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On first integrals of linear systems, Frobenius integrability theorem and linear representations of Lie algebras

Jean MOULIN OLLAGNIER*
Jean-Marie STRELCYN†

Abstract

A necessary condition to be satisfied by $n-1$ vector fields in $\mathbb{R}^n$ in order to have a common first integral is supplied by the compatibility condition of Frobenius integrability theorem. This condition is also generically sufficient for the local existence of such a common first integral. We study here the question of the existence of a global common first integral for compatible linear vector fields in $\mathbb{R}^n$.

For the dimension 3, we prove that any two compatible linear vector fields have a common global first integral.

On the contrary, we give an example for the dimension 4, in which three compatible linear vector fields cannot have a common global first integral.

This leads us to ask many simple and natural questions, some of them about representations of Lie algebras by Lie algebras of linear vector fields.

Some historical comments and abundant references are also provided.

1 Introduction

Let us consider two systems of homogeneous linear differential equations with constant coefficients in $\mathbb{R}^3$:

$$\frac{du}{dt} = L_1(u) = A_1 u, \quad \frac{du}{dt} = L_2(u) = A_2 u \tag{1}$$

where u belongs to $\mathbb{R}^3$ and where A_1 and A_2 are real 3×3 matrices.

The point of departure of the present paper is the problem of the existence of a common non-trivial first integral for both systems (1). This problem seems to

*Département de Mathématiques et Informatique, UA CNRS 742, C. S. P., Université Paris-Nord, Avenue J. B. Clément 93430 VILLETANEUSE, FRANCE

†Département de Mathématiques, Université de Rouen, B.P. 118, 76134 MONT-SAINT-AIGNAN CEDEX, FRANCE, UA CNRS 742 & 1378

have never been studied before. If F is such a common integral, the level surfaces $\{F = const.\}$ are tangent to both vector fields L_1 and L_2. The compatibility condition from the Frobenius integrability theorem is thus necessarily satisfied, i. e. at any point u of $\mathbb{R}^3$, the three vectors $L_1(u)$, $L_2(u)$ and $[L_1, L_2](u)$ are linearly dependent. As usual, $[L_1, L_2]$ denotes the Lie bracket of the two vector fields; here, as L_1 and L_2 are linear, one has $[L_1, L_2](u) = -[A_1, A_2](u)$, where $[A_1, A_2] = A_1A_2 - A_2A_1$ is the matrix commutator.

The compatibility condition is equivalent to the following one:

$$\det(L_1(u), L_2(u), [L_1, L_2](u)) = 0 \tag{2}$$

for every u in $\mathbb{R}^3$.

It is worth noting that this property does not imply that the three vector fields L_1, L_2 and $[L_1, L_2]$ are linearly dependent over $\mathbb{R}$.

In the following, any two, not necessarily linear, smooth vector fields satisfying condition (2) will be called *compatible*.

Although the Frobenius integrability theorem guarantees that two compatible vector fields have a common first integral around any point at which these vector fields are linearly independent, nothing can be said on the existence of a global first integral without a further study of the concrete framework.

Our first result asserts that two compatible linear vector fields defined on $\mathbb{R}^3$ always have a common first integral, typically with some singularities.

Let us note that a similar result was also obtained by P. Basarab-Horwath and S. Wojciechowski [4].

Let us give an example. Consider the two matrices A_1 and A_2

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The following function $F(x, y, z)$ is easily shown (cf. [77, 32]) to be a common global first integral for both systems (1) corresponding to the matrices A_1 and A_2

$$F(x, y, z) = \frac{y^2 - 2xz}{2z^2} + \log |\, z \,| \, .$$

Having succeeded in proving the existence of a common global first integral for systems (1), we learnt that the solution of this problem was in fact almost entirely, but implicitly, contained in the classical works of C. G. Jacobi [36] and D. Poisson (cf. [24]). Their result are clearly stated in the classical textbooks of E. Goursat [31] and E. L. Ince [35].

Our point of view is somewhat different so that our solution has some special features. In particular, we consider very carefully the question of the uniformity of our integrals; we also use the intrinsic, coordinate free, very economical approach with the differential forms.

After having sought a common first integral of two linear compatible vector fields in $\mathbb{R}^3$, we were naturally led to consider a similar problem in higher dimensions.

Let $L_1, \ldots, L_k$ be some smooth vector fields defined on $\mathbb{R}^n$, $2 \leq k < n$, satisfying for all indices i and j and every point u in $\mathbb{R}^n$ the compatibility condition of the Frobenius integrability theorem:

$$L_1(u), L_2(u), \ldots, L_k(u), [L_i, L_j](u) \text{ are linearly dependent.} \tag{3}$$

Recall that the Frobenius integrability theorem asserts that the compatibility condition (3) is sufficient to find, around any point u at which the k vectors $L_1(u), \ldots, L_k(u)$ are linearly independent, $n-k$ functionally independent common first integrals $\Phi_1, \ldots, \Phi_{n-k}$.

The level manifolds $\{\Phi_1 = c_1, \ldots, \Phi_{n-k} = c_{n-k}\}$ thus define a smooth k-dimensional foliation of some neighborhood of u. This local foliation can be extended to a global one [76].

The simplest higher dimensional problem is the following one. Is is true that three linear compatible vector fields in $\mathbb{R}^4$ always have a common global first integral, perhaps with some singularities ? The answer is surprisingly no and we give an example; this is our main result and the true "raison d'être" of this paper.

Anticipating a little, let us say that, in this example, all but two global leaves of the associated foliation are everywhere dense in $\mathbb{R}^4$.

Let us stress that such an example is of direct interest in control theory in relation with the notion of set of accessibility (cf. [28, 78]). Indeed, we have here an approximate controlability despite of the compatibilty (integrability) of our linear vector fields.

All this leads us to ask many simple and natural questions, some of them about representations of Lie algebras by Lie algebras of linear vector fields, that do not seem to have been formulated before.

Although completely independent and self-contained, the present paper is a sequel of [77] and of [32] where the compatibility condition (2) is used as an effective tool for the search of first integrals of some non-linear systems of three autonomous ordinary differential equations.

The paper is organized as follows. In section 2, we recall some facts from exterior calculus; in section 3, we describe, in the exterior form framework, the integrability results in $\mathbb{R}^3$ and give the outline of the proof. The complete proof of the result is presented in section 4 while section 5 consists of the description of our example of non-integrability in $\mathbb{R}^4$ and its easy extension to higher dimensional cases. In section 6, we formulate some questions and give additionnal remarks, mainly of historical nature.

The problems related to those studied in the present paper were intensively investigated by many people. See, for example, [1, 2, 3, 4, 8, 9, 10, 11, 13, 21, 22, 23, 24, 26, 27, 30, 31, 32, 35, 36, 37, 38, 39, 41, 42, 43, 44, 47, 50, 51, 54, 55, 56, 60, 61, 62, 63, 64, 65, 66, 68, 69, 70, 71, 77, 81], and this list is very far from being complete.

Let us stress that the important books by D. Cerveau and F. Mattei [11] and by J.-P. Jouanolou [37] are devoted to problems directly related to ours.

