Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

729

Ergodic Theory

Proceedings, Oberwolfach, Germany,
June 11–17, 1978

Edited by
M. Denker and K. Jacobs

Springer-Verlag
Berlin Heidelberg New York 1979

C o n t e n t s :

Dynamical systems of total orders

Jean Moulin Ollagnier

Didier Pinchon

In this short paper, we present some examples of dynamical systems of total orders in connexion with amenability problems.

First of all, we introduce the compact metrizable space T of all total orders on a countable group. The group G acts in a natural way on T by homeomorphisms. An ergodic minimax theorem in this dynamical system leads to the existence of an ameaning filter for a group with the fixed point property (Følner boxes in the ergodic folkore).

Among all G-invariant probability measures on T, one can select a special one π which is invariant under a wider group of homeomorphisms. This allows to obtain local results from mean results in the theory of information gain, hence giving a characterization of equilibrium measures in statistical mechanics.

We then consider the G-invariant subspace T_0 of T consisting of all total orders isomorphic to Z. The existence of an invariant Borel probability on this standard (non compact) space implies the amenability of the group.

One can indeed in this case, not only prove the fixed point property, but directly show the existence of an amea ning filter.

Total orders were first introduced in ergodic theory by Kieffer to show the L^1 convergence of the mean information.

1- The dynamical system (T,G).

a- The compact space T.

Let T be the set of all total orders on the set G. For a finite part F of G and a given total order t on F, we denote by O(F,t) the subset of T consisting of all total orders, the restriction of which to F is t. The set T, endowed with the topology generated by the O(F,t) is a compact, totally disconnected Hausdorff space. Moreover, when G is countable, T is metrizable.

b- The probability π.

Let B denote the group of all permutations of the set G and B_f the invariant subgroup of finite permutations (only moving a finite number of points in G).
Let us denote by b the homeomorphism of the compact set T induced by a given element b of B in the following way

$$x \,\tau\, y \iff b(x)\, b(\tau)\, b(y)$$

i.e. $$x\, b(\tau)\, y \iff b^{-1}(x)\, \tau\, b^{-1}(y)$$

One can easily verify that there exists one and only one Radon probability measure on T, invariant under all the homeomorphisms b induced by permutations of G. This proba bility gives the measure $1/|F|!$ to $O(F,t)$.
Right and left tranlations on G are permutations, hence proving the existence of an invariant probability for the dynamical system (X,T) without any assumption of amenability on the group G.

c- The cone $\Sigma(G)$.

Let us consider functions f from the set F(G) of all finite parts of G to R verifying the four following conditions

(1) $f(\emptyset) = 0$

(2) $f(Aa) = f(A)$ invariance under right translations

(3) $f(A \cup B) + f(A \cap B) \leqslant f(A) + f(B)$ strong subadditivity

(4) $\exists K > 0 \ , \ \forall a \in G \ , \ \forall A \in F(G),$
$f(A \cup a) - f(A) \geqslant -K$

The set of all such functions is a convex cone $\Sigma(G)$, on which we define a function q by

$$q(f) = \inf_{A \neq \emptyset} |A|^{-1} f(A)$$

The existence of the ameaning filter for G is nothing else than the result $q(m_D) = 0$ for every finite part D of G containing e where the element m_D of $\Sigma(G)$ is defined by

$$m_D(A) = \left|\{x \in A, \exists d \in D, dx \notin A\}\right|$$

For details, see [2] .

d- A minimax theorem in (T,G).

Let f belong to $\Sigma(G)$ and a belong to G. The function f_a defined by $f_a(A) = f(A \cup a) - f(A)$ is decreasing because of the strong subadditivity of f and therefore can be extended by monotonicity in an u.s.c. function on $\mathcal{P}(G)$ also denoted by f_a. Moreover, because of condition (4), f_a is bounded below.

We denote once more by f_a the u.s.c. function on T :
$f_a(\tau) = f_a(a_\tau^-)$ where a_τ^- stands for $\{x \in G,\ x \tau a,\ x \neq a\}$
When $f = m_D$, we denote the increment $(m_D)_e$ by i_D .

