i.e.* if curve $\{F = 0\}$ passes through point M. Points at which the multiplicity of a given F is 1 are the ordinary points of the curve and those where $\mu_M(F) > 1$ are multiple points of it. It is a well-known fact that an irreducible curve has only finitely many multiple points in the projective plane $\mathbf{P}^2(\mathbf{C})$ (*see* for instance [5], p. 69).

Moreover, if F and G are relatively prime homogeneous polynomials in $\mathbf{C}[x, y, z]$, the set of their common zeroes in $\mathbf{P}^2(\mathbf{C})$ is finite. More precisely, according to a theorem due to BÉZOUT (*see* [5], p. 112),

$$\sum_{M \in \mathbf{P}^2(\mathbf{C})} \mu_M(F)\,\mu_M(G) \leq \deg(F)\deg(G).$$

This result applies to polynomial F together with one of its non-zero partial derivatives to yield

$$(2.7) \qquad \sum_{M \in \mathbf{P}^2(\mathbf{C})} \mu_M(F)\,(\mu_M(F) - 1) \leq \deg(F)\,(\deg(F) - 1).$$

In fact, a stronger inequality holds for an irreducible homogeneous polynomial F (*see* [5] p. 117):

$$(2.8) \quad \sum_{M \in \mathbf{P}^2(\mathbf{C})} \mu_M(F)(\mu_M(F)-1) \leq (\deg(F)-1)(\deg(F)-2).$$

Strangely enough, inequality (2.7) is not sufficient to conclude the proof of Jouanolou's theorem; we need the full strength of inequality (2.8).

2.3. Darboux points

Let us now resume our study of equation (2.4); we are looking for an homogeneous irreducible non-trivial polynomial F of some degree m and an homogeneous polynomial P of degree $s-1$ such that equations (2.4) and (2.1) hold:

$$(2.9) \qquad d_V(F) = \sum_{i=1}^{n} V_i \frac{\partial F}{\partial x_i} = PF,$$

$$(2.10) \qquad \sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} = mF.$$

Adding the product of equation (2.9) by x_n and the product of equation (2.10) by $(-V_n)$, we get an equation in which the partial derivative of F with respect to the last variable x_n no longer appears:

$$(2.11) \qquad \sum_{i=1}^{n-1} (x_n V_i - x_i V_n) \frac{\partial F}{\partial x_i} = (x_n P - m V_n) F.$$

According to Euler's formula (2.10), equation (2.9) and (2.11) are in fact equivalent for homogeneous polynomials F of degree m.

A point $Z \in \mathbf{P}^{n-1}(\mathbf{C})$ will be called a *Darboux point* of derivation d_V if vector $V(z) = (V_1(z), \ldots, V_n(z))$ is proportional to vector $z = (z_1, \ldots, z_n)$ for every system z of homogeneous coordinates of Z.

Let then Z be a Darboux point of derivation d_V; without lost of generality, we can suppose that the last coordinate z_n of $z = (z_1, \ldots, z_n)$ is equal to 1. By the very definition of a Darboux point, all differences $V_i(z_1, \ldots, 1) - z_i V_n(z_1, \ldots, 1)$ vanish so that $[P(z_1, \ldots, 1) - m V_n(z_1, \ldots, 1)] F(z_1, \ldots, 1) = 0$. Let us stress the fact that we cannot *a priori* exclude the possibility that $F(z_1, \ldots, 1) \neq 0$.

Choose now the local affine coordinates $y_1, \ldots, y_{n-1}$ defined by $x_1 = z_1 + y_1, \ldots, x_{n-1} = z_{n-1} + y_{n-1}$. This change of coordinates sends the studied Darboux point Z to the origin of our new coordinate system.

In what follows, we will adopt the following convention: if some homogeneous polynomial in n variables $z_1, \ldots, z_n$ is denoted by a capital letter, we denote by the corresponding small letter the non-homogeneous polynomial in $n-1$ variables $y_1, \ldots, y_{n-1}$, that we get from the homogeneous polynomial in n variables. For instance, we define f by

$$f(y_1, \ldots, y_{n-1}) = F(z_1 + y_1, \ldots, z_{n-1} + y_{n-1}, 1). \tag{2.12}$$

In this local system of coordinates, equation (2.11) becomes

$$\sum_{i=1}^{n-1} (v_i - (z_i + y_i)\, v_n) \frac{\partial f}{\partial y_i} = (p - m v_n)\, f. \tag{2.13}$$

The study of this equation will be called the *local analysis* of our derivation d_V. Looking simultaneously at many or all such equations in various Darboux points and at their relationships will be called a *global analysis* of the derivation.

2.4. Local analysis

We are interested by equation (2.13), that we need study around the point $(0, \ldots, 0)$ of $\mathbf{C}^{n-1}$. The involved polynomials are in general non-homogeneous polynomials in $n-1$ variables and can be decomposed into their homogeneous components:

$$\phi = \sum_{i=0}^{\deg(\phi)} \phi_{(i)},$$

where polynomial $\phi_{(i)}$ is homogeneous of degree i; in particular, $\phi_{(0)}$ is the constant term of polynomial ϕ.

Let $\mu_Z(F)$ be the lowest integer such that $f_{(i)} \neq 0$, *i.e.* the multiplicity of F at point Z.

When $p(0) \neq m v_n(0)$, the minimal degree on the right-hand side of equation (2.13) is $\mu_Z(F)$ while it seems to be $\mu_Z(F) - 1$ on the left-hand side. The contradiction is only apparent since constant terms $(v_i - (z_i + y_i)\, v_n)_{(0)}$ are all 0. Indeed, Z is a Darboux point of d_V.

Comparing now the terms of minimal degree $\mu_Z(F)$ of both sides of equation (2.13) yields

$$\sum_{i=1}^{n-1} (v_i - (z_i + y_i)\, v_n)_{(1)} \frac{\partial h}{\partial y_i} = (p - m v_n)_{(0)}\, h, \tag{2.14}$$

where h is the non-trivial (*i.e.* non-zero) homogeneous component $f_{(\mu_Z(F))}$ of lowest degree of f.

In equation (2.14), partial derivatives of h are multiplied by linear homogeneous polynomials and h by a constant.

Then, homogeneous polynomial h is an eigenvector of a *linear derivation* (linear differential operator) $d_L : \mathbf{C}[t_1, \ldots, t_\nu] \to \mathbf{C}[t_1, \ldots, t_\nu]$ defined by

$$d_L(h) = \sum_{i=1}^{\nu} l_i \frac{\partial h}{\partial t_i} = \chi\, h, \tag{2.15}$$

where coefficients l_i are linear forms in variables $t_1, \ldots, t_\nu$; $l_i(t_1, \ldots, t_\nu) = \sum_{j=1}^{\nu} l_{ij}\, t_j$ and $L = (l_{ij})_{1 \le i, j \le \nu}$ is the $\nu \times \nu$ corresponding matrix.

Of course, in our case, $t_i = y_i$, $1 \le i \le n-1$, χ is the constant term $(p - m v_n)_{(0)}$ while the l_i are the linear components $(v_i - (z_i + y_i)\, v_n)_{(1)}$.

When the matrix L is diagonalizable, the following lemma is easy to be proved. We present below two different proofs of it in the general case.

LEMMA 2.3. – *Let h be a homogeneous polynomial eigenvector of derivation d_L defined in equation* (2.15) *where χ is the corresponding eigenvalue. Denote by $\rho_1, \ldots, \rho_\nu$ the ν eigenvalues of L.*

Then, there exist ν non-negative integers $i_1, \ldots, i_\nu$ such that

$$\begin{cases} \sum_{j=1}^{\nu} \rho_j\, i_j = \chi \\ \sum_{j=1}^{\nu} i_j = \deg(h) \end{cases} \tag{2.16}$$

First proof. – It is not difficult to see that a linear change of variables preserves the form of the problem in the following way: the eigenvalue χ remains the same while matrix L is replaced by a suitable conjugate L' of it.

We thus choose a new basis of $\mathbf{C}^\nu$ in which the matrix L' of the operator has the Jordan's form with the ones under the main diagonal.

