Physica A 163 (1990) 683–722
North-Holland

INTEGRALS OF QUADRATIC ORDINARY DIFFERENTIAL EQUATIONS IN $\mathbb{R}^3$: THE LOTKA–VOLTERRA SYSTEM

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Received 16 May 1989

A method already introduced by the last two authors for finding the integrable cases of three-dimensional autonomous ordinary differential equations based on the Frobenius integrability theorem is described in detail. Using this method and computer algebra, the so-called three-dimensional Lotka–Volterra system is studied. Many cases of integrability are thus found. The study of this system is completed by the application of Painlevé analysis and the Jacobi last multiplier method. The methods used are of general interest and can be applied to many other systems.

1. Introduction

Given a system of ordinary differential equations (ODEs) depending on parameters, the question arises, how to recognize the values of the parameters for which the equations have first integrals? Except for some simple cases, this problem is very hard and no satisfying methods to solve it are known. To date, the most successful approach is offered by the so-called Painlevé analysis (see refs. [1–4]), the roots of which can be found in the seminal work of S. Kovalevskaya on the rigid body problem [5, 6]. Unfortunately this method, of high practical value, is not based on a firmly established mathematial ground. Moreover, the Painlevé analysis method puts emphasis on complex analytic integrals and is not well adapted to the search of integrals in the real domain.

In ref. [7] a method for finding first integrals for ODEs in $\mathbb{R}^3$ based on the Frobenius integrability theorem was presented together with some simple examples. Anticipating a little, we can say that the main point of the method introduced in ref. [7] is to detect the values of the parameters for which the system can have first integrals which at the same time are integrals of some non-trivial, linear system of three differential equations with constant coefficients, i.e. first integrals of linear vector fields in $\mathbb{R}^3$. Surprisingly, at least for three-dimensional systems, such integrals occur much more frequently that one would a priori expect. In what follows, this method will be called linear compatibility analysis method.

This method can be considered as a kind of generalization of the well known Lie symmetry method for finding first integrals of differential equations when applied to three-dimensional systems (cf. refs. [8–11]).

The full power of this method, when applied to specific examples, is in general unattainable without use of computer algebra. Indeed the amount of simple and elementary computations is so big that it is impossible to perform it by hand computations.

The main purpose of this paper is to give a thorough examination through the linear compatibility analysis method of one of the most interesting examples among those studied in ref. [7], namely the so-called three-dimensional Lotka–Volterra system (3D L–V system) which is traditionally written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(Cy + z + \lambda)\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(x + Az + \mu)\,, \tag{1.1}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = z(Bx + y + \nu)\,,$$

where A, B, C, λ, μ and ν are real or, sometimes, complex parameters. Using computer algebra we manage to find all cases where the linear compatibility analysis method of ref. [7] can be applied to the 3D L–V system and in all cases except one, we find explicitly at least one first integral.

The first paper, to our knowledge, where the integrals of the 3D L–V system were systematically studied was ref. [4], where the Painlevé analysis approach was used to detect some integrable cases. But in fact, already as soon as December 1884, S. Kovalevskaya in a letter to G. Mittag-Leffler (letter 57 from ref. [12]) announced that she was working on the problem of the integrability for systems of three quadratic homogeneous ODEs in $\mathbb{R}^3$, in particular for systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xL_1(x, y, z)\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = yL_2(x, y, z)\,, \tag{1.2}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = zL_3(x, y, z)\,,$$

where L_1, L_2, L_3 are homogeneous linear forms in x, y, z. Unfortunately, she never published any paper about this problem.

The second goal of this paper is to complete the study of 3D L–V system begun in ref. [4] by writing down all the integrable cases which can be obtained by Painlevé analysis. For this purpose, we first present the so-called ARS algorithm [1, 2], and then apply it to the 3D L–V system. Our presentation can thus be used as an introduction to the Painlevé analysis method.

These two approaches, together with the Jacobi last multiplier method, enable us to complete our list of first integrals of 3D L–V system obtained before. Therefore, we supply the most complete list ever published of integrable cases together with the corresponding first integrals for 3D L–V system.

Finally, the third purpose of this paper is to serve as a model example for the investigation of integrability of three-dimensional systems. Thus the paper is written in a completely self-contained style and the knowledge of refs. [4, 7] is not assumed.

The paper is organized as follows. In section 2 the complete list of all known integrable cases for 3D L–V systems is provided in tables I–III together with their first integrals. In section 3 the method of linear compatibility analysis is presented while in section 4 we discuss the case of quadratic systems of ODEs in $\mathbb{R}^3$. The application of the above method to the study of 3D L–V system via computer algebra is described in section 5, where the obtained results are reported in tables IV and V. In section 6 examples of explicit computation of first integrals are given. In section 7 the Painlevé analysis approach is recalled and its application to 3D L–V system is presented. In section 8 we recall the method of the Jacobi last multiplier which has been used to complete the list of first integrals (table III). Some open problems and comments suggested by our study are formulated in section 9.

The main results of this paper are collected in tables I to V. To avoid errors, the contents of these tables were verified using the symbolic manipulation language REDUCE.

2. First integrals of the 3D L–V system

Let us consider a smooth C^2 vector field X defined on $\mathbb{R}^3$, or on some open subset of it. For simplicity we will consider exclusively the case of $\mathbb{R}^3$. As X is defined on a non-compact space, the escape to infinity in finite time along the orbits of X is not excluded. A subset $V \subset \mathbb{R}^3$ is called X-invariant (or invariant, for short) if it contains only complete trajectories of X.

By a first integral of X we understand any smooth function F defined on some open, dense, X-invariant subset $V \subset \mathbb{R}^3$, non-constant on any open subset of V and constant along any orbit of X, i.e. $XF \equiv 0$. The introduction of the subset V is necessary, because, as will be seen in the examples, the first integrals considered in this paper have typically singularities which are located in an X-invariant subset $\mathbb{R}^3 \setminus V$. Let us note that grad $F \neq 0$ on an open dense subset of $\mathbb{R}^3$.

We will now write down in tables I–III all values of parameters A, B, C, λ, μ and ν for which we know at least one first integral of the 3D L–V system. As we consider a three-dimensional system, the number of functionally independent first integrals is at most equal to two. Let us stress that in this paper we consider only time-independent first integrals, although in some cases (cf. ref. [4]) time-dependent first integrals also exist. Before presenting tables I–III, let us note that the 3D L–V system (1.1) is invariant with respect to the simultaneous cyclic permutation of A, B, C, of λ, μ, ν and of x, y, z. More precisely after a such a permutation the three equations (1.1) undergo a similar one. Thus if for some values of A, B, C and of λ, μ, ν we know some first integral F of (1.1), by cyclic permutation as above inside it, F transforms into a first integral $\tilde{F}$ of the transformed system. Consequently, to avoid useless repetitions in our listing of first integrals, we will write down the values of A, B, C and of λ, μ, ν up to simultaneous cyclic permutation only.

Let us note also the following important feature of the 3D L–V system. The coordinate planes $x = 0$, $y = 0$ and $z = 0$ are invariant, i.e. if some orbit has one point on one of these planes, then the whole orbit is contained in it. Consequently, the eight octants, i.e. the connected components of $\{(x, y, z) \in \mathbb{R}^3;\ xyz \neq 0\}$ are also invariant. Moreover, for some particular values of the parameters, there exist additional invariant surfaces (mostly planes) which, together with the coordinate planes, divide $\mathbb{R}^3$ into invariant regions. In particular, whenever the absolute value or the logarithm of some expression appears in any of the following tables, it will always be true that its value is finite and non-vanishing outside such invariant surfaces.

In table I we list all the known integrable cases for the 3D L–V system, for which there exists at least one first integral which is also a first integral of some linear field Z given at the right of the table and written down in tables IV and V. We also list the corresponding first integrals. All these first integrals were obtained by compatibitity analysis (cf. section 3) except for case 1 and the integral F_1 of case 4, which were first obtained by different methods. Still, they could easily have been deduced from compatibiity analysis.

This table is an exhaustive one except for the case $\lambda = \mu = \nu = 0$ and

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = Z_0 F = 0 , \tag{2.1}$$

i.e. the case where F is a first integral of the vector field $Z_{0_1}(x, y, z) = (x, y, z)$.

Table I
Up to simultaneous cyclic permutation of A, B, C, of λ, μ, ν and of x, y, z, this is a full list of all cases of integrability of the 3D L–V system obtained by linear compatibility analysis. Z is the compatible vector field (cf. tables IV, V).

No.	A, B, C	λ, μ, ν	First integrals	Z
1	$ABC+1=0$	$\lambda=\mu=\nu$	$F=\lvert x\rvert^{-AB}\lvert y\rvert^{B}\lvert z\rvert^{-1}\lvert x-Cy+ACz\rvert^{AB-B+1}$	Z_{0_1}
2	$A=B=1$, $C\neq 0$	$\lambda=\mu=\nu$	$F=\lvert x\rvert\lvert x-Cy\rvert^{-1}\lvert z-y\rvert^{C}\lvert z\rvert^{-C}$	Z_{0_1}
3	$A=B=1$, $C=0$	$\lambda=\mu=\nu$	$F=\dfrac{y}{x}+\log\left\lvert 1-\dfrac{y}{z}\right\rvert$	Z_{0_1}
4	$ABC+1=0$	$\nu=\mu B-\lambda AB$	$F_1=\lvert x\rvert^{AB}\lvert y\rvert^{-B}\lvert z\rvert$, $F_2=ABx+y-Az+\nu\log\lvert y\rvert-\mu\log\lvert z\rvert$	Z_{1_1} Z_{2_1}
5	$A=1$, $BC=-1$	$\lambda=\mu$	$F=Bx+y-z+\nu\log\lvert y+Bx\rvert-\lambda\log\lvert z\rvert$	Z_{2_2}
6	$A=1$, $B=2$, $C=-\frac{1}{2}$	$\lambda=\mu=\nu$	$F=2x+y-z-\lambda\log\left\lvert\dfrac{2xz+y^2-yz}{x^2}\right\rvert$	Z_{2_3}
7	$A=1$, $B=-2$, $C=\frac{1}{2}$	$\lambda=\mu=\nu$	$F=2x-y+z+\lambda\log\left\lvert\dfrac{4x^2+2xz+yz}{y^2}\right\rvert$	Z_{2_4}
8	$A\neq -1$, $A\neq 0$, $B=-\dfrac{1}{A+1}$, $C=-\dfrac{A+1}{A}$	$\lambda=\mu=\nu$	Let $u=Ax+(A+1)y+A(A+1)z$ and $K=u^2-4A(A+1)^2yz$, then $F_1=\sqrt{K}+\lambda(A+1)\log\left(\dfrac{u+\sqrt{K}}{u-\sqrt{K}}\right)$. If moreover $\tilde{u}=Ax+(A+1)y-A(A+1)z$, then $K=\tilde{u}^2+4A^2(A+1)xz$ and $F_2=\log\left(\dfrac{u+\sqrt{K}}{u-\sqrt{K}}\right)+A\log\left(\dfrac{\tilde{u}+\sqrt{K}}{\tilde{u}-\sqrt{K}}\right)$.	Z_{4_1} Z_{0_1}
9	$A=1$, $C\neq 0$	$\lambda=\mu$, $\nu=0$	$F=\lvert x\rvert^{-1}\lvert y\rvert^{-BC}\lvert Cy-x\rvert^{BC+1}\lvert z\rvert^{C}$	Z_{5_1}
10	$A=1$, $C=0$	$\lambda=\mu$, $\nu=0$	$F=\dfrac{y}{x}+B\log\left\lvert\dfrac{y}{x}\right\rvert-\log\lvert z\rvert$	Z_{5_1}
11	$A=1$, $B=0$	$\lambda=\mu$, $\nu=0$	$F=\dfrac{\lvert Cy-x\rvert\lvert z\rvert^{C}}{\lvert x\rvert}$	Z_{6_1}
12	$A=B=C=1$	$\lambda=\nu$, $2\lambda=\mu$	$F=\dfrac{(x-y)(y-z)}{y}+\lambda\log\left(\dfrac{(x-z)^2}{\lvert y\rvert}\right)$	Z_{7_1}
13	$A=B=1$	$\lambda=0$, $\mu=\nu$	$F=\dfrac{(x-Cy)(y-z)}{y}-C\mu\log\left\lvert\dfrac{y}{z}\right\rvert$	Z_{7_2}
14	$A=B=1$	$\lambda=\mu$, $\nu=0$	$F=\dfrac{(x-Cy)(y-z)}{y}+\lambda\log\left\lvert\dfrac{x}{y}\right\rvert$	Z_{7_3}
15	$A=1$, $B\neq -1$, $C=-\dfrac{1}{B+1}$	$\lambda=\mu=\nu=0$	$F=\dfrac{z[(B+1)^2x+By]}{(B+1)x+y}-B[(B+1)x+y]$	Z_8

Probably, here, some cases are missing, where the integral could be found in closed form.

