Six mathematical gems from the history of Distance Geometry

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Abstract

This is a partial account of the fascinating history of Distance Geometry. We make no claim to completeness, but we do promise a dazzling display of beautiful, elementary mathematics. We prove Heron's formula, Cauchy's theorem on the rigidity of polyhedra, Cayley's generalization of Heron's formula to higher dimensions, Menger's characterization of abstract semi-metric spaces, a result of Gödel on metric spaces on the sphere, and Schoenberg's equivalence of distance and positive semidefinite matrices, which is at the basis of Multidimensional Scaling.
Keywords: Euler's conjecture, Cayley-Menger determinants, Multidimensional scaling, Euclidean Distance Matrix

1 Introduction

Distance Geometry (DG) is the study of geometry with the basic entity being distance (instead of lines, planes, circles, polyhedra, conics, surfaces and varieties). As did much of Mathematics, it all began with the Greeks: specifically Heron, or Hero, of Alexandria, sometime between 150BC and 250AD, who showed how to compute the area of a triangle given its side lengths [36].

After a hiatus of almost two thousand years, we reach Arthur Cayley's: the first paper of volume I of his *Collected Papers*, dated 1841, is about the relationships between the distances of five points in space [7]. The gist of what he showed is that a tetrahedron can only exist in a plane if it is flat (in fact, he discussed the situation in one more dimension). This yields algebraic relations on the side lengths of the tetrahedron.

Hilbert's influence on foundations and axiomatization was very strong in the 1930s *Mitteleuropa* [24]. This pushed many people towards axiomatizing existing mathematical theories [23]. Karl Menger, a young professor of geometry at the University of Vienna and an attendee of the Vienna Circle, proposed in 1928 a new axiomatization of metric spaces using the concept of distance and the relation of congruence, and, using an extension of Cayley's algebraic machinery (which is now known as Cayley-Menger determinant), generalized Heron's theorem to compute the volume of arbitrary K-dimensional simplices using their side lengths [31].

The Vienna Circle was a group of philosophers and mathematicians which convened in Vienna's *Reichsrat café* around the nineteen-thirties to discuss philosophy, mathematics and, presumably, drink coffee. When the meetings became excessively politicized, Menger distanced himself from it, and organized instead a seminar series, which ran from 1929 to 1937 [34]. A notable name crops up in the intersection of Menger's geometry students, the Vienna Circle participants, and the speakers at Menger's *Kolloquium*: Kurt Gödel. Most of the papers Gödel published in the Kolloquium's proceedings are about logic and foundations, but two, dated 1933, are about the geometry of distances on spheres and surfaces. The first [34, 18 Feb. 1932, p. 198] answers a question posed at a previous seminar by Laura Klanfer, and shows that a set X of four points in any metric space, congruent to four non-coplanar points in $\mathbb{R}^3$, can be realized on the surface of a three-dimensional sphere using geodesic distances. The second [34, 17 May 1933, p. 252] shows that Cayley's relationship hold locally on certain surfaces which behave locally like Euclidean spaces.

The first public mention of Gödel's completeness theorem [18] (which was also the subject of his Ph.D. thesis) was given at the *Kolloquium* [34, 14 May 1930, p. 135], just three months after obtaining his doctorate from the University of Vienna. As for his incompleteness theorem [19], F. Alt recalls [34, Afterword] that Gödel's seminar [34, 22 Jan. 1931, p. 168] appears to have been the first oral presentation of its proof:

> There was the unforgettable quiet after Gödel's presentation, ended by what must be the understatement of the century: "That is very interesting. You should publish that." Then a question: "You use Peano's system of axioms. Will it work for other systems?" Gödel, after a few seconds of thought: "Yes, any system broad enough to define the field of integers." Olga Taussky (half-smiling): "The integers do not constitute a field!" Gödel, who knew this as well as anyone, and had only spoken carelessly: "Well, the. . . the. . . the domain of integrity of the integers." And final relaxing laughter.

The incompleteness theorem was first mentioned by Gödel during a meeting in Königsberg, in Sept. 1930. Menger, who was travelling, had been notified immediately: with John Von Neumann, he was one of the first to realize the importance of Gödel's result, and began lecturing about it immediately [34, Biographical introduction].

Figure 1: We cannot refrain from mentioning Gödel's incompleteness theorem.

The pace quickens: in 1935, Isaac Schoenberg published some remarks on a paper [39] by Fréchet on the *Annals of Mathematics*, and gave, among other things, an algebraic proof of equivalence between Euclidean Distance Matrices (EDM) and Gram matrices. This is almost the same proof which is nowadays given to show the validity of the classical Multidimensional Scaling (MDS) technique [5, § 12.1].

This brings us to the computer era, where the historical account ends and the contemporary treatment begins. Computers allow the efficient treatment of masses of data, some of which are incomplete and noisy. Many of these data concern, or can be reduced to, distances, and DG techniques are the subject of an application-oriented renaissance [27, 35]. Motivated by the Global Positioning System (GPS), for example, the old geographical concept of *trilateration* (a system for computing the position of a point given its distances from three known points) makes its way into DG in wireless sensor networks [12]. Wüthrich's Nobel Prize for using Nuclear Magnetic Resonance (NMR) techniques in the study of proteins brings DG to the forefront of structural bioinformatics research [22]. The massive use of robotics in mechanical production lines requires mathematical methods based on DG [38].

DG is also tightly connected with graph rigidity [21]. This is an abstract mathematical formulation of statics, the study of structures under the action of balanced forces [30], which is at the basis of architecture [44]. Rigidity of polyhedra gave rise to a conjecture of Euler's [13] about closed polyhedral surfaces, which was proved correct only for some polyhedra: strictly convex [6], convex and higher-dimensional [2], and generic (a polyhedron is generic if no algebraic relations on $\mathbb{Q}$ hold on the components of the vectors which represent its vertices) [17]. It was however disproved in general by means of a very special, non-generic nonconvex polyhedron [8].

The rest of this paper will focus on the following results, listed here in chronological order: Heron's theorem (Sect. 2), Euler's conjecture and Cauchy's proof for strictly convex polyhedra (Sect. 3), Cayley-Menger determinants (Sect. 4), Menger's axiomatization of geometry by means of distances (Sect. 5), a result by Gödel's concerning DG on the sphere (Sect. 6), and Schoenberg's equivalence (Sect. 7) between EDM and Positive Semidefinite Matrices (PSD). There are many more results in DG: this is simply a choice of our favourite results amongst those we knew best.

2 Heron's formula

Heron's formula, which is usually taught at school, relates the area $\mathcal{A}$ of a triangle to the length of its sides a, b, c and its semiperimeter $s = \frac{a+b+c}{2}$ as follows:

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}. \tag{1}$$

There are many ways to prove its validity. Shannon Umberger, a student of the "Foundations of Geometry I" course given at the University of Georgia in the fall of 2000, proposes, as part of his final project (`http://jwilson.coe.uga.edu/emt668/emat6680.2000/umberger/MATH7200/HeronFormulaProject/finalproject.html`), three detailed proofs: an algebraic one, a geometric one, and a trigonometric one. John Conway and Peter Doyle discuss Heron's formula proofs in a publically available email exchange (`https://math.dartmouth.edu/~doyle/docs/heron/heron.txt`) from 1997 to 2001.

Our favourite proof is based on complex numbers, and was submitted (`http://www.artofproblemsolving.com/Resources/Papers/Heron.pdf`) to the "Art of Problem Solving" online school for gifted mathematics students by Miles Edwards (also see `http://newsinfo.iu.edu/news/page/normal/13885.html` and `http://www.jstor.org/stable/10.4169/amer.math.monthly.121.02.149` for more recent career achievements of this gifted student) when he was studying at Lassiter High School in Marietta, Georgia.

2.1 Theorem (Heron's formula [36])
Let $\mathcal{A}$ be the area of a triangle with side lengths a, b, c and semiperimeter length $s = \frac{1}{2}(a+b+c)$. Then $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$.

