Combinatorial topology and the coloring of Kneser graphs

Frédéric Meunier

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Ecole des Ponts, France

Martin Kneser proposed in 1955 the following problem ("Aufgabe 360"):

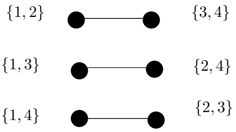
Let $\mathbf{k}$ *and* $\mathbf{n}$ *be two natural numbers,* $\mathbf{2k \leq n}$*; let* $\mathbf{N}$ *be a set with* $\mathbf{n}$ *elements,* $\mathbf{N_k}$ *the set of all subsets of* $\mathbf{N}$ *with exactly* $\mathbf{k}$ *elements; let* $\mathbf{f : N_k \rightarrow M}$ *with the property* $\mathbf{f(K_1) \neq f(K_2)}$ *if* $\mathbf{K_1 \cap K_2 = \emptyset}$*. Let* $\mathbf{m(k,n)}$ *be the minimal number of elements in* $\mathbf{M}$ *such that* $\mathbf{f}$ *exists. Prove that there are* $\mathbf{m_0(k)}$ *and* $\mathbf{n_0(k)}$ *such that* $\mathbf{m(k,n) = n - m_0(k)}$ *for* $\mathbf{n \geq n_0(k)}$*; here* $\mathbf{m_0(k) \geq 2k - 2}$ *and* $\mathbf{n_0(k) \geq 2k - 1}$*; both inequalities probably hold with equality.*

The *Kneser graph* $\mathbf{KG(n,k)}$:

- vertex set $\mathbf{V = \{A \subseteq [n] : |A| = k\}}$
- pairs of disjoint elements of $\mathbf{V}$ as edge set.

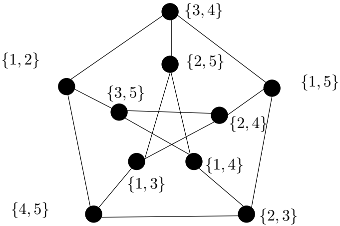
Examples of Kneser graphs

KG(4,2)



Matching

KG(5,2)



Petersen graph

"Aufgabe 360" becomes in the terminology of graphs

Conjecture (Kneser's conjecture)

For $\mathbf{n} \geq \mathbf{2k}$

$$\chi(\mathbf{KG(n,k)}) = \mathbf{n - 2k + 2}.$$

The proof of $\leq \mathbf{n - 2k + 2}$ has a simple proof:

$$\mathbf{F \mapsto \min(\min(F), n - 2k + 2))}$$

is a proper coloring.

Before 1979, only few cases were proved ($\mathbf{k} \leq \mathbf{3}$).

In 1979, Lovász found a suprising proof, using tools of algebraic topology.

Theorem (The Lovász-Kneser theorem)

For $\mathbf{n} \geq \mathbf{2k}$

$$\chi(\mathbf{KG}(\mathbf{n}, \mathbf{k})) = \mathbf{n} - \mathbf{2k} + \mathbf{2}.$$

One of the interest of Kneser graphs is – among many other properties – the gap between the *chromatic number* $\chi(\mathbf{KG(n,k)})$ and the *fractionnal chromatic number* $\chi_{\mathbf{f}}(\mathbf{KG(n,k)})$.

Fractionnal chromatic number of a graph $\mathbf{G=(V,E)}$: minimum of the fractions $\frac{\mathbf{a}}{\mathbf{b}}$ such that $\mathbf{V}$ can be covered by $\mathbf{a}$ independent sets in such a way that every vertex is covered at least $\mathbf{b}$ times.
By definition

$$\frac{|\mathbf{V}|}{\alpha(\mathbf{G})} \leq \chi_{\mathbf{f}}(\mathbf{G}) \leq \chi(\mathbf{G}).$$

We can prove that

$$\chi_{\mathbf{f}}(\mathbf{KG(n,k)}) = \frac{\mathbf{n}}{\mathbf{k}}.$$

Even if the proof by Lovász was simplified over the years (Barany 1979, Greene 2002), it remains purely topological.

In 2003, Matoušek proposed the first combinatorial proof of the Lovász theorem.

The main tool of the approach by Matoušek is Tucker's lemma.

Why looking for combinatorial proofs ?

