Online Appendix to: Functional Characterization of Deformation Fields

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1 EXTRINSIC VECTOR FIELDS AS OPERATOR

Proposition 1.1. *For any extrinsic vector field V there is a unique linear functional operator E^V that satisfies:*

$$\int_M \langle \nabla g, \nabla E^V(f) \rangle \mathrm{d}\mu = \int_M \mathcal{L}_V \mathbf{g}(\nabla g, \nabla f) \mathrm{d}\mu. \tag{1}$$

Moreover, this operator is linear in both the vector field V and function f.

Proof. Let $H_0^1(M)$ be the space of square integrable functions with $L^2(M)$ gradients and zero integrals:

$$H_0^1(M) = \left\{ f \in L^2(M) : \int_M \|\nabla f\|^2 \,\mathrm{d}\mu < +\infty, \int_M f \,\mathrm{d}\mu = 0 \right\}.$$

This space seems natural when studying this operator as E^V maps any constant function to zero. When equipped of the scalar product $\langle ., . \rangle_{L^2} + \langle ., . \rangle_{H_0^1}$, H_0^1 is a Hilbert space.

The bilinear form $(f, g) \mapsto \langle f, g \rangle_{H_0^1(M)}$ is continuous and coercive thanks to the Wirtinger's inequality [3]. Moreover, for a given function g in $H_0^1(M)$ the linear form $f \mapsto \int_M \mathcal{L}_V \mathbf{g}(\nabla g, \nabla f) \mathrm{d}\mu$ is continuous assuming that V is smooth enough (at least H_1) since we have the inequality:

$$\int_M \mathcal{L}_V \mathbf{g}(\nabla g, \nabla f) \mathrm{d}\mu \leq \|\nabla g\|_{L^2}^2 \|\mathcal{L}_V \mathbf{g}\|_{L^2}^2 \|\nabla f\|_{L^2}^2.$$

Thus all conditions of the Lax-Milgram theorem [3] are satisfied therefore for any function g there exists a unique $E^V(g)$ satisfying Eq. (1). □

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2 RELATION TO SHAPE DIFFERENCE OPERATORS

2.1 Unified Shape Difference

Definition 2.2. Assuming that $\varphi : N \to M$ is a diffeomorphism, the unified shape difference $D_I : C^\infty(M) \to C^\infty(M)$ is defined implicitly by:

$$\langle f, D_I(g) \rangle_{H_0^1(M)} := \int_M C_{\varphi^{-1}} \left(\langle \nabla C_\varphi(f), \nabla C_\varphi(g) \rangle \right) \mathrm{d}\mu^M.$$

Suppose that $\varphi : N \to M$ is a diffeomorphism. We denote $(\varphi_\star X)_{\varphi(p)} = \mathrm{d}\varphi_p X_p$ the pullback of a vector field and $\mathrm{d}\varphi_p : T_p N \to T_{\varphi(p)} M$ the linear map between tangent spaces. Moreover the pullback with respect to φ^{-1} of the metric field $\mathbf{g}^N : T_p N \times T_p N \to \mathbb{R}$ is given by $((\varphi^{-1})^\star \mathbf{g}^N)_p(X, Y) = \mathbf{g}^N_{\varphi^{-1}(p)}(\mathrm{d}\varphi^{-1} X, \mathrm{d}\varphi^{-1} Y)$. For the gradient of a function f on M at a point $q \in N$:

$$\nabla(f \circ \varphi)_q = (\mathrm{d}\varphi^{-1} \nabla f) \circ \varphi(q)$$

Let's note $\varphi^{-1}(p) = q \in N$, therefore the pullback metric reads

$$\begin{aligned} \langle \nabla(f \circ \varphi), \nabla(g \circ \varphi) \rangle_q &= \mathbf{g}^N_{\varphi(q)}(\mathrm{d}\varphi^{-1} \nabla f, \mathrm{d}\varphi^{-1} \nabla g) \\ &= ((\varphi^{-1})^\star \mathbf{g}^N)_{\varphi(q)}(\nabla f, \nabla g). \end{aligned}$$

We can now rewrite Definition 2.2 with respect to the pullback metric:

$$\begin{aligned} \int_M \langle \nabla f, \nabla D_I(g) \rangle \mathrm{d}\mu &= \int_M C_\varphi^{-1} \left(\langle \nabla C_\varphi(f), \nabla C_\varphi(g) \rangle \right) \mathrm{d}\mu \\ &= \int_M C_\varphi^{-1} \left(((\varphi^{-1})^\star \mathbf{g}^N)_{\varphi(p)}(\nabla f, \nabla g) \right) \mathrm{d}\mu \\ &= \int_M ((\varphi^{-1})^\star \mathbf{g}^N)(\nabla f, \nabla g) \mathrm{d}\mu \end{aligned}$$

This alternative definition leads to the characterization of the metric change:

Proposition 2.3. *$D_I(f) = f$ for all $f \in C^\infty(M)$ if and only if φ is an isometry.*

Proof. If φ is an isometry then $(\varphi^{-1})^\star \mathbf{g}^N = \mathbf{g}^M$ so

$$\int_M \langle \nabla f, \nabla D_I(g) \rangle \mathrm{d}\mu^M = \int_M \langle \nabla f, \nabla g \rangle \mathrm{d}\mu$$

Using the fundamental lemma of calculus of variations: $\Delta D_I(g) = \Delta g$.