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2 Some facts from exterior calculus

We recall now some well-known and useful definitions and results about exterior differentiation, inner products and volume form (cf. [7, 18, 53, 58, 76] for more details) as well as Euler's theorem on homogeneous functions.

All differential forms and vector fields are supposed to be defined and sufficiently differentiable on a non-empty open subset U of $\mathbb{R}^n$. We denote by w^r an exterior r-form.

The exterior differentiation. The exterior derivative is a linear map d from the set of differential forms into itself that increases the degree by 1 (so that $dw = 0$ for n-forms), whose square $d \circ d$ is the null map; moreover, d is an antiderivation with respect to the exterior product of differential forms, i. e.

$$d\,(w^p \wedge w^q) = (dw^p) \wedge w^q + (-1)^p\, w^p \wedge (dw^q). \tag{4}$$

The inner product. The inner product $i(X).(.)$ by a vector field X is a linear map from the set of differential forms into itself that decreases the degree by 1; thus $i(X).(f) = 0$ for 0-forms, i. e. functions. If w^p is a p-form, $i(X).(w^p)$ is the $(p-1)$-form given by

$$i(X).(w^p)\,(v_1,\ldots,v_{p-1}) = w^p(X,v_1,\ldots,v_{p-1})$$

where $v_1,\cdots,v_{p-1}$ are vector fields.

With respect to the differential forms, the inner product is an antiderivation:

$$i(X).(w^p \wedge w^q) = (i(X).(w^p)) \wedge w^q + (-1)^p\, w^p \wedge (i(X).(w^q)). \tag{5}$$

On the other hand, the inner product is obviously anticommutative with respect to the vector fields and, in particular, two successive inner products by the same vector field yield 0.

In coordinate form, the inner product $i(X).(w)$ of a 1-form $w = \sum_{i=1}^{n} w_i dx_i$ by a vector field $X = \sum_{i=1}^{n} X_i \partial/\partial x_i$ is equal to $\sum_{i=1}^{n} X_i w_i$.

In particular, a smooth function F is a first integral of a vector field X if it satisfies:

$$i(X).(dF) \;=\; X(dF) \;=\; \sum_{i=1}^{n} X_i\, \partial F/\partial x_i \;=\; 0.$$

Volume form. Denote by $x_1,\cdots,x_n$ the cartesian coordinates in $I\!R^n$. The volume form Ω is the exterior n-form $\Omega \;=\; dx_1\, \cdots\, dx_n$.

Given n vectors $X_1,\cdots,X_n$ in $I\!R^n$, $\Omega(X_1,\cdots,X_n)$ is equal to the determinant $\det((X_1,\cdots,X_n))$, where $(X_1,\cdots,X_n)$ is the $n \times n$ matrix, whose columns are the vectors $X_1,\cdots,X_n$.

Euler's theorem on homogeneous functions. A function f defined on $I\!R^n$ is said to be homogeneous of degree k if, for every point x in $I\!R^n$ and every positive real number t, $f(tx) = t^k f(x)$.

The famous Euler's theorem on homogeneous functions in $I\!R^n$ asserts that a smooth function f defined on $I\!R^n$ is homogeneous of degree k if and only if the following identity holds:

$$\sum_{i=1}^{n} x_i\, \partial f/\partial x_i \;=\; k\, f. \tag{6}$$

From the previous identity, a generalized Euler's formula can be deduced; if w is a p-form in $I\!R^n$, all of whose components are homogeneous functions of degree k, and if I is the so-called radial vector field $I = \sum_{i=1}^{n} x_i\, \partial/\partial x_i$, then the following identity holds:

$$i(I).(dw) + d\,(i(I).(w)) \;=\; (p+k)\, w. \tag{7}$$

Euler's original identity (6) is a special case, when p is equal to 0, of the generalized one and can then be written as:

$$i(I).(df) = k\,f.$$

3 Integrability in $\mathbb{R}^3$: outline of the proof

In this section, we give the outline of the proof of the following theorem:

Any two compatible linear vector fields in $\mathbb{R}^3$ have a global common first integral.

We must first carefully define the notion of a global first integral; let us give it in a general context, not only for linear fields.

Let us consider a smooth vector field X defined on $\mathbb{R}^n$ or on some open subset U of it. A *global first integral* F of X is a smooth function defined on a dense open X-invariant subset V of U, which satisfies the identity $XF = i(X).(dF) = 0$ at every point of V, and which is not constant on any open subset of V.

A subset E of U is said to be X-invariant if it consists of complete trajectories of the field X; this means that no segment of an X-trajectory can join a point of E to a point of $U \setminus E$. Equivalently, E and $U \setminus E$ are *locally X-invariant*, i. e. invariant under the local flow induced by X.

Let us remark that the escape to infinity in finite time cannot be generally excluded, so that the complete X-trajectories are not necessarily described by a time parameter going from $-\infty$ to $+\infty$. Nevertheless, this phenomenon never occurs with linear vector fields.

A *linear vector field* in $\mathbb{R}^n$ is a vector field whose components are homogeneous linear polynomials with respect to the space variables $x_1, \cdots, x_n$. In the case $n = 3$, we write naturally the variables x_1, x_2, x_3 as x, y and z.

Due to the algebraic aspect of the problem, the invariant subset V, on which we define the common first integral of two linear vector fields on $\mathbb{R}^3$ will be the complement of the set of zeros of a finite number of real polynomials, i. e. a dense Zariski open subset of $\mathbb{R}^3$.

Given two linear vector fields L_1 and L_2, denote by w the 1-form defined by $w = i(L_1).(i(L_2).(\Omega))$ where Ω is the volume 3-form $\Omega = dx\,dy\,dz$. These two vector fields are compatible if and only if w is *integrable* i. e. satisfies the integrability condition:

$$w \wedge dw = 0. \tag{8}$$

If w vanishes everywhere, L_1 and L_2 are either linearly dependent vector fields or multiples of the same constant vector field. In this case our theorem relies on the easily proven fact that a linear or constant vector field has always a global first integral.

We shall therefore only consider pairs (L_1, L_2) of compatible linear vector fields such that the 1-form w, whose coefficients are homogeneous quadratic polynomials in the space variables, does not identically vanish; this 1-form w is then different from 0 on a dense open subset U of $\mathbb{R}^3$.

The derivative dF of a common first integral of the two fields is everywhere colinear to w; indeed, consider the obvious identity $\Omega \wedge dF = 0$, take its inner product by L_2, then by L_1 (cf. (5)) to get $w \wedge dF = 0$, which means that the two 1-forms w and dF are colinear.

The first step then consists in finding an integrating factor for w, i. e. a function ϕ such that $d(\phi w) = 0$. A primitive F of this closed 1-form will then be the desired first integral, provided that F is uniform, i. e. univalued on its domain of definition.

The Frobenius integrability theorem yields the local existence of an integrating factor for a 1-form w satisfying the integrability condition (8). But we are interested in a global solution to the problem; we show that there exists a non-zero homogeneous cubic polynomial Q such that $1/Q$ is the desired integrating factor of w. Our proof is then in fact independent of the Frobenius theorem.