- to get a better insight
- to get (sometimes) shorter proofs
- to get new results
- to be constructive

Lemma (Tucker's lemma)

If for any set-pair $\mathbf{A}, \mathbf{B} \subseteq [\mathbf{n}]$ *with* $\mathbf{A} \cap \mathbf{B} = \emptyset$ *and* $\mathbf{A} \cup \mathbf{B} \neq \emptyset$ *we have a label* $\lambda(\mathbf{A}, \mathbf{B}) \in \{-1, +1, -2, +2, \ldots, -(\mathbf{n}-1), +(\mathbf{n}-1)\}$ *such that* $\lambda(\mathbf{A}, \mathbf{B}) + \lambda(\mathbf{B}, \mathbf{A}) = \mathbf{0}$*, then there exist two set-pairs* $(\mathbf{A}_1, \mathbf{B}_1)$ *and* $(\mathbf{A}_2, \mathbf{B}_2)$ *such that* $(\mathbf{A}_1, \mathbf{B}_1) \subseteq (\mathbf{A}_2, \mathbf{B}_2)$ *and* $\lambda(\mathbf{A}_1, \mathbf{B}_1) + \lambda(\mathbf{A}_2, \mathbf{B}_2) = \mathbf{0}$.

Case $\mathbf{n = 2}$

$$\lambda(\{1\}, \emptyset) = 1$$
$$\lambda(\{2\}, \emptyset) = 1$$
$$\lambda(\{1\}, \{2\}) = 1$$
$$\lambda(\{1, 2\}, \emptyset) = 1$$

The proof by Matoušek

Assume that $\mathbf{KG(n,k)}$ is properly colored by a map $\mathbf{c} : \binom{[\mathbf{n}]}{\mathbf{k}} \mapsto \{\mathbf{1}, \ldots, \mathbf{t}\}$.

Define

$$\lambda(\mathbf{A},\mathbf{B}) = \begin{cases} \pm(|\mathbf{A}| + |\mathbf{B}|) & \text{if } |\mathbf{A}| + |\mathbf{B}| \leq \mathbf{2k-2} \\ \pm(\mathbf{c(S)} + \mathbf{2k-2}) & \text{if not,} \end{cases}$$

where $\mathbf{S}$ is a $\mathbf{k}$-set $\subseteq \mathbf{A}$ or $\subseteq \mathbf{B}$ and such that $\mathbf{c(S)}$ takes the smallest possible value.

In the first case, the sign is $+$ is $\mathbf{min(A)} < \mathbf{min(B)}$ and $-$ if not.
In the second case, the sign is $+$ if $\mathbf{S} \subseteq \mathbf{A}$ and $-$ if not.

If $\mathbf{t} \leq \mathbf{n} - \mathbf{2k} + \mathbf{1}$, we would have a map $\boldsymbol{\lambda}$ satisfying exactly the requirement of Tucker's lemma. Hence, there are two set-pairs $(\mathbf{A}_1, \mathbf{B}_1)$ and $(\mathbf{A}_2, \mathbf{B}_2)$ such that $(\mathbf{A}_1, \mathbf{B}_1) \subseteq (\mathbf{A}_2, \mathbf{B}_2)$ and $\lambda(\mathbf{A}_1, \mathbf{B}_1) + \lambda(\mathbf{A}_2, \mathbf{B}_2) = \mathbf{0}$.

But this would mean that two disjoint $\mathbf{k}$-sets have the same color through $\mathbf{c}$.

Schrijver's theorem

A $\mathbf{k}$-set $\mathbf{A} \subseteq [\mathbf{n}]$ is said to be *stable* if it does not contain two adjacent elements modulo $\mathbf{n}$ (if $\mathbf{i} \in \mathbf{A}$, then $\mathbf{i}+\mathbf{1} \notin \mathbf{A}$, and if $\mathbf{n} \in \mathbf{A}$, then $\mathbf{1} \notin \mathbf{A}$).

The *Schrijver graph* $\mathbf{SG(n,k)}$:

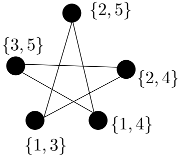
- vertex set $\mathbf{V} = \{\mathbf{A} \subseteq [\mathbf{n}] : |\mathbf{A}| = \mathbf{k} \text{ and } \mathbf{A} \text{ is stable}\}$
- pairs of disjoint elements of $\mathbf{V}$ as edge set.

Examples of Schrijver graphs

SG(4,2)

Matching

SG(5,2)



5-cycle

Theorem (Schrijver's theorem, 1979)

$\chi(\mathbf{SG}(\mathbf{n}, \mathbf{k})) = \mathbf{n} - \mathbf{2k} + \mathbf{2}$.

The proof was again topological.

Ziegler (2004) adapted Matoušek's idea to get a combinatorial proof of Schrijver's theorem. It was a rather long proof using *oriented matroids*.

Our goal is now to show that it is actually possible to modify slightly Matoušek's proof in order to get a short and combinatorial proof of Schrijver's theorem.