Let us call $u_1, \ldots, u_\nu$ the coordinates in the new basis. Interval $[1, \nu]$ divides in k subintervals $[1, \nu_1]$, $[\nu_1+1, \nu_2]$, $\ldots$, $[\nu_{k-1}+1, \nu_k = \nu]$, corresponding to the various Jordan's cells of L', in such a way that coefficients of matrix L' are given by

$$L'_{i,j} = \rho_i\,\delta\,(i, j) + \delta\,(i+1, j)\,\mathbf{1}_{\{i \neq i^+\}}$$

where δ is the Kronecker symbol, $\mathbf{1}_{\{\}}$ is the indicator of a subset and i^+ stands for the right end of the subinterval to which i belongs. Due to the form of Jordan's cells, ρ_i only depends on the its cell *i.e.* $\rho_i = \rho_{i^+}$.

Keeping the same name h for the transformed polynomial in the Jordan's basis, equation (2.15) becomes

$$\text{(2.17)} \qquad \sum_{i=1}^{\nu} \rho_j\, u_i\, \frac{\partial h}{\partial u_i} + \sum_{i \neq i^+} u_i\, \frac{\partial h}{\partial u_{i+1}} = \chi\, h.$$

Here $\sum_{i\neq i^+}$ denotes the sum extended over all indices that are not at the end of Jordan cells.

Let μ be the degree of the non-zero homogeneous polynomial h; h is a linear combination of monomials u^α of total degree μ. Such an α is a ν-tuple $(\alpha_1, \ldots, \alpha_\nu)$ of non-negative integers whose sum, noted $w(\alpha)$ and called the weight of α, is equal to μ; $u^{\underline{\alpha}}$ then stands for the product $u_1^{\alpha_1} \cdots u_\nu^{\alpha_\nu}$.

Polynomial h then writes in a unique way as

$$h = \sum_\alpha \lambda_\alpha\, u^\alpha$$

Let us now introduce some new notations to conclude the proof. Call e_i the unit ν-tuple with coordinate 1 in the i-th place and 0 elsewhere. Equation (2.17) leads to a linear system of equations, in which the unknowns are the λ_α

$$\chi\lambda_\alpha = \Big(\sum_{i=1}^{\nu} \rho_i\,\alpha_i\Big)\,\lambda_\alpha + \sum_{i\neq i^+,\,\alpha_i \neq 0} (\alpha_{i+1}+1)\,\lambda_{\alpha - e_i + e_{i+1}},$$

or equivalently

$$\text{(2.18)} \quad \Big(\chi - \sum_{i=1}^{\nu} \rho_i\,\alpha_i\Big)\,\lambda_\alpha = \sum_{i\neq i^+,\,\alpha_i \neq 0} (\alpha_{i+1}+1)\,\lambda_{\alpha - e_i + e_{i+1}}.$$

Call δ the defect function defined on ν-tuples of weight μ by

$$\delta(\alpha) = \sum_{i=1}^{\nu} \alpha_i (i^+ - i),$$

where the same notational convention as in (2.17) is used. A value 0 for the defect means that the ν-tuple has only non-zero coordinates at the places that are right ends of subintervals.

The above linear system (2.18) is then "triangular" with respect to defect δ, which means that the equation corresponding to α allows us to compute λ_α from some other unknowns with a strictly smaller defect of indices, provided that the corresponding coefficient $\chi - (\sum_{i=1}^{\nu} \rho_i \alpha_i)$ is not 0.

This system is supposed to have some non-trivial solution. Let then β be a ν-tuple of lowest defect among those for which $\lambda_\alpha \neq 0$. The equation of system (2.18) corresponding to this β can only be satisfied if the difference $\chi - \sum_{i=1}^{\nu} \rho_i \beta_i$ is 0. So β is the sought ν-tuple.

Second proof. – Let L' be the same matrix as in preceding proof. Then $L' = D + N$, where D is the diagonal matrix, N nilpotent one and $DN = ND$. This decomposition leads to the decomposition $d_{L'} = d_D + d_N$ of the linear derivation $d_{L'}$. Since N is nilpotent then it is not difficult to see that for every $f \in \mathbf{C}[t_1, \ldots, t_\nu]$ there exists such natural number n that $(d_N)^n (f) = 0$.

Now let $d_{L'}(h) = \chi h$ and let s be the smallest natural number such that $\tilde{h} = (d_N)^{s-1}(h) \neq 0$ and $d_N(\tilde{h}) = 0$. Then $\tilde{h}$ is a non-trivial homogeneous polynomial of the same degree as h. As $d_N d_D = d_D d_N$, then $d_D(\tilde{h}) = \chi \tilde{h}$. Hence, the problem is reduced to the easy diagonal case. ■

The following almost obvious proposition will be used in next sections. We leave its proof to the reader.

PROPOSITION 2.4. – *Let us consider the equation*

$$\sum_{i=1}^{\nu} \rho_i t_i \frac{\partial h}{\partial t_i} = \chi h, \tag{2.19}$$

where $\rho_1, \ldots, \rho_\nu, \chi \in \mathbf{C}$ *and* h *is a non-constant homogeneous polynomial.*

2.4.1. *Assume that* $h = \sum_{j=1}^{p} h_j$, *where* h_j *are monomials (of the same degree). Then for every* j, $1 \le j \le p$, h_j *also satisfies the equation* (2.19).

2.4.2. *If the unique solution in non-negative integers of the system* (2.16) *is such that* $i_{j_0} \neq 0$ *and* $i_j = 0$ *for all* $j \neq j_0$, $1 \le j \le \nu$, *then* $h = c x_{j_0}^{\deg(h)}$, *for some* $c \in \mathbf{C}$.

2.5. A USEFUL DETERMINANT

The following elementary fact, whose proof is omitted, is well known (*see* for example Section 60 of [13]).

Let $n \ge 2$ and let $\alpha_0, \ldots, \alpha_{n-1}$ belong to $\mathbf{C}$. Consider the $n \times n$ matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix}.$$

Its characteristic polynomial $P(\lambda) = \det(A - \lambda I)$ equals

$$P(\lambda) = (-1)^n (\alpha_0 + \cdots + \alpha_{n-1}\lambda^{n-1} + \lambda^n)$$

This immediately implies the following special case, which will be useful in the sequel: Let $n \ge 2$. Consider the $n \times n$ matrix M_n:

$$M_n = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.20}$$

Its characteristic polynomial $P(\lambda)$ equals

$$P(\lambda) = (-1)^n (\lambda^n + \lambda^{n-1} + \cdots + \lambda + 1) = (-1)^n \sum_{i=0}^{n} \lambda^i$$

so that the eigenvalues of matrix M_n are all $(n+1)$-th roots of 1, except 1 itself.

3. Proof of Jouanolou's theorem

In the present section, we prove that, for $s \geq 2$, the equation

$$z^s \frac{\partial F}{\partial x} + x^s \frac{\partial F}{\partial y} + y^s \frac{\partial F}{\partial z} = PF \tag{3.1}$$

does not admit a non-constant homogeneous polynomial solution F in $\mathbf{C}[x, y, z]$ for any homogeneous polynomial eigenvalue P of degree $s-1$ in $\mathbf{C}[x, y, z]$. As already noticed in Section 2.1 the limitation to the homogeneous F is not restrictive. Thanks to Lemma 2.2, the unknown F can be supposed to be irreducible without any restriction of generality.

The first step consists in finding all Darboux points of Jouanolou's derivation defined by (1.6), *i.e.* points of the projective complex plane where vectors (x, y, z) and (z^s, x^s, y^s) are proportional. That leads to the following three equations

$$x^{s+1} = yz^s; \qquad y^{s+1} = zx^s; \qquad z^{s+1} = xy^s$$

whose corresponding non-trivial solutions represent the coordinates of $S = s^2 + s + 1$ different points of $\mathbf{P}^2(\mathbf{C})$. An easy computation shows that the z-coordinate can be chosen equal to 1 for all these points and that they are represented by all triple $(\xi, \xi^{s+1}, 1)$, where ξ runs in the set U_S of all S-roots of unity.

In this particular situation, equation (2.11) writes

$$(z^{s+1} - xy^s)\frac{\partial F}{\partial x} + (zx^s - y^{s+1})\frac{\partial F}{\partial y} = (P\,z - my^s)\,F \tag{3.2}$$

where $m \geq 1$ is the degree of the sought homogeneous polynomial F. For such an F, equations (3.1) and (3.2) are equivalent.