If $D_I(f) = f$ then

$$\int_M \langle \nabla f, \nabla g \rangle \mathrm{d}\mu = \int_M ((\varphi^{-1})^\star \mathbf{g}^N)(\nabla f, \nabla g) \mathrm{d}\mu$$

Using a result from [6], it implies $\mathbf{g}^M = (\varphi^{-1})^\star \mathbf{g}^N$ so φ is an isometry. □

2.2 Infinitesimal Shape Difference Operators

To define the infinitesimal shape differences we first need to introduce the correct framework and notation. Let's assume that the family of oriented surfaces M_t without boundary of intrinsic dimension 2 are isometrically immersed in $\mathbb{R}^3$ by the local mappings $F_t : U \subset \mathbb{R}^2 \longrightarrow M_t \subset \mathbb{R}^3$. This family of manifolds is generated by the displacement of the points along the smooth vector field $V(p) \in T_pM \times T_pM^{\perp} \simeq \mathbb{R}^3$:

$$\frac{\partial F_t}{\partial t}(p) = V(p), \quad (p,t) \in M \times \mathbb{R}^+ \tag{2}$$

The metric of the embedded surface is by definition $g^t_{ij} = \langle \partial_i F_t, \partial_j F_t \rangle$ and the area form is $\mu^t = \sqrt{\det g^t}$. The Riemannian connection on the ambient space $\mathbb{R}^3$ is denoted $\bar{\nabla}$. As mentioned in [4], the projection of the ambient connection into the tangent space of M coincides the unique Levi-Civita connection on M. Therefore the connection ∇ on M is naturally extended to extrinsic vector fields by $\nabla_i V_j = \langle \partial_i V, \partial_j F_t \rangle$. Once the connection is defined other differential operator can extended to extrinsic vector fields for example the divergence is defined as the trace of the connection $\operatorname{div}(V) = g^{ij}\nabla_i V_j$.

We consider the family of diffeomorphisms $\varphi_t : M_t \longrightarrow M$ given by $\varphi_t(p) : F_t - t(p)V(p)$.

The derivative of local quantities links the Lie derivative with the Strain tensor.

Lemma 2.1. *Given a one parameter family of surfaces described in Eq.* (2), *for a fixed point p, the first-order change in the metric tensor* g *and in the local area element* $\mu = \sqrt{\det(g)}$ *are given as:*

$$\left.\frac{\partial g(t)}{\partial t}\right|_{t=0} = \mathcal{L}_V g \tag{3}$$

$$\left.\frac{\partial \mu(t)}{\partial t}\right|_{t=0} = \operatorname{div}(V)\mu. \tag{4}$$

Proof. Those properties are easily proven when using local coordinates. Given a family of diffeomorphisms φ_t the Lie derivative of the metric tensor with respect to the vector field V denoted $\mathcal{L}_V g$ is by definition:

$$\mathcal{L}_V g := \left.\frac{\partial}{\partial t}\left((\varphi_t^{-1})^{\star} g(t)\right)\right|_{t=0}.$$

Since the local immersion F_t use a common chart system, the coordinates of the pullback metric $((\varphi_t^{-1})^{\star} g^t)_{ij}$ are equal to the metric on M_t in local coordinates $g_{ij}(t) = \langle \partial_i F_t, \partial_j F_t \rangle$. The computation of derivative is then straigthforward:

$$\begin{aligned}\left.\frac{\partial}{\partial t}\left(((\varphi_t^{-1})^{\star} g(t))_{ij}\right)\right|_{t=0} &= \left.\frac{\partial}{\partial t}\left(\langle \partial_i F_t, \partial_j F_t \rangle\right)\right|_{t=0} \\ &= \nabla_i V_j + \nabla_j V_i\end{aligned}$$

From there, Eq. (4) is easily obtained:

$$\begin{aligned}\left.\frac{\partial \mu(t)}{\partial t}\right|_{t=0} &= \left.\frac{\partial}{\partial t}\left(\sqrt{\det(g(t))}\right)\right|_{t=0} \\ &= \frac{1}{2\mu}\det(g)g^{ij}(\nabla_i V_j + \nabla_j V_i) = \operatorname{div}(V)\mu\end{aligned}$$

□

We then obtain the derivative of the shape differences.

Proposition 2.2. *Let V be a smooth deformation field on M, the derivatives of D_A, D_C and D_I at time zero satisfy for all smooth functions f, g:*

$$\begin{aligned}\langle f, E_A^V(g)\rangle^M_{L^2} &= \int_M \operatorname{div}(V) f g \, d\mu, \\ \langle f, E_C^V(g)\rangle^M_{H^1_0} &= \int_M \operatorname{div}(V)\langle \nabla f, \nabla g\rangle - \mathcal{L}_V g(\nabla f, \nabla g)\, d\mu, \\ \langle f, E_I^V(g)\rangle^M_{H^1_0} &= -\int_M \mathcal{L}_V g(\nabla f, \nabla g)\, d\mu.\end{aligned}$$

Proof. The first statement is obtained by using (4):

$$\begin{aligned}\langle f, \partial_t D_A(g)\rangle_{L^2} &= \left.\frac{\partial}{\partial t}\left(\int_{M_t} C_t(f)C_t(g)d\mu^t\right)\right|_{t=0} \\ &= \left.\frac{\partial}{\partial t}\left(\int_M fg d((\varphi_t)_{\star}\mu^t)\right)\right|_{t=0} \\ &= \int_M \operatorname{div}(V) f g d\mu\end{aligned}$$