Proof. [11] Consider a triangle with sides a, b, c (opposite to the vertices A, B, C respectively) and its inscribed circle centered at O with radius r. The perpendiculars from O to the triangle sides split a into y, z, b into x, z and c into x, y as shown in Fig. 2. Let u, v, w be the segments joining O with A, B, C, respectively. First, we note that $2\alpha + 2\beta + 2\gamma = 2\pi$, which implies $\alpha + \beta + \gamma = \pi$. Next, the following

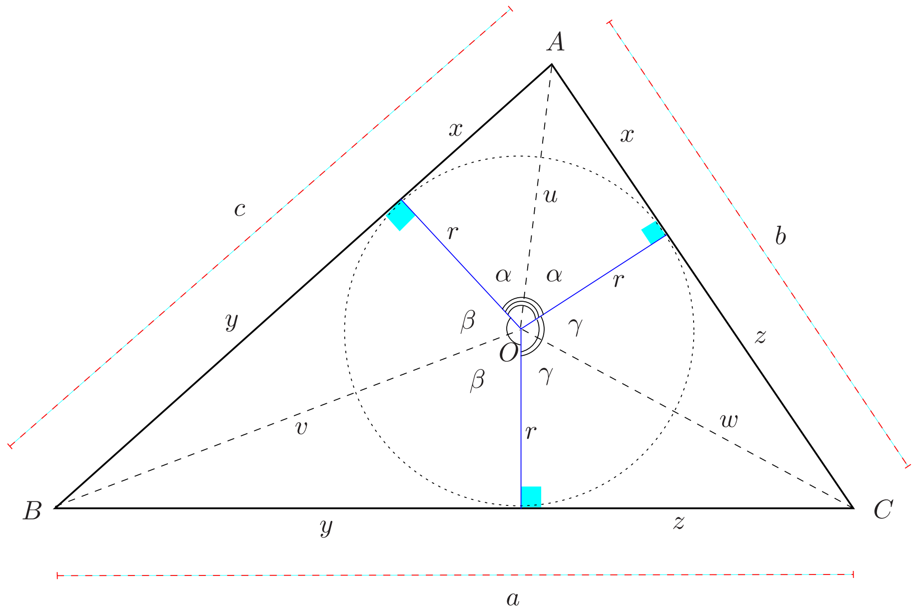


Figure 2: Heron's formula: a proof using complex numbers.

complex identities are easy to verify geometrically in Fig. 2:

$$\begin{aligned} r + ix &= ue^{i\alpha} \\ r + iy &= ve^{i\beta} \\ r + iz &= we^{i\gamma}. \end{aligned}$$

These imply:

$$(r + ix)(r + iy)(r + iz) = (uvw)e^{i(\alpha+\beta+\gamma)} = uvwe^{i\pi} = -uvw,$$

where the last step uses Euler's identity $e^{i\pi} + 1 = 0$ [16, I-VIII, § 138-140, p. 148]. Since $-uvw$ is real, the imaginary part of $(r + ix)(r + iy)(r + iz)$ must be zero. Expanding the product and rearranging terms, we get $r^2(x + y + z) = xyz$. Solving for r, we have the nonnegative root

$$r = \sqrt{\frac{xyz}{x + y + z}}. \tag{2}$$

We can write the semiperimeter of the triangle ABC as $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(y+z+x+z+x+y) = x+y+z$. Moreover,

$$\begin{aligned} s - a &= x + y + z - y - z = x \\ s - b &= x + y + z - x - z = y \\ s - c &= x + y + z - x - y = z, \end{aligned}$$

so $xyz = (s - a)(s - b)(s - c)$, which implies that Eq. (2) becomes:

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

We now write the area $\mathcal{A}$ of the triangle ABC by summing it over the areas of the three triangles AOB, BOC, COA, which yields:

$$\mathcal{A} = \frac{1}{2}(ra + rb + rc) = r\frac{a + b + c}{2} = rs = \sqrt{s(s - a)(s - b)(s - c)},$$

as claimed. □

3 Euler's conjecture and the rigidity of polyhedra

Consider a square with unit sides, in the plane. One can shrink two opposite angles and correspondingly widen the other two to obtains a rhombus (see Fig. 3), which has the same side lengths but a different shape: no sequence of rotations, translations or reflections can turn one into the other. In other words, a square is *flexible*. By contrast, a triangle is not flexible, or *rigid*.

Euler conjectured in 1766 [15] that all three-dimensional polyhedra are rigid. The conjecture appears at the end of the discussion about the problem *Invenire duas superficies, quarum alteram in alteram transformare liceat, ita ut in utraque singula puncta homologa easdem inter se teneat distantias*, i.e.:

> To find two surfaces for which it is possible to transform one into the other, in such a way that corresponding points on either keep the same pairwise distance. (†)

Towards the end of the paper, Euler writes *Statim enim atque figura undique est clausa, nullam amplius mutationem patitur*, which means "As soon as the shape is everywhere closed, it can no longer be transformed". Although the wording appears ambiguous by today's standards, scholars of Euler and rigidity agree: what Euler really meant is that 3D polyhedra are rigid [17].

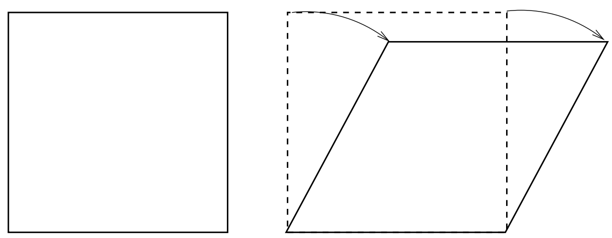


Figure 3: A square is flexed into a rhombus. The set of faces (the edges) are the same, and each maintains pairwise distances through the flexing, i.e. two points on the same edge have the same distance on the left as on the right figure.

To better understand this statement, we borrow from [1] the precise definition of a *polyhedron*: a family $\mathcal{K}$ of points, open segments and open triangles is a *triangulation* if (a) no two elements of $\mathcal{K}$ have common points, and (b) all sides and vertices of the closure of any triangle of $\mathcal{K}$, and both extreme points of the closure of any segment of $\mathcal{K}$ are all in $\mathcal{K}$ themselves. Note that this definition is different from the usual definition employed in convex analysis, i.e. that a polyhedron is an intersection of half-spaces; however, a convex polyhedron in the sense given here is the same as a polytope in the sense of convex analysis. Given a triangulation $\mathcal{K}$ in $\mathbb{R}^K$ (where $K \in \{1, 2, 3\}$), the union of all points of $\mathcal{K}$ with all points in the segments and triangles of $\mathcal{K}$ is called a *polyhedron*. Note that several triangular faces can belong to the same affine space, thereby forming polygonal faces.

Each polyhedron has an incidence structure of points on segments and segments on polygonal (not necessarily triangular) faces, which induces a partial order (p.o.) based on set inclusion. For example, the closure of the square $ABCD$ contains the closures of the segments AB, BC, CD, DA, each of which contains the corresponding adjacent points A, B, B, C, C, D, D, A. Accordingly, the p.o. is $A \subset AB, DA$; $B \subset AB, BC$; $C \subset BC, CD$; $D \subset CD, DA$; $AB, BC, CD, DA \subset ABCD$. Since this p.o. also has a bottom element (the empty set) and a top element (the whole polyhedron), it is a *lattice*. A lattice isomorphism is a bijective mapping between two lattices which preserves the p.o. Two polyhedra P, Q are *combinatorially equivalent* if their triangulations are lattice isomorphic. If, moreover, all the lattice isomorphic polygonal faces of P, Q are exactly equal, the polyhedra are said to be *facewise equal*.

Under the above definition, nothing prevents a polyhedron from being nonconvex (see Fig. 4). It is known that every closed surface, independently of the convexity of its interior, is homeomorphic (intuitively: smoothly deformable in) to some polyhedron (again [1, § 2.2]). This is why we can replace "surface" with "polyhedra".

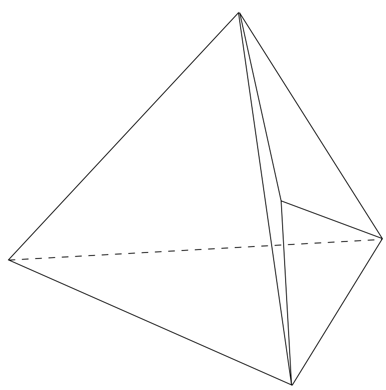


Figure 4: A nonconvex polyhedron.

The "rigidity" implicit in Euler's conjecture should be taken to mean that no point of the polyhedron can undergo a continuous motion under the constraint that the shape be the same at each point of the motion. As for the concept of "shape", it is linked to that of distance, as appears clear from (†). The following is therefore a formal restatement of Euler's conjecture: *two combinatorially equivalent facewise equal polyhedra must be isometric under the Euclidean distance*, i.e. each pair of points in one polyhedron is equidistant with the corresponding pair in the other.

A natural question about the Euler conjecture stems from generalizing the example in Fig. 3 to 3D (see Fig. 5). Does this not disprove the conjecture? The answer is no: all the polygonal faces in the cube

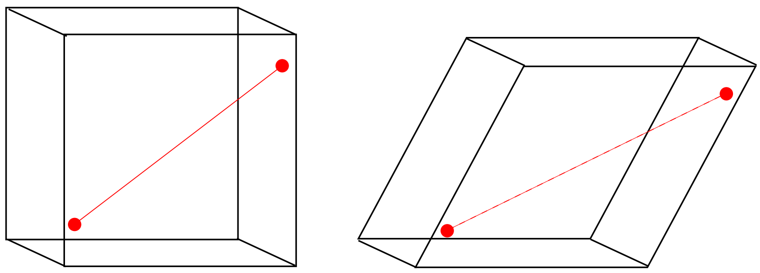


Figure 5: A cube can be transformed into a rhomboid, but the set of faces is not the same anymore (accordingly, corresponding point pairs may not preserve their distance, as shown).

are squares, but this does not hold in the rhomboid. The question is more complicated than it looks at first sight, which is why it took 211 years to disprove it.