3.1. Local analysis

Let us suppose that equation (3.2) admits a non-constant solution F in $\mathbf{C}[x, y, z]$ and let us fix such a solution.

We have now to perform a local analysis of equation (3.2) around every Darboux point $(\xi, \xi^{s+1}, 1)$ of our derivation.

Choose the following local affine system (u, v) of coordinates

$$x = \xi\,(1+u); \qquad y = \xi^{s+1}\,(1+v).$$

Although these coordinates are slightly different from those used in Section 2.4, all considerations from this section remain valid here with inessential modification. In this new system of coordinates, equation (3.2) becomes

$$\text{(3.3)}\quad \begin{cases} (1-\xi\,(1+u)\,[\xi^{s+1}\,(1+v)]^s)\,\xi^{-1}\,\dfrac{\partial f}{\partial u} \\ \qquad + ([\xi\,(1+u)]^s - [\xi^{s+1}\,(1+v)]^{s+1})\,\xi^{-(s+1)}\,\dfrac{\partial f}{\partial v} \\ \qquad\qquad = (p - m\,[\xi^{s+1}\,(1+v)]^s)\,f \end{cases}$$

where

$$\begin{cases} f\,(u,\,v) = F\,(x,\,y,\,1) = F\,(\xi\,(1+u),\,\xi^{s+1}\,(1+v),\,1) \\ p\,(u,\,v) = P\,(x,\,y,\,1) = P\,(\xi\,(1+u),\,\xi^{s+1}\,(1+v),\,1). \end{cases}$$

According to equation (2.14), we have to compute homogeneous components of degree 1 of the factors by which the partial derivatives of f are multiplied and the constant term of the right-hand side factor in equation (3.3).

$$\begin{cases} [(1-\xi\,(1+u)\,[\xi^{s+1}\,(1+v)]^s)\,\xi^{-1}]_{(1)} = -\xi^{-1}\,(u+sv) \\ [([\xi\,(1+u)]^s - [\xi^{s+1}\,(1+v)]^{s+1})\,\xi^{-(s+1)}]_{(1)} = \xi^{-1}\,(su-(s+1)\,v) \\ [(p-m\,[\xi^{s+1}\,(1+v)]^s)]_{(0)} = \xi^{-1}\,(\xi\,P\,(\xi,\,\xi^{s+1},\,1)\,-m). \end{cases}$$

Call now $\mu\,(\xi)$ the multiplicity of F at the Darboux point $(\xi,\,\xi^{s+1},\,1)$ and h the non-zero homogeneous component of f of degree $\mu\,(\xi) \le m$.

Polynomial h would satisfy equation (2.14). After a multiplication by the factor $-\xi$, this becomes

$$\text{(3.4)}\quad [u+sv]\,\frac{\partial h}{\partial u} + [-su+(s+1)\,v]\,\frac{\partial h}{\partial v} = [-\xi\,P\,(\xi,\,\xi^{s+1},\,1)+m]\,h.$$

The left-hand side of this equation does not dependent on ξ. Let ρ_1 and ρ_2 be the eigenvalues of the matrix corresponding to the linear differential operator from on the left-hand side of equation (3.4).

They are equal to the two complex conjugate numbers $\rho_1 = (s+2-is\sqrt{3})/2$ and $\rho_2 = (s+2+is\sqrt{3})/2$.

According to Lemma 2.3, there then exists two non-negative integers $i_1(\xi)$ and $i_2(\xi)$ such that

$$\text{(3.5)} \qquad \begin{cases} i_1(\xi) + i_2(\xi) = \mu(\xi), \\ \rho_1\, i_1(\xi) + \rho_2\, i_2(\xi) = -\xi\, P(\xi,\ \xi^{s+1},\ 1) + m. \end{cases}$$

The arithmetic relations (3.5) are the departure point for the global analysis of our derivation.

3.2. GLOBAL ANALYSIS

Denote now be I_1 and I_2 the sums of numbers $i_1(\xi)$ and $i_2(\xi)$ at various Darboux points of the studied derivation and call M the sum of the multiplicities of the Darboux polynomial F at all these points :

$$I_1 = \sum\nolimits_{\xi\in U_S} i_1(\xi), \qquad I_2 = \sum\nolimits_{\xi\in U_S} i_2(\xi), \qquad M = \sum\nolimits_{\xi\in U_S} \mu(\xi).$$

Summing now all identities (3.5) for all ξ in U_S yields

$$\text{(3.6)} \qquad I_1 + I_2 = M, \qquad \rho_1\, I_1 + \rho_2\, I_2 = S\, m = (s^2+s+1)\, m.$$

Indeed, P is an homogeneous polynomial in three variables and its degree is $s-1$. The corresponding one-variable polynomial $\xi\, P(\xi,\ \xi^{s+1},\ 1)$ has a degree at most s^2 and its constant term is 0 so that it writes

$$\xi\, P(\xi,\ \xi^{s+1},\ 1) = \sum\nolimits_{j=1}^{s^2} \beta_j\, \xi^j$$

and consequently

$$\sum\nolimits_{\xi\in U_S} \xi\, P(\xi,\ \xi^{s+1},\ 1) = \sum\nolimits_{j=1}^{s^2} \beta_j \big(\sum\nolimits_{\xi\in U_S} \xi^j\big) = 0.$$

Indeed, for every positive integer j smaller than S, $\sum\nolimits_{\xi\in U_S} \xi^j = 0$.

As I_1 and I_2 are integers, as eigenvalues ρ_1 and ρ_2 are complex conjugate numbers, and as $m \geq 1$, the second identity of (3.6) implies that $I_1 = I_2 \stackrel{\text{def}}{=} I$ and equations (3.6) write

$$\text{(3.7)} \qquad 2\, I = M, \qquad (s+2)\, I = (s^2+s+1)\, m.$$

Applying inequality (2.8) to F yields

$$(3.8)\quad \sum_{\xi\in U_s} \mu(\xi)(\mu(\xi)-1) \leq \sum_{Q\in \mathbf{P}^2(\mathbf{C})} \mu_Q(F)(\mu_Q(F)-1) \leq (m-1)(m-2)$$

so that

$$(3.9)\quad \sum_{\xi\in U_s} (\mu(\xi))^2 \leq (m-1)(m-2)+M.$$

Now, as the total number of all ξ is s^2+s+1,

$$M^2 = \Big(\sum_{\xi\in U_s} \mu(\xi)\Big)^2 \leq (s^2+s+1) \sum_{\xi\in U_s} (\mu(\xi))^2$$

which, together with (3.7) and (3.9) gives the following inequality involving I, m and s:

$$(3.10)\quad 4I^2 \leq (s^2+s+1)[(m-1)(m-2)+2I].$$

3.3. Conclusion of the proof: first case

According to (3.7), numbers I, m and s are also related by equality

$$(3.11)\quad (s+2)I = (s^2+s+1)m.$$

Numbers $s+2$ and s^2+s+1 may be relatively prime or not; if they are so, it will be rather easy to conclude that inequality (3.10) cannot hold, which will achieve the proof of Jouanolou's theorem for such s. If they are not, this proof needs supplementary arguments.

In the first case, where $s+2$ and s^2+s+1 are relatively prime, there exists a positive integer r such that $I = r(s^2+s+1)$ and $m = r(s+2)$. After substitutions, the inequality (3.10) becomes

$$T_s(r) = 3s^2r^2 - (2s^2-s-4)r - 2 \leq 0.$$

It is easy to see that the quadratic polynomial $T_s(r)$, where s is a strictly positive integer, takes strictly positive values for all strictly positive integers r. This contradiction completes the proof of Jouanolou's theorem in the case where numbers $s+2$ and s^2+s+1 are relatively prime, and then in particular in the crucial case $s=2$.

3.4. THE SECOND CASE

The greatest common divisor of $s+2$ and s^2+s+1 is easily seen to be either 1 or 3. It remains to be supposed that it is 3. In this case, there exists a natural number $q \geq 1$ such that

$$\begin{cases} s = 3\,q+1, \qquad s+2 = 3\,(q+1), \\ S = s^2+s+1 = 3\,(3\,q^2+3\,q+1) \end{cases} \tag{3.12}$$

and numbers $q+1$ and $3\,q^2+3\,q+1$ are relatively prime.