For the second statement let's start with the evolution of the point-wise scalar product between gradient:

$$\begin{aligned}\left.\frac{\partial}{\partial t}(\langle \nabla f, \nabla g\rangle)\right|_{t=0} &= \left.\frac{\partial}{\partial t}(g^{ik}\partial_k f)g_{ij}(g^{jl}\partial_l g)\right|_{t=0} \\ &= \left.\frac{\partial}{\partial t}\partial_i f g^{ij}\partial_j g\right|_{t=0} \\ &= -(g^{ik}\partial_k f)\left.\frac{\partial g_{ij}}{\partial t}\right|_{t=0}(g^{jl}\partial_l g) \\ &= -\langle \nabla f, \nabla_{\nabla g} V\rangle - \langle \nabla_{\nabla f} V, \nabla g\rangle.\end{aligned}$$

It follows that:

$$\begin{aligned}\langle f, \partial_t D_C(g)\rangle_{H^1_0} &= \left.\frac{\partial}{\partial t}\left(\int_{M_t}\langle \nabla C_t(f), \nabla C_t(g)\rangle d\mu^t\right)\right|_{t=0} \\ &= \left.\frac{\partial}{\partial t}\left(\int_M ((\varphi_t^{-1})^{\star} g^t)(\nabla f, \nabla g) d((\varphi_t)_{\star}\mu^t)\right)\right|_0 \\ &= \int_M \operatorname{div}(V)\langle \nabla f, \nabla g\rangle d\mu \\ &\quad - \int_M \left(\langle \nabla f, \nabla_{\nabla g} V\rangle + \langle \nabla_{\nabla f} V, \nabla g\rangle\right) d\mu\end{aligned}$$

Starting from Definition 2.2:

$$\begin{aligned}\langle f, \partial_t D_I(g)\rangle_{H^1_0} &= \left.\frac{\partial}{\partial t}\left(\int_M C_t^{-1}\left(\langle \nabla C_t(f), \nabla C_t(g)\rangle\right) d\mu\right)\right|_{t=0} \\ &= \left.\frac{\partial}{\partial t}\left(\int_M ((\varphi_t^{-1})^{\star} g^t)(\nabla f, \nabla g) d\mu\right)\right|_{t=0} \\ &= -\int_M \langle \nabla f, \nabla_{\nabla g} V\rangle + \langle \nabla_{\nabla f} V, \nabla g\rangle d\mu\end{aligned}$$

□

3 DISCRETE CONNECTION

The connection of the ambient space $\bar{\nabla}_u V$ where u is a tangent vector and V is an extrinsic vector field :

$$\begin{array}{cccc}\bar{\nabla} : & \mathbb{R}^{3|\mathcal{F}|} \times \mathbb{R}^{3|\mathcal{V}|} & \longrightarrow & \mathbb{R}^{3|\mathcal{F}|} \\ & (u, V) & \longmapsto & \bar{\nabla}_u V\end{array}$$

Recall that we build the connection $\bar{\nabla}$ using finite differences as follows. Since extrinsic vector fields are defined at vertices the differences are taken along the edges.

Definition 3.3. In a given triangle $T \in \mathcal{F}$ the ambient covariant derivative along the edge e_{ij} is defined by

$$\left(\bar{\nabla}_{\frac{e_{ij}}{\|e_{ij}\|}} V\right)_T = \frac{V_i - V_j}{\|e_{ij}\|}.$$

Thus the ambient connection in the directions $E = (e_{ij}, e_{jk})$ can be stored in a matrix

$$(\bar{\nabla}_E V)_T = \begin{pmatrix} V_i - V_j & V_j - V_k \end{pmatrix}.$$

Then, given any tangent vector $x = E\alpha$, the covariant derivative in its direction can be computed as $\bar{\nabla}_x V = (\bar{\nabla}_E V)\alpha$.

Given the expression above, the discrete Lie derivative of the metric at triangle T follows immediately. Namely for any pair of tangent vectors $x = E\alpha, y = E\beta$ in the triangle T, we have:

$$\mathcal{L}_V \mathrm{g}(x, y)_T = \langle x, (\bar{\nabla}_E V)\beta\rangle + \langle(\bar{\nabla}_E V)\alpha, y\rangle. \tag{5}$$

After integration we obtain the discrete infinitesimal shape difference:

$$f^\top W_M E^V g = -\sum_{T \in \mathcal{F}} \mathcal{L}_V \mathrm{g}(\nabla f, \nabla g)_T \mu(T).$$

The expression of the matrix $W_M E^V$ is more easily found using the derivative of the unified shape difference operator (see Section 4) and is proven later in Thm. 4.1. The same goes for the proof of Prop. 4 which is also postponed until Section 4.4.

4 DISCRETE INFINITESIMAL SHAPE DIFFERENCES

4.1 Discrete Unified Shape Differences

The discretization of the unified shape difference is straightforward when N and M are triangle meshes and share the same connectivity. In Definition 2.2 given above, the gradients and the point-wise scalar products are taken on N while the measure $\mathrm{d}\mu^M$ comes from M. Therefore the right hand side can be discretized by a modified cotangent weight formula:

$$W_M D_I = W_N^M, \text{ where}$$
$$(W_N^M)_{i,j} = \frac{1}{2}\left(\frac{\mu^M(T_\alpha)}{\mu^N(T_\alpha)} \cot \alpha_{ij}^N + \frac{\mu^M(T_\beta)}{\mu^N(T_\beta)} \cot \beta_{ij}^N\right). \tag{6}$$

In Section 4.1 we derived an infinitesimal shape difference from a discrete connection. This discrete can be done by a time derivative of the Eq. (6). To do so, however, we need to introduce an alternative formalism for the cotangent-weights Laplacian.