3.1 Strictly convex polyhedra: Cauchy's proof

Although Euler's conjecture is false in general, it is true for many important subclasses of polyhedra. Cauchy proved it true for strictly convex polyhedra (in fact Cauchy's proof contained two mistakes, corrected by Steinitz [28, p. 67] and Lebesgue). There are many accounts of Cauchy's proof: Cauchy's original text, still readable today [6]; Alexandrov's book [2], Lyusternik's book [28, § 20], Stoker's paper [43], Connelly's chapter [9] just to name a few. Here we follow the treatment given by Pak [37].

We consider two combinatorially equivalent, facewise equal strictly convex polyhedra P, Q, and aim to show that P and Q are isometric.

For a polyhedron P we consider its associated graph $G(P) = (V, E)$, where V are the points of P and E its segments. Note that $G(P)$ only depends on the incidence structure of the polygonal faces, segments and points of P. Since P, Q are combinatorially equivalent, $G(P) = G(Q)$. Consider the dihedral angles (i.e. the angle smaller than π formed by two half-planes in $\mathbb{R}^3$ intersecting on a line L) α_{uv}, β_{uv} on P, Q induced by the segment represented by the edge $\{u, v\} \in E$. We assign to each edge $\{u, v\} \in E$ a label $\ell_{uv} = \mathrm{sgn}(\beta_{uv} - \alpha_{uv})$ (so $\ell_{uv} \in \{-1, 0, 1\}$), and consider, for each $v \in V$, the edge sequence $\sigma_v = (\{u, v\} \mid u \in N(v))$, where $N(v)$ is the set of nodes adjacent to v. The order of the edges in σ_v is given by any circuit around the polygon $p(v)$ obtained by intersecting P with a plane γ which separates v from the other vertices in V (this is possible by strict convexity, see Fig. 6). It is easy to see that every edge $\{u, v\} \in \sigma_v$ corresponds to a vertex of $p(v)$. Therefore, a circuit over $p(v)$ defines an order over σ_v. We also assume that this order is periodic, i.e. its last element precedes the first one. Any such sequence σ_v naturally induces a sign sequence $s_V = (\ell_{uv} \mid \{u, v\} \in \sigma_v)$; we let $\bar{s}_v$ be the sequence s_v without the zeros, and we count the number m_v of sign changes in $\bar{s}_v$, including the sign change occurring between the last and first elements.

3.1 Lemma
For all $v \in V$, m_v is even.

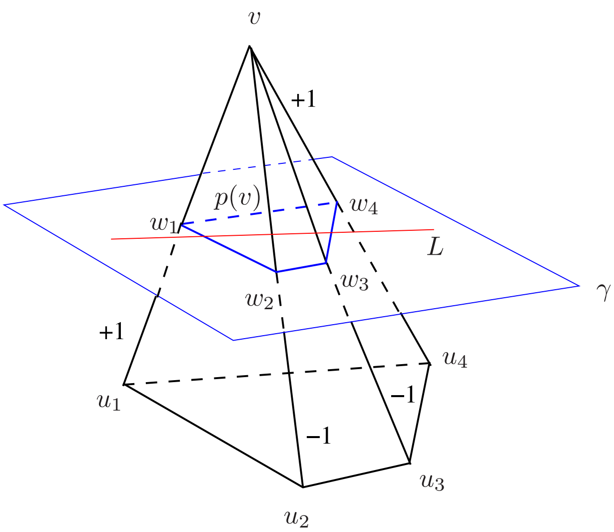


Figure 6: The plane γ separating v from the other vertices in the strictly convex polyhedron P, and the intersection polygon $p(v)$ defined by $w_1, \ldots, w_4$. The line L (lying in $p(v)$) separates the $+1$ and -1 labels applied to the points $w_1, \ldots, w_4$ of intersections between the edges of P and $p(v)$.

Proof. Suppose m_v is odd, and proceed by induction on m_v: if $m_v = 1$, then there is only one sign change. So, the first edge $\{u, v\}$ in σ_v to be labelled with $\ell_{uv} \neq 0$ has the property that, going around the periodic sequence with only one sign change, $\{u, v\}$ is also labelled with $-\ell_{uv}$, which yields $+1 = -1$, a contradiction. A trivial induction step yields the same contradiction for all odd m_v. □

We now state a fundamental technical lemma, and provide what is essentially Cauchy's proof, rephrased as in [28, Lemma 2 in § 20].

3.2 Lemma
If P is strictly convex, then for each $v \in V$ we have either $m_v = 0$ or $m_v \geq 4$.

Proof. By Lemma 3.1, for each $v \in V$ we have $m_v \notin \{1, 3\}$, so we aim to show that $m_v \neq 2$. Suppose, to get a contradiction, that $m_v = 2$, and consider the polygon $p(v)$ as in Fig. 6. By the correspondence between edges in σ_v and vertices of $p(v)$, the labels ℓ_{uv} are vertex labels in $p(v)$. Since there are only two sign changes, the sequence of vertex labels can be partitioned in two contiguous sets of $+1$ and -1 (possibly interspersed by zeros). By convexity, there exists a line L separating the $+1$ and the -1 vertices (see Fig. 6). Since all of the angles marked $+1$ strictly increase, the segment $\bar{L} = L \cap p(v)$ also strictly increases (this statement was also proved in Cauchy's paper [6], but this proof contained a serious flaw, later corrected by Steinitz); but, at the same time, all of the angles marked -1 strictly decrease, so the segment $\bar{L}$ also strictly decreases, which means that the same segment $\bar{L}$ both strictly increases and decreases, which is a contradiction (see Fig. 7). □

3.3 Theorem (Cauchy's Theorem [6])
If two closed convex polyhedra P, Q are combinatorially equivalent and facewise equal, they are isometric.

We only present the proof of the base case where

$$\forall v \in V \ (m_v > 0) \quad \vee \quad \forall v \in V \ (m_v = 0) \tag{3}$$

and $G(P) = G(Q)$ is a connected graph, and refer the reader to [37, p. 251] for the other cases (which are mostly variations of the ideas given in the proof below).

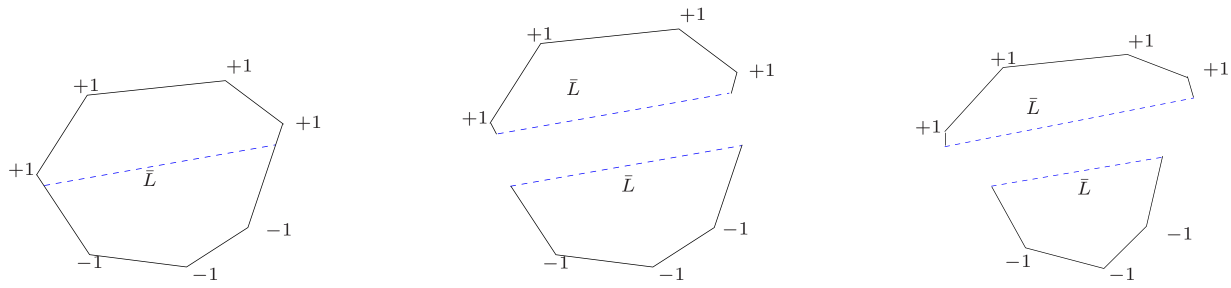


Figure 7: Visual representation of the contradiction in the proof of Lemma 3.2. The angles at all vertices labelled $+1$ increase their magnitude, and those at -1 decrease: it follows that $\bar{L}$ both increases and decreases its length, a contradiction.