For $\mathbf{A}, \mathbf{B} \subseteq [\mathbf{n}]$, define $\mathbf{alt(A, B)}$ to be the length of the longest increasing sequence $\mathbf{x_1, x_2, \ldots, x_l}$ such that $\mathbf{x_i} \in \mathbf{A} \cup \mathbf{B}$ for all $\mathbf{i}$ and such that if $\mathbf{x_i} \in \mathbf{A}$, then $\mathbf{x_{i+1}} \in \mathbf{B}$ and if $\mathbf{x_i} \in \mathbf{B}$, then $\mathbf{x_{i+1}} \in \mathbf{A}$.

$$\mathbf{alt(\{3\}, \{1, 6\}) = 3}$$

$$\mathbf{alt(\{1, 4\}, \{2, 5, 6\}) = 4}$$

$$\mathbf{alt(\{2, 3, 5, 11\}, \{1, 6, 8, 9, 16\}) = 5}$$

Combinatorial proof of Schrijver's theorem

Assume that $\mathbf{KG(n,k)}$ is properly colored by a map $\mathbf{c} : \binom{[n]}{k} \mapsto \{1, \ldots, t\}$.

Define

$$\lambda(A,B) = \begin{cases} \pm(\mathrm{alt}(A,B)) & \text{if } \mathrm{alt}(A,B) \leq 2k-1 \\ \pm(c(S)+2k-1) & \text{if not,} \end{cases}$$

where $\mathbf{S}$ is a $\mathbf{k}$-set $\subseteq \mathbf{A}$ or $\subseteq \mathbf{B}$ and such that $\mathbf{c(S)}$ takes the smallest possible value.

In the first case, the sign is $+$ is $\min(A) < \min(B)$ and $-$ if not.
In the second case, the sign is $+$ if $S \subseteq A$ and $-$ if not.

If $\mathbf{t} \leq \mathbf{n} - \mathbf{2k} + \mathbf{1}$, we would have a map $\boldsymbol{\lambda}$ satisfying exactly the requirement of Tucker's lemma. Hence, there are two set-pairs $(\mathbf{A_1}, \mathbf{B_1})$ and $(\mathbf{A_2}, \mathbf{B_2})$ such that $(\mathbf{A_1}, \mathbf{B_1}) \subseteq (\mathbf{A_2}, \mathbf{B_2})$ and $\lambda(\mathbf{A_1}, \mathbf{B_1}) + \lambda(\mathbf{A_2}, \mathbf{B_2}) = \mathbf{0}$.

But this would mean that two disjoint **k**-sets have the same color through **c**.

Hedetniemi's conjecture

The *tensorial product* $\mathbf{G} \times \mathbf{H}$ of two graphs $\mathbf{G}$ and $\mathbf{H}$ has vertex set $\mathbf{V}(\mathbf{G} \times \mathbf{H}) = \mathbf{V}(\mathbf{G}) \times \mathbf{V}(\mathbf{H})$ and edge set $\mathbf{E}(\mathbf{G} \times \mathbf{H}) = \{(\mathbf{v}, \mathbf{w}), (\mathbf{v}', \mathbf{w}') : \mathbf{v}\mathbf{v}' \in \mathbf{E}(\mathbf{G}),\ \mathbf{w}\mathbf{w}' \in \mathbf{E}(\mathbf{H})\}$.

Conjecture (Hedetniemi)

$$\chi(\mathbf{G} \times \mathbf{H}) = \min(\chi(\mathbf{G}), \chi(\mathbf{H}))$$

Proved for various families of graphs. Proved for Kneser and Schrijver graphs through advanced topological tools.

With the same kind of proof as before, again, we get a short combinatorial proof.

Theorem

$$\chi(\mathrm{SG}(n_1, k_1), \mathrm{SG}(n_2, k_2)) =$$

$$\min(\chi(\mathrm{SG}(n_1, k_1)), \chi(\mathrm{SG}(n_2, k_2)))$$

Let $\mathbf{n} := \mathbf{n_1} + \mathbf{n_2}$ and $\mathbf{k} := \mathbf{k_1} + \mathbf{k_2}$. Assume w.l.o.g. that $\mathbf{n_1} - 2\mathbf{k_1} \geq \mathbf{n_2} - 2\mathbf{k_2}$.

Assume that $\mathbf{SG(n_1, k_1)} \times \mathbf{SG(n_2, k_2)}$ is properly colored by a map $\mathbf{c} : \binom{[\mathbf{n_1}]}{\mathbf{k_1}} \times \binom{[\mathbf{n_2}]}{\mathbf{k_2}} \mapsto \{1, \ldots, \mathbf{t}\}$.