4.2 Cotangent weights alternative

The usual cotangent weight formula is not well-suited to carry out the computations. Therefore we use an alternative formulation which makes more apparent the link with continuous properties.

We denote the local basis $E = (e_{ij}, e_{jk})$ formed by edges of triangle of a triangle $T = \{x_i, x_j, x_k\} \in \mathcal{F}$ where e_{ij} is an oriented edge. We denote $\|e_{ij}\| = \ell_{ij}$ the edge length. Using this notation, the Finite Element gradient is given by the formula [2]:

$$\nabla f = \frac{1}{2\mu(T_{ijk})} \mathcal{R}^{90^\circ} E_{T_{ijk}} \begin{pmatrix} f_k - f_j \\ f_i - f_j \end{pmatrix}. \tag{7}$$

where $\mathcal{R}^{90^\circ}$ denotes the counter-clockwise rotation by 90°. One can remark that the gradient of a function can be expressed in an alternative way depending on the 2×2-symmetric matrix g_T per triangle:

$$\nabla f = E_T \mathrm{g}_T^{-1} \begin{pmatrix} f_j - f_i \\ f_k - f_j \end{pmatrix}. \tag{8}$$

This matrix will be referred to as *discrete metric tensor* in the local basis $E = (e_{ij}, e_{jk})$ of the triangle T:

$$\mathrm{g}_T := \frac{1}{2}\begin{pmatrix} 2\ell_{ij}^2 & \ell_{ki}^2 - \ell_{jk}^2 - \ell_{ij}^2 \\ \ell_{ki}^2 - \ell_{jk}^2 - \ell_{ij}^2 & 2\ell_{jk}^2 \end{pmatrix} = E^\top E. \tag{9}$$

Note that g_T is defined such that $\begin{pmatrix} 1 & 0 \end{pmatrix}^\top \mathrm{g}_T \begin{pmatrix} 1 & 0 \end{pmatrix} = \ell_{ij}^2$ so the bilinear form of two adjacent triangle agrees along the edges. Moreover using Heron's formula one can verify that $\det(\mathrm{g}_T) = 4\mu(T)^2$.

It follows an alternative expression for the standard cotangent formula:

$$f^\top W g = \sum_{T \in \mathcal{F}} \begin{pmatrix} f_j - f_i \\ f_k - f_j \end{pmatrix}^\top \mathrm{g}_T^{-1} \begin{pmatrix} g_j - g_i \\ g_k - g_j \end{pmatrix} \mu(T).$$

This formulation is equivalent to the one found in [1]:

$$f^\top W g = \sum_{T \in \mathcal{F}} \frac{1}{4\mu(T)} \begin{pmatrix} f_j - f_k \\ f_j - f_i \end{pmatrix}^\top \mathrm{g}_T \begin{pmatrix} g_j - g_k \\ g_j - g_i \end{pmatrix}, \tag{10}$$

by noting that any 2×2 invertible symmetric matrix is linked to its inverse by the formula:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^\top \mathrm{g}_T^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{4\mu(T)^2} \mathrm{g}_T. \tag{11}$$

The classical cotangent weight formula is recovered by noting that $\cot \alpha_{ij} = (-\ell_{ij}^2 + \ell_{jk}^2 + \ell_{ki}^2)/(4\mu(T))$.

4.3 Discrete Metric Derivative

First let's remark that the derivative of the discrete metric can be expressed with respect to discrete connection:

Lemma 4.2. *Given a one parameter family of meshes, the first-order change in the metric tensor $\mathrm{g}_T = E^\top E$ and in the area at a triangle $T \in \mathcal{F}$, is given as:*

$$\left.\frac{\partial_t \mathrm{g}_T}{\partial t}\right|_{t=0} = E^\top (\nabla_E V)_T + (\nabla_E V)_T^\top E,$$
$$\left.\frac{\partial_t \mu(T)}{\partial t}\right|_{t=0} = \mathrm{div}(u)_T \mu(T),$$

where the divergence at triangle T is defined as $\mathrm{div}(u)_T := \mathrm{Tr}\left(\mathrm{g}_T^{-1} E^\top (\nabla_E V)\right)$.

Proof. First let's remark that the derivative of the discrete metric can be expressed with respect to discrete connection:

$$\left.\frac{\partial \mathbf{g}_T^t}{\partial t}\right|_{t=0} = E^\top (\nabla_E V)_T + (\nabla_E V)_T^\top E.$$

Since the metric is linked to the triangle area by $\mu_t(T) = \frac{1}{2}\sqrt{\det(\mathbf{g}_T^t)}$ the statement obtained by a direct computation of the derivative. □

4.4 Discrete infinitesimal shape difference

Taking the derivative of the discrete unified shape difference in Eq. (6) might be challenging. However, using the formulation of Eq. (10) leads to an equivalent formulation of Eq. (6) is:

$$f^\top W D_I^t g := \sum_{T\in\mathcal{F}} \langle \nabla^t f, \nabla^t g\rangle_T^t \mu(T). \tag{12}$$

By taking the derivative of his expression at time $t = 0$, we obtain an alternative discretization of the infinitesimal shape difference E.