Proof. If $m_v = 0$ for all $v \in V$, it means that all of the dihedral angles in P are equals to those of Q, which implies isometry. So we assume the alternative w.r.t. Eq. (3) above: $\forall v \in V$ $(m_v > 0)$, and aim for a contradiction. Let $M = \sum_{v \in V} m_v$: by Lemma 3.2 and because $m_v > 0$ for each v, we have $M \geq 4|V|$, a lower bound for M. We now construct a contradicting upper bound for M. For every $h \geq 3$, we let F_h be the number of polygonal faces of P with h sides (or edges). The total number of polygonal faces in P (or Q) is $\mathcal{F} = \sum_h F_h$, and the total number of edges is therefore $\mathcal{E} = \frac{1}{2}\sum_h hF_h$ (we divide by 2 since each edge is counted twice in the sum — one per adjacent face — given that P, Q are closed). A simple term by term comparison of $\mathcal{F}$ and $\mathcal{E}$ yields $4\mathcal{E} - 4\mathcal{F} = \sum_h 2(h-2)F_h$. Since each polygonal face f of P is itself closed, the number c_f of sign changes of the quantities ℓ_{uv} over all edges $\{u, v\}$ adjacent to the face f is even, by the same argument given in Lemma 3.1. It follows that if the number h of edges adjacent to the face f is even, then $c_f \leq h$, and $c_f \leq h - 1$ if h is odd. This allows us to compute an upper bound on M:

$$\begin{aligned} M &\leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + 6F_7 + 8F_8 + \dots \\ &\leq 2F_3 + 4F_4 + 6F_5 + 8F_6 + 10F_7 + 12F_8 + \dots \\ &\leq 4\mathcal{E} - 4\mathcal{F} = 4|V| - 8. \end{aligned}$$

The middle step follows by simply increasing each coefficient. The last step is based on Euler's characteristic [14]: $|V| + \mathcal{F} - \mathcal{E} = 2$. Hence we have $4|V| \leq M \leq 4|V| - 8$, which is a contradiction. □

3.2 Euler was wrong: Connelly's counterexample

Proofs behind counterexamples can rarely be termed "beautiful" since they usually lack generality (as they are applied to one particular example). Counterexamples can nonetheless be dazzling by themselves. Connelly's counterexample [8] to the Euler's conjecture consists in a very special non-generic nonconvex polyhedron which flexes, while keeping combinatorial equivalence and facewise equality with all polyhedra in the flex. Some years later, Klaus Steffen produced a much simpler polyhedron with the same properties (see `http://demonstrations.wolfram.com/SteffensFlexiblePolyhedron/`). It is this polyhedron we exhibit in Fig. 8.

4 Cayley-Menger determinants and the simplex volume

The foundation of modern DG, as investigated by Menger [32] and Blumenthal [4], rests on the fact that:

> *the four-dimensional volume of a four-dimensional simplex embedded in three dimensional space is zero,* (∗)

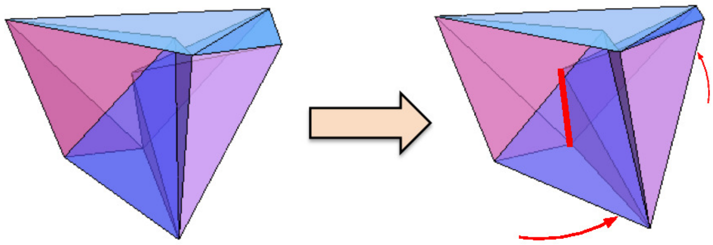


Figure 8: Steffen's polyhedron: the flex (these two images were obtained as snapshot from the *Mathematica* [45] demonstration cited on page 8). There is a rotation, in the direction showed by the arrows, around the edge which is emphasized on the right picture. The short upper right edge only appears shorter on the right because of perspective.

which we could also informally state as "flat simplices have zero volume". This is related to DG because the volume of a simplex can be expressed in terms of the lengths of the simplex sides, which yields a polynomial in the length of the simplex side lengths that can be equated to zero. If these lengths are expressed in function of the vertex positions as $\|x_u - x_v\|^2$, this yields a polynomial equation in the positions $x_1, \ldots, x_5$ of the simplex vertices in terms of its side lengths. Thus, if we know the positions of $x_1, \ldots, x_4$, we can compute the unknown position of x_5 or prove that no such position exists, through a process called *trilateration* [26].

The proof of $(*)$ was published by Arthur Cayley in 1841 [7], during his undergraduate studies. It is based on the following well-known lemma about determinants (stated without proof in Cayley's paper).

4.1 Lemma
If A, B are square matrices having the same size, $|AB| = |A||B|$.

4.2 Theorem (Cayley [7])
Given five points $x_1, \ldots, x_5 \in \mathbb{R}^4$ all belonging to an affine 3D subspace of $\mathbb{R}^4$, let $d_{ij} = \|x_i - x_j\|_2$ for each $i, j \le 5$. Then

$$\begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 & 1 \\ d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 & d_{25}^2 & 1 \\ d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 & d_{35}^2 & 1 \\ d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 & d_{45}^2 & 1 \\ d_{51}^2 & d_{52}^2 & d_{53}^2 & d_{54}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0. \tag{4}$$

We note that Cayley's theorem is expressed for $n = 5$ points in $\mathbb{R}^3$, but it also holds for $n \ge 3$ points in $\mathbb{R}^{n-2}$ [4]. Cayley explicitly remarks that it holds for the cases $n = 4$ and $n = 3$ (see [42, VIII, § 5] for the proof of general n). The determinant on the right-hand side of Eq. (4) is called *Cayley-Menger determinant*, denoted by Δ. We remark that in the proof below x_{ik} is the k-th component of x_i, for each $i \le 5, k \le 4$.

Proof. We follow Cayley's treatment. He pulls the following two matrices

$$A = \begin{pmatrix} \|x_1\|^2 & -2x_{11} & -2x_{12} & -2x_{13} & -2x_{14} & 1 \\ \|x_2\|^2 & -2x_{21} & -2x_{22} & -2x_{23} & -2x_{24} & 1 \\ \|x_3\|^2 & -2x_{31} & -2x_{32} & -2x_{33} & -2x_{34} & 1 \\ \|x_4\|^2 & -2x_{41} & -2x_{42} & -2x_{43} & -2x_{44} & 1 \\ \|x_5\|^2 & -2x_{51} & -2x_{52} & -2x_{53} & -2x_{54} & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ x_{11} & x_{21} & x_{31} & x_{41} & x_{51} & 0 \\ x_{12} & x_{22} & x_{32} & x_{42} & x_{52} & 0 \\ x_{13} & x_{23} & x_{33} & x_{43} & x_{53} & 0 \\ x_{14} & x_{24} & x_{34} & x_{44} & x_{54} & 0 \\ \|x_1\|^2 & \|x_2\|^2 & \|x_3\|^2 & \|x_4\|^2 & \|x_5\|^2 & 1 \end{pmatrix}$$

out of a magic hat. He performs the product AB, re-arranging and collecting terms, and obtains a 6×6 matrix where the last row and column are $(1, 1, 1, 1, 1, 0)$, and the (i, j)-th component is $\|x_i - x_j\|_2^2$ for every $i, j \le 5$. To see this, it suffices to carry out the computations using *Mathematica* [45]; by way of an example, the first diagonal component of AB is $\|x_1\|^2 - 2\sum_{k\le 4} x_{1k}x_{1k} + \|x_1\|^2 = 0$, and the component on the first row, second column of AB is $\|x_1\|^2 - 2\sum_{k\le 4} x_{1k}x_{2k} + \|x_2\|^2 = \|x_1 - x_2\|^2$. In other words, $|AB|$ is the Cayley-Menger determinant in Eq. (4). On the other hand, if we set $x_{4k} = 0$ for each $k \le 4$, effectively projecting the five four-dimensional points in three-dimensional space, it is easy to show that $|A| = |B| = 0$ since the 5-th columns of both A and the 5th row of B are zero. Hence we have $0 = |A||B| = |AB|$ by Lemma 4.1, and $|AB| = 0$ is precisely Eq. (4) as claimed. □

The missing link is the relationship of the Cayley-Menger determinant with the volume of an n-simplex. Since this is not part of Cayley's paper, we only establish the relationship for $n = 3$. Let $d_{12} = a$, $d_{13} = b$, $d_{23} = c$. Then:

$$\begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = a^4 - 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4 = -16(s(s-a)(s-b)(s-c)),$$

where $s = \frac{1}{2}(a+b+c)$ (this identity can be established by using e.g. *Mathematica* [45]). By Heron's theorem (Thm. 2.1 above) we know that the area of a triangle with side lengths a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$. So, for $n = 3$, the determinant on the left-hand side is proportional to the negative of the square of the triangle area. This result can be generalized to every value of n [4, II, § 40, p. 98]: it turns out that the n-dimensional volume V_n of an n-simplex in $\mathbb{R}^n$ with side length matrix $d = (d_{ij} \mid i, j \le n+1)$ is:

$$V_n^2 = \frac{(-1)^{n-1}}{2^n(n!)^2}\Delta.$$

The beauty of Cayley's proof is in its extreme compactness: it uses determinants to hide all the details of elimination theory which would be necessary otherwise. His paper also shows some of these details for the simplest case $n = 3$. The starting equations, as well as the symbolic manipulation steps, depend on n. Although Cayley's proof is only given for $n = 5$, Cayley's treatment goes through essentially unchanged for any number n of points in dimension $n - 2$.

5 Menger's characterization of abstract metric spaces

At a time where mathematicians were heeding Hilbert's call to formalization and axiomatization, Menger presented new axioms for geometry based on the notion of distance, and provided conditions for arbitrary sets to "look like" Euclidean spaces, at least distancewise [31, 32]. Menger's system allows a formal treatment of geometry based on distances as "internal coordinates". The starting point is to consider the relations of geometrical figures having proportional distances between pairs of corresponding points, i.e. congruence. Menger's definition of a congruence system is defined axiomatically, and the resulting characterization of abstract distance spaces with respect to subsets of Euclidean spaces (possibly his most important result) transforms a possibly infinite verification procedure (any subset of any number of points) into a finitistic one (any subset of $n+3$ points, where n is the dimension of the Euclidean space).