In two cases, 4 and 8, the compatibility analysis leads to two independent integrals. In the cases 2, 3, 9, 10, 11, 13, 14, 15, the Jacobi last multiplier method allows us to obtain the second integral (see table III). This is also true in some subcases of case 5 (which include cases 6 and 7). Except for case 8, all integrals of table I are real when A, B, C, λ, μ and ν are real. In fact, as we will discuss below, in case 8 also one can write down two real first integrals. The first integral is never defined in a unique way. For our tables we have chosen the form which seemed to us the most convenient one.

The first integrals of table I are written in an unambiguous way except in case 8. The ambiguity in the latter case is due to the fact that neither the quadratic polynomial K nor the expressions $(u+\sqrt{K})/(u-\sqrt{K})$ and $(\tilde{u}+\sqrt{K})/(\tilde{u}-\sqrt{K})$ are non-negative everywhere. The detailed discussion of these integrals will be given in section 6.

Among the above fifteen cases, only case 1 when $AB-B+1=0$ (which is also a subcase of case 4), the following four subcases of case 1 (up to cyclic permutations of A, B and C) $\{A, B, C\} = \{-3/2, -2, -1/3\}$, $\{-3, -1/2, -2/3\}$, $\{-2, -1, -1/2\}$ or $\{-1, -1, -1\}$ and case 8 can be detected by Painlevé analysis (see section 7). In table II we write down the five remaining cases which can be found by Painlevé analysis and for $\lambda=\mu=\nu=0$ we give the corresponding first integrals, which are homogeneous polynomials. This topic goes back to ref. [4], but the details are published here for the first time. The interesting feature of the above five cases is that for them A, B and C are truly complex numbers.

At this point we must stress that, as follows from above, if one considers only real A, B and C, the Painlevé analysis approach does not furnish any new cases in comparison with linear compatibility analysis.

Let us note that if A, B, C, λ, μ, ν are complex numbers and F is a corresponding integral of the 3D L–V system, then $\bar{F}$ is a first integral for the case $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$, where $\bar{z}$ denotes as usually the complex conjugation.

As usually, we denote $j=\frac{1}{2}(-1+\mathrm{i}\sqrt{3})$.

Let us observe that the measure m defined by $\mathrm{d}m = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z/xyz$ is invariant under the (local) flow induced by the 3D L–V system. Indeed $\operatorname{div}(fX)=0$, where $f(x, y, z) = 1/xyz$ and X denotes the 3D L–V vector field given by the right-hand side of eqs. (1.1) (cf. ref. [13] for details about invariant measures).

When the density of the invariant measure is known and when one also knows a first integral for three-dimensional systems, in some cases, the Jacobi last multiplier method (see section 8) allows to find a second functionally independent first integral. In table III are collected all the integrals obtained in this way, using the first integrals from table I, except the cases when two independent integrals were already known.

Table II
Up to simultaneous circular permutation of A, B, C and x, y, z and up to simultaneous complex conjugation of A, B and C, when $\lambda = \mu = \nu$, these five cases were obtained by Painlevé analysis. Here, we present the polynomial first integrals when $\lambda = \mu = \nu = 0$.

No.	A, B, C	First integrals for $\lambda = \mu = \nu = 0$
16	$A = -2 - j$, $B = -1 - j$, $C = (-2 - j)/3$	$x^3 + (2 + j)x^2y - 3jx^2z + (1 + j)xy^2 + (-1 + j)xyz - 3(1 + j)xz^2 + (1 + 2j)/9y^3 + y^2zj + (-1 + j)yz^2 - z^3$
17	$A = (-3 + \mathrm{i})/2$, $B = -1 + \mathrm{i}$, $C = (-2 + \mathrm{i})/5$	$4(-7 + 24\mathrm{i})x^4 + 16(2 + 11\mathrm{i})x^3y + 8(-31 + 17\mathrm{i})x^3z + 24(3 + 4\mathrm{i})x^2y^2 + 8(-13 - 9\mathrm{i})x^2yz + 12(-24 - 7\mathrm{i})x^2z^2 + 16(2 + \mathrm{i})xy^3 + 40(3 - \mathrm{i})xy^2z + 8(7 - 24\mathrm{i})xyz^2 + 4(-17 - 31\mathrm{i})xz^3 + 4y^4 + 8(3 - \mathrm{i})y^3z + 12(4 - 3\mathrm{i})y^2z^2 + 4(9 - 13\mathrm{i})yz^3 + (7 - 24\mathrm{i})z^4$
18	$A = -2 + \mathrm{i}$, $B = (-1 + \mathrm{i})/2$, $C = (-3 + \mathrm{i})/5$	$(7 + 24\mathrm{i})x^4 + 4(9 + 13\mathrm{i})x^3y + 4(-17 + 31\mathrm{i})x^3z + 12(4 + 3\mathrm{i})x^2y^2 + 8(11 + 2\mathrm{i})x^2yz + 12(-24 + 7\mathrm{i})x^2z^2 + 8(3 + \mathrm{i})xy^3 + 8(13 - 9\mathrm{i})xy^2z + 40(-1 - 7\mathrm{i})xyz^2 + 8(-31 - 17\mathrm{i})xz^3 + 4y^4 + 16(2 - \mathrm{i})y^3z + 24(3 - 4\mathrm{i})y^2z^2 + 16(2 - 11\mathrm{i})yz^3 + 4(-7 - 24\mathrm{i})z^4$
19	$A = (-4 + j)/3$, $B = -1 + j$, $C = (-2 + j)/7$	$27(37 + 360j)x^6 + 162(62 + 149j)x^5y + 54(-286 + 397j)x^5z + 405(39 + 55j)x^4y^2 + 324(-87 + 62j)x^4yz + 135(-323 + 37j)x^4z^2 + 540(19 + 18j)x^3y^3 + 54(94 + 71j)x^3y^2z + 378\ (-149 - 87j)x^3yz^2 + 60(-683 - 286j)x^3z^3 + 405(8 + 5j)x^2y^4 + 54(179 - 29j)x^2y^3z + 27(-104 - 947j)x^2y^2z^2 + 18(-1427 - 2074j)x^4yz^3 + 45(-360 - 323j)x^2z^4 + 162(3 + j)xy^5 + 162(18 - j)xy^4z + 108(52 - 41j)xy^3z^2 + 36(73 - 328j)xy^2z^3 + 126(-23 - 94j)xyz^4 + 6(-397 - 683j)xz^5 + 27y^6 + 54(4 - j)y^5z + 135(5 - 3j)y^4z^2 + 60(17 - 20j)y^3z^3 + 45(16 - 39j)y^2z^4 + 6(25 - 211j)yz^5 + (-37 - 360j)z^6$
20	$A = -2 + j$, $B = (-1 + j)/3$, $C = (-4 + j)/7$	$(323 + 360j)x^6 + 6(236 + 211j)x^5y + 6(286 + 683j)x^5z + 45(55 + 39j)x^4y^2 + 36(236 + 211j)x^4yz + 45(-37 + 323j)x^4z^2 + 60(37 + 20j)x^3y^3 + 126(94 + 23j)x^3y^2z + 18(-211 + 25j)x^3yz^2 + 60(-397 + 286j)x^3z^3 + 135(8 + 3j)x^2y^4 + 54(135 - 29j)x^2y^3z + 27(104 - 843j)x^2y^2z^2 + 54(-795 - 782j)x^2yz^3 + 135(-360 - 37j)x^2z^4 + 54(5 + j)xy^5 + 1134(2 - j)xy^4z + 108(26 - 125j)xy^3z^2 + 108(-167 - 424j)xy^2z^3 + 162(-323 - 360j)xyz^4 + 54(-683 - 397j)xz^5 + 27y^6 + 162(2 - j)y^5z + 405(3 - 5j)y^4z^2 + 540(1 - 18j)y^3z^3 + 405(-16 - 55j)y^2z^4 + 162(-87 - 149j)yz^5 + 27(-323 - 360j)z^6$

The integrals in table III were obtained assuming that the trajectory lies in an invariant region of $\mathbb{R}^3$ such that the quantities appearing through an absolute value (or a logarithm) in table I are all positive. For a different choice of the invariant region, the final form of the integral Φ may be different. Moreover, in each region, it is possible to find the initial integration point depending smoothly on F only, such that the integrand be finite throughout the integration domain. For example, in case 2 one can choose as initial point

Table III

Integrals obtained through the Jacobi last multiplier method. Note that in the last three cases, the integrals presented are the exponentials of those directly obtained. In the cases 2, 3, 5, 9, 10 and 11, F denotes the corresponding integral from table I. When performing the integration, F is considered as a constant under the integral, and only after the integration is F replaced by its expression in terms of x, y and z. Note that the integrals of cases 9, 10 and 11 can be expressed in terms of the incomplete beta and gamma functions.

No.	A, B, C	λ, μ, ν	First integrals from Jacobi last multiplier method
2	$A = B = 1$, $C \neq 0$	$\lambda = \mu = \nu$	$\Phi = \dfrac{(y-z)(x-Cy)}{y} - C\lambda \displaystyle\int^{1-y/z} \dfrac{r^C \, \mathrm{d}r}{(F-r^C)(1-r)}$
3	$A = B = 1$, $C = 0$	$\lambda = \mu = \nu$	$\Phi = \dfrac{x(y-z)}{y} - \lambda \displaystyle\int^{1-y/z} \dfrac{\mathrm{d}r}{(F-\log r)(1-r)}$
5	$A = 1$, $BC = -1$	$\lambda = \mu$	– if $\lambda = \mu = 0$ $\Phi = B \log \dfrac{x}{y} + \displaystyle\int^{y+Bx} \dfrac{\mathrm{d}r}{r + \nu \log r - F}$ – if $\lambda = \mu = \nu$ $\Phi = (x/y)^B z(Bx + y - z)^{-1}$ (subcase of case 1: compatible with Z_{u_1}) – if $\nu = 0$ $\Phi = (x/y)^B z$
6	$A = 1$, $B = 2$, $C = -\frac{1}{2}$	$\lambda = \mu = \nu$	These two cases are subcases of both case 1 and case 5. In these two cases, the quantities $F(5) - F(6)$ and $F(5) + F(7)$ respectively, where $F(n)$ means the integral of case n from table I, are functions of the new integral Φ of the second subcase of case 5, given above, of the form $\lambda \log(\alpha/\Phi + \beta)$, (α, β) being $(1, 4)$ and $(4, 1)$ respectively.
7	$A = 1$, $B = -2$, $C = \frac{1}{2}$	$\lambda = \mu = \nu$	
9	$A = 1$, $C \neq 0$	$\lambda = \mu$, $\nu = 0$	$\Phi = x - Cy + \lambda \log \dfrac{x}{y} + F^{1/C} \displaystyle\int^{C/x} r^{-1-1/C}(C-r)^{-B-1/C} \, \mathrm{d}r$
10	$A = 1$, $C = 0$	$\lambda = \mu$, $\nu = 0$	$\Phi = x - \lambda \log \dfrac{y}{x} - \mathrm{e}^{-F} \displaystyle\int^{y/x} r^{B-1} \mathrm{e}^r \, \mathrm{d}r$
11	$A = 1$, $B = 0$	$\lambda = \mu$, $\nu = 0$	$\Phi = \dfrac{x}{z^C} - C \displaystyle\int^{z} \dfrac{(r+\lambda)\,\mathrm{d}r}{(F + r^C)r}$
13	$A = B = 1$	$\lambda = 0$, $\mu = \nu$	$\Phi = \dfrac{xz}{y} \left\vert 1 - \dfrac{y}{z} \right\vert^{C+1}$
14	$A = B = 1$	$\lambda = \mu$, $\nu = 0$	$\Phi = \dfrac{xz}{y} \left\vert Cz - \dfrac{xz}{y} \right\vert^{-C-1}$
15	$A = 1$, $B \neq -1$, $C = -\dfrac{1}{B+1}$	$\lambda = \mu = \nu = 0$	$\Phi = \left(\dfrac{B+1}{z} + \dfrac{y}{xz} \right) \left\vert \dfrac{y}{x} \right\vert^{B}$

$(1 + F^{-1/C})^{-1}$. The obtained integral is thus smooth in each invariant region of $\mathbb{R}^3$.

Finally, comparing tables I and III, one sees that among the 16 integrable cases of table I we were able to find a second first integral for all cases but 1, 12 and the full case 5. It seems reasonable to think that the tables I–III do not contain all cases of integrability (in closed form) of the 3D L–V system and the problem of finding all such cases is largely open. The case when $\lambda = \mu = \nu = 0$ is already very interesting.