Theorem 4.1. *The discrete infinitesimal shape difference reads $E^V(u) = W_M^{-1}H$, where H is a Laplacian matrix whose weights depend on the extrinsic vector field:*

$$(H)_{ij} = \frac{1}{2}\sum_{j\sim i}(c(T_{\alpha_{ij}}) + c(T_{\beta_{ij}})),$$

$$c(T) = (\langle e_{jk}, V_j - V_i\rangle + \langle e_{ij}, V_j - V_k\rangle)\frac{1}{4\mu(T)} - \mathrm{div}(V)_T \frac{\langle e_{jk}, e_{ki}\rangle}{\mu(T)}.$$

Proof. Using the FEM gradient, e.g. Eq. (7), to discretize the unified shape difference written in Eq. (12) leads to:

$$f^\top W D_I^t g = \sum_{T\in\mathcal{F}} \frac{1}{4}\begin{pmatrix} f_j - f_k \\ f_j - f_i\end{pmatrix}^\top \frac{\mathbf{g}_T^t}{\mu_t(T)^2}\begin{pmatrix} g_j - g_k \\ g_j - g_i\end{pmatrix}\mu(T). \tag{13}$$

We can now compute the derivative with respect to time by using Lemma 4.2:

$$f^\top W E^V g = \sum_{T\in\mathcal{F}} \frac{1}{4\mu(T)}\begin{pmatrix} f_j - f_k \\ f_j - f_i\end{pmatrix}^\top \mathbf{L}_T \begin{pmatrix} g_j - g_k \\ g_j - g_i\end{pmatrix},$$

$$\mathbf{L}_T = E^\top(\nabla_E V)_T + (\nabla_E V)_T^\top E - 2\mathrm{Tr}\left(\mathbf{g}_T^{-1}E^\top(\nabla_E V)\right)\mathbf{g}_T.$$

The matrices $\mathbf{L}_T$ can be written in a form similar to the discrete metric (see Eq. (9)):

$$\mathbf{L}_T = \frac{1}{2}\begin{pmatrix} 2a_{ij} & a_{ki} - a_{jk} - a_{ij} \\ a_{ki} - a_{jk} - a_{ij} & 2a_{jk}\end{pmatrix},$$

$$\text{where } a_{ij} = \langle e_{ij}, V_i - V_j\rangle - 2\mathrm{div}(V)_T \ell_{ij}^2, \tag{14}$$

where the divergence is define as in Lemma 4.2. This leads to the point-wise formulation:

$$(W_M E^V)_{ij} = \frac{1}{2}\sum_{j\sim i}(c(T_{\alpha_{ij}}) + c(T_{\beta_{ij}})),$$

$$c(T) = \frac{-a_{ki} + a_{jk} + a_{ij}}{4\mu(T)}.$$

□

5 EQUIVALENCE OF THE TWO DISCRETIZATIONS

Proposition 5.5. *The discretization of E based on the discrete Levi-Civita connection is equivalent to the one obtained by differentiating the unified shape difference operator.*

Proof. In Eq. (5) the tangent vectors in a given triangle have be to expressed in the basis form by two edges of the triangle. Following the discussion in Section 4.2, the FEM gradient at a face T can be written in two equivalent ways:

$$\nabla f_T = \frac{1}{2\mu(T)}\mathcal{R}^{90^\circ} E\begin{pmatrix} f_j - f_k \\ f_j - f_i\end{pmatrix} = E\mathbf{g}_T^{-1}\begin{pmatrix} f_j - f_i \\ f_k - f_j\end{pmatrix}.$$

Therefore the discrete strain tensor at triangle T in Eq. (5) follows immediately:

$$\begin{aligned}\mathcal{L}_V \mathbf{g}(\nabla f, \nabla g) &= \begin{pmatrix} f_i - f_j \\ f_j - f_k\end{pmatrix}^\top \mathbf{g}_T^{-1}E^\top(\bar{\nabla}_E V)\mathbf{g}_T^{-1}\begin{pmatrix} g_i - g_j \\ g_j - g_k\end{pmatrix} \\ &+ \begin{pmatrix} f_i - f_j \\ f_j - f_k\end{pmatrix}^\top \mathbf{g}_T^{-1}(\bar{\nabla}_E V)^\top E\mathbf{g}_T^{-1}\begin{pmatrix} g_i - g_j \\ g_j - g_k\end{pmatrix} \\ &= \begin{pmatrix} f_i - f_j \\ f_j - f_k\end{pmatrix}^\top \left.\frac{\partial}{\partial t}\left(\mathbf{g}_T^{-1}\right)\right|_{t=0}\begin{pmatrix} g_i - g_j \\ g_j - g_k\end{pmatrix}.\end{aligned}$$

From Lemma 4.2, we recognize the term the derivative of the inverse metric. Using Eq. (11), one can further modified the expression to:

$$\mathcal{L}_V \mathbf{g}(\nabla f, \nabla g)_T = -\frac{1}{4}\begin{pmatrix} f_j - f_k \\ f_j - f_i\end{pmatrix}^\top \left.\frac{\partial}{\partial t}\left(\frac{\mathbf{g}_T^t}{\mu_t(T)^2}\right)\right|_{t=0}\begin{pmatrix} g_j - g_k \\ g_j - g_i\end{pmatrix}. \tag{15}$$

The right had side term appears in the discrete isometric shape difference as written in Eq. (13). This leads to the equality between the different discretization:

$$\left.\frac{\partial}{\partial t}\left(f^\top W D_I^t g\right)\right|_{t=0} = -\sum_{T\in\mathcal{F}} \mathcal{L}_V \mathbf{g}(\nabla f, \nabla g)_T \mu(T).$$

□

6 VECTOR FIELDS REPRESENTATION

Proposition 6.4. *For almost all triangle meshes M without boundary, the operator E^V uniquely defines the extrinsic vector field V up to rigid motion.*

Proof. The proof is organized as follow: we show that we can recover the matrices $\mathbf{L}_T$ from the infinitesimal shape difference in Eq. (14) then we use a standard results in combinatorix to prove that $\mathbf{L}_T = 0$ if and only if the extrinsic vector field is a rigid motion.