Let us first suppose that w is *irreducible* (and this is typically the case), which means that the components of w have no non-trivial polynomial common factor.

Consider the homogeneous cubic polynomial $P = i(I).(w)$, where I is the radial linear vector field $I = x\, \partial/\partial x + y\, \partial/\partial y + z\, \partial/\partial z$. If P does not vanish identically, let V be the dense open subset of U where P is different from 0. In this case, $1/P$ can be choosen as an integrating factor of w on V. Moreover P satisfies $dP \wedge r = 0$ where $r = dw$ is the exterior differential of w.

When $P = i(I).(w)$ vanishes identically, the inverse $1/Q$ of a cubic homogeneous non-zero polynomial Q is an integrating factor for w if and only if $dQ \wedge r = 0$; and such polynomials do exist; and, in this case, we call V the dense open subset of U where Q is different from 0.

We are then faced with two global problems. Knowing that w/Q is a closed 1-form defined on the dense open subset V of $\mathbb{R}^3$, on which w and Q do not vanish, we have to integrate it, i. e. to study the topology of the connected components of V; in order to show that V is natural with respect to our problem, we must also prove that it consists of complete trajectories of the two original linear vector fields L_1 and L_2.

To solve the first geometrical question, we apply the classification of the closed non-zero 2-forms r in $\mathbb{R}^3$, whose coefficients are homogeneous linear polynomials, under the action of $SL(3, \mathbb{R})$; and we give, in each case, a description of the vector space of all homogeneous cubic polynomials Q such that $dQ \wedge r = 0$.

In this way, besides the fact that such non-zero cubic polynomials always exist, it also appears that the connected components of the complement of the set of zeros of any such polynomial are either simply connected, or of degree of connectivity two, i. e. their fundamental group is isomorphic to the group $\mathbb{Z}$ of all relative integers.

On a simply connected component V_0 of V, a closed 1-form is exact and we get our first integral. Otherwise the integration of w/Q can lead us to consider a first integral with values in the circle $S^1 = \mathbb{R}/k\mathbb{Z}$ instead of the real line $\mathbb{R}$, where k is the smallest strictly positive jump of the integral on a closed non contractible curve in V_0.

This kind of multivalued first integral can nevertheles be considered as a good parametrization of the set of leaves of the foliation given by the 1-form. Let us note that whenever such a first integral F is known, $G(u) = \sin(\frac{2\pi}{k}u)$ defines a first integral in the usual sense.

On the other hand, to prove that the involved open set $V = \{Q \neq 0, w \neq 0\}$ consists of complete trajectories, we prove the local invariance of its complement under the two linear vectors fields. More precisely, we prove that the set $\{Q = 0, w \neq 0\}$ is locally invariant for any polynomial vector fields X such that $i(X).(w) = 0$. To prove the local X-invariance of $\{w = 0\}$, we use essentially the fact that w is irreducible; if w is not irreducible, the result follows from the consideration of some irreducible 1-forms of lower degree.

4 Integrability in $\mathbb{R}^3$: the proof

We begin this section with a classification of the closed 2-forms in $\mathbb{R}^3$, whose coefficients are real homogeneous linear polynomials, with respect to a linear change of variables in $\mathbb{R}^3$.

This classification relies on the corresponding classification of linear vector fields in $\mathbb{R}^3$, which in turn is nothing else but the well-known classification of linear mappings from $\mathbb{R}^3$ to itself. In what follows, we do not distinguish between linear vector fields and linear mappings.

In $\mathbb{R}^3$, it can indeed be easily verified that the mapping ϕ

$$l_x\,\partial/\partial x + l_y\,\partial/\partial y + l_z\,\partial/\partial z \xrightarrow{\phi} i(l_x\,\partial/\partial x + l_y\,\partial/\partial y + l_z\,\partial/\partial z).(\Omega)$$

establishes an isomorphism between the vector space of all linear vector fields $L = l_x\,\partial/\partial x + l_y\,\partial/\partial y + l_z\,\partial/\partial z$ in $\mathbb{R}^3$ and the vector space of all 2-forms whose coefficients are homogeneous linear polynomials in $\mathbb{R}^3$.

Moreover, this mapping ϕ commutes with a linear change of variables, provided that this change belongs to the special linear group $SL(3,\mathbb{R})$, which preserves the volume form Ω.

Indeed, let ρ_L be the image $\rho_L = \phi(L) = i(L).(\Omega)$ of a linear vector field L

under ϕ. Recall that, for any point u of $\mathbb{R}^3$ and any two vectors A and B of $\mathbb{R}^3$

$$\rho_L(u)(A,B) = \det(Lu, A, B).$$

Now, if T is an invertible linear mapping from $\mathbb{R}^3$ to itself, it can easily be shown that

$$\rho_L(Tu)(TA,TB) = \det(T)\,\rho_{T^{-1}LT}(u)(A,B).$$

Thus, up to a non-zero multiplicative constant, the classification of our 2-forms under the action of $GL(3,\mathbb{R})$ is the same as the real linear classification of 3×3 real matrices; and, similarly, the classification under the action of $SL(3,\mathbb{R})$ of the 2-forms, whose coefficients are linear polynomials, agrees with the corresponding classification of matrices.

Moreover, the 2-forms r, that we are interested in, are closed; this corresponds to the vanishing trace of the linear mappings $L = \phi^{-1}(r)$. Let us now state a general remark: a smooth function Q is a first integral of a smooth vector field X defined on $\mathbb{R}^3$ if and only if $dQ \wedge (i(X).(\Omega)) = 0$.

As we are interested in the description of the vector space of all cubic homogeneous polynomials Q which are first integrals of L, i. e. such that $dQ \wedge r = 0$, our classification is more detailled than the linear classification of vanishing trace linear mappings.

Proposition 1 *Consider the following nine canonical forms $L_1, \cdots, L_9$ of vanishing trace linear mappings of $\mathbb{R}^3$*

$$L_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \begin{array}{l} \lambda \neq 0, \mu \neq 0, \nu \neq 0, \lambda \neq \mu \neq \nu \neq \lambda \\ \lambda + \mu + \nu = 0 \end{array}$$

$$L_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda \neq 0$$

$$L_3 = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \quad \begin{array}{l} a \neq 0 \\ b \neq 0 \end{array}, \qquad L_4 = \begin{pmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad b \neq 0$$

$$L_5 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \quad a \neq 0, \qquad L_6 = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \quad a \neq 0$$

$$L_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let L be a vanishing trace linear mapping of $\mathbb{R}^3$. Then L is conjugate in $GL(3,\mathbb{R})$ with exactly one of the canonical forms. Moreover, if $L = TL_iT^{-1}$, where L_i is one of the canonical form, mapping T can be choosen in $SL(3,\mathbb{R})$.

Proof. We built this classification according to the multiplicity and non-nullity of eigenvalues.

