Sequent Calculus: Cut elimination and proof search (Lecture 3)

Dale Miller

INRIA Saclay & Ecole Polytechnique, France

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Sequent calculus for classical and intuitionistic logics.

How does the “single-conclusion” restriction restrict rules?

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(2) In the implication left rule, the rhs of the conclusion must be the rhs of the right premise. That is, the inference rule

$$\frac{\Sigma\colon \Gamma_1 \longrightarrow \Delta_1, B \qquad \Sigma\colon \Gamma_2, C \longrightarrow \Delta_2}{\Sigma\colon \Gamma_1, \Gamma_2, B \supset C \longrightarrow \Delta_1, \Delta_2} \supset L$$

is really the inference rule

$$\frac{\Sigma\colon \Gamma_1 \longrightarrow B \qquad \Sigma\colon \Gamma_2, C \longrightarrow E}{\Sigma\colon \Gamma_1, \Gamma_2, B \supset C \longrightarrow E} \supset L$$

Cut elimination: permuting a cut up

$$
\dfrac{\dfrac{\dfrac{\Xi_1}{\Sigma\colon \Gamma_1 \vdash A_1, \Delta_1} \quad \dfrac{\Xi_2}{\Sigma\colon \Gamma_1 \vdash A_2, \Delta_1}}{\Sigma\colon \Gamma_1 \vdash A_1 \wedge A_2, \Delta_1} \wedge R \quad \dfrac{\dfrac{\Xi_3}{\Sigma\colon \Gamma_2, A_i \vdash \Delta_2}}{\Sigma\colon \Gamma_2, A_1 \wedge A_2 \vdash \Delta_2} \wedge L}{\Sigma\colon \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \; cut
$$

Here, $i \in \{1, 2\}$. Change this fragment to

$$
\dfrac{\dfrac{\Xi_i}{\Sigma\colon \Gamma_1 \vdash A_i, \Delta_1} \quad \dfrac{\Xi_3}{\Sigma\colon \Gamma_2, A_i \vdash \Delta_2}}{\Sigma\colon \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \; cut
$$

The cut rule is on a smaller formula.

Cut elimination: permuting a cut up

$$
\dfrac{\dfrac{\begin{array}{c}\Xi_1\\ \Sigma\colon \Gamma_1, A_1 \vdash A_2, \Delta_1\end{array}}{\Sigma\colon \Gamma_1 \vdash A_1 \supset A_2, \Delta_1}\supset R \qquad \dfrac{\begin{array}{c}\Xi_2\\ \Sigma\colon \Gamma_2 \vdash A_1, \Delta_2\end{array} \quad \begin{array}{c}\Xi_3\\ \Sigma\colon \Gamma_3, A_2 \vdash \Delta_3\end{array}}{\Sigma\colon \Gamma_2, \Gamma_3, A_1 \supset A_2 \vdash \Delta_2, \Delta_3}\supset L}{\Sigma\colon \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\ \mathit{cut}
$$

This part of the proof can be changed locally to

$$
\dfrac{\dfrac{\begin{array}{c}\Xi_2\\ \Sigma\colon \Gamma_2 \vdash A_1, \Delta_2\end{array} \quad \begin{array}{c}\Xi_1\\ \Sigma\colon \Gamma_1, A_1 \vdash A_2, \Delta_1\end{array}}{\Sigma\colon \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A_2}\ \mathit{cut} \qquad \begin{array}{c}\Xi_3\\ \Sigma\colon \Gamma_3, A_2 \vdash \Delta_3\end{array}}{\Sigma\colon \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3}\ \mathit{cut}
$$

Although there are now two cut rules, they are on smaller formulas.