It is remarkable that almost none of the results below offers an intuitive geometrical grasp, such as the proofs of Heron's formula and Cayley's theorem do. As formal mathematics has it, part of the beauty in Menger's work consists in turning the "visual" geometrical proofs based on intuition into formal symbolic arguments based on sets and relations. On the other hand, Menger himself gave a geometric intuition of his results in [33, p. 335], which we comment in Sect. 5.4 below.

5.1 Menger's axioms

Let $\mathcal{S}$ be a system of sets, and for any set $S \in \mathcal{S}$ and any two (not necessarily distinct) points $p, q \in S$, denote the couple (p, q) by pq. Menger defines a relation $\approx$ by means of the following axioms.

1. $\forall S, T \in \mathcal{S}$, $\forall p, q \in S$ and $\forall r, s \in T$, we have either $pq \approx rs$ or $pq \not\approx rs$ but not both.
2. $\forall S \in \mathcal{S}$ and $\forall p, q \in S$ we have $pq \approx qp$.
3. $\forall S, T \in \mathcal{S}$, $\forall p \in S$ and $\forall r, s \in T$, we have $pp \approx rs$ if and only if $r = s$.
4. $\forall S, T \in \mathcal{S}$, $\forall p, q \in S$ and $\forall r, s \in T$, if $pq \approx rs$ then $rs \approx pq$.
5. $\forall S, T, U \in \mathcal{S}$, $\forall p, q \in S$, $\forall r, s \in T$ and $\forall t, u \in U$, if $pq \approx rs$ and $pq \approx tu$ then $rs \approx tu$.

The couple $(\mathcal{S}, \approx)$ is called a *congruence system*, and the $\approx$ relation is called *congruence*.

Today, we are used to think of *relations* as defined on a single set. We remark that in Menger's treatment, congruence is a binary relation defined on sets of ordered pairs of points, where each point in each pair belongs to the same set as the other, yet left-hand and right-hand side terms may belong to different sets. We now interpret each axiom from a more contemporary point of view.

1. Axiom 1 states that Menger's congruence relation is in fact a *partial* relation on $\mathscr{S} = (\bigcup \mathcal{S})^2$ (the Cartesian product of the union of all sets $S \in \mathcal{S}$ by itself), which is only defined for a couple $pq \in \mathscr{S}$ whenever $\exists S \in \mathcal{S}$ such that $p, q \in S$.
2. By axiom 2, the $\approx$ relation acts on sets of *unordered* pairs of (not necessarily distinct) points; we call $\bar{\mathscr{S}}$ the set of all unordered pairs of points from all sets $S \in \mathcal{S}$.
3. By axiom 3, rs is congruent to a pair pq where $p = q$ if and only if $r = s$.
4. Axiom 4 states that $\approx$ is a symmetric relation.
5. Axiom 5 states that $\approx$ is a transitive relation.

Note that $\approx$ is also reflexive (i.e. $pq \approx pq$) since $pq \approx qp \approx pq$ by two successive applications of Axiom 2. So, using today's terminology, $\approx$ is an equivalence relation defined on a subset of $\bar{\mathscr{S}}$.

5.2 A model for the axioms

Menger's model for his axioms is a *semi-metric* space S, i.e. a set S of points such that to each unordered pair $\{p, q\}$ of points in S we assign a nonnegative real number d_{pq} which we call *distance* between p and q. Under this interpretation, Axiom 2 tells us that $d_{pq} = d_{qp}$ for each pair of points p, q, and Axiom 3 tells us that rs is congruent to a single point if and only if $d_{rs} = 0$, which, together with nonnegativity, are the defining properties of *semi-metrics* (the remaining property, the triangular inequality, tells semi-metrics apart from *metrics*). Thus, the set $\mathcal{S}$ of all semi-metric spaces together with the relation given by $pq \approx rs \leftrightarrow d_{pq} = d_{rs}$ is a congruence system.

5.3 A finitistic characterization of semi-metric spaces

Two sets $S, T \in \mathcal{S}$ are *congruent* if there is a map (called *congruence map*) $\phi : S \to T$, such that $pq \approx \phi(p)\phi(q)$ for all $p, q \in S$. We denote this relation by $S \approx_\phi T$, dropping the ϕ if it is clear from the context.

5.1 Lemma
Any congruence map $\phi : S \to T$ is injective.

Proof. Suppose, to get a contradiction, that $\exists p, q \in S$ with $p \neq q$ and $\phi(p) = \phi(q)$: then $pq \approx \phi(p)\phi(q) = \phi(p)\phi(p)$ and so, by Axiom 3, $p = q$ against assumption. □

If S is congruent to a subset of T, then we say that S is *congruently embeddable* in T.

5.3.1 Congruence order

Now consider a set $S \in \mathcal{S}$ and an integer $n \geq 0$ with the following property: for any $T \in \mathcal{S}$, if all n-point subsets of T are congruent to an n-point subset of S, then T is congruently embeddable in S. If this property holds, then S is said to have *congruence order* n. Formally, the property is written as follows:

$$\forall T \in \mathcal{S}\, \forall T' \subseteq T \quad (\ (|T'| = n \to \exists S' \subseteq S\ (|S'| = n \wedge T' \approx S')) \quad \longrightarrow \quad \exists R \subseteq S\ (T \approx R)\ ). \tag{5}$$

If $|S| < n$ for some positive integer n, then S can have congruence order n, since the definition is vacuously satisfied. So we assume in the following that $|S| \geq n$.

5.2 Proposition
If S has congruence order n in $\mathcal{S}$, then it also has congruence order m for each $m > n$.

Proof. By hypothesis, for every $T \in \mathcal{S}$, if every n-point subset T' of T is congruent to an n-point subset of S, then there is a subset R of S such that $T \approx_\phi R$. Now any m-point subset of S is mapped by ϕ to a congruent m-point subset of S, and again $T \approx R$, so Eq. (5) is satisfied for S and m. □

In view of Prop. 5.2, it becomes important to find the minimum congruence order of a given metric space.

5.3 Proposition
$\mathbb{R}^0$ *(i.e. the Euclidean space which simply consists of the origin) has minimum congruence order* 2 *in* $\mathcal{S}$.

Proof. Pick any $T \in \mathcal{S}$ with $|T| > 1$. None of its 2-point subsets is congruent to any 2-point subset of $\mathbb{R}^0$, since none exists. Moreover, T itself cannot be congruently embedded in $\mathbb{R}^0$, since $|T| > 1 = |\mathbb{R}^0|$ and no injective congruence map can be defined, against Lemma 5.1. So the integer 2 certainly (vacuously) satisfies Eq. (5) for $S = \mathbb{R}^0$, which means that $\mathbb{R}^0$ has congruence order 2. In view of Prop. 5.2, it also has congruence order m for each $m > 2$. Hence we have to show next that the integer 1 cannot be a congruence order for $\mathbb{R}^0$. To reach a contradiction, suppose the contrart, and let T be as above. By Axiom 3, every singleton subset of T is congruent to a subset of $\mathbb{R}^0$, namely the subset containing the origin. Thus, by Eq. (5), T must be congruent to a subset of $\mathbb{R}^0$; but, again, $|T| > 1 = |\mathbb{R}^0|$ contradicts Lemma 5.1: so T cannot be congruently embedded in $\mathbb{R}^0$, which negates Eq. (5). Hence 1 cannot be a congruence order for $\mathbb{R}^0$, as claimed. □

5.3.2 Menger's fundamental result

The fundamental result proved by Menger in 1928 [31] is that the Euclidean space $\mathbb{R}^n$ has congruence order $n+3$ but not $n+2$ for each $n > 0$ in the family $\mathcal{S}$ of all semi-metric spaces. The important implication of Menger's result is that in order to verify whether an abstract semi-metric space is congruent to a subset of a Euclidean space, we only need to verify congruence of each of its $n+3$ point subsets.

We follow Blumenthal's treatment [4], based on the following preliminary definitions and properties, which we shall not prove:

1. A congruent mapping of a semi-metric space onto itself is called a *motion*;
2. $n+1$ points in $\mathbb{R}^n$ are *independent* if they are not affinely dependent (i.e. if they do not all belong to a single hyperplane in $\mathbb{R}^n$);
3. two congruent $(n+1)$-point subsets of $\mathbb{R}^n$ are either both independent or both dependent;
4. there is at most one point of $\mathbb{R}^n$ with given distances from an independent $(n+1)$-point subset;
5. any congruence between any two subsets of $\mathbb{R}^n$ can be extended to a motion;
6. any congruence between any two independent $(n+1)$-point subsets of $\mathbb{R}^n$ can be extended to a unique motion.