The nine cases then correspond to:

1. three different real non-zero eigenvalues,
2. three different real eigenvalues, one of them is 0,
3. three different non-zero eigenvalues, with only one of them real,
4. three different eigenvalues, one of them 0 and the two other conjugate,
5. a double (real, non-zero) eigenvalue, diagonalizable case,
6. a double (real, non-zero) eigenvalue, non-diagonalizable case,
7. a triple 0 eigenvalue, rank 2,
8. a triple 0 eigenvalue, rank 1,
9. a triple 0 eigenvalue, rank 0, i. e. the 0 matrix.

It is not difficult to verify, in each case, that matrices with an arbritrary real determinant exist in the commutant of a canonical matrix; then, the element T of $GL(3,\mathbb{R})$ such that $L = TL_iT^{-1}$ can be choosen in $SL(3,\mathbb{R})$. ■

The above proposition implies immediately the following one.

Proposition 2 *Let r be a non-zero closed 2-form in $\mathbb{R}^3$, the coefficients of which are homogeneous linear polynomials. Under a linear change of variables belonging to $SL(3,\mathbb{R})$, r is conjugate to one and only one of the following canonical forms:*

1. $r = \lambda\, x\, dy \wedge dz + \mu\, y\, dz \wedge dx + \nu\, z\, dx \wedge dy$
   with $\lambda \neq 0, \mu \neq 0, \nu \neq 0, \lambda \neq \mu \neq \nu \neq \lambda$ *and* $\lambda + \mu + \nu = 0$
2. $r = \lambda\, x\, dy \wedge dz - \lambda\, y\, dz \wedge dx$, *with* $\lambda \neq 0$
3. $r = (a\, x - b\, y)\, dy \wedge dz + (b\, x + a\, y)\, dz \wedge dx - 2a\, z\, dx \wedge dy$
   with $a \neq 0, b \neq 0$
4. $r = -b\, y\, dy \wedge dz + b\, x\, dz \wedge dx$, *with* $b \neq 0$
5. $r = a\, x\, dy \wedge dz + a\, y\, dz \wedge dx - 2\, a\, z\, dx \wedge dy$, *with* $a \neq 0$
6. $r = (a\, x + y)\, dy \wedge dz + a\, y\, dz \wedge dx - 2\, a\, z\, dx \wedge dy$, *with* $a \neq 0$
7. $r = y\, dy \wedge dz + z\, dz \wedge dx$
8. $r = z dy \wedge dz$

It is now easy, although slightly cumbersome, to compute the general form of a cubic homogeneous polynomial Q such that $dQ \wedge r = 0$ in each of the previous cases. These computations are summarized in the following proposition.

Proposition 3 *In each case of the previous classification, the corresponding vector space of all third degree homogeneous real polynomials Q such that $dQ \wedge r = 0$ is generated by the following polynomials:*

1. $Q_1 = xyz$,
2. $Q_1 = xyz$ *and* $Q_2 = z^3$,
3. $Q_1 = (x^2 + y^2)z$,
4. $Q_1 = (x^2 + y^2)z$ *and* $Q_2 = z^3$,
5. $Q_1 = x^2 z$, $Q_2 = xyz$ *and* $Q_3 = y^2 z$,
6. $Q_1 = y^2 z$,
7. $Q_1 = z^3$ *and* $Q_2 = z(y^2 - 2xz)$,
8. $Q_1 = y^3$, $Q_2 = y^2 z$, $Q_3 = yz^2$ *and* $Q_4 = z^3$.

After these algebraic preliminaries, we pass on to the heart of the matter.

Let us denote by w an arbitary exterior 1-form defined in $\mathbb{R}^3$, whose coefficients are homogeneous quadratic polynomials in the space variables x, y and z; r stands for the closed 2-form dw and the integrability condition $w \wedge r = 0$ holds.

Let us now describe how inverses of cubic polynomials can be used as integrating factors for such exterior 1-forms.

Proposition 4 *If the inverse of a non-zero homogeneous polynomial Q is a integrating factor for w, then Q satisfies $dQ \wedge r = 0$.*

Proof. The hypothesis means that the 1-form w/Q is closed on the open set $W = \{u \in \mathbb{R}^3, Q(u) \neq 0\}$, i. e. that the following identity holds on W (cf. (4))

$$d\,(w/Q) = (1/Q^2)(Qdw - dQ \wedge w) = 0.$$

Differentiating the numerator yields the result

$$0 = d\,(Qdw - dQ \wedge w) = 2dQ \wedge r.$$

That completes the proof. ∎

Proposition 5 *Let P be the cubic homogeneous polynomial $P = i(I).(w)$. If P does not vanish identically, then $1/P$ is a integrating factor for w; if $P = 0$, then for every non-zero cubic homogeneous polynomial Q such that $dQ \wedge r = 0$, $1/Q$ is a integrating factor for w.*

Proof. When P is different from 0, it suffices, in order to show that $1/P$ is an integrating factor w, to prove that the numerator of $d(w/P)$ is equal to 0, i. e. that $Pdw = dP \wedge w$. Thanks to the generalized Euler's formula (7), applied to the homogeneous 1-form w, the following identity holds

$$3w = i(I).(r) + dP. \tag{9}$$

Exterior multiplication by w yields

$$0 = i(I).(r) \wedge w + dP \wedge w.$$

The desired equality then follows from the inner product by I of the identity $w \wedge r = 0$ (cf. (5))

$$0 = i(I).(w \wedge r) = i(I).(w)\, r + i(I).(r) \wedge w.$$

Comparing the last two equalities, one obtains $P\,dw = dP \wedge w$ as needed.

When P is equal to 0, formula (9) allows us to define w from its exterior differential $dw = r$ by $3w = i(I).(r)$. To prove that the inverse $1/Q$ is an integrating factor for w it suffices to show that

$$Qr = dQ \wedge w.$$

To prove this identity, we apply the inner product by I to equality $dQ \wedge r = 0$ (cf. (5) and (6)):

$$0 = i(I).(dQ \wedge r) = i(I).(dQ)\, r - dQ \wedge i(I).(r) = 3Q\, r - 3dQ \wedge w.$$

And the proof is now complete. ∎

The following proposition is the key result to show that an exterior integrable 1-form, whose coefficients are homogeneous quadratic polynomials, has a global first integral. To formulate this proposition in a concise manner, as explained at the end of section 3, by functions we will not only understand real-valued functions, but also circle-valued ones.

Proposition 6 *Let r be a non-zero closed 2-form defined in $I\!R^3$, whose coefficients are linear homogeneous polynomials, and let Q be a non-zero cubic homogeneous polynomial such that $dQ \wedge r = 0$.*

Then the closed form $(1/Q)i(I).(r)$ is exact on every connected component of the open dense set $W = \{u \in I\!R^3, Q(u) \neq 0\}$.

Proof. The above mentioned property does not depend on a linear change of variables; it then suffices to show it for any canonical form of r listed in Proposition 2.

In all cases, the connected components of W are easily shown either to be simply connected, or to have a fundamental group isomorphic to the group $\mathbb{Z}$ of all relative integers.