3. First integrals via Frobenius integrability theorem

If a smooth vector field in $\mathbb{R}^3$ admits a first integral F then $\mathbb{R}^3$ is foliated into two-dimensional level surfaces of F and a typical leaf of this foliation is an invariant manifold of this vector field. Let X and Y be two smooth vector fields on $\mathbb{R}^3$ such that the set U of points $x \in \mathbb{R}^3$ where the vectors $X(x)$ and $Y(x)$ are linearly independent is open and dense in $\mathbb{R}^3$. Let us suppose that F is a common first integral of both X and Y. Without any restriction of generality, one can suppose that $F(x) \neq 0$ for every $x \in \mathrm{U}$. Then, by the Frobenius integrability theorem, the vectors $X(p)$, $Y(p)$ and $[X, Y](p)$ are linearly dependent at every point $p = (x, y, z) \in \mathrm{U}$, that is

$$[X, Y] = \alpha X + \beta Y \tag{3.1}$$

and consequently, by continuity of the vector fields X, Y and $[X, Y]$, for every $p \in \mathbb{R}^3$

$$\det\{X(p), Y(p), [X, Y](p)\} = 0\,. \tag{3.2}$$

Here, $[X, Y]$ denotes the commutator of X and Y while α and β are some functions of p. This last condition is, at least locally in a sufficiently small neighborhood of any point $x \in \mathrm{U}$, equivalent to (3.1), with some smooth functions α and β. We shall say that two vector fields X and Y are *compatible* if and only if condition (3.2) holds.

Conversely, if the compatibility condition (3.2) is satisfied for two vector fields X and Y, then every point $p \in \mathbb{R}^3$ such that $X(p)$ and $Y(p)$ are linearly independent has a neighbourhood in which there exists a two-dimensional foliation tangent to X and Y. Each leaf of such a local foliation can be extended to a maximal connected one. However, the two-dimensional foliation thus obtained is not, in general, globally defined by level surfaces of some function. In what follows we shall consider only the case when such a function can be explicitly constructed and restrict ourselves to the algebraic aspect of the method.

Let $u(p)$ and $v(p)$ be two functionally independent first integrals of Y, i.e. $Yu = Yv = 0$. In fact on some two-dimensional submanifold, grad u and grad v

are allowed to be parallel (in particular one or both of them can be equal to zero). In some neighbourhood of a point $p \in \mathbb{R}^3$ where grad u and grad v are not parallel, any other first integral of Y is of the form $f(u, v)$ for some function f of two variables.

Now, let us return to the vector field X compatible with Y. If $X(p)$ and $Y(p)$ are linearly independent, then the two-dimensional foliation tangent to X and Y and defined in some neighbourhood of the point p is of the form $F = \text{const.}$ for some smooth function F. As this foliation is tangent to both vector fields X and Y, F is a first integral of X and Y. Thus by the above remark, X admits (locally) a first integral of the form $F = F(u, v)$.

How does one find $F = F(u, v)$? We know that F satisfies

$$\begin{aligned} 0 = XF &= \frac{\partial F}{\partial u} Xu + \frac{\partial F}{\partial v} Xv = (Xv)\left(\frac{\partial F}{\partial u}\frac{Xu}{Xv} + \frac{\partial F}{\partial v}\right) \\ &= (Xv)\left(\frac{\partial F}{\partial u} G(u, v) + \frac{\partial F}{\partial v}\right). \end{aligned} \tag{3.3}$$

The coefficient $G(u, v) = Xu/Xv$ is a priori a function of $p = (x, y, z)$ but in fact, at least locally, it is a function of u, v only. Indeed, condition (3.1) implies

$$\begin{aligned} Y(Xu/Xv) &= (Xv)^{-2}\{(YXu)(Xv) - (Xu)(YXv)\} \\ &= (Xv)^{-2}\{(\alpha Xv)(Xu) - (Xv)(\alpha Xu)\} = 0 \end{aligned}$$

and, therefore, Xu/Xv is also a first integral of Y. But for three-dimensional autonomous systems, any three first integrals are funtionally dependent and this implies our assertion.

Solving eq. (3.3) is thus equivalent to finding a first integral $F = F(u, v)$ of the two-dimensional system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = G(u, v)\,, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = 1\,. \tag{3.4}$$

More precisely if $F(u, v)$ is a first integral of (3.4) then

$$F(x, y, z) = F(u(x, y, z), v(x, y, z))$$

is a first integral of the vector field X.

The function G can admit singularities arising from the zeroes of the vector field Y. Outside these singularities, the system (3.4) admits at least locally a first integral. Indeed, this is a direct consequence of the standard linearisation theorem of vector fields around non singular points (cf. ref. [14], section 7).

However, from the local existence of a first integral one cannot deduce its global existence (cf. ref. [15] for a particularly interesting example of this kind, see also ref. [16]).

In all examples considered here

$$G(u, v) = \frac{P(u, v)}{Q(u, v)},$$

where P and Q are some polynomials. Thus instead of system (3.4) it is more convenient to consider the system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = P(u, v), \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = Q(u, v), \tag{3.5}$$

the first integrals of which are exactly the same as for system (3.4). There is no rule for finding the first integrals of system (3.4) or (3.5) and the success in their search is a matter of skill.

A class of vector fields in $\mathbb{R}^3$ the first integrals of which are completely understood is the class of affine vector fields. In fact, for the 3D L–V system, we performed not only the linear, but also the affine compatibility analysis i.e. the compatibility analysis with an affine vector field $Y(p) = Zp + q$, where $Z \in \mathbb{L}(\mathbb{R}^3, \mathbb{R}^3)$ and $q \in \mathbb{R}^3$ is a fixed vector. Although some cases of compatibilty with genuinely affine vector fields were found, this did not provide any new functionally independent first integral compared to those listed in table I and obtained by studying the compatibility with a linear vector field Z, i.e. for $q = 0$. For this reason, in what follows, we will not consider the affine case, but we will limit ourselves to the linear case.

Finally, let us remark that if in (3.1) $\beta = 0$, i.e. if $[X, Y] = \alpha X$, we recover the Lie symmetry condition for the vector field X with "symmetry" Y (cf. refs. [8–11]).

4. Quadratic systems of ODEs in $\mathbb{R}^3$

First, let us consider a vector field X in $\mathbb{R}^3$, all components of which are homogeneous polynomials of order m. Such a vector field X is always compatible with the linear vector field $Z_{0_1}(x, y, z) = (x, y, z)$, as a consequence of the Euler theorem on homogeneous functions, namely

$$[X, Z_{0_1}] = (1 - m)X$$

and therefore the compatibility condition (3.1) is satisfied. For the vector field

Z_{0_1} the simplest choice of first integrals is

$$u = x/z \quad \text{and} \quad v = y/z \, .$$

Thus according to section 3 we look for an integral of the form

$$F(u, v) = F(x/z, y/z) \, .$$

We write down the corresponding system of eqs. (3.5) and look for first integrals of it. But even in the simplest case $m = 2$, when one considers homogeneous quadratic vector fields, for instance the homogeneous Lotka–Volterra vector field,

$$X(x, y, z) = \{x(Cy + z), y(x + Az), z(Bx + y)\} \, , \tag{4.1}$$

only in very exceptional cases are we able to find the first integral F of system (3.5) in closed form.

Let us consider now the general inhomogeneous quadratic vector field X without constant term, i.e. $X = K + L$, where K is a homogeneous quadratic vector field and L is a linear homogeneous one. Let us write down the compatibility condition (3.2) where instead of a general vector field Y we consider a linear vector field Z.

Eq. (3.2) now writes

$$\det(K + L, Z, [K + L, Z]) = 0 \, . \tag{4.2}$$

This determinant is a non-homogeneous fifth-degree polynomial in x, y and z. Thus all its homogeneous polynomial terms are equal to zero. The equations for the fifth-, fourth- and third-order terms respectively write

$$\det(K, Z, [K, Z]) = 0 \, , \tag{4.3}$$

$$\det(K, Z, [L, Z]) + \det(L, Z, [K, Z]) = 0 \, , \tag{4.4}$$

$$\det(L, Z, [L, Z]) = 0 \, . \tag{4.5}$$

Now, (4.3) and (4.5) imply that the linear vector field Z is compatible separately with each of the homogeneous terms K and L. Moreover eq. (4.3), which is independent of L, suggests that the first step to obtain Z is to find a linear vector field compatible with the homogeneous quadratic vector field K.

Our main problem is now the following: Under what condition on parameters A, B, C, λ, μ and ν of the Lotka–Volterra system (1.1) can one find a non-trivial compatible linear vector field Z. Whenever this happens, compute the first integral of the corresponding 3D L–V system (if possible). As already stated, the Lotka–Volterra system is invariant with respect to the simultaneous cyclic permutation of x, y, z, of A, B, C and of λ, μ, ν. Thus if one knows a first integral for some values of A, B, C and of λ, μ, ν, then one can also find it for C, A, B, ν, λ, μ and for B, C, A, μ, ν, λ.

5. Compatible linear vector fields

In this section, we will exceptionally denote $x = x_1$, $y = x_2$, $z = x_3$. We will study the linear vector fields compatible with the vector field corresponding to the 3D L–V system. Let us note indistinctly by Z a linear mapping in $\mathbb{R}^3$ or the matrix $Z = \{z_{ij}\}_{1 \leq i, j \leq 3}$ corresponding to this mapping in the canonical basis of $\mathbb{R}^3$.

First let us consider the homogeneous 3D L–V system, and the corresponding homogeneous quadratic vector field K. Condition (3.2) gives

$$\det\begin{pmatrix} x_1(Cx_2 + x_3) & z_{11}x_1 + z_{12}x_2 + z_{13}x_3 & P(x_1, x_2, x_3) \\ x_2(x_1 + Ax_3) & z_{21}x_1 + z_{22}x_2 + z_{23}x_3 & Q(x_1, x_2, x_3) \\ x_3(Bx_1 + x_2) & z_{31}x_1 + z_{32}x_2 + z_{33}x_3 & R(x_1, x_2, x_3) \end{pmatrix} = 0 \tag{5.1}$$

for all x_1, x_2, $x_3 \in \mathbb{R}$, where P, Q and R are homogeneous quadratic polynomials defined by $[K, Z](x) = \{P(x), Q(x), R(x)\}$. We do not write down here the explicit expressions for P, Q and R as they are too cumbersome. Moreover it is practically impossible to compute determinant (5.1) by hand without error. Thus, from now on all reported results concerning eq. (5.1) and other equations of this type will be obtained using computer algebra.

We already know one non-trivial solution of (5.1) namely $Z = \alpha Z_{0_1}$, $\alpha \in \mathbb{R}$, i.e. $z_{ij} = \alpha\delta_{ij}$, $1 \leq i, j \leq 3$ (δ_{ij} is the Kronecker delta). To find all solutions of (5.1) we will proceed as follows. Eq. (5.1), when written in the form

$$\sum_{\substack{0 \leq i_1, i_2, i_3 \leq 5 \\ i_1 + i_2 + i_3 = 5}} M_{i_1 i_2 i_3}(Z, A, B, C)x_1^{i_1}x_2^{i_2}x_3^{i_3} = 0$$

for every x_1, x_2, $x_3 \in \mathbb{R}$, where the coefficients $M_{i_1 i_2 i_3}$ are quadratic homogeneous polynomials on the $\{z_{ij}\}$, is equivalent to the following system of 21 equations:

$$M_{i_1 i_2 i_3}(Z, A, B, C) = 0\,, \quad 0 \leq i_1, i_2, i_3 \leq 5,\; i_1 + i_2 + i_3 = 5\,. \tag{5.2}$$

In fact we have here only 18 non-trivial equations, because M_{500}, M_{050} and M_{005} vanish identically.

We ask for which values of the parameters A, B and C this system has a non-trivial solution $\{z_{ij}\}$. We look for 12-uples $(A, B, C, \{z_{ij}\})$, in which the z_{ij} are not all equal to zero, that verify the 18 nonlinear equations (5.2).