Thus, by (3.11), there would exist some positive integer r such that $I = r\,(3\,q^2+3\,q+1)$ and $m = r\,(q+1)$ and inequality (3.10) would be

$$\tilde{T}_q\,(r) = (9\,q^2+6\,q+1)\,r^2 - 3\,(6\,q^2+3\,q-1)\,r - 6 \leq 0.$$

For every natural number q, $\tilde{T}_q\,(1) < 0$ and $\tilde{T}_q\,(r) > 0$ for every natural number $r \geq 2$. Thus we get a contradiction for $r \geq 2$; the unique case which is not excluded up to now is the one of a Darboux polynomial of degree $q+1$. We need supplementary arguments to prove that such a polynomial cannot exist.

When $r = 1$, $I = 3\,q^2+3\,q+1$ and the total multiplicity M of F at all Darboux points of the derivation, which, according to (3.7), is equal to

$$M = 2\,I = 2\,(3\,q^2+3\,q+1), \tag{3.13}$$

can be decomposed as follows

$$M = \sum\nolimits_{\{\xi \in U_S,\, \mu\,(\xi)\geq 1\}} 1 + \sum\nolimits_{\{\xi \in U_S,\, \mu\,(\xi)\geq 2\}} (\mu\,(\xi)-1).$$

The first term is simply the total number R of points of curve $\{F=0\}$ among all Darboux points. Thanks to inequality (3.8), the second term is bounded above by $\dfrac{(m-1)\,(m-2)}{2} = \dfrac{q\,(q-1)}{2}$, which, together with (3.13), yields a lower bound for R:

$$R \geq 2\,(3\,q^2+3\,q+1) - \frac{q\,(q-1)}{2}. \tag{3.14}$$

On the other hand, consider the decomposition of the non-trivial homogeneous polynomial F as a sum of monomials:

$$F = \sum\nolimits_{i+j+k=m=q+1} f_{i,\,j,\,k}\, x^i\, y^j\, z^k$$

The one-variable polynomial ϕ defined by (*see* (3.12))

$$\phi(\xi) = F(\xi, \xi^{s+1}, 1) = F(\xi, \xi^{3q+2}, 1) = \sum_{i+j+k=m=q+1} f_{i,j,k}\, \xi^{(3q+2)\,i+j}$$

vanishes for R different values of ξ in the finite set U_S.

The map $(i, j, k) \to (i + (3q+2)j)$ is easily shown to be injective from the set of all triples of non-negative integers whose sum is $q+1$ to the interval $[0, (3q+2)(q+1)]$ of integers. As one of the coefficients $f_{i,j,k}$ at least is not 0, polynomial ϕ is not zero. Its degree is then at most $(3q+2)(q+1)$.

As ϕ vanishes in R different points at least, we get an upper bound for R:

$$R \leq (3q+2)(q+1). \tag{3.15}$$

The lower (3.14) and upper (3.15) bounds on R are contradictory: for a natural number q, the double inequality

$$2(3q^2+3q+1) - \frac{q(q-1)}{2} \leq R \leq (3q+2)(q+1)$$

leads to the inequality

$$(3q+2)(q+1) - 2(3q^2+3q+1) + \frac{q(q-1)}{2} = -\frac{q}{2}(5q+3) \geq 0.$$

But this last inequality only holds for $q = 0$. This contradiction concludes the proof of Jouanolou's theorem.

Let us note that the case $q = 0$ is not excluded by the proof. It correspond to $s = 1$ for which we know that some Darboux curves do exist, as explained in Section 1.2.

3.5. Comments and remarks

3.5.1. – As shown in [14], in three variables, Darboux curves are not only related to the existence of rational first integrals of an homogeneous polynomial vector field, but also to the existence of first integrals in a wider class of functions.

If there are not sufficiently many Darboux curves, it would indeed be impossible to find a homogeneous Liouvillian first integral for the vector field.

The notion of a Liouvillian function in the multivariate case was introduced by Michael SINGER [18]; it is a way to make precise what can be defined "in finite terms", *i.e.* by means of algebraic extensions, primitives, and exponentials.

In particular, the Jouanolou's system does not admit any Liouvillian homogeneous first integral.

Let us incidently note that another very interesting way to give a precise meaning to "finite terms" can be found in in A. G. KHOVANSKII's works (*see* [9], [10], where other references can be found).

3.5.2. – In algebraic terms, Jouanolou's theorem is equivalent to the fact that derivation d from $\mathbf{C}[x, y, z]$ to itself defined by (1.6) does not admit any non-trivial principal differential ideal, *i.e.* an ideal $A \in \mathbf{C}[x, y, z]$ which is generated by exactly one element of $\mathbf{C}[x, y, z]$ and such that $d(A) \subset A$.

Nevertheless, derivation d has some non-trivial differential ideals, as for instance, the two-generator ideal $A = (y - x, z - x)$.

3.5.3. – Let us note that from Jouanolou's theorem (by considering the terms of highest degree on the both sides of (2.4)) one deduces immediately the non-existence of Darboux polynomials for any $\mathbf{C}$-derivation of the form

$$\begin{aligned} d(x) &= z^s + f(x, y, z) \\ d(y) &= x^s + g(x, y, z) \\ d(z) &= y^s + h(x, y, z), \end{aligned}$$

where $s \geq 2$, $g, h \in \mathbf{C}[x, y, z]$, $\deg(f) < s$, $\deg(g) < s$, $\deg(h) < s$.

3.5.4. – Let d be a $\mathbf{C}$-derivation of $\mathbf{C}[x_1, \ldots, x_n]$ and let σ be a $\mathbf{C}$-automorphism of $\mathbf{C}[x_1, \ldots, x_n]$. Define the derivation $\delta = \sigma\, d \sigma^{-1}$ of $\mathbf{C}[x_1, \ldots, x_n]$. Then the non-existence of non-trivial constants or of Darboux polynomials for d is equivalent to their non-existence for δ.

As an example, let us apply this remark to Jouanolou's derivation d (for $s = 2$) and to the linear $\mathbf{C}$-automorphism $\sigma(x, y, z) = (y + z, x + z, x + y)$. Then we obtain the non-existence of Darboux polynomials for $\delta_1 = \sigma\, d\sigma^{-1}$ and $\delta_2 = \sigma^{-1}\, d\sigma$. The computations gives

$$\begin{cases} \delta_1(x) = z^2 + xz - xy + yz \\ \delta_1(y) = x^2 + yx - yz + zx \\ \delta_1(z) = y^2 + zy - zx + xy \end{cases}$$

$$\begin{cases} \delta_2(x) = z^2 + (x-y)^2 \\ \delta_2(y) = x^2 + (y-z)^2 \\ \delta_2(z) = y^2 + (z-x)^2 \end{cases}.$$

3.5.5. – In the proof of Jouanolou's theorem, the non-trivial homogeneous component of lowest degree is used. In some cases, the consideration of components of highest degree may also lead to the non-existence proof.

As an example, let us consider derivation $\tilde{d}$ from $\mathbf{C}[x, y]$ to itself defined by

$$\tilde{d}(x) = 1 - xy^s, \qquad \tilde{d}(y) = x^s - y^{s+1},$$

where $s \geq 1$ is a natural number. This derivation arises from the left-hand side of equation (3.2) when $z = 1$.

Let us now prove that, for $s \geq 1$, derivation $\tilde{d}$ does not admit any non-trivial constant, *i.e.* that $\mathbf{C}[x, y]^{\tilde{d}} = \mathbf{C}$. Indeed, let us suppose some $f \in \mathbf{C}[x, y] \setminus \mathbf{C}$ satisfies $\tilde{d}(f) = 0$ and let us denote by f^+ the homogeneous component of highest degree of f. Then $f^+ \neq 0$ and

$$xy^s \frac{\partial f^+}{\partial x} + y^{s+1} \frac{\partial f^+}{\partial y} = 0.$$

The Euler's theorem on homogeneous functions yields

$$0 = y^s \left(x \frac{\partial f^+}{\partial x} + y \frac{\partial f^+}{\partial y} \right) = y^s (\deg f^+),$$

which implies that $f^+ = 0$. This contradiction concludes the proof.