Kernel of $\mathbf{L}_T \mapsto E^V$. The information about the extrinsic vector field is solely contained by the matrices $\mathbf{L}_T$. Like the discrete metric those matrices agrees across edges so they can be reduced to the vector $a \in \mathbb{R}^{|\mathcal{E}|}$ as defined in Eq. (14). The application $a \mapsto E^V$ is linear and we will prove it is almost always invertible.

Extracting elements of WE^V corresponding to edges on M yields a linear operator $B : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}^{|\mathcal{E}|}$ with matrix

$$B_{ij} = \frac{1}{8}\begin{cases} \mu(T_i)^{-1} + \mu(T_i')^{-1} & \text{if } i = j \\ -\mu(T)^{-1} & \text{if } i, j \text{ are edges of } T \\ 0 & \text{otherwise.}\end{cases}$$

Here, indices i, j refer to edges on M; for a given edge i, we label its adjacent triangles T_i and T_i'. Remark that B can be written as a weighted sum: $B = \sum_k \frac{1}{8}\mu(T_k)^{-1}B^k$, where each B^k is a matrix such that:

$$B^k_{ij} = \begin{cases} 1 & \text{when } i = j, \text{ and } i \text{ belongs to triangle } k. \\ -1 & \text{when } i, j \text{ are edges of triangle } k. \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the intersection of the kernels of all B^k is empty. Moreover, by considering the determinant of B as a multivariate polynomial with real coefficients, we conclude that B is either singular for any choice of values of $\mu(T_k)^{-1}$, or for a finite set of coefficients, which thus have measure zero. The proof of the claim follows by noting that for a closed manifold mesh there must exist a non-singular weighted sum, which can be obtained by iteratively adding triangles, while maintaining that the corresponding matrix sum is non-singular on the incident edges.

Rigidity Theorem. As shown previously the kernel of $a \mapsto E^V$ is almost always reduced to the zero element. Going back to the matrices $\mathbf{L}_T$, the extrinsic vector field in the kernel should satisfy:

$$\mathbf{g}_T^{-1}E^\top(\nabla_E V)_T + \mathbf{g}_T^{-1}(\nabla_E V)_T^\top E - 2\mathrm{Tr}\left(\mathbf{g}_T^{-1}E^\top(\nabla_E V)\right)\mathrm{Id} = 0.$$

Taking the trace in both sides implies that $\mathrm{div}(V)_T = \mathrm{Tr}(\mathbf{g}_T^{-1}\partial_t \mathbf{g}_T^t|_0)$ should vanish so it is equivalent to have all matrices $E^\top(\nabla_E V)_T + (\nabla_E V)_T^\top E$ equal zero. It follows that the extrinsic vector field satisfies $\langle e_{ij}, V_i - V_j\rangle = 0$ at all edges. It has been proved in [5] that almost all simply connected closed surfaces only admit rigid deformation as solution of this equation. □

7 CONSTRUCTION FOR TETRAHEDRAL MESHES

Let's consider the case of a mesh whose constitutive elements are only tetrahedra. The set of tetrahedra is denoted $\mathcal{S}$. Figure 1 illustrates a tetrahedron S defined by 4 vertices $(x_1, \ldots, x_4)$. The first 3 edges form the local basis: $E_S = (x_2 - x_1, \ldots, x_4 - x_1)$. The discrete metric tensor, now denoted $\mathbf{g}_S = E_S^\top E_S$, is expressed locally using the edge lengths:

$$\mathbf{g}_{ij} = \begin{cases} \ell_{1i}^2, & i = j \\ \frac{1}{2}(\ell_{ij}^2 - \ell_{1i}^2 - \ell_{1j}^2), & i \neq j. \end{cases}, \qquad 1 \leq i, j \leq k.$$

The volume of the simplex is accessible through the determinant of the metric by $\mu(S) = \sqrt{\det(\mathbf{g}_S)}/6$. The gradient of a piece-wide linear function inside the simplex S is now computed, by analogy with Eq. (8), with the formula:

$$\nabla f = E_S \mathbf{g}_S^{-1}\begin{pmatrix} f_2 - f_1 \\ \ldots \\ f_4 - f_1 \end{pmatrix}. \tag{16}$$

This formula is strictly equivalent to other more classical FEM formulation as it relies only on computing the derivative of a piece-wide linear function.