Six among these 18 equations, corresponding to M_{410}, M_{401}, M_{140}, M_{041}, M_{104}, M_{014} are particularly simple. Indeed,

$$\begin{aligned}
M_{410} &= z_{31}(z_{31} + BCz_{21} - Bz_{11}) = z_{31}L_1 = 0\,,\\
M_{401} &= z_{21}(z_{31} + BCz_{21} - Bz_{11}) = z_{21}L_1 = 0\,,\\
M_{140} &= z_{32}(ACz_{32} - Cz_{22} + z_{12}) = z_{32}L_2 = 0\,,\\
M_{041} &= z_{12}(ACz_{32} - Cz_{22} + z_{12}) = z_{12}L_2 = 0\,,\\
M_{104} &= z_{23}(Az_{33} - z_{23} - ABz_{13}) = z_{23}L_3 = 0\,,\\
M_{014} &= z_{13}(Az_{33} - z_{23} - ABz_{13}) = z_{13}L_3 = 0\,,
\end{aligned} \tag{5.3}$$

where $L_k = L_k(A, B, C, \{z_{ij}\})$, $1 \leq k \leq 3$. The set of all solutions of system (5.3) can be divided into 8 mutually disjoint subsets $P_1 - P_8$ defined as follows:

$$\begin{aligned}
&\mathrm{P}_1: \quad L_1 = 0,\ L_2 = 0,\ L_3 = 0\,,\\
&\mathrm{P}_2: \quad L_1 \neq 0,\ L_2 = 0,\ L_3 = 0\,,\\
&\mathrm{P}_3: \quad L_1 = 0,\ L_2 \neq 0,\ L_3 = 0\,,\\
&\mathrm{P}_4: \quad L_1 = 0,\ L_2 = 0,\ L_3 \neq 0\,,\\
&\mathrm{P}_5: \quad L_1 \neq 0,\ L_2 \neq 0,\ L_3 = 0\,,\\
&\mathrm{P}_6: \quad L_1 \neq 0,\ L_2 = 0,\ L_3 \neq 0\,,\\
&\mathrm{P}_7: \quad L_1 = 0,\ L_2 \neq 0,\ L_3 \neq 0\,,\\
&\mathrm{P}_8: \quad L_1 \neq 0,\ L_2 \neq 0,\ L_3 \neq 0\,.
\end{aligned}$$

Up to a simultaneous cyclic permutation of the indices of the z_{ij} and of the (A, B, C) there are only four different cases to be considered among these 8, which are described below.

Case P_1. In this case one has

$$
\begin{aligned}
z_{31} &= Bz_{11} - BCz_{21} , \\
z_{12} &= Cz_{22} - ACz_{32} , \\
z_{23} &= Az_{33} - ABz_{13} ,
\end{aligned}
\tag{5.4}
$$

where A, B, C, z_{11}, z_{13}, z_{21}, z_{22}, z_{32} and z_{33} are arbitrary.

Case P_4. In this case one has

$$
\begin{aligned}
z_{31} &= Bz_{11} - BCz_{21} , \\
z_{12} &= Cz_{22} - ACz_{32} , \\
z_{23} &= z_{13} = 0 .
\end{aligned}
\tag{5.5a}
$$

As $L_3 \neq 0$, this implies that

$$
Az_{33} \neq 0 . \tag{5.5b}
$$

Thus (5.5a, b) describe completely the solutions of (5.3) when B, C, z_{11}, z_{21}, z_{22}, z_{32} are arbitrary and where $A \neq 0$ and $z_{33} \neq 0$ are also arbitrary.

Case P_7. In this case one has

$$
\begin{aligned}
z_{31} &= Bz_{11} - BCz_{21} , \\
z_{12} &= z_{13} = z_{23} = z_{32} = 0 .
\end{aligned}
\tag{5.6a}
$$

As $L_2 \neq 0$ and $L_3 \neq 0$, this implies that

$$
Az_{33} \neq 0 \quad \text{and} \quad Cz_{22} \neq 0 . \tag{5.6b}
$$

Thus (5.6a, b) describe completely the solutions of (5.3) when B, z_{11} and z_{21} are arbitrary and where $A \neq 0$, $C \neq 0$, $z_{22} \neq 0$ and $z_{33} \neq 0$ are also arbitrary.

Case P_8. In this case one has

$$
z_{12} = z_{13} = z_{21} = z_{23} = z_{31} = z_{32} = 0 . \tag{5.7a}
$$

As $L_1 \neq 0$, $L_2 \neq 0$ and $L_3 \neq 0$, this implies that

$$Az_{33} \neq 0 , \quad Cz_{22} \neq 0 \quad \text{and} \quad Bz_{11} \neq 0 . \tag{5.7b}$$

Thus (5.7a, b) describe completely the solutions of (5.3) when $A \neq 0$, $B \neq 0$, $C \neq 0$, $z_{11} \neq 0$, $z_{22} \neq 0$ and $z_{33} \neq 0$ are also arbitrary.

In this way the set of all solutions of the system (5.3) is completely described. It is worth noting that by this procedure we never get the same solution twice.

Now, in order to solve the full system of 18 non-trivial equations (5.2) we consider separately each of the 8 cases P_1–P_8. In fact it is sufficient to consider only the above 4 cases P_1, P_4, P_7 and P_8. Let us consider for example case P_1. Substituting expressions (5.4) into the remaining equations, one obtains a system of non-linear algebraic equations with unknowns A, B, C and z_{11}, z_{13}, z_{21}, z_{22}, z_{31}, z_{33}. The computer algebra program written in MACSYMA gives the factorization of these equations into factors of lower degree. Indeed, we use the standard factorization algorithm for polynomials in several variables with integer coefficients (ref. [17], section 4.2.2). Some of these factors are linear with respect to at least one unknown. This leads to equations similar to (5.4), (5.5a), (5.6a) and (5.7a). The substitution procedure used in the solution of system (5.3) can also be applied here. In this way the number of unknowns decreases and we keep treating the remaining equations in exactly the same way. Finally, by considering all possible cases that appear in this procedure we obtain the complete solution of system (5.2) in case P_1. The same work was done also for the remaining cases P_4, P_7 and P_8.

The tedious analysis of taking into account the invariance of the problem under the simultaneous cyclic permutations of the variables x_1, x_2, x_3, the parameters A, B, and C, and the coefficients z_{ij}'s was performed by hand.

In this painstaking way we obtain the solution of system (5.2). The fact that we were able to describe, by this procedure, completely and efficiently the set of all solutions of our system of 18 nonlinear equations with 12 unknown A, B, C and $\{z_{ij}\}$, $1 \leqslant i, j \leqslant 3$, is very astonishing. It would be quite interesting to understand the reason for this phenomenon.

Up to cyclic permutations of x_1, x_2, x_3 and of A, B, C the complete answer is given in table IV. In this table, the parameters z_{ij}'s are arbitrary real or complex numbers. Unless a restriction is explicitly given, the parameters A, B, C are also arbitrary. To organize this table in a coherent and simple way, we completely drop inequalities like (5.5b), (5.6b) and (5.7b) (except for Z_4 and Z_8 for consistency reasons). For this reason some different cases have a non-empty intersection.

Let us now pass to the full non-homogeneous 3D L–V system (1.1). As before, we search for compatible linear vector fields.

In this case beside the identity (4.3) we have also to satisfy the identities (4.4) and (4.5). Thus in total we now have 46 homogeneous quadratic equations on $\{z_{ij}\}$. Indeed beside the 21 equations corresponding to (4.3) we have also 15 equations corresponding to (4.4) and 10 equations corresponding to (4.5). We ask for which values of the parameters these 46 equations admit a non-trivial solution. As the first 21 equations are the same as for homogeneous

Table IV
All cases of compatibility of a linear vector field Z with 3D L–V system with $\lambda = \mu = \nu = 0$ (up to simultaneous circular permutation of A, B, C and x, y, z, respectively). Let us note that the matrices Z_0, Z_5, Z_8 depend on only one parameter, the matrices Z_1, Z_3, Z_6, Z_7 depend on two parameters, Z_4 on three parameters and Z_2 on six parameters. Since for a three-dimensional system we can have at most two functionally independent first integrals, the richness of parameters for Z_4 and Z_2 is rather spurious, because in any of such family we will use no more than two different matrices. We report the first integrals u and v of the vector field Z_8 as well as the first integrals of the corresponding 3D L–V system written as functions of u and v. (No. 15 of Table I.)

A, B, C	Linear compatible vector field Z	First integrals
A, B, C arbitrary	$Z_0 = \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & z_{11} & 0 \\ 0 & 0 & z_{11} \end{bmatrix}$	
$ABC + 1 = 0$	$Z_1 = \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & z_{22} & 0 \\ 0 & 0 & -ABz_{11} + Bz_{22} \end{bmatrix}$	
$ABC + 1 = 0$	$Z_2 = \begin{bmatrix} z_{11} & Cz_{22} - ACz_{32} & z_{13} \\ z_{21} & z_{22} & Az_{33} - ABz_{13} \\ Bz_{11} - BCz_{21} & z_{32} & z_{33} \end{bmatrix}$	
$A = -1$, $B = 1$, $C = 1$ (and thus $ABC + 1 = 0$)	$Z_3 = \begin{bmatrix} -z_{11} & z_{12} & 0 \\ 0 & 0 & 0 \\ 0 & z_{12} & z_{11} \end{bmatrix}$	
$A \neq -1$, $A \neq 0$, $B = -\dfrac{1}{A+1}$, $\left(\text{thus } C = -\dfrac{1}{B+1},\ A = -\dfrac{1}{C+1} \text{ and } ABC = 1\right)$	$Z_4 = \begin{bmatrix} Cz_{21} - Az_{32} & -BCz_{13} & z_{13} \\ z_{21} & Az_{32} - Bz_{13} & -ACz_{21} \\ -ABz_{32} & z_{32} & Bz_{13} - Cz_{21} \end{bmatrix}$	
$A = 1$	$Z_5 = \begin{bmatrix} z_{11} & 0 & 0 \\ 0 & z_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
$A = 1$, $B = 0$	$Z_6 = \begin{bmatrix} z_{22} + Cz_{33} & 0 & 0 \\ z_{33} & z_{22} & 0 \\ 0 & 0 & z_{33} \end{bmatrix}$	
$A = B = 1$	$Z_7 = \begin{bmatrix} z_{11} & Cz_{22} & 0 \\ 0 & z_{11} + z_{22} & 0 \\ 0 & z_{11} & z_{22} \end{bmatrix}$	
$A = 1$, $B \neq -1$, $C = -\dfrac{1}{B+1}$	$Z_8 = \begin{bmatrix} -(B+1)z_{11} & -\dfrac{Bz_{11}}{B+1} & 0 \\ (B+1)^2 z_{11} & Bz_{11} & 0 \\ 0 & 0 & z_{11} \end{bmatrix}$	$u = z[(B+1)^2x + By]$, $v = (B+1)x + y$, $F(u, v) = \dfrac{u}{v} - Bv$

Table V
All cases of compatibility of a linear vector vector field Z with a 3D L–V system with some non-vanishing λ, μ, ν (up to simultaneous circular permutation of A, B, C, of λ, μ, ν and of x, y, z respectively). The matrix Z_{a_b} is always a subcase of matrix Z_a from table III. We report the first integrals u and v of Z as well as the first integrals, written in terms of u and v, of the 3D L–V system except in the case of compatibility with the vector field Z_{0_1}.