Nevertheless, for $s = 1$, derivation $\tilde{d}$ has a non-trivial Darboux polynomial. Indeed $\tilde{d}(f) = pf$, where $f = x + y + 1$ and $p = 1 - y$.

3.5.6. – The analogue of Jouanolou's theorem fails in positive characteristic. Let indeed R be any commutative ring of prime characteristic $p > 0$ and take $s = p$. In this case,

$$\begin{aligned} d(x+y+z) = x^p + y^p + z^p &= (x+y+z)^p \\ &= (x+y+z)^{p-1}(x+y+z), \end{aligned}$$

where the derivation d is defined by (1.6).

3.5.7. – A natural question arises: what happens when more three variables are considered. More precisely, let us consider the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$ and its derivation d defined by

$$d(x_1) = x_n^s \quad \text{and} \quad d(x_i) = x_{i-1}^s \qquad \text{for} \quad 2 \le i \le n.$$

What about the solvability of equation $d(F) = PF$ for F and P in $\mathbf{C}[x_1, \ldots, x_n]$?

4. Factorisable derivations

In this section, we describe the first of the three examples in which the above method, together with specific arguments, leads to the proof that a typical homogeneous factorisable derivation has no non-trivial polynomial, or even rational, constants.

Let us recall the following well known notion. Let T be a subset of $\mathbf{C}$ and let E be a complex vector space. A finite subset $\{e_1, \ldots, e_k\}$ of E will be called *T-independent* if the equality $\sum_{i=1}^{k} t_i e_i = 0$, where $t_1, \ldots, t_k \in T$, implies that $t_1 = \cdots = t_k = 0$. In what follows we will consider exclusively the cases when $T = \mathbf{Z}$ or $T = \mathbf{Z}^+$, where $\mathbf{Z}^+$ denotes the set of non-negative integers.

Let $n \ge 2$ and let $W_1, \ldots, W_n \in \mathbf{C}[x_1, \ldots, x_n]$ be homogeneous $\mathbf{Z}$-independent polynomials of the same degree $s \ge 1$.

The $\mathbf{C}$-derivation

$$d(x_i) = x_i W_i, \qquad 1 \le i \le n, \tag{4.1}$$

as well as the corresponding system of ordinary differential equations is called *factorisable.*

Let us note that usually in the definition of factorisable systems the conditions of homogeneity and of $\mathbf{Z}$-independence are not required. The factorisable systems of ordinary differential equations was intensively studied from a long time; *see* for example [7] and [6], where many references on this subject can be found.

One of the main features of factorisable derivations is the fact that the polynomials $x_1, \ldots, x_n$ are always Darboux polynomials of it.

Consequently any polynomial of the form

$$C \prod_{i=1}^{n} x_i^{\alpha_i}, \tag{4.2}$$

where $C \neq 0$ and $\alpha_1, \ldots, \alpha_n$ are non-negative integers, is also a Darboux polynomial of it.

First let us prove the following statement which will be useful in the next section.

Proposition 4.1. – *Let d be a factorisable derivation defined by* (4.1). *Let us suppose that all its homogeneous Darboux polynomials are of the form* (4.2). *Then:*

(4.1.1) *All its Darboux polynomials are also of this form,*
(4.1.2) $\mathbf{C}(x_1, \ldots, x_n)^d = \mathbf{C}$.

Proof. – (4.1.1). Let $F \in \mathbf{C}[x_1, \ldots, x_n]$ be a Darboux polynomial, *i.e.* $d(F) = PF$ for some homogeneous $P \in \mathbf{C}[x_1, \ldots, x_n]$. Let $F = \sum F_i$ be the homogeneous decomposition of F.

If $F_i \neq 0$ for only one i, our conclusion is evident. If this is not the case, one can find two different indices i and j, such that $F_i \neq 0$ and $F_j \neq 0$. From Lemma 2.1 we know that

$$d(F_i) = PF_i \quad \text{and} \quad d(F_j) = PF_j. \tag{4.3}$$

In virtue of our assumptions we know that $F_i = a x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $a \neq 0$ and that $F_j = b x_1^{\beta_1} \ldots x_n^{\beta_n}$, $b \neq 0$, where

$$(\alpha_1, \ldots, \alpha_n) \neq (\beta_1, \ldots, \beta_n). \tag{4.4}$$

As F_i and F_j are Darboux polynomials of d, then one immediately obtains that

$$d(F_i) = a\left(\sum \alpha_k W_k\right) x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$
$$d(F_j) = b\left(\sum \beta_k W_k\right) x_1^{\beta_1} \ldots x_n^{\beta_n}.$$

Now, (4.3) implies that

$$P = \sum \alpha_k W_k = \sum \beta_k W_k$$

and thus $\sum(\alpha_k - \beta_k) W_k = 0$. From the assumption on $\mathbf{Z}$-independence of $W_1, \ldots, W_n$ one deduces that $\alpha_k = \beta_k$ for $1 \le k \le n$. This contradicts (4.4). Then $F_i \neq 0$ for only one i and (4.1.1) is proved.

(4.1.2) Let us suppose that $F/G \in \mathbf{C}(x_1, \ldots, x_n)^d$, where F, $G \in \mathbf{C}[x_1, \ldots, x_n]$ and F, G are relatively prime. Then, as already noticed in Section 1.2 (*see* (1.9)), $d(F) = PF$ and $d(G) = PG$ for some $P \in \mathbf{C}[x_1, \ldots, x_n]$. We known from (4.1.1) that $F = C_1 \prod x_i^{\alpha_i}$ and $G = C_2 \prod x_i^{\beta_i}$ for some $C_1 \neq 0$, $C_2 \neq 0$ and some non-negative integers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$. From the proof of (4.1.1) we know that $\alpha_i = \beta_i$, $1 \le i \le n$, and thus F and G are not relatively prime. This contradiction finishes the proof of (4.1.2). ■

Let us now introduce a new notation. If W is a homogeneous polynomial of degree s, then $W^{(k)}$ denotes the coefficient of the monomial x_k^s which appears in W.

THEOREM 4.2. – *Let* d *be a factorisable derivation defined by* (4.1).

(4.2.1) *If for some* k, $1 \le k \le n$, *the numbers* $W_1^{(k)}, \ldots, W_n^{(k)}$ *are* $\mathbf{Z}^+$*-independent, then*

$$\mathbf{C}[x_1, \ldots, x_n]^d = \mathbf{C}. \tag{4.5}$$

Equivalently, the system of differential equations

$$\frac{dx_i}{dt} = x_i W_i, \qquad 1 \le i \le n, \tag{4.6}$$

does not admit a non-constant polynomial first integral.

(4.2.2) *If for some* k, $1 \le k \le n$, *the numbers* $W_1^{(k)}, \ldots, W_n^{(k)}$ *are* $\mathbf{Z}$*-independent, then*

$$\mathbf{C}(x_1, \ldots, x_n)^d = \mathbf{C}. \tag{4.7}$$

Equivalently, the system (4.6) *does not admit a non-constant rational first integral.*

Proof. – (4.2.1) Suppose that for some $F \neq 0$, $F \in \mathbf{C}[x_1, \ldots, x_n]$, $\deg(F) = m \ge 1$ one has $d(F) = 0$. As we have previously noticed, we can assume that F is homogeneous of degree m.

Moreover, without any restriction of generality we can suppose that $k = n$, *i.e.* that the numbers $W_1^{(n)}, \ldots, W_n^{(n)}$ are $\mathbf{Z}^+$-independent.

Let us consider the Darboux point $z = (0, \ldots, 0, 1) \in \mathbf{P}^{n-1}(\mathbf{C})$ of the derivation d and introduce local affine coordinates $(y_1, \ldots, y_{n-1})$ around z, defined by

$$x_i = y_i \qquad \text{for} \quad 1 \le i \le n-1$$

and $x_n = 1$.

