Sequent Calculus: overview (Lecture 2)

Dale Miller

INRIA Saclay & Ecole Polytechnique, France

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Sequent calculus for classical and intuitionistic logics.

Terms and formulas

Formally, we use Church's Simple Theory of Types [1940] to encode terms and formulas.

Informally, terms and formulas are first-order with occasional and natural uses of higher-order abstractions via λ-abstraction.

Equality via α and η-conversion useful for comparing formulas.

Equality via β-conversion useful for specifying substitution.

Sequents

Sequents are triples $\Sigma : \Gamma \vdash \Delta$ where

- Σ, the *signature* of the sequent, is a set of (eigen) variables (with scope over the sequent);
- Γ, the *left-hand-side*, is a multiset of formulas; and
- Δ, the *right-hand-side*, is a multiset of formulas.

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There are three kinds of inference rules.

1. structural rules,
2. identity rules, and
3. introduction rules.

Inference rules: two structural rules

There are two sets of these: *contraction*, *weakening*.

$$\frac{\Sigma : \Gamma, B, B \vdash \Delta}{\Sigma : \Gamma, B \vdash \Delta}\ cL \qquad \frac{\Sigma : \Gamma \vdash \Delta, B, B}{\Sigma : \Gamma \vdash \Delta, B}\ cR$$

$$\frac{\Sigma : \Gamma \vdash \Delta}{\Sigma : \Gamma, B \vdash \Delta}\ wL \qquad \frac{\Sigma : \Gamma \vdash \Delta}{\Sigma : \Gamma \vdash \Delta, B}\ wR$$

NB: Gentzen's use of lists of formulas required him to also have an *exchange* rule.

Inference rules: two identity rules

There are exactly two: *initial, cut*.

$$\frac{}{\Sigma : B \vdash B}\ init \qquad \frac{\Sigma : \Gamma_1 \vdash \Delta_1, B \qquad \Sigma : B, \Gamma_2 \vdash \Delta_2}{\Sigma : \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

Notice the repeated use of the variable B in these rules.

In general: all instances of both of these rules can be *eliminated* except for *init* when B is atomic.

Inference rules: introduction rules (some examples)

$$\frac{\Sigma\colon \Gamma, B_i \vdash \Delta}{\Sigma\colon \Gamma, B_1 \wedge B_2 \vdash \Delta}\ \wedge L \qquad \frac{\Sigma\colon \Gamma \vdash \Delta, B \qquad \Sigma\colon \Gamma \vdash \Delta, C}{\Sigma\colon \Gamma \vdash \Delta, B \wedge C}\ \wedge R$$

$$\frac{\Sigma\colon \Gamma, B \vdash \Delta \qquad \Sigma\colon \Gamma, C \vdash \Delta}{\Sigma\colon \Gamma, B \vee C \vdash \Delta}\ \vee L \qquad \frac{\Sigma\colon \Gamma \vdash \Delta, B_i}{\Sigma\colon \Gamma \vdash \Delta, B_1 \vee B_2}\ \vee R$$

$$\frac{\Sigma\colon \Gamma_1 \vdash \Delta_1, B \qquad \Sigma\colon \Gamma_2, C \vdash \Delta_2}{\Sigma\colon \Gamma_1, \Gamma_2, B \supset C \vdash \Delta_1, \Delta_2}\ \supset L \qquad \frac{\Sigma\colon \Gamma, B \vdash \Delta, C}{\Sigma\colon \Gamma \vdash \Delta, B \supset C}\ \supset R$$

$$\frac{\Sigma \vdash t : \tau \qquad \Sigma\colon \Gamma, B[t/x] \vdash \Delta}{\Sigma\colon \Gamma, \forall_\tau x\, B \vdash \Delta}\ \forall L \qquad \frac{\Sigma, y : \tau \ :\ \Gamma \vdash \Delta, B[y/x]}{\Sigma\colon \Gamma \vdash \Delta, \forall_\tau x\, B}\ \forall R$$

$$\frac{\Sigma, y : \tau \ :\ \Gamma, B[y/x] \vdash \Delta}{\Sigma\colon \Gamma, \exists_\tau x\, B \vdash \Delta}\ \exists L \qquad \frac{\Sigma \vdash t : \tau \qquad \Sigma\colon \Gamma \vdash \Delta, B[t/x]}{\Sigma\colon \Gamma \vdash \Delta, \exists_\tau x\, B}\ \exists R$$