A, B, C	λ, μ, ν	Z	First integrals
A, B, C arbitrary	$\lambda = \mu = \nu$	$Z_{0_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$u = \frac{x}{z}$, $v = \frac{y}{z}$
$ABC + 1 = 0$	$\nu = \mu B - \lambda AB$	$Z_{1_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B \end{bmatrix}$	$u = x$, $v = \frac{\lvert y\rvert^B}{z}$, $F = \lvert u\rvert^{AB}\lvert v\rvert^{-1}$
$ABC + 1 = 0$	$\nu = \mu B - \lambda AB$	$Z_{2_1} = \begin{bmatrix} 0 & BC\mu & \nu \\ 0 & B\mu & 0 \\ 0 & 0 & B\nu \end{bmatrix}$	$u = Bx - BCy - z$, $v = \lvert y\rvert^{\nu}\lvert z\rvert^{-\mu}$, $F = Au + \log\lvert v\rvert$
$A = 1$, $BC = -1$ (thus $ABC + 1 = 0$)	$\mu = \lambda$	$Z_{2_2} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ -Bz_{11} & -Bz_{12} & -Bz_{13} \\ 0 & 0 & 0 \end{bmatrix}$	$u = Bx + y$, $v = z$, $F = u - v + \log\left(\frac{\lvert u\rvert^{\nu}}{\lvert v\rvert^{\lambda}}\right)$
$A = 1$, $B = 2$, $C = -\frac{1}{2}$ (thus $ABC + 1 = 0$)	$\lambda = \mu = \nu$	$Z_{2_3} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$u = 2x + y - z$, $v = \frac{2x - y}{x^2}$, $F = u - \lambda \log\lvert 4 - uv\rvert$
$A = 1$, $B = -2$, $C = \frac{1}{2}$ (thus $ABC + 1 = 0$)	$\lambda = \mu = \nu$	$Z_{2_4} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ -8 & 0 & 0 \end{bmatrix}$	$u = 2x - y + z$, $v = \frac{2x + y}{y^2}$, $F = u + \lambda \log\lvert uv + 1\rvert$
$A \neq -1$, $A \neq 0$, $B = -\frac{1}{A+1}$, $C = -\frac{A+1}{A}$, (thus $ABC - 1 = 0$)	$\lambda = \mu = \nu$	$Z_{4_1} = \begin{bmatrix} 0 & -\frac{1}{A} & 1 \\ 0 & \frac{1}{A+1} & 0 \\ 0 & 0 & -\frac{1}{A+1} \end{bmatrix}$	$u = Ax + (A+1)y + A(A+1)z$, $v = yz$, $F = \sqrt{K} + \lambda(A+1)\log\left(\frac{u+\sqrt{K}}{u-\sqrt{K}}\right)$ where $K = u^2 - 4A(A+1)^2 v$
$A \neq -1$, $A \neq 0$, $B = -\frac{1}{A+1}$, $C = -\frac{A+1}{A}$, (thus $ABC - 1 = 0$)	$\lambda = \mu = \nu$	$Z_{4_2} = \begin{bmatrix} -\frac{1}{A} & 0 & 0 \\ \frac{1}{A+1} & 0 & 1 \\ 0 & 0 & \frac{1}{A} \end{bmatrix}$	$u = Ax + (A+1)y - A(A+1)z$, $v = xz$, $F = \sqrt{K} - \lambda A(A+1)\log\left(\frac{u+\sqrt{K}}{u-\sqrt{K}}\right)$ where $K = u^2 + 4A^2(A+1)v$
$A = 1$	$\lambda = \mu$, $\nu = 0$	$Z_{5_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$u = x/y$, $v = z$ if $C \neq 0$ $F = \lvert C - u\rvert^{BC-1}\frac{\lvert v\rvert^{C}}{\lvert u\rvert}$, if $C = 0$ $F = \frac{1}{u} - B\log\lvert u\rvert - \log\lvert v\rvert$.
$A = 1$, $B = 0$	$\lambda = \mu$, $\nu = 0$	$Z_{6_1} = \begin{bmatrix} C & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$u = x - Cy$, $v = \frac{\lvert z\rvert^{C}}{x}$, $F = uv$
$A = B = C = 1$ (thus $ABC - 1 = 0$)	$\nu = \lambda$, $\mu = 2\lambda$	$Z_{7_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$u = (x - y)(y - z)/y$, $v = (x - z)^2/y$, $F = u + \lambda \log\lvert v\rvert$
$A = B = 1$	$\lambda = 0$, $\mu = \nu$	$Z_{7_2} = \begin{bmatrix} 0 & C & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$u = x - Cy$, $v = z/y$, $F = u(1 - v) + \mu C \log\lvert v\rvert$
$A = B = 1$	$\lambda = \mu$, $\nu = 0$	$Z_{7_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$u = y - z$, $v = x/y$, $F = u(v - C) + \lambda \log\lvert v\rvert$

case, we know already that for the solutions $(A, B, C, \lambda, \mu, \nu)$ with λ, μ or ν non-vanishing, A, B and C necessarily satisfy one of the nine conditions from table IV and that the corresponding matrix Z is a particular case of matrices Z_0–Z_8. Exactly in the same way as in the homogeneous case, using a large amount of computer algebra we were able to solve completely this system of equations. The results are presented in table V, where, for one-parameter families of matrices Z, the value of the parameter (denoted by z_{11} in table IV) is taken here equal to one.

Let us note that the affine compatibility analysis was performed along exactly the same lines as the linear one. However, the amount of computations needed was substantially larger than the one involved in the linear case.

It is worth repeating that the fact that by this procedure we are able to describe completely and efficiently the set of all solutions of such complicated systems of nonlinear equations seems to be a real miracle.

Finally let us stress that the success of our method in solving the above systems of non-linear equations is intimately related to the good choice of substitutions as given by (5.4), (5.5a), (5.6a) and (5.7a) at each stage. A bad choice may lead to very long subsequent computations. In principle our method can be completely automatized by the use of computer algebra. Nevertheless the problem of finding the good choice of substitutions is decisive for the practical implementation of this work.

6. The determination of first integrals

We will now indicate how, through the prescription from section 3, together with the tables IV and V one can find the first integrals presented in table I, using the compatibility with a non-trivial linear vector field. Except for case 8, all other cases are of elementary character and do not present difficulties. Thus we will describe in detail the search for integrals in case 8 and also, as an example, in case 6. In all other cases the computations are of the same type as in case 6, except cases 3 and 9 where one finds the integrals by differentiating with respect to C those of cases 2 and 8 respectively. Some remarks concerning case 1 are also given.

Case 6: $A = 1$, $B = 2$, $C = -\frac{1}{2}$, $\lambda = \mu = \nu$

In this case, the 3D L–V vector field is compatible with the linear field corresponding to the matrix

$$Z_{2_3} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

The corresponding system of linear ODEs

$$\dot{x} = x\,, \quad \dot{y} = -2x + 2y\,, \quad \dot{z} = 2y$$

has two first integrals

$$u(x, y, z) = 2x + y - z$$

and

$$v(x, y, z) = \frac{2x - y}{x^2}\,.$$

Using the 3D L–V equations, one finds immediately that

$$u' = \lambda u\,, \qquad \frac{v'}{v} = \lambda v - uv + 4\,, \tag{6.1}$$

where $f' = \mathrm{d}f/\mathrm{d}t$ denotes derivation according to the 3D L–V vector field.
Let us denote $w = 4 - uv$. Then it is easy to see from (6.1) that

$$w' = uw\,,$$

i.e. that

$$u' = \lambda(\log|w|)'\,.$$

This implies that $\phi(u, v) = u - \lambda \log|w| = u - \lambda \log|4 - uv|$ is a first integral of (6.1) and thus that (cf. section 3)

$$F(x, y, z) = \phi(u(x, y, z), v(x, y, z)) = 2x + y - z - \lambda \log\left|\frac{2xz + y^2 - yz}{x^2}\right|$$

is a first integral of 3D L–V system in case 6.

Case 1

Let us note that although case 1 can also be treated as above, we found its first integral F for the first time by an another method. Indeed we deduced the integral F of case 1 from the knowledge of integrals F_1 and F_2 of case 4 when $\lambda = \mu = \nu = 0$. For this we need the following well known and easy to prove fact, which will be used again in the next section.

Let $(x(\tau), y(\tau), z(\tau))$ be a solution of the 3D L–V system with $\lambda = \mu = \nu = 0$. Let $\tilde{x}(t) = \lambda\, e^{\lambda t} x(e^{\lambda t})$, $\tilde{y}(t) = \lambda\, e^{\lambda t} y(e^{\lambda t})$ and $\tilde{z}(t) = \lambda\, e^{\lambda t} z(e^{\lambda t})$ with $\lambda \neq 0$. Then $(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$ is a solution of the 3D L–V system with $\lambda = \mu = \nu$.

Now let us replace x, y and z in the expressions for the first integrals F_1 and F_2 of case 4 with $\lambda = \mu = \nu = 0$ by $x\, e^{-\lambda t}/\lambda$, $y\, e^{-\lambda t}/\lambda$ and $z\, e^{-\lambda t}/\lambda$ respectively. One easily verifies that the obtained expressions are two time-dependent first integrals for case 1, and with the same time dependence. Their quotient gives the integral F.

Case 8: $A \neq -1$, $A \neq 0$, $B = -\dfrac{1}{A+1}$, $C = \dfrac{A+1}{A}$, $\lambda = \mu = \nu$

In this case, the 3D L–V vector field is compatible with the following two linear vector fields corresponding to the matrices (cf. table V)

$$Z_{4_1} = \begin{bmatrix} 0 & -\dfrac{1}{A} & 1 \\ 0 & \dfrac{1}{A+1} & 0 \\ 0 & 0 & -\dfrac{1}{A+1} \end{bmatrix}$$

and

$$Z_{4_2} = \begin{bmatrix} -\dfrac{1}{A} & 0 & 0 \\ \dfrac{1}{A+1} & 0 & 1 \\ 0 & 0 & \dfrac{1}{A} \end{bmatrix}.$$

As first integrals of the linear vector field corresponding to Z_{4_1} one chooses

$$u(x, y, z) = Ax + (A+1)y + A(A+1)z$$

and

$$v(x, y, z) = yz\,.$$

Instead of working directly with the first integrals u and v it is more convenient to introduce the quantity

$$K = u^2 - 4A(A+1)^2 v \tag{6.2}$$

and to work with u and K. An easy computation gives

$$u' = \lambda u + 2A(A+1)v\,,$$
$$K' = 2\lambda K\,. \tag{6.3}$$

Let us note by the way that the last equality implies that for $\lambda = 0$, K is a first integral of the 3D L–V system. This integral was already found in ref. [7].

From (6.2) one obtains that $v = (u^2 - K)/4A(A+1)^2$ and thus the system (6.3) becomes

$$u' = \lambda u + \frac{1}{2(A+1)}(u^2 - K)\,,$$
$$K' = 2\lambda K\,. \tag{6.4}$$

Let us put $L = \sqrt{K}$. Then $\mathrm{d}L/\mathrm{d}t = \lambda L$ and thus the system (6.4) is transformed into

$$u' = \lambda u + \frac{1}{2(A+1)}(u^2 - L^2)\,,$$
$$L' = \lambda L\,. \tag{6.5}$$

Now it is not difficult to find a first integral of system (6.5). By adding and subtracting eqs. (6.5) one obtains

$$[\log(u+L)]' = \frac{(u+L)'}{u+L} = \lambda + \frac{1}{2(A+1)}(u - L)$$

and

$$[\log(u-L)]' = \frac{(u-L)'}{u-L} = \lambda + \frac{1}{2(A+1)}(u + L)\,,$$

consequently

$$\left[\log\left(\frac{u+L}{u-L}\right)\right]' = -\frac{L}{A+1} = -\frac{L'}{\lambda(A+1)}\,,$$

and finally one obtains the first integral

$$F_1 = \sqrt{K} + \lambda(A+1)\log\left(\frac{u+\sqrt{K}}{u-\sqrt{K}}\right).$$

Let us consider now the linear vector field corresponding to the matrix Z_{4_2}. As its first integrals one chooses $\tilde{u}(x, y, z) = Ax + (A+1)y - A(A+1)z$ and $\tilde{v}(x, y, z) = xz$. Let us note that

$$K = \tilde{u}^2 + 4A^2(A+1)\tilde{v} .$$

As above one obtains the equations

$$\tilde{u}' = \lambda \tilde{u} - \frac{1}{2(A+1)}(\tilde{u}^2 - L^2) ,$$

$$L' = \lambda L$$

and the first integral

$$\Omega = \sqrt{K} - \lambda A(A+1) \log\left(\frac{\tilde{u} + \sqrt{K}}{\tilde{u} - \sqrt{K}}\right) .$$

The integral F_2 from table I is equal to $(F_1 - \Omega)/\lambda(A+1)$. As F_2 is a homogeneous function of degree zero, i.e. $F_2(x, y, z) = F_2(kx, ky, kz)$ for any $k \neq 0$, F_2 is also a first integral of the vector field Z_{0_1}. Now using the complex (in general not uniform) integrals F_1 and F_2, we will write two real, functionally independent integrals Φ_1 and Φ_2 for case 8.

First of all, let us note that since

$$K = u^2 - 4A(A+1)^2 yz = \tilde{u}^2 + 4A^2(A+1)xz ,$$

whenever $xyz \neq 0$ we have $u \neq \pm\sqrt{K}$ and $\tilde{u} \neq \pm\sqrt{K}$. Moreover (cf. (6.3)) $\mathrm{d}K/\mathrm{d}t = 2\lambda K$. Thus the conical surface $C_K = \{(x, y, z) \in \mathbb{R}^3;\ K(x, y, z) = 0\}$ is invariant for the 3D L–V system, and thus both the interior and the exterior of C_K are also invariant. Let us define $C_K^+ = \{(x, y, z) \in \mathbb{R}^3;\ K(x, y, z) > 0\}$ and $C_K^- = \{(x, y, z) \in \mathbb{R}^3;\ K(x, y, z) < 0\}$. Thus, in C_K^+ and C_K^-, $|(u + \sqrt{K})/(u - \sqrt{K})|$ and $|(\tilde{u} + \sqrt{K})/(\tilde{u} - \sqrt{K})|$ are always finite and non-zero, and moreover C_K, C_K^+ and C_K^- are always non-empty. Taking into account that for $w \in \mathbb{C}\backslash\{0\}$, $\log w = \log|w| + \mathrm{i} \arg w$, by considering the real parts of integrals F_1 and F_2 respectively, one obtains immediately that inside C_K^+

$$\Phi_1 = \sqrt{K} + \lambda(A+1) \log\left|\frac{u + \sqrt{K}}{u - \sqrt{K}}\right|$$

and

$$\Phi_2 = \left|\frac{u+\sqrt{K}}{u-\sqrt{K}}\right| \cdot \left|\frac{\tilde{u}+\sqrt{K}}{\tilde{u}-\sqrt{K}}\right|^{A+1}$$

are the first integrals. Inside C_K^-, i.e. where $K<0$, these formulas do not define first integrals.