In the first case, a closed 1-form is exact, which means that a real-valued function F such that $dF = (1/Q)i(I).(r)$ does exist. The same is true in the second case, if the value k of the integral of $(1/Q)i(I).(r)$ on a closed curve corresponding to a generator of the fundamental group is equal to 0. If this value k is not zero, the 1-form is the derivative of a function from W to the circle $\mathbb{R}/k\mathbb{Z}$. This situation cannot be avoided in some cases of the classification (cases 3 and 5). ∎

We need now a proposition to ensure that, if the cubic polynomial Q yields an integrating factor for a 1-form w, then the subset $\{Q \neq 0, w \neq 0\}$ consists of global leaves of the foliation defined by w.

Proposition 7 *Let w be a smooth integrable 1-form defined on some open subset U of $\mathbb{R}^3$ on which it does not vanish. Let Q be a smooth function defined on U such that $Q\,dw = dQ \wedge w$, i. e. such that $1/Q$ is an integrating factor of w on the set $\{u \in U, Q(u) \neq 0\}$. Let m and m' be the beginning and the end of a smooth path lying in some leaf of the foliation of U defined by w and suppose that $Q(m) = 0$. Then, $Q(m')$ also vanishes.*

Proof. Because w does not vanish on U, a smooth vector field X such that $i(X).(w) = 1$ can be defined on U. The inner product by X of the identity $Q\,dw = dQ \wedge w$ yields (cf. (5))

$$Q\,i(X).(dw) = i(X).(dQ)\,w - dQ. \tag{10}$$

Consider now a smooth path lying in the leaf of m from m to m'. This path is a smooth mapping ϕ from some real interval $[0,a]$ to U. Denote by ψ the function $Q \circ \phi$.

Because the tangent vector $d\phi/dt$, $0 \leq t \leq a$, is everywhere tangent to the leaf, the previous equality (10) yields by an inner product by $d\phi/dt$

$$\psi(t)\,i(d\phi/dt).(i(X[\phi(t)]).(dw[\phi(t)])) = -d\psi/dt.$$

The continuous function $i(d\phi/dt).(i(X[\phi(t)]).(dw[\phi(t)]))$ is bounded on the compact interval $[0,a]$ and we deduce an a priori estimate

$$|\,d\psi/dt\,| \leq C\,|\,\psi\,| \tag{11}$$

everywhere on the interval $[0,a]$ with some positive constant C.

But as $Q(m) = 0$, $\psi(0) = 0$ and, thanks to a Gronwall lemma, $\psi(a)$ is also 0, i. e. $Q(m') = 0$, which completes the proof. ∎

Given an integrable 1-form w, a *first integral* of w is any smooth function Φ such that $d\Phi = \alpha w$ for some function α and such that Φ is not constant on any open set.

We can now conclude with two theorems.

Theorem 1 *Let w be an integrable non-zero 1-form defined in $\mathbb{R}^3$, whose coefficients are homogeneous quadratic polynomials. This form has a global first integral defined on a open dense subset V of $\mathbb{R}^3$ consisting of global leaves of the foliation given by w.*

Proof. It follows from the identity (9) and from Propositions 3–7. More precisely, let as usual P be the inner product $P = i(I).(w)$.

If $P \neq 0$, $1/P$ is an integrating factor for w and w/P can be written

$$w/P = (1/3)(dP/P + (i(I).(r))/P)$$

where $r = dw$.

The closed form dP/P is exact and the 1-form $(i(I).(r))/P$ is exact on the connected components of the open set $V = \{u \in \mathbb{R}^3, P(u) \neq 0, w(u) \neq 0\}$.

If $P = 0$, then w is equal to $(1/3)i(I).(r)$ (cf. (9)), where $r = dw$. Let then Q be a non-zero homogeneous cubic polynomial such that $dQ \wedge r = 0$; r does not vanish identically and such polynomials do exist according to the previously described classification given in Proposition 3.

Function $1/Q$ is then an integrating factor of w on the open set V defined by $V = \{u \in \mathbb{R}^3, Q(u) \neq 0, w(u) \neq 0\}$ and the 1-form $i(I).(r)/Q$ is exact on the connected components of V.

In both cases, a closed form can be integrated up to a real or circle valued function on the connected components of the set V.

Finally Proposition 7 shows that the open set V consists of global leaves of the foliation defined by w on the open set $U = \{u \in \mathbb{R}^3, w(u) \neq 0\}$. ■

Let us now recall that a non-zero 1-form $w = w_x dx + w_y dy + w_z dz$ defined in $\mathbb{R}^3$, whose coefficients are homogeneous real polynomials, is irreducible if w_x, w_y and w_z have no non-trivial polynomial common factor.

If a polynomial S is a common factor of the coefficients of w, then the reduced form w/S defines the same foliation as w, but perhaps on a larger open subset of $\mathbb{R}^3$; it is therefore natural to consider irreducible 1-forms.

The next proposition shows the interest of this assumption for the trajectories of vector fields that are orthogonal to such a form.

Proposition 8 *Let w be an integrable 1-form defined on $\mathbb{R}^3$, whose coefficients are homogeneous polynomials of the same degree. Let X be a non-zero vector field defined on $\mathbb{R}^3$, whose coefficients are homogeneous polynomials of the same* *degree; suppose that the identity* $i(X).(w) = 0$ *holds on* $\mathbb{R}^3$ *and that* w *is irreducible.*

Then, the subset $\{w = 0\}$ *is locally* X*-invariant, so that the open subset* $\{w \neq 0\}$ *consists of complete trajectories of the field* X*.*

Proof. Consider the integrability relation $w \wedge dw = 0$ and take its inner product by X to get $w \wedge (i(X).(dw)) = 0$.

The polynomial 1-forms w and $i(X).(dw)$ are then colinear on some non-empty open subset of $\mathbb{R}^3$, and there exists an irreducible rational function N/D such that $i(X).(dw) = (N/D)\,w$, i. e. such that $N\,w = D\,i(X).(dw)$.

Because w is irreducible, the polynomial D is a constant and we thus get

$$i(X).(dw) = N_1\,w \tag{12}$$

where N_1 is some polynomial.

Recall now the well known formula for the Lie derivative $\mathcal{L}_X w$ of an exterior form with respect to a vector field X (cf. [53, 58])

$$\mathcal{L}_X w = i(X).(dw) + d[i(X).(w)].$$

As $i(X).(w) = 0$, $i(X).(dw)$ is the Lie derivative $\mathcal{L}_X w$. Taking into account (12), an a priori estimate, like (11), can then be established showing that, if $w(m) = 0$, then w remains equal to 0 along the trajectory of the field X passing through m, which proves the result.

As proven by easy examples, the irreducibility assumption is essential here. ∎

Theorem 2 *Any two compatible linear vector fields* L_1 *and* L_2 *defined in* $\mathbb{R}^3$ *have a common global first integral defined on a open dense subset of* $\mathbb{R}^3$ *consisting of complete trajectories of both fields.*

Proof. Consider the 1-form $w = i(L_1).(i(L_2).(\Omega))$. If $w = 0$, the fields are proportional and it is a well known fact that a global first integral exists for a linear field.