Additive vs multiplicative inference rules

Inference rules with two or more premises are classified as follows:

Additive: side formulas are the same in premises and conclusion.

$$\frac{\Sigma\colon \Gamma, B \vdash \Delta \qquad \Sigma\colon \Gamma, C \vdash \Delta}{\Sigma\colon \Gamma, B \vee C \vdash \Delta}\ \vee L$$

Multiplicative: side formulas in premises accumulate.

$$\frac{\Sigma\colon \Gamma_1 \vdash \Delta_1, B \quad \Sigma\colon \Gamma_2, C \vdash \Delta_2}{\Sigma\colon \Gamma_1, \Gamma_2, B \supset C \vdash \Delta_1, \Delta_2}\ \supset L$$

These versions are inter-admissible in the presence of contraction and weakening. In linear logic, these adjectives applied to connectives as well.

Permutations of inference rules

$$\dfrac{\dfrac{\Sigma\colon \Gamma, p, r \vdash s, \Delta \qquad \Sigma\colon \Gamma, q, r \vdash s, \Delta}{\Sigma\colon \Gamma, p \vee q, r \vdash s, \Delta}\ \vee L}{\Sigma\colon \Gamma, p \vee q \vdash r \supset s, \Delta}\ \supset R$$

$$\dfrac{\dfrac{\Sigma\colon \Gamma, p, r \vdash s, \Delta}{\Sigma\colon \Gamma, p \vdash r \supset s, \Delta}\ \supset R \qquad \dfrac{\Sigma\colon \Gamma, q, r \vdash s, \Delta}{\Sigma\colon \Gamma, q \vdash r \supset s, \Delta}\ \supset R}{\Sigma\colon \Gamma, p \vee q \vdash r \supset s, \Delta}\ \vee L$$

Permutations of inference rules (continued)

$$\dfrac{\dfrac{\Sigma\colon \Gamma_1, r \vdash \Delta_1, p \qquad \Sigma\colon \Gamma_2, q \vdash \Delta_2, s}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q, r \vdash \Delta_1, \Delta_2, s}\ \supset\!L}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q \vdash \Delta_1, \Delta_2, r \supset s}\ \supset\!R$$

To switch the order of these two inference rules requires introducting weakenings and a contraction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Sigma\colon \Gamma_1, r \vdash \Delta_1, p}{\Sigma\colon \Gamma_1, r \vdash \Delta_1, p, s}\ wR}{\Sigma\colon \Gamma_1 \vdash \Delta_1, p, r \supset s}\ \supset\!R \qquad \dfrac{\dfrac{\Sigma\colon \Gamma_2, q \vdash \Delta_2, s}{\Sigma\colon \Gamma_2, q, r \vdash \Delta_2, s}\ wL}{\Sigma\colon \Gamma_2, q \vdash \Delta_2, r \supset s}\ \supset\!R}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q \vdash \Delta_1, \Delta_2, r \supset s, r \supset s}\ \supset\!L}{\Sigma\colon \Gamma_1, \Gamma_2, p \supset q \vdash \Delta_1, \Delta_2, r \supset s}\ cR$$

Provability defined

A **C**-proof (*classical proof*) is any proof using these inference rules.

An **I**-proof (*intuitionistic proof*) is a **C**-proof in which the right-hand side of all sequents contain either 0 or 1 formula.

Let Σ be a given first-order signature over S, let Δ be a finite set of Σ-formulas, and let B be a Σ-formula.

Write $\Sigma; \Delta \vdash_C B$ and $\Sigma; \Delta \vdash_I B$ if the sequent $\Sigma : \Delta \longrightarrow B$ has, respectively, a **C**-proof or an **I**-proof.

Some Exercises

Provide a **C**-proof only if there is no **I**-proof.

1. $[p \wedge (p \supset q) \wedge ((p \wedge q) \supset r)] \supset r$
2. $(p \supset q) \supset (\neg q \supset \neg p)$
3. $(\neg q \supset \neg p) \supset (p \supset q)$
4. $p \vee (p \supset q)$
5. $((r\ a \wedge r\ b) \supset q) \supset \exists x(r\ x \supset q)$
6. $((p \supset q) \supset p) \supset p$ (Pierce's formula)
7. $\exists y \forall x\ (r\ x \supset r\ y)$
8. $\forall x \forall y\ (s\ x\ y) \supset \forall z\ (s\ z\ z)$

N.B. Negation is defined: $\neg B = (B \supset f)$.