For $\mathbf{A_i}, \mathbf{B_i} \subseteq [\mathbf{n_i}]$, define

$$\lambda(\mathbf{A_1}, \mathbf{B_1}, \mathbf{A_2}, \mathbf{B_2}) = \begin{cases} \pm(\mathrm{alt}(\mathbf{A_1}, \mathbf{B_1}) + \mathrm{alt}(\mathbf{A_2}, \mathbf{B_2})) \\ \text{if } \mathrm{alt}(\mathbf{A_1}, \mathbf{B_1}) + \mathrm{alt}(\mathbf{A_2}, \mathbf{B_2}) \leq \mathbf{n_1} + 2\mathbf{k_2} - 2 \\ \pm(\mathbf{c}(\mathbf{S_1}, \mathbf{S_2}) + \mathbf{n_1} + 2\mathbf{k_2} - 2) \\ \text{if not,} \end{cases}$$

where $\mathbf{S_i}$ is a $\mathbf{k_i}$-set $\subseteq \mathbf{A_i}$ or $\subseteq \mathbf{B_i}$ and such that $\mathbf{c(S_1, S_2)}$ takes the smallest possible value.

With $\mathbf{A} := \mathbf{A_1} \uplus \mathbf{A_2}$ and $\mathbf{B} := \mathbf{B_1} \uplus \mathbf{B_2}$, λ satisfies the requirements of Tucker's lemma: if $\mathbf{t} = \mathbf{n_2} - 2\mathbf{k_2} + 1$, the maximal value taken by λ is $\mathbf{n_2} - 2\mathbf{k_2} + 1 + \mathbf{n_1} + 2\mathbf{k_2} - 2 = \mathbf{n} - 1$.

Kneser hypergraphs

The *Kneser hypergraph* $\mathbf{KG(n,k,r)}$:

- vertex set $\mathbf{V = \{A \subseteq [n] : |A| = k\}}$
- $\mathbf{r}$-uples of disjoint elements of $\mathbf{V}$ as edge set.

Conjectured by Erdös in 1976, proved by Alon, Frankl and Lovász in 1986

Theorem

$$\chi(\mathbf{KG(n,k,r)}) = \left\lceil \frac{\mathbf{n-(k-1)r}}{\mathbf{r-1}} \right\rceil$$

Again, the proof was completely topological. Ziegler gave in 2004 a combinatorial proof of it, very similar to the one proposed by Matoušek, but this time with a $\mathbf{Z_p}$-Tucker lemma.

Main tool: the $\mathbf{Z_p}$-Tucker lemma

Lemma ($\mathbf{Z_p}$-Tucker lemma)

Let p *be a prime,* $n, m \geq 1$, $\alpha \leq m$ *and for* $X = (X_1, X_2, \ldots, X_p)$ *define*

$$\lambda(X) = (\lambda_1(X), \lambda_2(X)) \in Z_p \times [m]$$

to be Z_p*-equivariant map and satisfying the following properties:*

- *for all* $X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, *if* $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, *then* $\lambda_1(X^{(1)}) = \lambda_1(X^{(2)})$;
- *for all* $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, *if* $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \geq \alpha + 1$, *then the* $\lambda_1(X^{(i)})$ *are not pairwise distinct for* $i = 1, \ldots, p$.

Then $\alpha + (m - \alpha)(p - 1) \geq n$.

Conjecture (Alon-Ziegler)

$$\chi(\mathbf{KG}(\mathbf{n},\mathbf{k},\mathbf{r})_{\mathbf{r}\text{-}stab}) = \left\lceil \frac{\mathbf{n}-(\mathbf{k}-1)\mathbf{r}}{\mathbf{r}-1} \right\rceil$$

where "$\mathbf{r}$-stab" means that the elements of the $\mathbf{k}$-subsets $\subseteq [\mathbf{n}]$ are at distance $\mathbf{r}$ (modulo $\mathbf{n}$) to each other.

In particular $\mathbf{KG}(\mathbf{n},\mathbf{k},2)_{2\text{-stab}} = \mathbf{SG}(\mathbf{n},\mathbf{k})$.