Cut elimination: permuting a cut away

$$\dfrac{\dfrac{\Xi}{\Sigma\colon \Gamma_1 \vdash \Delta, B} \qquad \dfrac{}{\Sigma\colon \Gamma_2, B \vdash B}\ init}{\Sigma\colon \Gamma_1, \Gamma_2 \vdash \Delta, B}\ cut$$

Rewrite this proof to the following.

$$\dfrac{\dfrac{\Xi}{\Sigma\colon \Gamma_1 \vdash \Delta_1, B}}{\Sigma\colon \Gamma_1, \Gamma_2 \vdash \Delta_1, B}\ wL$$

We have removed one occurrence of the cut rule.

N.B. Notice that if Γ_2 is empty then the *wL* is not needed.

Cut elimination

Theorem. If a sequent has a **C**-proof (respectively, **I**-proof) then it has a cut-free **C**-proof (respectively, **I**-proof).

This theorem was stated and proved by Gentzen 1935.

Gentzen invented the sequent calculus so that he could formulate one proof of this *Hauptsatz* for both classical *and* intuitionistic logic.

Structural rules were key to describing the difference between these two logics.

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Think to all the other ways you know for describing the difference between them (excluded middle, constructive vs non-constructive, Kripke semantics, etc).

Consequences of cut elimination

Theorem. Logic is consistency: It is impossible for there to be a proof of B and $\neg B$.

Proof. Assume that $\vdash B$ and $B \vdash$ have proofs. By cut, $\vdash$ has a proof. Thus, it also has a cut-free proof, but this is impossible.

Theorem. A cut-free proof system of a sequent is composed only of subformula of formulas in the root sequent.

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Should I eliminate cuts in general? **NO!** Cut-free proofs of interesting mathematical statement often do not exists in nature.

If you are using cut-free proofs, you are probably modeling computation or model checking.

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Solution: Good question. We concentrate on this issue next using *focused proof systems.* .

Some "focusing" behavior

Given the inference figure (a variant of $\supset$L), where A is atomic.

$$\frac{\Gamma \longrightarrow G \qquad \Gamma, D \overset{\Xi}{\longrightarrow} A}{\Gamma \longrightarrow A}\text{, provided } G \supset D \in \Gamma$$

can we restrict what is the last inference rule in Ξ?
In intuitionistic logic, we can insist that Ξ ends with either

- an introduction rule for D (if D is not atomic) or
- an initial rule with $A = D$ (if D is atomic).

Backchaining as focusing behavior

Let D be the formula (for atomic A')

$$\forall\bar{x}_1(G_1 \supset \forall\bar{x}_2(G_2 \supset \cdots \forall\bar{x}_n(G_n \supset A')\ldots))$$

and consider the sequent $\Sigma : \Gamma, D \vdash A$, for atomic A.

We can insist that if one applies a left introduction rule on D, the it cascades into a series of $\forall$L, $\supset$L, and initial rule.

That is, there is a substitution θ such that $A = A'\theta$ and $\Sigma : \Gamma \vdash G_i\theta$ are provable $(i = 1, \ldots, n)$.

This cascade of introduction rules will be called a "focus".

Backward and Forward Chaining

$$\frac{\Gamma \longrightarrow a \qquad \Gamma, b \longrightarrow G}{\Gamma, a \supset b \longrightarrow G}$$ a, b are atoms, focus on $a \supset b$

Negative atoms: The right branch is trivial; i.e., $b = G$. Continue with $\Gamma \longrightarrow a$ (backward chaining).
Positive atoms: The left branch is trivial; i.e., $\Gamma = \Gamma', a$. Continue with $\Gamma', a, b \longrightarrow G$ (forward chaining).

Let G be $\mathit{fib}(n, f)$ and let Γ contain $\mathit{fib}(0, 0)$, $\mathit{fib}(1, 1)$, and

$$\forall n \forall f \forall f' [\mathit{fib}(n, f) \supset \mathit{fib}(n+1, f') \supset \mathit{fib}(n+2, f+f')].$$

The nth Fibonacci number is F iff $\Gamma \vdash G$.
If $\mathit{fib}(\cdot, \cdot)$ is negative then the unique proof is *exponential* in n.
If $\mathit{fib}(\cdot, \cdot)$ is positive then the shortest proof is *linear* in n.