5.4 Theorem (Menger [31])
A non-empty semi-metric space S is congruently embeddable in $\mathbb{R}^n$ (but not in any $\mathbb{R}^r$ for $r<n$) if and only if: (a) S contains an $(n+1)$-point subset S' which is congruent with an independent $(n+1)$-point subset of $\mathbb{R}^n$; and (b) each $(n+3)$-point subset U of S containing S' is congruent to an $(n+3)$-point subset of $\mathbb{R}^n$.

The proof of Menger's theorem is very formal (see below) and somewhat difficult to follow. It is nonetheless a good example of a proof in an axiomatic setting, where logical reasoning is based on syntactical transformations induced by inference rules on the given axioms. An intuitive discussion is provided in Sect. 5.4.

Proof. ($\Rightarrow$) Assume first that $S \approx_\phi T \subseteq \mathbb{R}^n$, where the affine closure of T has dimension n. Then T must contain an independent subset T' with $|T'| = n+1$, which we can map back to a subset $S' \subseteq S$ using ϕ^{-1}. Since ϕ, ϕ^{-1} are injective, $|S'| \le |T'|$, and by Axiom 3 we have $|S'| \ge |T'|$, so $|S'| = n+1$, which establishes (a). Now take any $U \subseteq S$ with $|U| = n+3$ and $U \supset S'$: this can be mapped via ϕ to a subset $W \subseteq T$: Lemma 5.1 ensures injectivity of ϕ and hence $|W| = n+3$, establishing (b).
($\Leftarrow$) Conversely, assume (a) and (b) hold. By (a), let $S' \subseteq S$ with $|S'| = n+1$ and $S' \approx_\phi T' \subseteq \mathbb{R}^n$, with T' independent and $|T'| = n+1$. We claim that ϕ can be extended to a mapping of S into $\mathbb{R}^n$. Take any $q \in S \smallsetminus S'$: by (b), $S' \cup \{q\} \approx_\psi W \subseteq \mathbb{R}^n$ with $|W| = n+2$. Note that $T' \approx_\omega W \smallsetminus \{\psi(q)\}$ by Axiom 5, which implies that for any $p \in S'$, we have $\omega\phi(p) = \psi(p)$. Moreover, by Property 3 above, $W \smallsetminus \{\psi(q)\}$ is independent and has cardinality $n+1$, which by Property 6 above implies that ω can be extended to a unique motion in $\mathbb{R}^n$. So the action of ω is extended to q, and we can define $\phi(q) = \omega^{-1}\psi(q)$. We now show that this extension of ϕ is a congruence. Let $p, q \in S$: we aim to prove that $pq = \phi(p)\phi(q)$. Consider the set $U = S' \cup \{p, q\}$: since $|U| \le n+3$, by (b) there is $W \subset \mathbb{R}^n$ with $|W| = |U|$ such that $U \approx_\psi W$. As above, we note that there is a subset $W' \subseteq W$ such that $|W'| = n+1$ and $T' \approx_\omega W'$, that $\omega\phi(r) = \psi(r)$ for each $r \in S'$, and that ω is a motion of $\mathbb{R}^n$. Hence $pq = \psi(p)\psi(q) = \omega^{-1}\phi(p)\omega^{-1}\phi(q) = \phi(p)\phi(q)$, as claimed. □

5.4 An intuitive interpretation

Although we stated initially that part of the the beauty of the formal treatment of geometry is that it is based on symbolic manipulation rather than visual intuition, we quote from a survey paper which Menger himself wrote (in Italian, with the help of L. Geymonat) to disseminate the work carried out at his seminar [33].

> *Affinché uno spazio metrico reale R sia applicabile a un insieme parziale di $\mathbb{R}^n$ è necessario e sufficiente che per ogni $n+3$ e per ogni $n+2$ punti di esso sia $\Delta = 0$ e inoltre che ogni $n+1$ punti di R siano applicabili a punti di $\mathbb{R}^n$.*

The translation is "a real metric space R is embeddable in a subset of $\mathbb{R}^n$ if and only if $\Delta = 0$ for each $(n+3)$- and $(n+2)$-point subsets or R, and that each $(n+1)$-point subset of R is embeddable in $\mathbb{R}^n$."

Since we know that Δ, the Cayley-Menger determinant of the pairwise distances of a set S of points (see Eq. (4)), is proportional to the volume of the simplex on S embedded in $|S|-1$ dimensions, what Menger is saying is that his result on the congruence order of Euclidean spaces can be intuitively interpreted as follows.

> An abstract semi-metric space R is congruently embeddable in $\mathbb{R}^n$ if and only if: (i) there are $n+1$ points in R which are congruently embeddable in $\mathbb{R}^n$; (ii) the volume of the simplex on each $n+2$ points of R is zero; (iii) the volume of the simplex on each $n+3$ points of R is zero.

This result is exploited in the algorithm for computing point positions from distances given in [41, p. 2284].

6 Gödel on spherical distances

Kurt Gödel's name is attached to what is possibly the most revolutionary result in all of mathematics, i.e. Gödel's incompleteness theorem, according to which any formal axiomatic system sufficient to encode the integers is either inconsistent (it proves A and $\neg A$) or incomplete (there is some true statement A which the system cannot prove). This shattered Hilbert's dream of a formal system in which every true mathematical statement could be proved. Few people know that Gödel, who attended the Vienna Circle, Menger's course in geometry, and Menger's seminar, also contributed two results which are completely outside of the domain of logic. These results only appeared in the proceedings of Menger's seminar [34], and concern DG on a spherical surface.

6.1 Four points on the surface of a sphere

The result we discuss here is a proof to the following theorem, conjectured at a previous seminar session by Laura Klanfer. We remark that a sphere in $\mathbb{R}^3$ is a semi-metric space whenever it is endowed with a distance corresponding to the length of a geodesic curve joining two points.

6.1 Theorem (Gödel [20])
Given a semi-metric space S of four points, congruently embeddable in $\mathbb{R}^3$ but not $\mathbb{R}^2$, is also congruently embeddable on the surface of a sphere in $\mathbb{R}^3$.

Gödel's proof looks at the circumscribed sphere around a tetrahedron in $\mathbb{R}^3$, and analyses the relationship of the geodesics, their corresponding chords, and the sphere radius. It then uses a fixed point argument to find the radius which corresponds to geodesics which are as long as the given sides.

Proof. The congruence embedding of S in $\mathbb{R}^3$ defines a tetrahedron T having six (straight) sides with lengths $a_1, \ldots, a_6$. Let r be the radius of the sphere circumscribed around T (i.e. the smallest sphere containing T). We shall now consider a family of tetrahedra $\tau(x)$, parametrized on a scalar $x > 0$, defined as follows: $\tau(x)$ is the tetrahedron in $\mathbb{R}^3$ having side lengths $c_x(a_1), \ldots, c_x(a_6)$, where $c_x(\alpha)$ is the length of the chord subtending a geodesic having length α on a sphere of radius $\frac{1}{x}$. As x tends towards zero, each $c_x(a_i)$ tends towards a_i (for each $i \leq 6$), since the radius of the sphere tends towards infinity and each geodesic length tends towards the length of the subtending chord. This means that $\tau(x)$ tends towards T, since T is precisely the tetrahedron having side lengths $a_1, \ldots, a_6$. For each $x > 0$, let $\phi(x)$ be the inverse of the radius of the sphere circumscribed about $\tau(x)$. Since $\tau(x) \to T$ as $x \to 0$, and the radius circumscribed about T is r, it follows that $\phi(x) \to \frac{1}{r}$ as $x \to 0$. Also, since T exists by hypothesis, we can define $\tau(0) = T$ and $\phi(0) = \frac{1}{r}$. Also note that it is well known by elementary spherical geometry that:

$$c_x(\alpha) = \frac{2}{x} \sin \frac{\alpha x}{2}. \tag{6}$$

Claim: if $a' = \max\{a_1, \ldots, a_6\}$ then ϕ has a fixed point in the open interval $I = (0, \frac{\pi}{a'})$.
Proof of the claim. First of all notice that $\tau(0)$ exists, and $c_x(\alpha)$ is a continuous function for $x > 0$ for each α (by Eq. (6)). Since $\tau(x)$ is defined by the chord lengths $c_x(a_1), \ldots, c_x(a_6)$, this also means that $\tau(x)$ varies continuously for x in some open interval $J = (0, \varepsilon)$ (for some constant $\varepsilon > 0$). In turn, this implies that $\bar{x} = \max\{y \in I \mid \tau(y) \text{ exists}\}$ exists by continuity. There are two cases: either $\bar{x}$ is at the upper extremum of I, or it is not.