When the group G has the fixed point property, the following minimax theorem holds

Theorem Let f belong to $\Sigma(G)$ and a belong to G.

$$q(f) = \sup \lambda(f_a)$$

where the supremum is taken over all probabilities on T, invariant under the homeomorphisms of T induced by right tranlations of G.
For the proof, see [2] .

e- Invariant measures of the i_D.

Let us recall that we denote by i_D the u.s.c. function $(m_D)_e$. In fact i_D is more regular. This function is continuous on T since it only depends on the restriction of the total order τ to a finite part in the following explicit manner

$$i_D(\tau) = 1 - 1_{(e = \sup D)} - \sum_{d \in D-e} 1_{(e = \sup Dd^{-1})}$$

where $(e = \sup E)$ denotes the subset of T $:\{\forall d \in E,\ d \ \tau\ e\}$

For every probability λ on T, invariant under homeomorphisms induced by right translations, we get

$$\lambda(i_D) = 1 - \lambda(e = \sup D) - \sum_{d \in D-e} \lambda(d = \sup D)$$

$\lambda(i_D) = 0$ since $\{(d = \sup D),\ d \in D\}$ is a partition of T.

Therefore, in the case where G has the fixed point property, $q(m_D) = \sup \lambda(i_D) = 0$ hence showing the existence of an ameaning filter for G.

Let us remark that the fixed point property is neither used for showing the existence of an invariant probability on T nor for proving that $\lambda(i_D) = 0$ for every invariant .

The crucial use of the fixed point property is the minimax ergodic theorem, which is a kind of invariant Hahn-Banach theorem.

2- A lemma about amenability.

To characterize equilibrium measures in statistical mechanics as invariant Gibbs measures, it is necessary to obtain local results from mean results. See [3] .

The following lemma, using the probability π on T, allows such an operation.

Lemma Let f be a positive, increasing, right invariant function on the set F(G) of all finite parts of G so that

$$\lim_{\mathcal{M}} |A|^{-1} f(A) = 0$$

where $\mathcal{M}$ is the ameaning filter on F(G).

Then, for an arbitrary given x in G,

$$\inf \lim_{A \to G-x} (f(A \cup x) - f(A)) = 0$$

Proof. There is no restriction to suppose $f(\emptyset) = 0$, so that we make this assumption.

For every A in F(G) and every total order t on A :

$$f(A) = \sum_{x \in A} f(x \cup x_t^-) - f(x_t^-)$$

whence

$$f(A) = 1/|A|! \sum_t \sum_{x \in A} f(x \cup x_t^-) - f(x_t^-)$$

The function f_x defined on F(G) by

$$f_x(A) = \inf_{A' \supset A} f(A' \cup x) - f(A')$$

is an increasing function of A.

It is then possible to extend f_x to an increasing positive

Borel function on $\mathcal{P}(G)$ and then to a positive Borel function on T by $f_x(\tau) = f_x(x_{\tau}^{-})$.

One can easily verify that

$$f(A) \geq \sum_{x \in A} d\pi(\tau)\ f_x(\tau) = |A| \int d\pi(\tau)\ f_e(\tau)$$

because of the invariance of π.

Then, thanks to the assumption $\lim_{\mathcal{M}} |A|^{-1} f(A) = 0$, there comes

$$\int d\pi(\tau)\ f_x(\tau) = 0$$

If we call ν the probability measure on $\mathcal{P}(G-x)$, image of π by the application $\tau \longrightarrow x_{\tau}^{-}$, we get

$$\int d\nu(A)\ f_x(A) = 0$$

f_x is then a positive, increasing function of A in $\mathcal{P}(G-x)$, so we have necessarily

$$\inf \{ f(A' \cup x) - f(A') , A' \in F(G-x) , A' \supset A \} = 0$$

since ν give to the cylinder $A' \supset A$ of $\mathcal{P}(G-x)$ the measure $1/(|A| + 1)$.

Whence the result

$$\inf \lim_{A \to G-x} [f(A \cup x) - f(A)] = 0$$

3- The dynamical system $(T_\circ, G)$.

Let us consider the subset $T_\circ$ of T consisting of all total orders isomorphic to the order of Z when G is countable.