If $w \neq 0$, Theorem 1 shows that an integrating factor $1/Q$ exists for w. As $i(L_i).(w) = 0$ for $i = 1, 2$, the function F, defined on $V = \{w \neq 0, Q \neq 0\}$ and such that dF is proportional to w, is a first integral for both fields.

To prove that this integral is a global one, it remains to be shown that the open dense subset V consists of complete trajectories of L_1 and L_2. It therefore suffices to show that the subsets $\{Q = 0, w \neq 0\}$ and $\{w = 0\}$ are locally L-invariant, L being one of the two linear vector fields L_1, L_2, or more generally some vector field with polynomial coefficients such that $i(L).(w) = 0$.

As far as the first subset is concerned, take the inner product by L of the usual equality $Q\,dw = dQ \wedge w$ to get

$$Q\, i(L).(dw) \;=\; i(L).(dQ)\, w.$$

As w is supposed to be different from 0 in some neighborhood of a point m at which $Q(m) = 0$, this relation yields one more time an a priori estimate like (11) and Q vanishes on the L-trajectory around m.

As far as the second subset is concerned, Proposition 8 gives the proof if w is irreducible.

If w is not irreducible, we replace it by a simpler irreducible 1-form $w' = w/S$, whose coefficients are of the same degree 0 or 1. To complete the proof in this case, we must find an integrating factor $1/Q$ for w such that S is a factor of Q. Then the open subset on which the first integral is defined is $\{Q' \neq 0, w' \neq 0\}$ where $Q' = Q/S$.

In the case where $Q = i(I).(w) \neq 0$, S is of course a factor of Q. Otherwise, $(1/3)\, i(I).(r)$ is irreducible in each of the eight cases but two (5 and 8) of the classification given by Proposition 2. These are precisely the two cases in which one of the eigenvalues has a two-dimensional eigenspace. It is nevertheless possible to choose, in the vector space of all cubic polynomials Q such that $dQ \wedge r = 0$, a non-zero polynomial which is a multiple of the greatest common divisor of the coefficients of $i(I).(r)$. ∎

Let us finally note that the present proof of the existence of a common first integral also gives an algorithm to find it. For each of the canonical forms $L_1, \cdots, L_8$ of $r = dw$, it is easy to see that the corresponding first integral is expressed in finite terms. In what concerns case L_9, it follows from identity (9) that the cubic homogeneous polynomial $P = i(I).(w)$ is a common global first integral.

Let us underline that the appearence of an *arctan* means that the integral is in fact circle-valued.

5 An example of non-integrability in $\mathbb{R}^4$

We shall now find three compatible linear vector fields L_1, L_2 and L_3 in $\mathbb{R}^4$ which generate together a three dimensional foliation with dense leaves. This foliation is also defined by the 1-form $w = i(L_1).(i(L_2).(i(L_3).(\Omega)))$, where Ω is the volume 4-form $\Omega = dx\,dy\,dz\,dt$. The coefficients of w are cubic homogeneous polynomials and w satisfies the integrability condition $w \wedge dw = 0$.

In our example, a local integrating factor for w is easily found: it is equal to $1/P$, where $P = i(I).(w)$, I being the radial vector field $I = x\,\partial/\partial x + y\,\partial/\partial y + z\,\partial/\partial z + t\,\partial/\partial t$.

The local situation is thus very similar to the three dimensional one.

On the other side, the global non-integrability relies on the density of all non-singular leaves of the foliation, which impedes the existence of a continuous common first integral for the three fields.

The example. Let α and β be two rationally independent real numbers. Consider now the three linear vector fields L_1, L_2 and L_3:

$$\begin{aligned} L_1 &= (x-y)\,\partial/\partial x + (x+y)\,\partial/\partial y \\ L_2 &= (z-t)\,\partial/\partial z + (z+t)\,\partial/\partial t \\ L_3 &= -\alpha y\,\partial/\partial x + \alpha x\,\partial/\partial y - \beta t\,\partial/\partial z + \beta z\,\partial/\partial t. \end{aligned}$$

These three vector fields commute with one another, i. e. the three pairewise Lie brackets vanish, and they are thus compatible.

The 1-form $w = i(L_1).(i(L_2).(i(L_3).(\Omega)))$ is equal to

$$\begin{aligned} w &= \beta\,(z^2+t^2)((x+y)\,dx + (y-x)\,dy) \\ &- \alpha\,(x^2+y^2)((z+t)\,dz + (t-z)\,dt). \end{aligned}$$

Let then P be the fourth degree homogeneous polynomial

$$P = i(I).(w) = (\beta - \alpha)\,(x^2+y^2)(z^2+t^2).$$

This polynomial is different from 0 outside of the two 2-dimensional planes $x^2+y^2=0$ and $z^2+t^2=0$. Function $(\beta-\alpha)/P$ is an integrating factor of w on the complement of these two planes in $I\!R^4$.

The closed form $(\beta-\alpha)\,w/P$ is equal to

$$\begin{aligned} (\beta-\alpha)\,w/P &= \beta\,((x+y)\,dx + (y-x)\,dy)/(x^2+y^2) \\ &- \alpha\,((z+t)\,dz + (t-z)\,dt)/(z^2+t^2) \end{aligned}$$

and it can be formally integrated up to

$$\frac{\beta}{2}\log(x^2+y^2) - \beta\arctan(y/x) - \frac{\alpha}{2}\log(z^2+t^2) + \alpha\arctan(t/z).$$

The rational independence of α and β then prevents us from giving any signification to the sum "$\beta\arctan(y/x) - \alpha\arctan(t/z)$".

We now give a more precise explanation of this fact by showing that every leaf of the foliation of the open invariant set $V = \{m \in I\!R^4,\ P(m) \neq 0\}$ defined by the 1-form w is dense in V.

To prove this, it suffices, given two points m and m' of V, to find a path starting from m, lying in the leaf of the foliation passing through m, and ending in a point arbitrarily close to m'. Denote by (x,y,z,t) the coordinates of m and by (x',y',z',t') those of m'. Such a path then consists of three pieces.

The first piece is a path from (x, y, z, t) to (x_1, y_1, z, t), where ${x_1}^2 + {y_1}^2 = x'^2 + y'^2$, the second one is a path from (x_1, y_1, z, t) to (x_1, y_1, z_2, t_2), where ${z_2}^2 + {t_2}^2 = z'^2 + t'^2$; and the third path goes from (x_1, y_1, z_2, t_2) to (x_3, y_3, z_3, t_3), where ${x_3}^2 + {y_3}^2 = x'^2 + y'^2$, ${z_3}^2 + {t_3}^2 = z'^2 + t'^2$, and where (x_3, y_3, z_3, t_3) is close to (x', y', z', t').