Inside C_K the situation is a little bit more complicated. Let us note by O_1, $O_2, \ldots, O_8$, the eight invariant octants which are the connected components of $\{(x, y, z) \in \mathbb{R}^3;\ xyz \neq 0\}$. It is easy to see that C_K is a cone the top of which is in $O \in \mathbb{R}^3$, intersecting exactly two opposite octants of $\mathbb{R}^3$, depending on the sign of A. The interior of this cone coincides with C_K^+, its exterior coincides with C_K. In what follows we will suppose that the octants are enumerated in such way that $C_K^- \subset O_1 \cup O_2$ and $C_K \supset \bigcup_{j=3}^8 O_j$. Let us note

$$\psi_1(x, y, z) = \frac{u(x, y, z) + \mathrm{i}\sqrt{-K(x, y, z)}}{u(x, y, z) - \mathrm{i}\sqrt{-K(x, y, z)}}$$

and

$$\psi_2(x, y, z) = \frac{\tilde{u}(x, y, z) + \mathrm{i}\sqrt{-K(x, y, z)}}{\tilde{u}(x, y, z) - \mathrm{i}\sqrt{-K(x, y, z)}}\ ,$$

ψ_i: $O_j \to \mathbb{C}\backslash\{0\}$, where $i = 1, 2$ and $3 \leq j \leq 8$, are continuous mappings. As the O_j's are simply connected and connected by arcs, the same is true for $\psi_i(O_j)$ and consequently one can define uniformly the logarithm on $\psi_i(O_j)$. Thus for $3 \leq j \leq 8$,

$$\Phi_1 = \sqrt{-K} + \lambda(A+1) \arg_{1j}\left(\frac{u+\mathrm{i}\sqrt{-K}}{u-\mathrm{i}\sqrt{-K}}\right)$$

and

$$\Phi_2 = \arg_{1j}\left(\frac{u+\mathrm{i}\sqrt{-K}}{u-\mathrm{i}\sqrt{-K}}\right) + A \arg_{2j}\left(\frac{\tilde{u}+\mathrm{i}\sqrt{-K}}{\tilde{u}-\mathrm{i}\sqrt{-K}}\right)$$

are the first integrals on O_j, for a well choosen continuous branch of the argument function on $\psi_i(O_j)$ denoted here by $\arg_{ij}$ respectively, $i = 1, 2$ and $3 \leq j \leq 8$.

As far as the remaining invariant sets $C_K^- \cap O_1$ and $C_K^- \cap O_2$ are concerned, we note that any of these sets is built from three simply connected, connected by arcs components. Indeed, the cone C_K is tangent to the coordinate planes $x=0$, $y=0$ and $z=0$ along the straight lines $y - Az = 0$, $x + (A+1)z = 0$ and $Ax + (A+1)z = 0$ and therefore, by the same reasons as above, the integrals Φ_1 and Φ_2 remain valid also on the sets $C_K^- \cap O_1$ and $C_K^- \cap O_2$.

7. Painlevé analysis of the 3D L–V system

In ref. [4] the Painlevé analysis of the 3D L–V system has been quickly sketched and our presentation will draw from these results. For a more detailed description of the Painlevé analysis see refs. [1, 2], where this method was introduced, as well as ref. [3] and references therein. In what follows we will only consider the simplest framework, i.e. the case of a system of first-order autonomous ODEs with entire (i.e. holomorphic on the whole complex space $\mathbb{C}^n$) right sides. Let us consider such a system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n), \quad 1 \leq i \leq n, \tag{7.1}$$

where $(x_1, \ldots, x_n) \in \mathbb{C}^n$, $t \in \mathbb{C}$ and $\{f_i\}_{1 \leq i \leq n}$, are entire functions. As it is well known, for any choice of initial conditions

$$x_i(t_0) = x_{i_0}, \quad 1 \leq i \leq n, \tag{7.2}$$

system (7.1) admits a unique solution satisfying (7.2) and this solution is defined and analytic at least in some neighbourhood of t_0. In what follows, when speaking about a solution of the initial value problem (7.1)–(7.2) we understand the maximal analytical prolongation of such a solution (spread over its Riemann surface).

Inasmuch as we consider only autonomous systems of ODEs the Painlevé property for such systems (7.1) means nothing else that all its solutions, i.e. the functions $x_1, \ldots, x_n$, are meromorphic in the whole complex plane $\mathbb{C}$.

The so-called ARS (Ablowitz–Ramani–Segur) conjecture in its simplest form says that if a system (7.1) enjoys the Painlevé property then this is an indication that the system may be integrable (perhaps with time-dependent integrals). In fact no counterexamples to the conjecture are known to date.

Usually the system (7.1) depends on some parameters and one asks for which values of these parameters the system is integrable. Following the ARS conjecture we are thus interested to know the values of the parameters for which the system enjoys the Painlevé property. Since it is in general very difficult to prove that a given system has the Painlevé property, one usually uses instead the so-called ARS algorithm. Let us consider the solution of the initial value problem (7.1)–(7.2). It may happen that such a solution has a singularity. The ARS algorithm detects the values of the parameters such that around any singularity allowed by system (7.1), the latter admits a non-trivial formal Laurent series solutions with only a finite number of negative powers. It is a priori not clear whether the systems detected by the ARS algorithm coincide with those satisfying the Painlevé property. Nevertheless, if some system is detected by the ARS algorithm then this is also an indication that it may be integrable.

The ARS algorithm is divided into three steps:

(a) the determination of the possible degrees of virtual poles of solutions, and of the leading coefficients;

(b) the determination of so called resonances, and finally,

(c) checking the compatibility conditions at the resonances.

We will now describe in detail the three steps of the ARS algorithm applied to the 3D L–V system and clarify the notions introduced above. As expected all the cases detected are integrable and the corresponding integrals are explicitly written in table I (case 1 for $AB - B + 1 = 0$, four other subcases of case 1 listed in table VI, and case 8) and in table II.

(a) We start with the 3D L–V system

$$\begin{aligned} \dot{x} &= x(Cy + z + \lambda)\,, \\ \dot{y} &= y(x + Az + \mu)\,, \\ \dot{z} &= z(Bx + y + v) \end{aligned} \tag{1.1}$$

with complex time t. Let us suppose that a solution $(x(t), y(t), z(t))$ of (1.1) has a pole at $t = t_0$, i.e. that at least one of the functions $x(t)$, $y(t)$, $z(t)$ has a pole at $t = t_0$. We denote $\tau = t - t_0$. Let us write the (formal) Laurent expansions of x, y and z in the form

$$x(t) = \tau^{p_x} \sum_{r \geq 0} x_r \tau^r\,, \quad y(t) = \tau^{p_y} \sum_{r \geq 0} y_r \tau^r\,, \quad z(t) = \tau^{p_z} \sum_{r \geq 0} z_r \tau^r\,, \tag{7.3}$$

where $x_0 \neq 0$, $y_0 \neq 0$ and $z_0 \neq 0$ and where at least one of the numbers p_x, p_y and p_z is negative. Substituting expansions (7.3) into eqs. (1.1), by straightforward analysis one finds that only two possibilities occur:

(i) $p_x = p_y = p_z = -1$ (and $x_0 \neq 0$, $y_0 \neq 0$, $z_0 \neq 0$), i.e. t_0 is a simple pole for x, y and z.

(ii) Two functions among x, y and z have a simple pole at t_0 and the third one is analytic around t_0.

We will now try to determine the conditions ensuring the (simultaneous) existence of solutions of (1.1) by formal Laurent series expansions (7.3) for both cases (i) and (ii) (when such expansions exist).

First let us try to determine the constants x_0, y_0 and z_0 for case (i). By substituting the Laurent expansions (7.3) into (1.1) and comparing the coefficients of τ^{-2} one obtains the linear system of equations

$$\begin{pmatrix} 0 & C & 1 \\ 1 & 0 & A \\ B & 1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{7.4}$$

with determinant equal to $ABC + 1$.

When $ABC + 1 = 0$ (case (i_1)), the necessary and sufficient condition for the existence of non-trivial solutions of (7.4) (with non-vanishing x_0, y_0, z_0) is

$$AB - B + 1 = 0\,, \quad BC - C + 1 = 0\,, \quad CA - A + 1 = 0\,. \tag{7.5}$$

In fact as $ABC + 1 = 0$, any one of conditions (7.5) implies the two others.

When $ABC + 1 \neq 0$ (case (i_2)) (7.4) has a unique solution. Because we require that $x_0 \neq 0$, $y_0 \neq 0$ and $z_0 \neq 0$ then $AB - B + 1 \neq 0$, $BC - C + 1 \neq 0$, $CA - A + 1 \neq 0$.

Let us consider now case (ii). Let us suppose that for example x is (formally) analytic around t_0, i.e. $p_x \geq 0$ in expansion (7.3). Let us try now to determine p_x. Substituting the corresponding Laurent expansions (7.3) into eqs. (1.1) and comparing the coefficients of the leading term ($\tau^{p_x - 1}$ for the first equation, τ^{-2} for the second and third one) leads to the following relations:

$$Cy_0 + z_0 = p_x\,, \quad Az_0 = -1 \quad \text{and} \quad y_0 = -1\,. \tag{7.6}$$

Thus $A \neq 0$ and finally for this solution the non-negative integer p_x is equal to

$$p_x = -C - 1/A\,.$$

By the same token, considering the two remaining solutions where y (or z) are respectively (formally) analytic around t_0 while x and z (or x and y) have simple poles at t_0, one deduces that for these solutions one has $p_y = -A - 1/B$ and $p_z = -B - 1/C$ respectively. Summarizing, all three numbers

$$\alpha = -C - 1/A\,, \quad \beta = -A - 1/B\,, \quad \gamma = -B - 1/C \tag{7.7}$$

must be non-negative integers.

Let us note that, when $ABC + 1 = 0$, conditions (7.5) and (7.7) cannot be satisfied simultaneously. Thus when (7.5) is satisfied (which implies $ABC + 1 = 0$), only singularities of type (i) are allowed by the 3D L–V system, and when $ABC + 1 = 0$ and (7.7) is satisfied, only singularities of type (ii) are allowed. The only possible values for A, B, C satisfying (7.7) with $ABC + 1 = 0$ are listed in table VI.

Table VI
All values of A, B and C (up to cyclic permutations) satisfying (7.7) with $ABC+1=0$.

A	B	C	α	β	γ
$-3/2$	-2	$-1/3$	1	2	5
-3	$-1/2$	$-2/3$	1	5	2
-2	-1	$-1/2$	1	3	3
-1	-1	-1	2	2	2

Indeed, condition (7.7) with $ABC+1=0$ is equivalent to

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}+\frac{1}{\gamma+1}=1\,,$$

which can be easily solved for non-negative integers. On the other hand, when $ABC+1\neq 0$, singularities of both types (i) and (ii) are simultaneously allowed for some particular values of the parameters A, B, C, λ, μ, ν, and in what follows, when $ABC+1\neq 0$, we will be interested exclusively in them.

(b) We now proceed to compute the coefficients in expansion (7.3) by substituting the latter into system (1.1) and by obtaining the recurrence relation for these coefficients allowing their recursive computation.

A natural number $r\geqslant 1$ is called a resonance if the coefficients (x_r, y_r, z_r) of expansion (7.3) are not uniquely determined, when knowing the coefficients x_k, y_k and z_k for $0\leqslant k\leqslant r-1$. If (x_0, y_0, z_0) are not uniquely determined, 0 is also called a resonance.