Equation (2.13) now writes

$$\sum\nolimits_{i=1}^{n-1} y_i \left(W_i(\tilde{y}) - W_n(\tilde{y})\right) \frac{\partial f}{\partial y_i} = -m\, W_n(\tilde{y})\, f,$$

where $\tilde{y} = (y_1, \ldots, y_{n-1}, 1)$, and equation (2.14) writes

$$\sum\nolimits_{i=1}^{n-1} (W_i^{(n)} - W_n^{(n)})\, y_i \frac{\partial h}{\partial y_i} = -m\, W_n^{(n)}\, h.$$

In virtue of Lemma 2.3, there exist non-negative integers $i_1, \ldots, i_{n-1}$ such that

$$\sum\nolimits_{j=1}^{n-1} i_j (W_j^{(n)} - W_n^{(n)}) = -m\, W_n^{(n)} \tag{4.8}$$

and

$$0 \le \sum\nolimits_{j=1}^{n-1} i_j = \mu \le m. \tag{4.9}$$

Let us note $i_n = m - \mu \ge 0$ so that (4.8) yields

$$\sum\nolimits_{j=1}^{n} i_j\, W_j^{(n)} = 0.$$

This equality together with (4.9) contradicts the $\mathbf{Z}^+$-independence of $W_1^{(n)}, \ldots, W_n^{(n)}$, which achieves the proof of (4.2.1).

(4.2.2) From Section 1.2 (*see* (1.9)), it follows that, in order to prove (4.7), it suffices to prove that, for a given $P \in \mathbf{C}[x_1, \ldots, x_n]$, if two non-zero polynomials $F_1, F_2 \in \mathbf{C}[x_1, \ldots, x_n]$ are such that $d(F_i) = PF_i$, $i = 1, 2$, then they are proportional.

First we will consider the particular case when F_1 and F_2 are homogeneous polynomials.

In agreement with the notations from Section 2.4 we will denote by h_i the non-trivial homogeneous component of the lowest degree of f_i, where (*see* (2.12))

$$f_i(y_1, \ldots, y_{n-1}) = F_i(y_1, \ldots, y_{n-1}, 1), \qquad i = 1, 2.$$

Exactly in the same way as in the proof of (4.2.1), passing through the equality (2.13) we obtain the existence of non-negative integers $i_1, \ldots, i_n$ and $k_1, \ldots, k_n$ such that

$$\sum_{j=1}^{n} i_j W_j^{(n)} = \sum_{j=1}^{n} k_j W_j^{(n)} = P(0, 0, \ldots, 0, 1), \tag{4.10}$$

and

$$\sum_{j=1}^{n} i_j = \deg F_1, \qquad \sum_{j=1}^{n} k_j = \deg F_2.$$

Indeed, this follows from Lemma 2.3 applied to h_1 and h_2 respectively.

From (4.10) it follows that $\sum_{j=1}^{n} (i_j - k_j) W_j^{(n)} = 0$. Our assumption of $\mathbf{Z}$-independence of $W_1^{(n)}, \ldots, W_n^{(n)}$ implies that $i_j = k_j$, for $1 \leq j \leq n$, and that these numbers are the unique non-negative integer numbers satisfying (4.10).

Consequently h_1 and h_2 are two proportional monomials, that is, $h_1 = r h_2$ for some $r \in \mathbf{C} \setminus \{0\}$.

Let us consider now the polynomial $F_3 = F_1 - r F_2$. We have two possibilities; either $F_3 \neq 0$ or $F_3 = 0$. In the first case F_3 is a homogeneous Darboux polynomial of d with the same P as for F_1 and F_2. It is easy to see that the degree of the lowest homogeneous component h_3 of corresponding polynomial f_3 is greater then $\deg(h_1)$. Repeating now the same arguments as above, but with respect to the polynomials F_1 and F_3 we conclude that $\deg(h_1) = \deg(h_3)$. This contradiction proves that $F_3 = 0$, *i.e.* that F_1 and F_2 are proportional.

Let us pass now to the general case when F_1 and F_2 are not supposed to be homogeneous. In virtue of Lemma 2.1 all non-zero homogeneous components of F_1 and F_2 are Darboux polynomials for d with the same P as above. Thus from the first part of our proof we deduce that all these homogeneous components are of the same degree and mutually proportional. This concludes the proof (4.2.2). ■

Before finishing this section let us note that the assumptions of Theorem 4.2, even in strengthened form, cannot exclude the existence of Darboux polynomials which are not of the form (4.2).

Indeed, let $A = -A^t$ be a skew-symmetric $n \times n$ matrix with complex entries; $A = [a_{ij}]$. Let $p_1, \ldots, p_n \in \mathbf{C}$ and let $b_{ij} = a_{ij} + p_j$, $1 \leq i, j \leq n$.

Consider linear polynomials

$$W_i = \sum_{j=1}^{n} b_{ij}\, x_j, \qquad 1 \leq i \leq n$$

together with the corresponding factorisable derivation defined by (4.1).

Then an easy reasoning proves that one can always find a non-trivial homogenous linear polynomial F such that $d(F) = PF$ with $P = \sum_{i=1}^{n} p_i\, x_i$.

5. The $x_i\, x_{i+1}$ system

In this section, we describe a factorisable derivation for which Theorem 4.2 of the preceding section cannot be applied. Nevertheless the above method, together with specific arguments, leads to the proof that derivation has no polynomial constants and even no rational constants.

Let us consider the $\mathbf{C}$-derivation d of $\mathbf{C}(x_1, \ldots, x_n)$ defined for $n \geq 2$ by

$$d(x_i) = x_i\, x_{i+1}, \qquad 1 \leq i \leq n, \tag{5.1}$$

where the index $n+1$ is identified with the index 1, *i.e.* $x_{n+1} = x_1$.

For $n = 2$, polynomial $x_1 - x_2$ is a non-trivial constant of the derivation d, and thus the subring $\mathbf{C}[x_1, x_2]^d$ of constants of d is larger than $\mathbf{C}$.

For $n \geq 3$, we will now prove the following

THEOREM 5.1. – *Let d be the derivation defined in* (5.1) *where $n \geq 3$; then d does not admit any other Darboux polynomial than the products of powers of coordinate functions,* i.e. *the equation*

$$d(F) = PF \tag{5.2}$$

has no other solution than $P = \sum \lambda_i x_{i+1}$, $F = C \prod x_i^{\lambda_i}$ *for some element* $C \in \mathbf{C}$ *and some* n*-tuple* $(\lambda_1, \ldots, \lambda_n)$ *of non-negative integers.*

As a consequence, the subfield $\mathbf{C}(x_1, \ldots, x_n)^d$ *of constants of* d *is equal to* $\mathbf{C}$ *or equivalently, the system of differential equations*

$$\frac{d x_i}{dt} = x_i x_{i+1}, \qquad 1 \le i \le n, \tag{5.3}$$

where $x_{n+1} = x_1$*, has no non-trivial rational first integral.*

Proof. – The proof divides in two parts.

We will first prove that the subring $\mathbf{C}[x_1, \ldots, x_n]^d$ of constants of d reduces to $\mathbf{C}$, *i.e.* that equation (5.2) has no non-trivial solution F for the eigenvalue $P = 0$. This means that the system (5.3) has no polynomial first integral. In the second part we will study Darboux polynomials of d.

The first part of the proof relies on the local analysis around the Darboux point (1, .., 1) and yields the non-existence of a non-trivial homogeneous polynomial first integral.

Following the procedure described in Section 2.3, let us introduce local affine coordinates $(y_1, \ldots, y_{n-1})$ around the Darboux point (1, .., 1) of $\mathbf{P}^{n-1}(\mathbf{C})$. These coordinates are defined by

$$x_i = 1 + y_i \qquad \text{for} \quad 1 \le i \le n-1, \tag{5.4}$$

and $x_n = 1$.

According to (2.12), a homogeneous polynomial first integral F would become a (non-homogeneous) polynomial $f(y_1, \ldots, y_{n-1})$ of degree at most m that satisfies equation (2.13), which writes

$$\sum\nolimits_{i=1}^{n-1} [(1+y_i)(1+y_{i+1}) - (1+y_i)(1+y_1)] \frac{\partial f}{\partial y_i} = -m(1+y_1) f,$$

where $y_n = 0$.