This conjecture is still open, but we have

Theorem (M., 2010)

$$\chi(\mathbf{KG}(\mathbf{n},\mathbf{k},\mathbf{r})_{\textit{quasi-stab}}) = \left\lceil \frac{\mathbf{n}-(\mathbf{k}-\mathbf{1})\mathbf{r}}{\mathbf{r}-\mathbf{1}} \right\rceil$$

where "quasi-stab" means that elements of the $\mathbf{k}$-subsets $\subseteq [\mathbf{n}]$ are at distance $\mathbf{2}$ to each other (but $\mathbf{n}$ and $\mathbf{1}$ can be together, $\neq$ $\mathbf{2}$-stab) (notion defined by Aigner and De Longueville).
The proof is combinatorial, uses the generalization of the $\mathbf{Z_r}$-Tucker lemma by Ziegler, and a map λ defined with $\mathbf{alt}(\mathbf{A^1},\mathbf{A^2},\ldots,\mathbf{A^r})$, which is the length of the longest increasing sequence of elements of the $\mathbf{A^i}$, two consecutive terms being always in two different $\mathbf{A^i}$'s.

No topological proof is known !

A map $\mathbf{c}: \mathbf{V} \to [\mathbf{p}]$ is *$(\mathbf{p}, \mathbf{q})$-coloring* of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ if $\mathbf{q} \leq |\mathbf{c}(\mathbf{v}) - \mathbf{c}(\mathbf{u})| \leq \mathbf{p} - \mathbf{q}$ for all $\mathbf{uv} \in \mathbf{E}$. The *circular chromatic number* $\chi_\mathbf{c}(\mathbf{G})$ is the minimum of $\mathbf{p}/\mathbf{q}$ such that there exists a $(\mathbf{p}, \mathbf{q})$-coloring.

Another example of theorem whose proof has no topological version is the following (conjectured by Johnson, Holroyd and Stahl in 1997)

Theorem

$$\chi_\mathbf{c}(\mathbf{KG}(\mathbf{n}, \mathbf{k})) = \mathbf{n} - 2\mathbf{k} + 2$$

Proved combinatorially by Chen (2010). The case $\mathbf{n}$ even was proved through topological arguments in 2006.

Chen proved first a version of Tucker's lemma with increasing $\boldsymbol{\lambda}$.

A new conjecture concerning Kneser hypergraphs

Conjecture

Let $\mathbf{n}, \mathbf{k}, \mathbf{r}, \mathbf{s}$ *be positive integers such that* $\mathbf{n} \geq \mathbf{rk}$ *and* $\mathbf{s} \geq \mathbf{r}$. *Then*

$$\chi\left(KG(\mathbf{n}, \mathbf{k}, \mathbf{r})_{\mathbf{s}\text{-}stab}\right) = \left\lceil \frac{\mathbf{n} - (\mathbf{k} - 1)\mathbf{s}}{\mathbf{r} - 1} \right\rceil .$$

Facts

- The easy direction is proved as usual.
- It contains the Alon-Ziegler conjecture as a special case.
- It is enough to prove it when
  - $\mathbf{r} = \mathbf{s}$ (Alon-Ziegler conjecture) and
  - $\mathbf{r}$ and $\mathbf{s}$ coprime

A proposition

Proposition

Let $\mathbf{k}$ *and* $\mathbf{s}$ *be two positive integers such that* $\mathbf{s} \geq \mathbf{2}$. *We have*

$$\chi\left(KG(\mathbf{ks}+\mathbf{1},\mathbf{k},\mathbf{2})_{\mathbf{s}\text{-stab}}\right)=\mathbf{s}+\mathbf{1}.$$

Conjectures have been checked for...

The Alon-Ziegler conjecture has been checked with a computer for

- $n \leq 9$, $k = 2$, $r = 3$.
- $n \leq 12$, $k = 3$, $r = 3$.
- $n \leq 14$, $k = 4$, $r = 3$.
- $n \leq 13$, $k = 2$, $r = 5$.
- $n \leq 16$, $k = 3$, $r = 5$.
- $n \leq 21$, $k = 4$, $r = 5$.

Conjectures have been checked for...

The new conjecture has been checked with a computer for

- $n \leq 9$, $k = 2$, $r = 2$, $s = 3$.
- $n \leq 10$, $k = 2$, $r = 2$, $s = 4$.
- $n \leq 11$, $k = 3$, $r = 2$, $s = 3$.
- $n \leq 13$, $k = 3$, $r = 2$, $s = 4$.
- $n \leq 14$, $k = 4$, $r = 2$, $s = 3$.
- $n \leq 17$, $k = 4$, $r = 2$, $s = 4$.
- $n \leq 11$, $k = 2$, $r = 3$, $s = 4$.
- $n \leq 14$, $k = 3$, $r = 3$, $s = 4$.
- $n \leq 12$, $k = 2$, $r = 3$, $s = 5$.
- $n \leq 13$, $k = 2$, $r = 4$, $s = 5$.

Thank you