We will now compute the resonances of the 3D L–V system. Let us consider first case (i), i.e. the case $p_x=p_y=p_z=-1$. Let us substitute expansions (7.3) into the 3D L–V system (1.1). Comparison of the coefficients of τ^{r-2} for $r\geqslant 2$ gives the identities

$$\begin{aligned}
(r-1)x_r &= x_0(Cy_r+z_r)+x_r(Cy_0+z_0)+M_{r-1}\,,\\
(r-1)y_r &= y_0(x_r+Az_r)+y_r(x_0+Az_0)+N_{r-1}\,,\\
(r-1)z_r &= z_0(Bx_r+y_r)+z_r(Bx_0+y_0)+P_{r-1}\,,
\end{aligned}\tag{7.8}$$

where M_{r-1}, N_{r-1} and P_{r-1} depend only on x_k, y_k and z_k with $1\leqslant k\leqslant r-1$ and, of course, on A, B, C, λ, μ, ν. Let us note that (7.8) remains also valid for $r=1$ with $M_0=N_0=P_0=0$. Taking into account relations (7.4), identities (7.8) become

$$rx_r - Cx_0y_r - x_0z_r = M_{r-1}\,,$$
$$-y_0x_r + ry_r - Ay_0z_r = N_{r-1}\,, \tag{7.9}$$
$$-Bz_0x_r - z_0y_r + rz_r = P_{r-1}\,.$$

The characteristic polynomial $\Delta(r)$ of system (7.9) with unknowns x_r, y_r and z_r is equal to

$$\Delta(r) = r^3 - r(Ay_0z_0 + Bx_0z_0 + Cx_0y_0) - (1 + ABC)x_0y_0z_0 = 0\,. \tag{7.10}$$

Relations (7.4) now imply that

$$\Delta(r) = (r+1)[r^2 - r - (1 + ABC)x_0y_0z_0] = 0\,. \tag{7.11}$$

Thus $r = -1$ is always a root of eq. (7.11). This is a general feature related to the arbitrariness of t_0.

In case (i_1), when $ABC + 1 = 0$, eq. (7.11) reduces to

$$\Delta(r) = r^3 - r = 0$$

and the two resonances are $r = 0$ and $r = 1$. Let us note that the resonance $r = 0$ corresponds to the non-uniqueness of (x_0, y_0, z_0).

Let us now consider case (i_2), when $ABC + 1 \neq 0$. We denote by r_1 and r_2 the remaining two roots of (7.11), i.e. the roots of $r^2 - r - (1 + ABC) \times x_0y_0z_0 = 0$, for which we have $r_1 + r_2 = 1$, $r_1r_2 = -(1 + ABC)x_0y_0z_0$. As we ask for integer roots r_1 and r_2, then $r_1 = m$ and $r_2 = 1 - m$, where $m \geqslant 2$ is some natural number. Moreover

$$m(m-1) = (1 + ABC)x_0y_0z_0 = \phi(A, B, C)\,,$$

because now, x_0, y_0 and z_0 can be explicitly computed from eqs. (7.4). A straightforward computation shows that the equality $m(m-1) = \phi(A, B, C)$ is equivalent to the equality (cf. (7.7))

$$\frac{1}{m(m-1)} = \frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} - 1\,, \tag{7.12}$$

and we already know (see (7.7)) that such α, β and γ are non-negative integers. It is not difficult to find all such solutions of (7.12) for $m \geqslant 2$ and the corresponding values of A, B and C. They are listed in table VII, where we have again $j = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$.

Table VII
All values of A, B and C (up to cyclic permutations and complex conjugations) when a resonance $m \geqslant 2$ exists. The numbers refer to tables I and II.

No.	A	B	C	m	α	β	γ
8	$\neq 0, -1$	$-1/(A+1)$	$-(A+1)/A$	2	1	1	1
16	$-2-j$	$-1-j$	$(-2-j)/3$	3	1	2	2
17	$(-3+\mathrm{i})/2$	$-1+\mathrm{i}$	$(-2+\mathrm{i})/5$	4	1	2	3
18	$-2+\mathrm{i}$	$(-1+\mathrm{i})/2$	$(-3+\mathrm{i})/5$	4	1	3	2
19	$(-4+j)/3$	$-1+j$	$(-2+j)/7$	6	1	2	4
20	$-2+j$	$(-1+j)/3$	$(-4+j)/7$	6	1	4	2

In the remaining case (ii), taking into account relation (7.6), one obtains that, as in case (i_1), the resonances are $r=0$ and $r=1$. Let us note that here the resonance $r=0$ corresponds to the non-uniqueness of x_0 (respectively y_0 or z_0) if $p_x \geqslant 0$ (respectively $p_y \geqslant 0$ or $p_z \geqslant 0$).

(c) The compatibility conditions at the resonances are the conditions depending only on the parameters A, B, C, λ, μ, ν ensuring the solvability of the system of linear equations (7.9) for all values of (x_0, y_0, z_0) in spite of the nullity of its determinant.

We now go on to determine them. First let us consider case (ii). As already stated, the resonance $r=1$ corresponds to the non-uniqueness of x_1, y_1 and z_1, knowing x_0, y_0 and z_0. Let us suppose that for example x is analytic. From table VII one sees that in all cases $p_x \geqslant 1$ and consequently $x(t)/(t-t_0) = x(t)/\tau$ does not have singularities at $t=t_0$, i.e. at $\tau=0$. Let us substitute expansion (7.3)

$$y(t) = \frac{y_0}{\tau} + y_1 + y_2\tau + \cdots$$

and

$$z(t) = \frac{z_0}{\tau} + z_1 + z_2\tau + \cdots$$

with $y_0 \neq 0$ and $z_0 \neq 0$ into the second and third equations (1.1). Comparing the coefficients of τ^{-1} on both sides one obtains the equalities

$$y_0(Az_1 + \mu) + Ay_1z_0 = 0\,,$$

$$z_0(y_1 + \upsilon) + y_0z_1 = 0\,.$$

Taking in account (7.6), one thus obtains

$$(Az_1 + \mu) + y_1 = 0 ,$$

$$(y_1 + v) + Az_1 = 0$$

and consequently $\mu = \nu$. The case when y is analytic implies $\lambda = \nu$ and finally $\lambda = \mu = \nu$, which is the compatibility condition in case (ii).

This compatibility condition is sufficient for all cases of table VI, since they only allow type (ii) singularities.

In case (i_1) the compatibility condition for the resonance $r = 0$ was already written in (7.5). A straightforward calculation for the resonance $r = 1$ leads again to the condition $\lambda = \mu = \nu$.

Let us consider now the remaining six cases (i_2) the characteristic feature of which is to have a unique strictly positive resonance $m \geq 2$. We will prove that in these cases supplementary compatibility conditions do not arise. In other words, as soon as $\lambda = \mu = \nu$ we can always find the expansions (7.3) satisfying the 3D L–V system in these six cases.

First let us note that when $\lambda = \mu = \nu = 0$ this is indeed so. Since m is the unique strictly positive resonance, then M_r, N_r and P_r (cf. (7.8)–(7.9)) vanish identically for $1 \leq r \leq m - 1$. Consequently x_r, y_r and z_r vanish identically for $1 \leq r \leq m - 1$ and therefore M_m, N_m and P_m also vanish. Thus, there is no problem to find a solution $(x_m, y_m, z_m) \neq (0, 0, 0)$, and we obtain the expansions

$$\begin{aligned} x(t) &= \frac{x_0}{t - t_0} + x_m(t - t_0)^{m-1} + \cdots , \\ y(t) &= \frac{y_0}{t - t_0} + y_m(t - t_0)^{m-1} + \cdots , \\ z(t) &= \frac{z_0}{t - t_0} + z_m(t - t_0)^{m-1} + \cdots \end{aligned} \tag{7.13}$$

satisfying the 3D L–V system with $\lambda = \mu = \nu = 0$. We can always suppose that $t_0 \neq 0$. Now, let $\lambda \neq 0$. As $\mathrm{e}^{\lambda t} = \Sigma_{k \geq 0} (\lambda t)^k/k!$, one can consider the expansions $\lambda\, \mathrm{e}^{\lambda t}\, x(\mathrm{e}^{\lambda t})$, $\lambda\, \mathrm{e}^{\lambda t}\, y(\mathrm{e}^{\lambda t})$ and $\lambda\, \mathrm{e}^{\lambda t}\, z(\mathrm{e}^{\lambda t})$, where x, y and z are defined by (7.13).

From the remark in case 1 of section 6 we know that they satisfy the 3D L–V system with $\lambda = \mu = \nu$. As $t_0 \neq 0$, they have simple poles at any v_0 such that $\mathrm{e}^{\lambda v_0} = t_0$. Thus, in case ($\mathrm{i}_2$), even if $\lambda = \mu = \nu \neq 0$, we do not need any supplementary compatibility condition for the existence of solutions of type (7.13). This concludes the last step of the ARS algorithm.

In summary, the ARS algorithm selects eleven cases, in all of which $\lambda = \mu = \nu$. The conditions on A, B and C are given by (7.5) for the first case, while for the remaining cases they are listed in tables VI and VII.

The first case is in fact a subcase of both cases 1 and 4 from table I, obtained already by compatibility analysis together with two independent integrals. The four cases of table VI are also subcases of case 1 from table I, with one integral compatible with a linear vector field. One can wonder whether a second independent integral exists in these cases.

Among the remaining six cases only the first one, corresponding to $m = 2$, was obtained by compatibility analysis (case 8 from table I). The five other cases are really new.

To find the first integrals of the above five cases from table II corresponding to the resonance $m \geqslant 3$, we proceed as follows.

First, one remarks that for the 3D L–V system this resonance is also a so-called Kovalevskaya exponent of this system (cf. ref. [18]). Theorem 1 from ref. [18] suggests that perhaps one can find a first integral in the form of a homogeneous polynomial on x, y and z of degree m. The hand computations are not feasible here, but the use of elementary computer algebra fully justifies this hope when $\lambda = \mu = \nu = 0$.

As shown by the case $m = 2$ (case 8 from table I) obtaining time-independent integrals in the case $\lambda = \mu = \nu \neq 0$ is really a non-trivial problem. In the remaining five cases when $m \geqslant 3$ and $\lambda = \mu = \nu \neq 0$, we were unable to find time-independent first integrals, but we hope that in these cases such integrals still exist. It will be interesting to clarify this question.

8. The Jacobi last multiplier method and its applications to the 3D L–V system

Roughly speaking, when considering a system of ODEs in $\mathbb{R}^n$ with $n - 2$ functionally first integrals known, the Jacobi last multiplier method allows us in some very particular cases to find the $(n-1)$th independent first integral. Although this method has been treated in many places (see for example refs. [19, 20]) we will give here a concise treatment of this classical topic, slightly different from the usual ones.

Let us consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x)\,, \tag{8.1}$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathrm{U} \subset \mathbb{R}^n$$

and

$$F(x)=\begin{pmatrix} f_1(x)\\ \vdots\\ f_n(x)\end{pmatrix}$$

is a smooth C^1 non-vanishing vector field defined on the open subset U of $\mathbb{R}^n$.

If

$$x(t,z)=\begin{pmatrix} x_1(t,z)\\ \vdots\\ x_n(t,z)\end{pmatrix}$$

is the solution of system (8.1) such that $x(0,z)=z$, where

$$z=\begin{pmatrix} z_1\\ \vdots\\ z_n\end{pmatrix}\in \mathrm{U}\ ,$$

then one defines the (local) flow $\{x^t\}$ induced by system (8.1) by $x^t(z)=x(t,z)$.

Let M be a non-negative C^1 function defined in U and non-identically vanishing on any open subset of U. M is called the density of invariant measure, or the last Jacobi multiplier, for system (8.1) if and only if the n-dimensional volume element

$$\omega(x)=M(x)\,\mathrm{d}x\ ,$$

where $\mathrm{d}x=\mathrm{d}x^1\ldots\mathrm{d}x^n$, is invariant with respect to the (local) flow $\{x^t\}$ induced by system (8.1), i.e. that the measure

$$m(A)=\int_A M(x)\,\mathrm{d}x$$

is invariant with respect to this flow.

Let us introduce in the open subset $\mathrm{V}\subset\mathrm{U}$ another system of coordinates $y=(y_1,\ldots,y_n)$, i.e. $x=G(y)$ with non-vanishing determinant of the Jacobian $\partial G/\partial y$. Then the volume element ω in the coordinates y is of the form

$$\omega(y)=M(G(y))\left|\det\left(\frac{\partial G}{\partial y}(y)\right)\right|\mathrm{d}y\ . \tag{8.2}$$

In particular, M is a density of invariant measure for system (8.1) if and only if for every $z\in\mathrm{U}$ and every $t\in\mathbb{R}$ small enough

$$M(z) = M(x^t(z)) \det\left(\frac{\partial x^t(z)}{\partial z}\right), \tag{8.3}$$

because it is always true that $\det[\partial x^t(z)/\partial z] > 0$.

It is well known, and not difficult to prove (cf. theorem 1 from ref. [13], ch. 2, section 2) that M is a density of invariant measure if and only if, on U,

$$\sum_{i=1}^{n} \frac{\partial(MF_i)}{\partial x_i} = 0\,. \tag{8.4}$$

We now pass to the heart of the matter. Let $\phi_1, \ldots, \phi_{n-1}$ be functionally independent first integrals of system (8.1) defined on U. Thus, if one identifies by use of the standard scalar product (,) on $\mathbb{R}^n$ the derivative $\mathrm{d}\phi_k(z) \in (\mathbb{R}^n)^*$ with the column vector grad $\phi_k(z) \in \mathbb{R}^n$, the vectors grad $\phi_k(z)$ and $F(z)$ are orthogonal for $z \in \mathrm{U}$ and $1 \leq k \leq n-1$.