7.1 Connection for Tetrahedral Meshes

Now, the connection of the ambient space $\bar{\nabla}_u V$ where u is a vector inside a tetrahedron and V is an extrinsic vector field assigning a

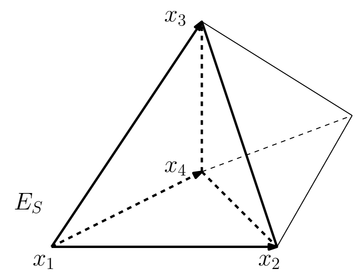

Fig. 1. Local basis in a tetrahedron.

vector per vertex:

$$\begin{array}{llll} \bar{\nabla}: & \mathbb{R}^{3|\mathcal{S}|} \times \mathbb{R}^{3|\mathcal{V}|} & \longrightarrow & \mathbb{R}^{3|\mathcal{S}|} \\ & (u, V) & \mapsto & \bar{\nabla}_u V \end{array}$$

We build the connection $\bar{\nabla}$ by analogy with triangle mesh case in Section 3 leading to add an extra vector in the local basis E_S.

Definition 7.4. In a given tetrahedron $S \in \mathcal{S}$ the ambient covariant derivative along the edge e_{i1} is defined by

$$\left(\bar{\nabla}_{\frac{e_{i1}}{\|e_{i1}\|}} V\right)_T = \frac{V_i - V_1}{\|e_{i1}\|}.$$

Thus the ambient connection in the directions E_S can be stored in a matrix

$$(\bar{\nabla}_{E_S} V)_T = \begin{pmatrix} V_2 - V_1 & \ldots & V_4 - V_1 \end{pmatrix}.$$

Then, given any tangent vector $x = E_S\alpha$, the covariant derivative in its direction can be computed as $\bar{\nabla}_x V = (\bar{\nabla}_{E_S} V)\alpha$.

Given the expression above, the discrete Lie derivative of the metric at simplex S follows immediately. Namely for any pair of tangent vectors $x = E\alpha, y = E\beta$ in the simplex S, we have:

$$\mathcal{L}_V \mathbf{g}(x, y)_S = \langle x, (\bar{\nabla}_{E_S} V)\beta\rangle + \langle(\bar{\nabla}_{E_S} V)\alpha, y\rangle.$$

Considering the local expression of the gradient in Eq. (16), we obtain:

$$\begin{aligned} \mathcal{L}_V \mathbf{g}(\nabla f, \nabla g)_S = &\begin{pmatrix} f_2 - f_1 \\ \ldots \\ f_4 - f_1 \end{pmatrix}^\top \mathbf{g}_S^{-1}(\bar{\nabla}_{E_S} V)\mathbf{g}_S^{-1}\begin{pmatrix} g_2 - g_1 \\ \ldots \\ g_4 - g_1 \end{pmatrix} \\ &+ \begin{pmatrix} f_2 - f_1 \\ \ldots \\ f_4 - f_1 \end{pmatrix}^\top \mathbf{g}_S^{-1}(\bar{\nabla}_{E_S} V)^\top \mathbf{g}_S^{-1}\begin{pmatrix} g_2 - g_1 \\ \ldots \\ g_4 - g_1 \end{pmatrix} \end{aligned}$$

After integration we obtain the discrete infinitesimal shape difference on a tetrahedral mesh:

$$f^\top W_M E^V g = -\sum_{S \in \mathcal{S}} \mathcal{L}_V \mathbf{g}(\nabla f, \nabla g)_S \mu(S).$$

8 FUNCTIONAL MAP INFERENCE

Proposition 8.6. *Given a pair of surfaces M, N embedded in 3D, and a diffeomorphism $\varphi : N \longrightarrow M$, let C be the corresponding functional map $\mathcal{F}_M \longrightarrow \mathcal{F}_M$. Then M and N are related by a rigid motion in space if and only if:*

$$\|C_\varphi \Delta_M - \Delta_N C_\varphi\| + \|C_\varphi E_M^n - E_N^n C_\varphi\| = 0,$$

where Δ are the LB operators, while E^n are functional deformation fields arising from the normal fields.

Proof. *Necessary condition.* The commutativity of the functional map with the Laplace-Beltrami operators immediately implies that φ is an isometry.

The condition $C_\varphi E^n_M = E^n_N C_\varphi$ will provide an equality between the second fundamental form as $\mathcal{L}_n \mathbf{g} = -2\mathbf{h}$. Let f, g be functions on M. Taking the inner product of the left hand side with the function $C_\varphi(f)$ and using the isometry property, allows to uncover the second fundamental form of M:

$$\begin{aligned}\langle C_\varphi(f), E^n_N C_\varphi(g)\rangle_{H^1_0(N)} &= \langle C_\varphi(f), C_\varphi E^n_M(g)\rangle_{H^1_0(N)} \\ &= \langle f, E^n_M(g)\rangle_{H^1_0(M)} \\ &= -2\int_M \mathbf{h}_M(\nabla f, \nabla g)\mathrm{d}\mu^M.\end{aligned}$$

The right hand side leads to the pullback of the second fundamental form from N to M:

$$\begin{aligned}\langle C_\varphi(f), E^n_N C_\varphi(g)\rangle_{H^1_0(N)} &= -2\int_N \mathbf{h}_N(\nabla C_\varphi(f), \nabla C_\varphi(g))\mathrm{d}\mu^N \\ &= -2\int_M ((\varphi^{-1})^\star \mathbf{h}_N)(\nabla f, \nabla g)\mathrm{d}(\varphi_\star \mu^N) \\ &= -2\int_M ((\varphi^{-1})^\star \mathbf{h}_N)(\nabla f, \nabla g)\mathrm{d}\mu^M.\end{aligned}$$

Thus for all functions f, g, we have:

$$\int_M \mathbf{h}_M(\nabla f, \nabla g)\mathrm{d}\mu^M = \int_M ((\varphi^{-1})^\star \mathbf{h}_N)(\nabla f, \nabla g)\mathrm{d}\mu^M.$$

Therefore, using a result from [6], it implies $\mathbf{h}^M = (\varphi^{-1})^\star \mathbf{h}^N$.