Each of the three pieces follows the trajectory of one of the three linear vector fields L_1, L_2 an L_3. Taking polar coordinates (ρ, θ) in th xy-plane and (ρ', θ') in the zt-plane, we get

$$\begin{aligned} L_1 &= \rho\, \partial/\partial\rho + \partial/\partial\theta \\ L_2 &= \rho'\, \partial/\partial\rho' + \partial/\partial\theta' \\ L_3 &= \alpha\, \partial/\partial\theta + \beta\, \partial/\partial\theta'. \end{aligned}$$

Trajectories of L_1 are the logarithmic spirals $\{\rho = Ce^{\theta},\ z = z_0,\ t = t_0\}$ and those of L_2 are the logarithmic spirals $\{\rho' = Ce^{\theta'},\ x = x_0,\ y = y_0\}$ while, due to the rational independence of α and β, every trajectory of L_3 is dense in the two dimensional torus $\{\rho = \rho_0,\ \rho' = \rho'_0\}$ in which it lies.

As suggested by P. Cartier, a similar construction can be done in higher dimensions. Indeed, there exist $n + 1$ compatible linear vector fields in $I\!R^{2n}$, which define together a foliation of an invariant dense open subset V of $I\!R^{2n}$, whose every leaf is dense.

Therefore choose cartesian coordinates x_i, y_i, for i between 1 and n and corresponding polar coordinates ρ_i, θ_i and consider n rationally independent real numbers $\alpha_1, \cdots, \alpha_n$. Define then n "spiral" linear vector fields $L_1, \cdots, L_n$ by $L_i = \rho_i\, \partial/\partial\rho_i + \partial/\partial\theta_i$ and another linear vector field L by $L = \sum \alpha_i\, \partial/\partial\theta_i$. The open set V is the set where all ρ_i are positive.

Following trajectories of the L_i, we can draw a path from an arbitrary point m of V to some point m' of an arbitrary fixed n-dimensional torus $\{\rho_1 = r_1, \cdots, \rho_n = r_n\}$ without leaving the leaf of m; thereafter, the L-trajectory passing through m' approaches arbitrarily any given point m'' of the torus.

This construction can be used to show that there exist $n+2$ compatible linear vector fields in $I\!R^{2n+1}$, which define together a foliation of a dense open subset V of $I\!R^{2n+1}$, whose every leaf is dense. For this aim, it suffices to consider the $(2n + 1)$-th coordinate t and to add the $(n + 2)$-th linear vector field $L' = t\, \partial/\partial t$ to the previous ones. In this case, V is the invariant dense open set where all ρ_i are positive and where $t \neq 0$ and the n-dimensional torus are defined by $\{\rho_1 = r_1, \cdots, \rho_n = r_n, t = t_0\}$.

6 Questions and final remarks

The positive result on the integrability of two arbitrary compatible linear vector fields in $I\!R^3$ and the negative result that non-integrability can occur for thre compatible (even commuting) linear vector fields in $I\!R^4$ lead in a natural way to many interesting questions. Let us formulate some of them.

Let us fix natural numbers k and n, $2 \leq k < n$. Describe, or rather classify, the set of all k-tuples of compatible linear vector fields in $I\!R^n$, which are linearly independent at some point of $I\!R^n$. Such a k-tuple will be noted shortly CLVF.

It will also be interesting to study the set of all k-tuples of CLVF, viewed as an algebraic manifold.

In fact we are rather interested in the global foliation of $I\!R^n$ (of the $(n-1)$-dimensional real projective space) corresponding to such a k-tuple in virtue of Frobenius integrability theorem. For $k = 2$ and $n = 3$, such a classification can in principle be deduced from our results. But in general the problem seems to be quite intricate (cf. [37]).

Let us note, by the way, that the important Hermann-Nagano refinement of Frobenius integrability theorem ([33, 57], cf. also [28] and Sec. 3.1 of [78]) applies to our framework.

Given a k-tuple $\mathcal{A}$ of CLVF in $I\!R^n$, we define its index, ind($\mathcal{A}$), as the number of its global functionally independent first integrals ($0 \leq \text{ind}(\mathcal{A}) \leq n-k$). More precisely, the preceeding problem can be stated as a problem of the description of the level set of the function ind, when k and n are fixed.

From a more algebraic point of view, it seems that particular attention should be paid to the first integrals whose gradients consist of rational functions. As proved before, this is always the case when $k = 2$ and $n = 3$. It is rather doubtful that this is a general feature (cf. [13, 32]); nevertheless such integrals seem to appear quite frequently.

Given a k-tuple of CLVF in $I\!R^n$, one can ask for a maximal dimension of the closure of an individual leaf of the assiociated k-dimensional foliation and its relation to the number of functionally independent global first integrals.

For a given $n \geq 4$ one can also ask for a minimal number $k = k(n) \geq 2$ such that there exists a k-tuple of CLVF with dense leaves. As follows from Sec. 5,

$$k(n) \leq \begin{cases} p+1 & \text{if } n = 2p \\ p+2 & \text{if } n = 2p+1 \end{cases}$$

where $p \geq 2$. Are $p+1$ and $p+2$ the true lower bounds ? If not, determine $k(n)$. In particular, is it true that two CLVF in $I\!R^4$ always have a global first integral ?

Although the description of all k-tuples of CLVF in $I\!R^n$ seems to be quite complicated, there is a natural subclass of them, which seems to be much easier to handle.

A k-tuple $(A_1, \cdots, A_k)$ of CLVF in $I\!R^n$ will be called $I\!R$-compatible ($I\!R$-CLVF) if it generates a k-dimensional real Lie algebra, i. e. if for every $1 \leq i < j \leq k$

$$[A_i, A_j] = \sum_{r=1}^{k} C_{i,j}^r A_r$$

for some real numbers $\{C_{i,j}^r\}$.

The foliation associated to k-tuples of $I\!R$-CLVF have a simple description in terms of linear representations of Lie algebras.

Let $\mathcal{A}$ be a k-dimensional real Lie algebra and let $\pi : \mathcal{A} \longrightarrow \mathcal{L}(I\!R^n)$ be an injective linear representation of $\mathcal{A}$ in the space $\mathcal{L}(I\!R^n)$ of all linear mappings of $I\!R^n$ into itself. To any basis $(a_1, \cdots, a_k)$ of $\mathcal{A}$ we assiociate the k-tuple $(A_1, \cdots, A_k)$ of $I\!R$-CLVF defined on $I\!R^n$ by

$$A_k(u) = \pi(a_k)(u)$$

for $u \in I\!R^n$.

Although $(A_1, \cdots, A_k)$ depends on the choice of the basis $(a_1, \cdots, a_k)$, the associated global k-dimensional foliation of $I\!R^n$ (at least of an open dense subset of it) does not depend on it, but only depends on the linear representation π.

When π in not injective, one considers the global foliation of $I\!R^n$ associated to the induced representation

$$\tilde{\pi} : \mathcal{A}/\mathrm{Ker}(\pi) \longrightarrow \mathcal{L}(I\!R^n)$$

which is injective.

Thus to any linear representation π of a real Lie algebra $\mathcal{A}$, one associates in a canonical way a foliation of the space in which the representation acts.

Consequently, all properties of these foliations can be described in terms of algebraic properties of $\mathcal{A}$ and π, in particular the density of leaves, the existence or non-existence of first integrals and so on.