After cancellations, this becomes

$$\sum\nolimits_{i=1}^{n-2} (1+y_i)(y_{i+1} - y_1) \frac{\partial f}{\partial y_i} - y_1 (1+y_{n-1}) \frac{\partial f}{\partial y_{n-1}} = -m(1+y_1) f. \tag{5.5}$$

Let h be the non-trivial homogeneous component of lowest degree of f; call then μ the degree of h. Polynomial h has to satisfy the following equation

$$\sum_{i=1}^{n-2} (y_{i+1} - y_1) \frac{\partial h}{\partial y_i} - y_1 \frac{\partial h}{\partial y_{n-1}} = -mh. \tag{5.6}$$

By Lemma 2.3, there would exist non-negative integers $\alpha_1, \ldots, \alpha_{n-1}$ such that

$$\sum_{i=1}^{n-1} \alpha_i \rho_i = -m \tag{5.7}$$

and

$$\sum_{i=1}^{n-1} \alpha_i = \mu \tag{5.8}$$

where the ρ_i are the eigenvalues of the corresponding square matrix of size $n-1$. This matrix coincides with the matrix M_{n-1} defined by (2.20), so that its eigenvalues are all n-th roots of unity except 1.

As $\mu = \deg(h) \le \deg(f) \le m$, then from (5.7) and (5.8) one obtains that

$$m = |\sum_{i=1}^{n-1} \alpha_i \rho_i| \le \sum_{i=1}^{n-1} \alpha_i = \mu, \tag{5.9}$$

and consequently that $\mu = m$, *i.e.* that $\deg(h) = \deg(f)$, and consequently that $h = f$. We will consider separately two cases; when n is even and when n is odd.

If n is odd, then $\rho_i \notin \mathbf{R}$ for $1 \le i \le n-1$, and thus from (5.9) one deduces than $m < \mu$, which contradicts $\mu \le m$.

If n is even, as among the eigenvalues $\rho_1, \ldots, \rho_{n-1}$ there is exactely one real which is equal -1, say $\rho_{i_0} = -1$, then from (5.9) one deduces that $\alpha_i = 0$ for all $i \ne i_0$. Passing by the basis in which the matrix M_{n-1} is diagonal, from Proposition 2.4 one easily deduces that

$$h = f = (l_1 y_1 + \cdots + l_{n-1} y_{n-1})^m, \tag{5.10}$$

where $l_1, \ldots, l_{n-1} \in \mathbf{C}$. Substituting (5.10) to (5.6) yields

$$l_1 = -l_2 = l_3 = \cdots = (-1)^n l_{n-1}. \tag{5.11}$$

From the other side substituting (5.10) to (5.5) one obtains that

$$\sum_{i=1}^{n-2} (1+y_i)\,(y_{i+1}-y_1)\,l_i - y_1\,(1+y_{n-1})\,l_{n-1} = -(1+y_1)\,h.$$

Putting here $y_1 = y_2 = \cdots = y_{n-1} = 1$, one deduces that $l_{n-1} = 0$. Taking in account (5.11), we obtain that $h = 0$, which contradicts $h \neq 0$. This finishes the first part of the proof.

Let us pass now to the second part of the proof; *i.e.* to the proof that a Darboux polynomial of d, which is not divisible by any of the coordinate polynomials x_i, has to be a constant of the derivation d.

We will prove that all Darboux polynomials of the derivation d are of the form (4.2). In virtue of the Proposition 4.1.1, it is sufficient to consider only the homogeneous Darboux polynomials F. As the factors of Darboux polynomials are also Darboux polynomials then, without any restriction of generality, one can suppose that F is not divisible by any of the polynomials x_i.

The polynomial P, such that $d\,(F) = PF$, is a linear form $P = \sum_{i=1}^{n} \lambda_j\,x_j$ with coefficients λ_i in $\mathbf{C}$. Denote by G_i the polynomial obtained from F by setting $x_i = 0$. For every value of index i, the defining identity

$$d\,(F) = \sum_{j=1}^{n} x_j\,x_{j+1}\,\frac{\partial F}{\partial x_j} = PF = \Big(\sum_{j=1}^{n} \lambda_j\,x_j\Big)\,F$$

can be ordered with respect to variable x_i and the corresponding constant term yields the following equation for polynomial G_i (in all variables except x_i)

$$\sum_{j\neq i-1,\,j\neq i} x_j\,x_{j+1}\,\frac{\partial G_i}{\partial x_j} = \Big(\sum_{j\neq i} \lambda_j\,x_j\Big)\,G_i$$

Both members of this equation can now be ordered with respect to variable x_{i+1}; looking to the homogeneous part of highest degree in x_{i+1} of the last equation, one deduces $\lambda_{i+1} = 0$.

And, as that is true for all indices, the polynomial P has to be 0, which concludes the second part of the proof. ■

6. The $(x_i + x_{i+1})^s$ system

Let $s \geq 1$ be a natural number. In this section, we will consider the **C**-derivation d of $\mathbf{C}[x_1, \ldots, x_n]$ defined for $n \geq 2$ by

$$d(x_i) = (x_i + x_{i+1})^s, \qquad 1 \leq i \leq n, \tag{6.1}$$

where, like in Section 5, the index $n+1$ is identified with the index 1, *i.e.* $x_{n+1} = x_1$.

For $n = 2$, $x_1 - x_2 \in \mathbf{C}[x_1, x_2]^d$.

THEOREM 6.1. – *Let* d *be the derivation defined by* (6.1). *Then for all* $s \geq 1$ *and* $n \geq 3$

$$\mathbf{C}[x_1, \ldots, x_n]^d = \mathbf{C}.$$

Equivalently, the system of differential equations

$$\frac{dx_i}{dt} = (x_i + x_{i+1})^s, \qquad 1 \leq i \leq n$$

has no non-constant polynomial first integral.

Proof. – The proof is along the same line as the proof of Theorem 5.1. Thus we will only sketch it. It is based on the local analysis around the Darboux point $(1, \ldots, 1)$ of $\mathbf{P}^{n-1}(\mathbf{C})$.

Let F be a non-trivial homogeneous polynomial of degree $m \geq 1$ such that

$$\sum_{i=1}^{n} (x_j + x_{i+1})^s \frac{\partial F}{\partial x_i} = 0.$$

Consequently (*see* (2.11))

$$\sum_{i=1}^{n-1} \left(x_n (x_i + x_{i+1})^s - x_i (x_n + x_1)^s\right) \frac{\partial F}{\partial x_i} = -m (x_n + x_1)^s F. \tag{6.2}$$

In the local coordinates $(y_1, \ldots, y_{n-1})$ defined by (5.4), the equation (2.13) applied to (6.2) writes

$$\sum_{i=1}^{n-2} \left((y_i + y_{i+1} + 2)^s - (1 + y_i)(2 + y_1)^s\right) \frac{\partial F}{\partial y_i} \tag{6.3}$$

$$+((y_{n-1}+2)^s-(1+y_{n-1})\,(2+y_1)^s\,)\,\frac{\partial F}{\partial y_{n-1}}$$
$$=-m\,(2+y_1)^s\,f.$$

Let h be the non-trivial homogeneous component of lowest degree of f, $\deg(h)=\mu\geq 0$. Polynomial h satisfies the following equation (*see* 2.14))

$$(6.4)\quad (-2\,y_1+sy_2)\,\frac{\partial h}{\partial y_1}+\sum_{i=2}^{n-2}(-sy_1+(s-2)\,y_i+sy_{i+1})\,\frac{\partial h}{\partial y_i}$$
$$+(-sy_1+(s-2)\,y_{n-1})\,\frac{\partial h}{\partial y_{n-1}}=-2\,mh.$$

Applying one time more Euler's theorem on homogeneous functions, one deduces from (6.4) that

$$(6.5)\qquad \sum_{i=1}^{n-2}(y_{i+1}-y_i)\,\frac{\partial h}{\partial y_i}+y_1\,\frac{\partial h}{\partial y_{n-1}}=-\frac{2\,m+(s-2)\,\mu}{s}\,h.$$

Now we will apply Lemma 2.3 to the equation (6.5). The corresponding matrix coincides with the M_{n-1} defined by (2.20), whose eigenvalues are $\rho_1,\,\ldots\rho_{n-1}$. For some non-negative integers $\alpha_1,\,\ldots\alpha_{n-1}$ one has

$$(6.6)\qquad \sum_{i=1}^{n-1}\alpha_i\,\rho_i=-\frac{2\,m+(s-2)\,\mu}{s}.$$

as well as the equality (5.8).

As $0\leq\mu\leq m$, then taking in account (6.6) and (5.8) one obtains that

$$\mu\leq\frac{2\,m+(s-2)\,\mu}{s}=|\sum_{i=1}^{n-1}\alpha_i\,\rho_i|\leq\sum_{i=1}^{n-1}\alpha_i=\mu.$$

Consequently $\mu=m$ and thus $h=f$.