The first and second fundamental forms of N and M agree, so as a consequence of the fundamental theorem of surface theory the two manifolds must relate by a rigid motion.

Sufficient condition. If N, M are equal up to a rigid motion then the first and second fundamental forms are equal. It immediately implies that $C_\varphi \Delta_M = \Delta_N C_\varphi$. We show the second equality by reusing the computation done for the necessary condition:

$$\begin{aligned}\langle C_\varphi(f), C_\varphi E^n_M(g)\rangle_{H^1_0(N)} &= \langle f, E^n_M(g)\rangle_{H^1_0(M)} \\ &= -2\int_M \mathbf{h}_M(\nabla f, \nabla g)\mathrm{d}\mu^M \\ &= -2\int_M ((\varphi^{-1})^\star \mathbf{h}_N)(\nabla f, \nabla g)\mathrm{d}\mu^M \\ &= \langle C_\varphi(f), E^n_N C_\varphi(g)\rangle_{H^1_0(N)}.\end{aligned}$$

Thus, we have $C_\varphi E^n_M = E^n_N C_\varphi$. □

9 INTRINSIC SYMMETRIZATION

The unified shape difference of composition of mapping can be computed from functional maps and shape differences of the independent maps as shown be the following Lemma.

Lemma 9.1. *Assume that $D^\varphi_I : H^1_0(M) \to H^1_0(M)$ represents the distortion of the metric between the surfaces M and P induced by the diffeomorphism $\varphi : P \to M$ and $D^\phi_I : H^1_0(P) \to H^1_0(P)$ the distortion between the surfaces P and N linked through $\phi : N \to P$. The distortion $D^{\varphi\circ\phi}_I : H^1_0(M) \to H^1_0(M)$ associated to $\varphi \circ \phi : N \to M$ is given by*

$$D^{\varphi\circ\phi}_I = D^\varphi_I \circ C^{-1}_\varphi \circ D^\phi_I \circ C_\varphi.$$

Proof. The proof relies only on Definition 2.2:

$$\begin{aligned}&\int_P C_\varphi \left(\left\langle \nabla f, \nabla D^{\varphi\circ\phi}_I(g)\right\rangle\right) \mathrm{d}\mu \\ &= \int_P C^{-1}_\phi \left(\langle \nabla(f \circ \varphi \circ \phi), \nabla(g \circ \varphi \circ \phi)\rangle\right) \mathrm{d}\mu \\ &= \int_P \left\langle \nabla(f \circ \varphi), \nabla D^\phi_I(g \circ \varphi)\right\rangle \mathrm{d}\mu \\ &= \int_P C_\varphi \left(\left\langle \nabla f, \nabla D^\varphi_I \left(D^\phi_I(g \circ \varphi) \circ \varphi^{-1}\right)\right\rangle\right) \mathrm{d}\mu.\end{aligned}$$

This yields the equality $D^{\varphi\circ\phi}_I(g) = D^\varphi_I \left(D^\phi_I(g \circ \varphi) \circ \varphi^{-1}\right)$ for all $g \in H^1_0(M)$. □

Lemma 9.1 is used to compute the defining condition for intrinsic symmetrization. Namely, we are looking for the diffeomorphism $\varphi : M' \to M$ such that the self-map $\psi = \varphi^{-1} \circ \pi \circ \varphi : M' \to M'$ is an isometry or equivalently the unified shape difference D^ψ_I, computed with the map ψ, should be equal to identity. Using Prop. 9.1, D^ψ_I becomes:

$$\begin{aligned}D^\psi_I &= D^{\varphi^{-1}}_I C_\varphi D^{\pi\circ\varphi}_I C^{-1}_\varphi \\ &= D^{\varphi^{-1}}_I C_\varphi D^\pi_I C^{-1}_\pi D^\varphi_I C_\pi C^{-1}_\varphi \\ &= C_\varphi \left(D^\varphi_I\right)^{-1} D^\pi_I C^{-1}_\pi D^\varphi_I C_\pi C^{-1}_\varphi.\end{aligned}$$

So the condition $D^\psi_I = I$ is equivalent to:

$$D^\pi_I C^{-1}_\pi D^\varphi_I C_\pi = D^\varphi_I.$$

REFERENCES

[1] Davide Boscaini, Davide Eynard, Drosos Kourounis, and Michael M Bronstein. 2015. Shape-from-Operator: Recovering Shapes from Intrinsic Operators. In *Computer Graphics Forum*, Vol. 34. Wiley Online Library, 265–274.

[2] Mario Botsch, Leif Kobbelt, Mark Pauly, Pierre Alliez, and Bruno Lévy. 2010. *Polygon mesh processing*. CRC press.

[3] Haim Brezis. 2010. *Functional analysis, Sobolev spaces and partial differential equations*. Springer.

[4] M.P. do Carmo. 2013. *Riemannian Geometry*. Birkhäuser Boston.

[5] Herman Gluck. 1975. Almost all simply connected closed surfaces are rigid. In *Geometric topology*. Springer, 225–239.

[6] Henrik Schumacher. 2013. *Conformal maps and p-Dirichlet energies*. Technical Report. Citeseer.