Explanation of these points is a very important problem. As an example, let us note that the commuting linear vector fields on $I\!R^n$ with dense leaves described in Sec. 5 arise from a representation of the commutative Lie algebra $I\!R^n$, but some other linear representations of the same Lie algebra have many independent first integrals.

The study of $I\!R$-CLVF would only be a first step because, already in $I\!R^3$, there exist foliations corresponding to some pairs of CLVF that do not correspond to any pair (M, N) of $I\!R$-CLVF.

Consider indeed such an example, due to W. Hebisch: the foliation in spheres centered at the origin. This foliation corresponds, for instance, to the pair (L_x, L_y) of CLVF, where $L_x = y\,\partial/\partial z - z\,\partial/\partial y$ and $L_y = z\,\partial/\partial x - x\,\partial/\partial z$, which generate the rotations around the x-axis and the y-axis respectively.

Let us note that a linear vector field whose trajectories lie on spheres centered at 0 is always given by a matrix in so$(3, I\!R)$. Consider now two non-proportional such linear vector fields M and N. Then, their Lie bracket $[M, N]$ is not a linear combination of them. This proves that the foliation in spheres cannot correspond to a pair of $I\!R$-CLVF.

It will be interesting to distinguish geometrically the global foliations corresponding to $I\!R$-CLVF among those corresponding to CLVF.

A careful examination proves that pairs (A, B) of non-commuting $I\!R$-CLVF in $I\!R^3$ are rather rare. Indeed, taking into account that there exists only one non-commutative 2-dimensionnal Lie algebra, one can assume, without any restriction of generality that $[A, B] = A$. Then, $\mathrm{tr}(A) = 0$ and B can only be found in cases L_7 and L_8 of the classification of Prop. 1.

It will be interesting to obtain a detailled analysis of the same phenomenon in higher dimension.

It is also natural to ask if there exist some classes of Lie algebras of dimension at least three for which the global foliation corresponding to an arbitrary injective linear representation of any algebra of the class always has a first integral.

As we were informed by W. Hebisch, an example of such a class is given by compact semisimple real Lie algebras, where a quadratic first integral always exists.

Indeed, according to H. Weyl's theorem, for an injective linear representation ϕ of such an algebra L, any connected Lie group corresponding to $\phi(L)$ is compact. The result now follows from a standard argument: integration of the translates of some positive definite quadratic form with respect to the Haar measure (see [82, 72]).

It is worth noting that the above remark admits a far reaching development in invariant theory, a topic intimately related to ours (see, for instance, Chap. 5 of [25] and also [67] for a development in another direction).

Although up to now the complete classification of all Lie algebras does not exist, it exists for small dimensions [59]. In particular, one has exactly nine types of three dimensional Lie algebras (see [59] and also [18]). The first stage in the realization of the above program will be the careful study of the possibilities occuring for their linear representations in low dimensional Euclidean spaces.

Three variations of our topics arize in a natural way.

First, instead of considering real linear vector fields, one can study affine vector fields, i. e. vector fields $A(u) = Bu + b$ where $A \in \mathcal{L}(I\!R^n)$ and $b \in I\!R^n$.

Secondly, the complex space C^n and the complex time can be considered instead of their real counterparts.

Finally, as the Frobenius integrability theorem remains valid in infinite dimensional setting (cf. [5, 7, 15, 19, 48]), it is natural to ask similar questions about it.

In fact, such problems can be considered in every specific framework where a counterpart of Frobenius integrability theorem is proved. See, for example, Sec. 9.4 of [5] for manifolds over valued fields with a non-zero characteristic.

Let us finish with some scattered remarks, mainly of historical nature.

We apologize for the complete lack of any explicit mention of contributions of differential algebra to the problem of the integration in finite terms of ordinary differential equations.

Our main tool was the search of an integrating factor written in finite terms, for integrable 1-forms. The method of the integrating factor goes back to L. Euler ([22, 23], cf. also [75]).

In particular, for $n = 2$, he knew the first part of our proposition 5. The second one is implicit in C. G. Jacobi's paper [36]. The search of integrating factors and the related problem of the search of a first integral for 1-forms, mainly in two dimensions, were very intensively studied during the nineteenth century. This is clearly shown in the treatises of A. R. Forsyth [26] and E. Goursat [30]. Let us stress the wealth of examples collected in the problems at the end of the corresponding chapters of these treatises.

This research culminated at the end of the nineteenth century in the fundamental works of G. Darboux [13], P. Painlevé [60, 61, 62, 63, 64, 65] and A. N. Korkine [41, 42, 43, 44]. Already in Darboux's paper [13] the close relations with algebraic geometry appears. This aspect was strongly emphazised by P. Painlevé [60, 61, 62, 63, 64, 65] and H. Poincaré [68, 69, 70].

After this fruitful period, partly as a consequence of the growing influence of the qualitative methods, this research was (almost) completely abandoned, although many open problems remain.

It is a pity that nowadays no published survey is available on the history and results obtained in this area up to the second world war.

One of the inherent difficulties in the preparation of such a survey is the fact that many interesting texts in this and related fields were published in Russian, some in journals and books which are now very difficult to find. It is worth noting that the strong activity of russian mathematicians in this area during the second half of the nineteenth century was the direct consequence of the great interest of P. L. Chebyshev in these problems [12, 29].

The lack of such a survey is only partly compensated by the historical notes at the end of B. M. Koialovich's book [38], which provides a very interesting annotated bibliography. The same remains true for D. Morduhai-Boltovskoi's book [55] and E. Vessiot's survey [80].

On the other hand, the unpublished thesis [52] of N. V. Lokot' is a very comprehensive study of the history of integration in finite terms of elementary functions.

Let us also quote [49] and the recent books [2, 16, 17] and [40]. In a forthcoming paper [6], we will publish an incomplete but nonetheless quite extensive bibliography of the subject up to the second world war.

Until recent times, integration in finite terms and related topics seemed to be marginal compared to the main stream of mathematics. But now, with a revival of the interest in the explicit integration of differential equations (cf. [58, 73, 34]), in the problem of non-integrability (cf. [46]) and above all with the development of the applications of computer algebra to the automatised study of differential equations (cf. [45, 20, 74, 79]), we hope that many of these, now almost completely forgotten works, will regain their importance and will find a contemporary understanding and development.

Let us stress that the Liouville theory of integration in finite terms resulted one hundred and fifty years later in the computer algebra programs for the integration of elementary functions (cf. [14] and bibliography therein).

The Liouville theory, together with the ideas of S. Lie, E. Picard and E. Vessiot on the Galois theory of differential equations finally resulted one century later in the computer algebra programs for the integration of second order linear differential equations with rational coefficients (cf. [20] and [45]).

As the algorithmic search for integrating factors written in finite terms is capital in the effective study of differential equations, one can look forward to an algorithm and then to a computer algebra program for an automatised search for them.

A. N. Korkine's papers and B. M. Koialovich's book will surely be very useful for this purpose. Recently the algorithm of B. M. Koialovich was substantial for the elaboration of a computer algebra program used for the discovery of new cases of integrability of Abel's differential equation of second kind $yy' - y = R(x)$ for hundreds of appropriate functions R [83].

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