(i) If $\bar{x} = \frac{\pi}{a'}$, then $\tau(\bar{x})$ exists, its longest edge has length $c_{\bar{x}}(a') = \frac{2a'}{\pi}$, so, by elementary spherical geometry, the radius of the sphere circumscribed around $\tau(\bar{x})$ is greater than $\frac{c_{\bar{x}}(a')}{2}$, i.e. greater than $\frac{a'}{\pi} = \frac{1}{\bar{x}}$. Thus $\phi(\bar{x}) < \bar{x}$. We also have, however, that $\phi(0) = \frac{1}{r} > 0$, so by the intermediate value theorem there must be some $x \in (0, \bar{x})$ with $\phi(x) = x$.

(ii) Assume now $\bar{x} < \frac{\pi}{a'}$ and suppose $\tau(\bar{x})$ is non-planar. Then for each y in an arbitrary small neighbourhood around $\bar{x}$, $\tau(y)$ must exist by continuity: in particular, there must be some $y > \bar{x}$ where $\tau(y)$ exists, which contradicts the definition of $\bar{x}$. So $\tau(\bar{x})$ is planar: this means that each geodesic is contained in the same plane, which implies that the geodesics are linear segments. It follows that the circumscribed sphere has infinite radius, or, equivalently, that $\phi(\bar{x}) = 0 < \bar{x}$. Again, by $\phi(0) > 0$ and the intermediate value theorem, there must be some $x \in (0, \bar{x})$ with $\phi(x) = x$.

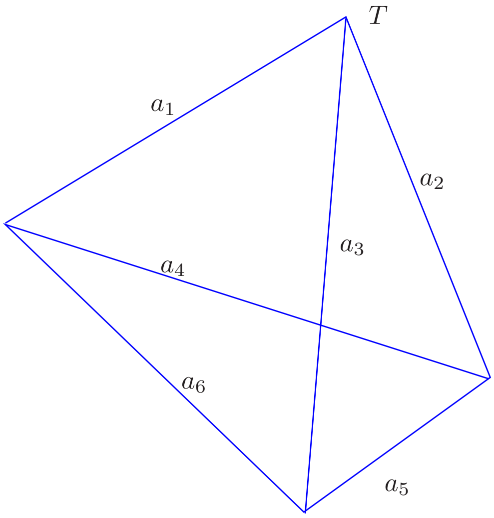
This concludes the proof of the claim.
So now let y be the fixed point of ϕ. The tetrahedron $\tau(y)$ has side lengths $c_y(a_i)$ for each $i \leq 6$, and is circumscribed by a sphere σ with radius $\frac{1}{y}$. It follows that, on the sphere σ, the geodesics corresponding to the chords given by the tetrahedron sides have lengths a_i (for $i \leq 6$), as claimed. □

6.2 Gödel's devilish genius

Gödel's proof exhibits an unusual peak of devilish genius. At first sight, it is a one-dimensional fixed-point argument which employs a couple of elementary notions in spherical geometry. Underneath the surface, the fixed-point argument eschews a misleading visual intuition.

T is a given tetrahedron in $\mathbb{R}^3$ which is assumed to be non-planar and circumscribed by a sphere of finite positive radius r (see Fig. 9, left). The map τ sends a scalar x to the tetrahedron having as side lengths the chords subtending the geodesics of length a_i ($i \leq 6$) on a sphere of radius $\frac{1}{x}$ (see Fig. 9, right). The map τ is such that $\tau(0) = T$ since for $x = 0$ the radius is infinite, which means that the geodesics are equal to their chords. Moreover, the map ϕ sends x to the inverse of the radius of the sphere circumscribing $\tau(x)$. Since every geodesic on the sphere is a portion of a great circle, it would appear from Fig. 9 (right) that the radius $\frac{1}{x}$ used to compute $c_x(a_i)$ ($i \leq 6$) is the same as the radius $\frac{1}{\phi(x)}$ of the sphere circumscribing $\tau(x)$, which would immediately yield $\phi(x) = x$ for every x — making the proof trivial. There is something inconsistent, however, in the visual interpretation of Fig. 9: the given tetrahedron T corresponds to the case $\tau(x) = T$, which happens when $x = 0$, i.e. the radius of the sphere circumscribed around T is ∞. But this would yield T to be a planar tetrahedron, which is a contradiction with an assumption of the theorem. Moreover, if $\phi(x)$ were equal to x for each x, this would yield $0 = \phi(0) = \frac{1}{r} > 0$, another contradiction.

The misleading concept is hidden in the picture in Fig. 9 (right). It shows a tetrahedron inscribed in a sphere, and a spherical tetrahedron *on the same vertices*. This is not true in general, i.e. the spherical tetrahedron with the given curved side lengths $a_1, \ldots, a_6$ cannot, in general, be embedded in the surface of a sphere of *any* radius. For example, the case $x = 0$ yields geodesics with infinite curvatures (i.e. straight



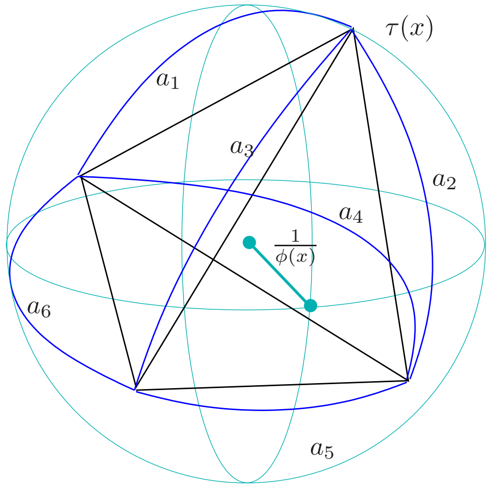


Figure 9: The given tetrahedron T (left), and the tetrahedron $\tau(x)$ (right). Beware of this visual interpretation: it may yield misleading insights (see Sect. 6.2).

lines laying in a plane), but $\phi(x) = \frac{1}{r} > 0$, and there is no flat tetrahedron with the same distances as those of T. The sense of Gödel's proof is that the function c_x simply transforms a set of geodesic distances into a set of linear distances, i.e. it maps scalars to scalars rather than geodesics to segments, whereas Fig. 9 (right) shows the special case where the geodesics are mapped to the corresponding segments, with intersections at the same points (namely the distances $a_1, \ldots, a_6$ can be embedded on the particular sphere shown in the picture). More specifically, the geodesic curves may or may not be realizable on a sphere of radius $\frac{1}{\phi(x)}$. Gödel's proof shows exactly that there must be some x for which $\phi(x) = x$, i.e. the geodesic curves become realizable.

6.3 Existential vs. constructive proofs

Like many existential proofs based on fixed-point theorems, this proof is beautiful because it asserts the truth of the theorem without any certificates other than its own logical validity. An alternative, constructive proof of Thm. 6.1 is given in [39, Thm. 3']. The tools used in that proof, Cayley-Menger determinants and positive semidefiniteness, are discussed in Sect. 7 below.

7 The equivalence of EDM and PSD matrices

Many fundamental innovations stem from what are essentially footnotes to apparently deeper or more important work. Isaac Schoenberg, better known as the inventor of splines [40], published a paper in 1935 titled *Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe d'espace distanciés vectoriellement applicable sur l'espace de Hilbert"* [39]. The impact of Schoenberg's remarks far exceeds that of the original paper: these remarks encode what amounts to the basis of the well-known MDS techniques for visualizing high-dimensional data [10], as well as all the solution techniques for Distance Geometry Problems (DGP) based on Semidefinite Programming (SDP) [29, 3].

A not altogether dissimilar situation arose for the Johnson-Lindenstrauss (JL) lemma [25]: the paper is concerned with extending a mapping from n-point subsets of a metric space to the whole metric space in such a way that the Lipschitz constant of the extension is bounded by at most a constant factor. Johnson and Lindenstrauss state on page 1 that "The main tool for proving Theorem 1 is a simply stated elementary geometric lemma". This lemma is now known as the *JL lemma*, and postulates the existence of low-distortion projection matrices which map to Euclidean spaces of logarithmically fewer dimensions. The impact of the lemma far exceeds that of the main result.

Figure 10: "The impact of Schoenberg's remarks far exceeds that of the original paper": this is something that happens quite often in mathematics.

7.1 Schoenberg's problem

Schoenberg poses the following problem, relevant to Menger's treatment of distance geometry [32, p. 737].

> Given an $n \times n$ symmetric matrix D, what are necessary and sufficient conditions such that D is a EDM corresponding to n points in $\mathbb{R}^r$, with $1 \leq r \leq n$ minimum?

Menger's solution is based on Cayley-Menger determinants; Schoenberg's solution is much simpler and more elegant, and rests upon the following theorem. Recall that a matrix is PSD if and only if all its eigenvalues are nonnegative.

7.1 Theorem (Schoenberg [39])
The $n \times n$ symmetric matrix $D = (d_{ij})$ is the EDM of a set of n points $x = \{x_1, \ldots, x_n\} \subset \mathbb{R}^r$ (with r minimum) if and only if the matrix $G = \frac{1}{2}(d_{1i}^2 + d_{1j}^2 - d_{ij}^2 \mid 2 \leq i, j \leq n)$ is PSD of rank r.