To conclude we proceed now exactly in the same way as in Section 5. We distinguish two cases of n even and of n odd.

When n is even we repeat word for word the argument from Section 5.

When n is odd, like in Section 5, we obtain the formula (5.10). Substituting (5.10) to (6.3) and putting $y_1=-2$, one obtains that for all $y_2,\,\ldots,\,y_{n-1}\in\mathbf{C}$ one has

$$l_1\,y_2^s+\sum_{i=2}^{n-2}l_i\,(y_i+y_{i+1}+2)^s+l_{n-1}\,(y_{n-1}+2)^s=0,$$

which easily implies that $l_1 = l_2 = \cdots = l_{n-1} = 0$.

Thus $h = 0$ which is a contradiction. ■

Let us note that for $s = 1$, the derivation (6.1) admits $F = x_1 + \cdots + x_n$ as a Darboux polynomial. Indeed $d(F) = 2F$.

7. An algebraic supplement

Let R be a commutative ring without zero divisors, which contains the ring $\mathbf{Z}$ of integers.

We will now show *the transfer principle*, already announced in Section 1, that all our results remains valid when instead of the field $\mathbf{C}$ of complex numbers, one considers the above ring R.

The proof is based on the well known fact (*see* for example [11]):

Let K be a field of characteristic zero, i.e. *it contains the field $\mathbf{Q}$ of rational numbers as a subfield. Let S be a finite subset of K. Consider the smallest subfield of K, noted by $\mathbf{Q}(S)$, which contains S. Then there exists a field embedding of $\mathbf{Q}(S)$ in $\mathbf{C}$.*

PROPOSITION 7.1. – *Let $d : \mathbf{Z}[x_1, \ldots, x_n] \to \mathbf{Z}[x_1, \ldots, x_n]$ be the derivation defined by*

$$d = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i},$$

where $f_1, \ldots, f_n \in \mathbf{Z}[x_1, \ldots, x_n]$.

Consider the equation

$$d(F) = PF. \tag{7.1}$$

If this equation does not admit a solution $F, P \in \mathbf{C}[x_1, \ldots, x_n]$ with $F \notin \mathbf{C}$, then this equation has no solutions $F, P \in R[x_1, \ldots, x_n]$ with $F \notin R$, where R is a ring as above.

Proof. – Assume that $F, P \in R[x_1, \ldots, x_n]$, $F \notin R$ is a solution of the equation (7.1). Denote by R_0 the field of fractions of the ring R. Consider now the finite set $S \subset R$ of all coefficients of polynomials F, P and the field $K = \mathbf{Q}(S)$, the smallest subfield of R_0 containing S.

It is clear that the polynomials F and P are the solutions of the equation (7.1) in $K[x_1, \ldots, x_n] \subset R_0[x_1, \ldots, x_n]$.

In virtue of the above mentioned fact we may assume $K \subset \mathbf{C}$. Therefore we obtain a contradiction, because by our assumption the equation (7.1) does not admit any solution in $\mathbf{C}[x_1, \ldots, x_n]$ with F non-constant. ■

The just proved transfer principle applies to the derivation studied in Section 3. The exactly same argument also works in what concerns equation $d(F) = 0$. Thus, the transfer principle also applies to the derivation studied in Section 6.

In what concern the factorisable derivations studied in Sections 4 and 5, the situation is slightly different, because they always have non-trivial Darboux polynomials. Moreover, in Section 4 the coefficients of polynomials $W_1, \ldots, W_n$ defining the derivation, are not necessary integer.

Nevertheless the transfer principle adapted to factorisable systems can be easily formulated and proved along the same line as above, but now the set S also contains all coefficients of polynomials $W_1, \ldots, W_n$.

Acknowledgements. – We sincerely thank Jean-Pierre FRANÇOISE (Université Paris-VI) who made us discover Jouanolou's theorem some years ago. We thank also sincerely Henryk ZOŁĄDEK (Warsaw University) who kindly send us his preprint [19] as well as Andrzej MACIEJEWSKI (Toruń University) for many inspiring discussions.

The third author acknowledges the Institute of Mathematics of Toruń University for his hospitality and excellent working conditions during December 1992 and July 1993.

REFERENCES

[1] CERVEAU (D.) and LINS-NETO (A.). – Holomorphic foliations in $\mathbf{C}P(2)$ having an invariant algebraic curve, *Ann. Inst. Fourier*, Vol. 41, (4), 1991, p. 883-903.

[2] DARBOUX (G.). – Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré, *Bull. Sc. Math.*, 2e série, T. 2, 1878, p. 60-96, 123-144, 151-200.

[3] EULER (L.). – De aequationis differentialibus secundi gradus, *Nov. Comm. Acad. Sci. Petrop.*, Vol. 7, 1761, p. 163-202.

[4] EULER (L.). – *Institutiones calculi integralis*, Vol. 3. – Petropoli, 1770; reprinted in *Opera Mathematica*, Vol. 13. – Leipzig, B. G. Teubner, 1914.

[5] FULTON (W.). – *Algebraic Curves, An Introduction to Algebraic Geometry*. – Reading, Addison-Wesley, 1969 (*Advanced Book Classics*).

[6] GRAMMATICOS (B.), MOULIN OLLAGNIER (J.), RAMANI (A.), STRELCYN (J.-M.) and WOJCIECHOWSKI (S.). – Integrals of quadratic ordinary differential equations in $\mathbf{R}^3$: the Lotka-Volterra system, *Physica A*, Vol. 163, 1990, p. 683-722.

[7] HOFBAUER (J.) and SIGMUND (K.). – *The Theory of Evolution and Dynamical Systems. Mathematical Aspects of Selection.* – Cambridge, Cambridge University Press, 1988. (*London Math. Society Student Text,* 7).

[8] JOUANOLOU (J.-P.). – *Équations de Pfaff algébriques.* – Berlin, Springer-Verlag, 1979 (*Lect. Notes in Math.,* No. 708).

[9] KHOVANSKII (A. G.). – Fewnomials and Pfaff manifolds, in *Proc. of the Internat. Congress of Math.* [August 16-24, 1983, Warszawa], Vol. 1, p. 549-564. – Warszawa, Amsterdam, New York, Oxford, Polish Scient. Publ. and North-Holland, 1984.

[10] KHOVANSKII (A. G.). – *Fewnomials.* – Providence, American Mathematical Society, 1991 (*Translations of Mathematical Monographs*, Vol. 88)

[11] LANG (S.). – *Algebra.* – Addison-Weseley Publ. Comp., 1965.

[12] LINS-NETO (A.). – Algebraic solutions of polynomial differential equations and foliations in dimension two, in "*Holomorphic dynamics*" , p. 193-232. – Berlin, Springer-Verlag, 1988 (*Lect. Notes in Math.,* No. 1345).

[13] MAL'CEV (A. I.). – *Foundations of Linear Algebra.* – San Francisco, W. H. Freeman and Company, 1963.

[14] MOULIN OLLAGNIER (J.). – Liouvillian first integrals of polynomial vector fields (preprint 1992, 17 p.).

[15] POINCARÉ (H.). – Sur l'intégration algébrique des équations différentielles, *C. R. Acad. Sc. Paris*, T. 112, 1891, p. 761-764; reprinted in *Œuvres*, T. III, p. 32-34. – Paris, Gauthier-Villars, 1965.

[16] POINCARÉ (H.). – Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré, *Rendic. Circ. Matem. Palermo*, Vol. 5, 1891, p. 161-191; reprinted in *Œuvres*, T. III, p. 35-58. – Paris, Gauthier-Villars, 1965.

[17] POINCARÉ (H.). – Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré, *Rendic. Circ. Matem. Palermo*, Vol. 11, 1897, p. 193-239; reprinted in *Œuvres*, T. III, p. 59-94. – Paris, Gauthier-Villars, 1965.

[18] SINGER (M. F.). – Liouvillan first integrals of differential equations, *Trans. Amer. Math. Soc.*, Vol. 333, 1992, p. 673-688.

[19] ZOŁĄDEK (H.). – On algebraic solutions of algebraic Pfaff equations, *Studia Mathematica* (to be published), 10 p.