It is rather simple to verify $\pi(T_\circ) = 0$ since, for total orders in $T_\circ$, given two elements of G, a and b, there are only a finite number of elements between a and b.

$$T_\circ \subset \bigcup_{D \in F(G)} \{ \tau \,,\, \forall x \notin D \,,\quad x \;\tau\; \inf(a,b) \text{ or } \sup(a,b) \;\tau\; x \} .$$

Let X be the set of all bijections from Z to G sending 0 on the unit element e of G. For a given x in X, $\bar{x}$ denotes the inverse map. X can be considered as a subset of G^Z and also as a subset of Z^G.

Setting on G^Z and Z^G the product topologies of discrete topologies, we make these sets become Polish spaces.

X is then the closed subspace of $G^Z \times Z^G$ defined by

$$X = \{ (x,y) \in G^Z \times Z^G \,,\, x \circ y = Id_G \,,\, y \circ x = Id_Z \,,\, x(0) = e \}$$

and so X is a Polish space.

It is possible to define an action of G on X by homeomorphisms, $x \longmapsto h(x)$, where h(x) is the element of X defined by

$$\forall m \in Z \,, \qquad h(x)(m) = x(m + \bar{x}(h)).h^{-1}$$

Let us consider the map τ from B to T

$$a \;\tau(x)\; b \iff \bar{x}(a) \leq \bar{x}(b) \quad \text{for the usual order on } Z.$$

This map is a continuous injection the image of which is T_o. Moreover, we have

$$\tau(h(x)) = h(\tau(x))$$

if we denote by h the homeomorphism of T induced by the right translation by h^{-1}.

The map τ is not an homeomorphism since T_o is not a Polish space. One can indeed verify that T_o does not possess the Baire property. Nevertheless, the Borel σ-algebras are the same since X is Polish.

So we can consider the standard dynamical systems (X,G) or (T_o, G).

<u>Theorem</u> If there exists a Borel probability measure P on X, invariant by the action of G, then G is amenable.

Proof. It is sufficient to prove the existence of an ameaning filter, to show that q on $\Sigma(G)$ verifies

$$q(f) = \int d\pi(\tau) \; f_e(\tau)$$

Using P we obtain a linear map $f \longrightarrow \varphi(f)$ from $\Sigma(G)$ to $\Sigma(Z)$

$$\varphi(A) = \int dP(x) \; f(x(A))$$

We have

$$q(\varphi) \geqslant q(f) \geqslant \pi(f) = \int_{T(G)} d\pi(\tau) \; f_e(\tau)$$

Since Z is amenable

$$q(\varphi) = \pi(\varphi) = \int_{T(Z)} d\pi(\tau) \; \varphi_0(\tau)$$

To achieve the proof, it remains to show $\pi(f) = \pi(\varphi)$.

$$\varphi_O(O_\tau^-) = \int dP(x)\ f_e(x(O_\tau^-))$$

Whence, by using Fubini's theorem

$$\pi(\varphi) = \int_{T(Z)} d\pi(\tau) \int_X dP(x)\ f_e(x(O_\tau^-)) =$$

$$\int_X dP(x) \int_{T(Z)} d\pi(\tau)\ f_C(x(O_\tau^-))$$

The image of π on T(Z) by an x is π on T(G) and

$$\pi(\varphi) = \int_X dP(x) \int_{T(G)} d\pi(\tau)\ f_e(e_\tau^-) = \pi(f).$$

<u>Remark</u>. It is easy to see that the existence of an hyperfinite action of G on a probability space implies the existence of P on X.

<u>References</u>

[1] KIEFFER J.C. A generalized Shannon-McMillan theorem for the action of an amenable group on a probability space. Annals of Proba. Vol.3 Nb 6 (1975) 1031-1037.

[2] MOULIN OLLAGNIER J. et PINCHON D. Une nouvelle démonstration du théorème de E.Følner. C.R.A.S. 287 (1978)

[3] ———— Riemann integration and the variational principle for amenable groups. Preprint 1978.

J.M.O. Département de Mathématiques, Université Paris-Nord, Avenue J.B.Clément F 93430 VILLETANEUSE

D.P. Laboratoire de Probabilité, Université P.et M. Curie, 4, place Jussieu F 75230 PARIS CEDEX 05