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A NOTE ABOUT HEDLUND'S THEOREM

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In this note, we propose a new proof of Hedlund's theorem for an arbitrary group without using skew-products.

Let X be a compact space and T a minimal homeomorphism of X. Hedlund proved the following theorem (1)

Theorem

Let f be a continuous function on X. If the sums $\sum_{i=0}^{n-1} f \circ T^{-i}$ are uniformly bounded there exists a continuous function F such that

$$f = F \circ T^{-1} - F$$

One can remark that the sums of iterates appearing in this theorem are the values of a cocycle for the powers of transformation T. More generally, let X be a compact space and G a group of homeomorphisms of X. An application V from G to the space C(X) of continuous functions on X ($g \mapsto V_g$) is said to be a cocycle if

$$\forall h,g \in G \qquad V_{gh} = V_g + V_h \circ g^{-1}$$

Cocycles appear in the study of quasi-invariant probabilities (2) and in the construction of skew-products.

Theorem

Let X be a compact space and G a group of homeomorphisms of X whose action is minimal i.e. every orbit is dense.

Let V be a cocycle on X for the action of G.

If the functions V_g are uniformly bounded there exists a continuous function F such that

$$\forall g \in G \qquad V_g = F \circ g^{-1} - F$$

Proof

1. Let a be a point of X and O(a) its orbit under G (which is dense).
 Let us define on O(a) a function f by $f(g^{-1}a) = V_g(a)$
 We can do it because $V_g(a)$ does not depend on g but only on the point $g^{-1}a$ of O(a) :
 If $g^{-1}a = k^{-1}a$, $k = sg$ where s belongs to the group St(a) of all s such that $s(a) = a$.
 Then
$$V_{sg}(a) = V_s(a) + V_g(a)$$
 But the application from St(a) to R, $s \mapsto V_s(a)$, is a group homomorphism and from the hypothesis of boundedness is the zero homomorphism.
 The function f is bounded.

2. Let F the lower semi-continuous function defined by
$$F(x) = \inf \lim_{y \to x,\ y \in O(a)} f(x)$$
 F is bounded.
 Let us show that the increment of F is V i.e.
$$\forall h \in G \qquad F \circ h^{-1} - F = V_h$$
 This relation is true for f for all points in O(a) :
$$f(h^{-1}b) - f(b) = V_{gh}(a) - V_g(a) = V_h(g^{-1}a) = V_h(b) \quad \text{where} \quad b = g^{-1}a$$
 Thus
$$f(h^{-1}b) = f(b) + V_h(b) \qquad \forall\, b \in O(a) \qquad (1)$$
$$F(h^{-1}x) = \inf \lim_{y \to h^{-1}x,\ y \in O(a)} f(y) = \inf \lim_{b \to x,\ b \in O(a)} f(h^{-1}b)$$
 with $b = hy$, because h is continuous.
 Taking the inf limit in (1)
$$F(h^{-1}x) \geqslant F(x) + V_h(x)$$
 and
$$\forall x \in X \quad \forall h \in G \qquad F(h^{-1}x) - F(x) \geqslant V_h(x)$$
 Changing h in h^{-1} and x in $h^{-1}x$: $F(hh^{-1}x) - F(h^{-1}x) \geqslant V_h\,(h^{-1}x)$
 So $F(h^{-1}x) - F(x) \leqslant V_h(x)$ Whence the equality.

3. Let OF be the oscillation function of F
$$OF(c) = \limsup_{x,y \to c} F(x) - F(y)$$
 OF is a bounded upper semi-continuous function.

$$OF(h^{-1}c) = \sup_{x,y \to c} \lim \; F(h^{-1}x) - F(h^{-1}y) \quad \text{for } h^{-1} \text{ is continuous.}$$

But as $F(x) - F(y) = F(h^{-1}x) - F(h^{-1}y) - (V_h(x) - V_h(y))$

Taking the sup limit $OF(c) \leqslant OF(h^{-1}c)$ because the oscillation of the continuous function V_h is zero.

That shows that the function OF is invariant under G.

By minimality, an upper semi-continuous function invariant is a constant.

4. To complete the proof of the theorem it remains to show that the constant function OF is zero. This result comes from the lemma :

Let g a lower semi-continuous function on a topological space E, Og its oscillation. If Og(a) is finite then $\inf_{x \to a} \lim \; Og(x) = 0$

Proof g being a l.s.c. function

$$Og(x) = \sup_{y \to x} \lim g(y) - \inf_{y \to x} \lim g(y) = \sup_{y \to x} \lim g(y) - g(x)$$

If Og(a) is finite there exists an open neighbourhood V_o of a in which g is bounded. So that for every open neighbourhood $V \subset V_o$, for every ε , there exists an x_V in V with

$$g(x_V) \geqslant \sup_{y \in V} g(y) - \varepsilon$$

Then

$$Og(x_V) \leqslant \sup_{y \; V} g(y) - g(x_V) \leqslant \varepsilon \qquad \text{q.e.d.}$$

References

(1) GOTTSCHALK W. and HEDLUND G. Topological dynamics. A.M.S. Col. Vol. 38 New York 1955

(2) MOULIN OLLAGNIER J. et PINCHON D. Systèmes dynamiques topologiques 1. Etudes des limites de cobords. To appear in "Bulletin de la S.M.F."

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