Let us begin with the following simple observation. The function

$$M(z) = \left|\det\left(\operatorname{grad}\phi_1(z), \operatorname{grad}\phi_2(z), \ldots, \operatorname{grad}\phi_{n-1}(z), \frac{F(z)}{\|F(z)\|^2}\right)\right|, \tag{8.5}$$

where $\|F(z)\|^2 = \Sigma_{k=1}^{n} |f_k(z)|^2$, is the density of an invariant measure for system (8.1).

Indeed, as ϕ_k is a first integral of (8.1) then $\phi_k(x^t(z)) = \phi_k(z)$ and thus

$$\mathrm{d}\phi_k(x^t(z))\, \partial x^t(z)/\partial z = \mathrm{d}\phi_k(z)\,,$$

i.e.

$$A(t) \operatorname{grad}\phi_k(x^t(z)) = \operatorname{grad}\phi_k(z)\,, \quad 1 \leq k \leq n-1\,,$$

where $A(t)$ is the matrix transposed to the matrix $\partial x^t(z)/\partial z$. Moreover,

$$\frac{\partial x^t(z)}{\partial z} F(z) = F(x^t(z))\,. \tag{8.6}$$

We will now prove that M defined by (8.5) sastisfies (8.3). Indeed from the above remarks one has

$$M(z) = \left|\det\left(A(t)\operatorname{grad}\phi_1(x^t(z)), \ldots, A(t)\operatorname{grad}\phi_{n-1}(x^t(z)), \frac{F(z)}{\|F(z)\|^2}\right)\right|$$

$$= \det\left(\frac{\partial x^t(z)}{\partial z}\right)\left|\det\left(\operatorname{grad}\phi_1(x^t(z)), \ldots, \operatorname{grad}\phi_{n-1}(x^t(z)), (A(t))^{-1}\frac{F(z)}{\|F(z)\|^2}\right)\right|.$$

To complete the proof of (8.3) it remains now only to prove that

$$A(t)^{-1}\frac{F(z)}{\|F(z)\|^2}=\frac{F(x'(z))}{\|F(x'(z))\|^2}+\sum_{k=1}^{n-1}\alpha_k(z)\,\mathrm{grad}\,\phi_k(x'(z)) \tag{8.7}$$

for some coefficients $\alpha_k(z)$, $1\leq k\leq n-1$. As the integrals $\phi_1,\ldots,\phi_{n-1}$ are functionally independent, then grad $\phi_1(z),\ldots,$grad ϕ_{n-1} span the subspace $F^{\perp}(z)=\{y\in\mathbb{R}^n;\ (y,F(z))=0\}$. Thus (8.7) is equivalent to

$$\left(A(t)^{-1}\frac{F(z)}{\|F(z)\|^2}-\frac{F(x'(z))}{\|F(x'(z))\|^2},\,F(x'(z))\right)=0\,,$$

i.e. to

$$\left(\frac{F(z)}{\|F(z)\|^2},\left(\frac{\partial x'(z)}{\partial z}\right)^{-1}F(x'(z))\right)=\left(\frac{F(x'(z))}{\|F(x'(z))\|^2},\,F(x'(z))\right)$$

and this is evident by (8.6). Moreover, $M>0$ everywhere on U.

The Jacobi last multiplier method is a reciprocal of the remark just stated. When given $n-2$ functionally independent first integrals $\phi_1,\ldots,\phi_{n-2}$ of system (8.1) together with a density of invariant measure M, one can write explicitly, at least locally, an integral formula for a $(n-1)$th first integral ϕ_{n-1} functionally independent of the $n-2$ ones already known, in the suitable coordinate system $y=(y_1,\ldots,y_n)$.

Let us describe now this in detail. As $\phi_1,\ldots,\phi_{n-2}$ are functionally independent on U, changing if necessary the numbering of the coordinate variables $(x_1,\ldots,x_n)$, one can suppose without any restriction of generality that in some sufficiently small open ball $\mathrm{V}\subset\mathrm{U}$,

$$\det\left(\frac{\partial(\phi_1,\ldots,\phi_{n-2})}{\partial(x_1,\ldots,x_{n-2})}(x)\right)\neq 0\,, \tag{8.8}$$

and that $y_k=\phi_k(x)$ for $1\leq k\leq n-2$, $y_{n-1}=x_{n-1}$ and $y_n=x_n$ is a new coordinate system $y=(y_1,\ldots,y_n)$ on V. Let us note

$$H(x)=(\phi_1(x),\ldots,\phi_{n-2}(x)\,,\ x_{n-1},x_n)\,.$$

In this new system of coordinates system (8.1) becomes the following one:

$$\begin{aligned}\frac{\mathrm{d}y_k}{\mathrm{d}t}&=0 &&\text{for } 1\leq k\leq n-2\,,\\ \frac{\mathrm{d}y_k}{\mathrm{d}t}&=f_k(y) &&\text{for } n-1\leq k\leq n\,,\end{aligned} \tag{8.9}$$

with some functions f_{n-1} and f_n.

According to (8.2), in the coordinate system $y = (y_1, \ldots, y_n) = H(x)$, the density M takes the form

$$N(y) = \frac{M(x)}{|\det[\partial H(x)/\partial x]|} . \tag{8.10}$$

Let us now write, in the coordinate system $(y_1, \ldots, y_n)$, relation (8.5) with $(y_1, \ldots, y_{n-2})$ as the $n-2$ first integrals and with the unknown $(n-1)$th column equal to

$$\begin{pmatrix} \omega_1(y) \\ \vdots \\ \omega_n(y) \end{pmatrix} .$$

One obtains

$$N(y) = \left| \det \begin{pmatrix} 1 & & 0 & \omega_1(y) & \dfrac{f_1(y)}{\|f(y)\|^2} \\ & 1 & & \omega_2(y) & \dfrac{f_2(y)}{\|f(y)\|^2} \\ 0 & & \ddots & \vdots & \vdots \\ & & 1 & \omega_{n-2}(y) & \dfrac{f_{n-2}(y)}{\|f(y)\|^2} \\ & 0 & & \omega_{n-1}(y) & \dfrac{f_{n-1}(y)}{\|f(y)\|^2} \\ & & & \omega_n(y) & \dfrac{f_n(y)}{\|f(y)\|^2} \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} \omega_{n-1}(y) & f_{n-1}(y)/\|f(y)\|^2 \\ \omega_n(y) & f_n(y)/\|f(y)\|^2 \end{pmatrix} \right| . \tag{8.11}$$

We look for a function ϕ_{n-1} such that $\partial \phi_{n-1}/\partial y_k = \omega_k$ for $1 \leqslant k \leqslant n$. Now (8.4) and (8.9) imply that

$$\frac{\partial(Nf_{n-1})}{\partial y_{n-1}} + \frac{\partial(Nf_n)}{\partial y_n} = 0 .$$

Thus, at least locally

$$\omega_{n-1} = Nf_n = \frac{\partial \phi}{\partial y_{n-1}} \quad \text{and} \quad \omega_n = -Nf_{n-1} = \frac{\partial \phi}{\partial y_n}$$

for some function $\phi = \phi(y_1, \ldots, y_{n-2}, y_{n-1}, y_n)$ which is uniquely defined up to the additive term of the form $g(y_1, \ldots, y_{n-2})$. Any such function ϕ is a first integral of our system restricted to the surface

$$\phi_1(x) = y_1, \ldots, \phi_{n-2}(x) = y_{n-2} .$$

Let us now consider a point $y^0 = (y_1^0, \ldots, y_n^0)$ where $y^0 = H(x^0)$. In a sufficiently small convex neighbourhood of y^0 in $\mathbb{R}^n$, the formula

$$\phi_{n-1}(y) = \int_{\gamma_y} \omega_{n-1}(y)\, \mathrm{d}y_{n-1} + \omega_n(y)\, \mathrm{d}y_n + g(y_1, \ldots, y_{n-2})$$

defines the general form of the first integral of our system such that $\partial\phi_{n-1}/\partial y_{n-1} = \omega_{n-1}$ and $\partial\phi_{n-1}/\partial y_n = \omega_n$. In the above expression, g is an arbitrary smooth function, and γ_y is the straight line interval going from $(y_1, \ldots, y_{n-2}, y_{n-1}^0, y_n^0)$ to $(y_1, \ldots, y_{n-2}, y_{n-1}, y_n)$. Moreover, as $N(y) > 0$, (8.11) implies that the integrals $\phi_1, \ldots, \phi_{n-1}$ are functionally independent.

The above definition of the integral ϕ_{n-1} leads, under condition (8.8), to the following Jacobi formula for it. From the equations

$$\phi_k(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) = a_r , \quad 1 \leqslant k \leqslant n-2 ,$$

$(x_1, \ldots, x_{n-2})$ can be expressed as functions of x_{n-1} and x_n, when $a_1, \ldots, a_{n-2}$ are given. In what follows, we denote by $\hat{f}(x) = \hat{f}(x_1, \ldots, x_n)$ the function f considered as a function of variables $(a_1, \ldots, a_{n-2}, x_{n-1}, x_n)$. Now

$$\hat{\phi}_{n-1}(x) = \int \frac{\hat{M}(x)}{\hat{\Delta}(x)} (\hat{f}_n\, \mathrm{d}x_{n-1} - \hat{f}_{n-1}\, \mathrm{d}x_n) , \tag{8.12}$$

where $\Delta(x) = |\det[\partial H(x)/\partial x]|$ is, at least locally, a new first integral of system (8.1).

In order to illustrate the usefulness of this formula we first apply it to the case of a linear first integral of the 3D L–V system, corresponding to case 4 of table I $(ABC + 1 = 0)$ in the subcase where $\lambda = \mu = \nu = 0$. We start from

$$F_2 = ABx + y - Az . \tag{8.13}$$

Here $\Delta = \partial F_2/\partial y = 1$ and

$$\hat{\Phi} = \int \frac{1}{xyz}\,[x(Cy+z)\,\mathrm{d}z - z(Bx+y)\,\mathrm{d}x]\,, \tag{8.14}$$

where one must substitute for y the expression $F_2 + Az - ABx$ from eq. (8.13). Thus $\hat{\Phi}$ rewrites

$$\hat{\Phi} = \int \frac{C\,\mathrm{d}z}{z} - \frac{\mathrm{d}x}{x} + \frac{\mathrm{d}z - B\,\mathrm{d}x}{F_2 + Az - ABx}\,,$$

which integrates to

$$\hat{\Phi} = C\log|z| - \log|x| + \frac{1}{A}\log|F_2 + Az - ABx|\,.$$

Multiplying by $-AB$ and exponentiating, we recover integral F_1 from table I. On the other hand, had we started from F_1, we would have recovered F_2 just as easily.

This method can also be applied to the case of all the other first integrals from table I from which one can explicitly solve for one of the variables x, y or z. The results can be found in table III. In most cases the second integral is given in terms of the quadrature of a complicated argument.

9. Final remarks

The results presented here, together with those of ref. [7], show that the compatibility analysis is an efficient method for the search of the cases of integrability of three-dimensional autonomous systems of ODEs. However, in order to realize the full extent of the possibilities of this method, further studies concerning other systems are necessary.

Usually, one associates non-integrability with the occurrence of very complicated (chaotic) orbits. Although such orbits do appear, at least numerically, in some systems of type (1.2) with linear nonhomogeneous forms L_1–L_3 (cf. refs. [21, 22]), it is not known, to our knowledge, whether such orbits can arise in the 3D L–V system (1.1) for some value of the parmeters A, B, C, λ, μ and ν. Some insight into this problem can be found in refs. [23, 24].

It is also worth noting that the usual methods to prove non-integrability deal with the non-existence of integrals analytic or meromorphic in domains related in an obvious way to the equations. Therefore even for equations the non-integrability of which is considered as proven, it is doubtful whether such proofs would exclude the existence of integrals of the form of many of those given in tables I and III.

Finally, the question of integrability notwithstanding, we can formulate the following conjecture concerning three-dimensional systems of ODEs: whenever such systems are compatible with a linear (or affine) vector field, their orbits do not exhibit chaotic behavior.

Acknowledgements

We are grateful to L. Brenig, A. Chenciner, J.-P. Françoise, R. Moussu and H. Yoshida for very useful discussions. The fourth author (J.-M.S.) wants to thank the Department of Mathematics of the University of Linköping for the exellent working conditions during his stay there in September 1988. We are also grateful to the Laboratoire d'Informatique Théorique et de Programmation de l'Université Paris VI, for granting free access to its VAX computer.

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