Instead of providing Schoenberg's proof, we follow a more modern treatment, which also unearths the important link of this theorem with classical MDS [10, § 2.2.1], an approximate method for finding sets of points $x = \{x_1, \ldots, x_n\}$ having EDM which approximates a given symmetric matrix. MDS is one of the cornerstones of the modern science of data analysis.

7.2 The proof of Schoenberg's theorem

Given a set $x = \{x_1, \ldots, x_n\}$ of points in $\mathbb{R}^r$, we can write x as an $r \times n$ matrix having x_i as i-th column. The matrix $G = xx^\top$ having the scalar product $x_i x_j$ as its (i,j)-th component is called the *Gram matrix* or *Gramian* of x. The proof of Thm. 7.1 works by exhibiting a 1-1 correspondence between squared EDMs and Gram matrices, and then by proving that a matrix is Gram if and only if it is PSD.

Without loss of generality, we can assume that the barycenter of the points in x is at the origin:

$$\sum_{i \leq n} x_i = 0. \tag{7}$$

Now we remark that, for each $i, j \leq n$, we have:

$$d_{ij}^2 = \|x_i - x_j\|^2 = (x_i - x_j)(x_i - x_j) = x_i x_i + x_j x_j - 2 x_i x_j. \tag{8}$$

7.2.1 The Gram matrix in function of the EDM

We “invert” Eq. (8) to compute the matrix $G = xx^\top = (x_i x_j)$ in function of the matrix $D^2 = (d_{ij}^2)$. We sum Eq. (8) over all values of $i \in \{1, \ldots, n\}$, obtaining:

$$\sum_{i \le n} d_{ij}^2 = \sum_{i \le n} (x_i x_i) + n(x_j x_j) - 2 \left(\sum_{i \le n} x_i \right) x_j. \tag{9}$$

By Eq. (7), the negative term in the right hand side of Eq. (9) is zero. On dividing through by n, we have

$$\frac{1}{n} \sum_{i \le n} d_{ij}^2 = \frac{1}{n} \sum_{i \le n} (x_i x_i) + x_j x_j. \tag{10}$$

Similarly for $j \in \{1, \ldots, n\}$, we obtain:

$$\frac{1}{n} \sum_{j \le n} d_{ij}^2 = x_i x_i + \frac{1}{n} \sum_{j \le n} (x_j x_j). \tag{11}$$

We now sum Eq. (10) over all j, getting:

$$\frac{1}{n} \sum_{\substack{i \le n \\ j \le n}} d_{ij}^2 = n \frac{1}{n} \sum_{i \le n} (x_i x_i) + \sum_{j \le n} (x_j x_j) = 2 \sum_{i \le n} (x_i x_i) \tag{12}$$

(the last equality in Eq. (12) holds because the same quantity $f(k) = x_k x_k$ is being summed over the same range $\{1, \ldots, n\}$, with the symbol k replaced by the symbol i first and j next). We then divide through by n to get:

$$\frac{1}{n^2} \sum_{\substack{i \le n \\ j \le n}} d_{ij}^2 = \frac{2}{n} \sum_{i \le n} (x_i x_i). \tag{13}$$

We now rearrange Eq. (8), (11), (10) as follows:

$$\begin{aligned} 2x_i x_j &= x_i x_i + x_j x_j - d_{ij}^2 && (14) \\ x_i x_i &= \frac{1}{n} \sum_{j \le n} d_{ij}^2 - \frac{1}{n} \sum_{j \le n} (x_j x_j) && (15) \\ x_j x_j &= \frac{1}{n} \sum_{i \le n} d_{ij}^2 - \frac{1}{n} \sum_{i \le n} (x_i x_i), && (16) \end{aligned}$$

and replace the left hand side terms of Eq. (15)-(16) into Eq. (14) to obtain:

$$2x_i x_j = \frac{1}{n} \sum_{k \le n} d_{ik}^2 + \frac{1}{n} \sum_{k \le n} d_{kj}^2 - d_{ij}^2 - \frac{2}{n} \sum_{k \le n} (x_k x_k), \tag{17}$$

whence, on substituting the last term using Eq. (13), we have:

$$2x_i x_j = \frac{1}{n} \sum_{k \le n} (d_{ik}^2 + d_{kj}^2) - d_{ij}^2 - \frac{1}{n^2} \sum_{\substack{h \le n \\ k \le n}} d_{hk}^2. \tag{18}$$

It turns out that Eq. (18) can be written in matrix form as:

$$G = -\frac{1}{2} J D^2 J, \tag{19}$$

where $J = I_n - \frac{1}{n} \mathbf{1}\mathbf{1}^\top$ and $\mathbf{1} = \underbrace{(1, \ldots, 1)}_{n}$.

7.2.2 Gram matrices are PSD matrices

Any Gram matrix $G = xx^\top$ derived by a point sequence (also called a *realization*) $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^K$ for some non-negative integer K has two important properties: (i) the rank of G is equal to the rank of x; and (ii) G is PSD, i.e. $y^\top G y \geq 0$ for all $y \in \mathbb{R}^n$. For simplicity, we only prove these properties in the case when $x = (x_1, \ldots, x_n)$ is a $1 \times n$ matrix, i.e. $x \in \mathbb{R}^n$, and x_i is a scalar for all $i \leq n$ (this is the case $r = 1$ in Schoenberg's problem above).

(i) The i-th column of G is the vector x multiplied by the scalar x_i, which means that every column of G is a scalar multiple of a single column vector, and hence that $\mathsf{rk}\, G = 1$;

(ii) For any vector y, $y^\top G y = y^\top (xx^\top) y = (y^\top x)(x^\top y) = (x^\top y)^2 \geq 0$.

Moreover, G is a Gram matrix *only if* it is PSD. Let M be a PSD matrix. By spectral decomposition there is a unitary matrix Y such that $M = Y \Lambda Y^\top$, where Λ is diagonal. By positive semidefiniteness, $\Lambda_{ii} \geq 0$ for each i, so $\sqrt{Y\Lambda}$ exists. Hence $M = \sqrt{Y\Lambda}(\sqrt{Y\Lambda})^\top$, which makes M the Gram matrix of the vector $\sqrt{Y\Lambda}$. This concludes the proof of Thm. 7.1.

7.3 Finding the realization of a Gramian

Having computed the Gram matrix G from the EDM D in Sect. 7.2, we obtain the corresponding realization x as follows. This is essentially the same reasoning used above to show the equivalence of Gramians and PSD matrices, but we give a few more details.

Let $\Lambda = \mathsf{diag}(\lambda_1, \ldots, \lambda_r)$ be the $r \times r$ matrix with the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_r$ along the diagonal and zeroes everywhere else, and let Y be the $n \times r$ matrix having the eigenvector corresponding to the eigenvalue λ_j as its j-th column (for $j \leq r$), chosen so that Y consists of orthogonal columns. Then $G = Y \Lambda Y^\top$. Since Λ is a diagonal matrix and all its diagonal entries are nonnegative (by positive semidefiniteness of G), we can write Λ as $\sqrt{\Lambda}\sqrt{\Lambda}$, where $\sqrt{\Lambda} = \mathsf{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r})$. Now, since $G = xx^\top$,

$$xx^\top = (Y\sqrt{\Lambda})(\sqrt{\Lambda}Y^\top),$$

which implies that

$$x = Y\sqrt{\Lambda} \tag{20}$$

is a realization of G in $\mathbb{R}^r$.

7.4 Multidimensional Scaling

MDS can be used to find realizations of approximate distance matrices $\tilde{D}$. As above, we compute $\tilde{G} = -\frac{1}{2} J \tilde{D}^2 J$. Since $\tilde{D}$ is not a EDM, $\tilde{G}$ will probably fail to be a Gram matrix, and as such might have negative eigenvalues. But it suffices to let Y be the eigenvectors corresponding to the H positive eigenvalues $\lambda_1, \ldots, \lambda_H$, to recover an approximate realization x of $\tilde{D}$ in $\mathbb{R}^H$.

Another interesting feature of MDS is that the dimensionality H of the ambient space of x is actually determined by D (or $\tilde{D}$) rather than given as a problem input. In other words, MDS finds the "inherent dimensionality" of a set of (approximate) pairwise distances.

8 Conclusion

We presented what we feel are the most important and/or beautiful theorems in DG (Heron's, Cauchy's, Cayley's, Menger's, Gödel's and Schoenberg's). Three of them (Heron's, Cayley's, Menger's) have to do with the volume of simplices given its side lengths, which appears to be *the* central concept in DG. We think Cauchy's proof is as beautiful as a piece of classical art, whereas Gödel's proof, though less important, is stunning. Last but not least, Schoenberg's theorem is the fundamental link between the history of DG and its contemporary treatment.

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